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Generalization of the Chekanov Theorem. Diameters of Immersed Manifolds and Wave Fronts

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INTRODUCTION

The Chekanov theorem [8] generalizes the classical Lyusternik–Shnirel'man and Morse theorems concerning critical points of a smooth function on a closed manifold. A Legendrian submanifold Λ of the space of 1-jets of the functions on a manifold M defines a many-valued function whose graph is the projection of Λ in $J^0M = M \times \mathbf{R}$. The Chekanov theorem asserts that if Λ is homotopic to the 1-graph of a smooth function in the class of embedded Legendrian manifolds, then such a graph of a many-valued function must have a lot of points (their number is determined by the topology of M) at which the tangent plane to the graph is parallel to $M \times 0$.

In the present paper a similar estimate is proved for a wider class of Legendrian manifolds. We consider Legendrian manifolds which are homotopic (in the class of embedded Legendrian manifolds) to Legendrian manifolds specified by generating families. Another generalization of the Chekanov theorem can be found in [13] (see also [10, 12, 17]).

As in [13], the proposed generalization of the Chekanov theorem is applied to investigating the wave front geometry. An immersed manifold M in $\mathbf{R}^n$ determines a Legendrian submanifold in the space $ST^*\mathbf{R}^n$ of the spherization of the cotangent bundle of $\mathbf{R}^n$ which can be defined by a generating family. The explicit form of the generating family makes it possible to obtain lower-bound estimates for the number of diameters (the diameter of M is a segment connecting different points of M and orthogonal to M at its endpoints) of the immersed manifolds M. The estimates obtained are stronger than those in [16]. It is proved that the number of diameters (counted with the multiplicities) is minorated by $\frac{1}{2}(B^2 + (\dim M - 1)B)$, where $B = \sum \dim H_*(M, \mathbf{Z}_2)$.

The problem of diameters (double normals) of a generic immersed submanifold in an Euclidean space was considered by Takens and White in [16]. It was proved that the number of diameters of a closed manifold M^k of dimension k embedded in an Euclidean space is at least

$$TW(M^k, K) = 2\left(\left[\frac{d_0}{2}\right] + \left[\frac{d_1 - d_0}{2}\right] + \left[\frac{d_2 - d_1 + d_0}{2}\right] + \dots \right.$$
$$\left. \dots + \left[\frac{d_{2k} - d_{2k-1} + d_{2k-2} - \dots + d_0}{2}\right]\right) - \left[\frac{d_{2k} - d_{2k-1} + d_{2k-2} - \dots + d_0}{2}\right].$$

Here, $d_i = \dim H_i(M^k \times M^k, \Delta; K)$, Δ is the diagonal in $M^k \times M^k$, K is the field of coefficients, and $[a]$ is the smallest integer no less than a. The Takens–White estimate satisfies the inequalities

$$\frac{1}{2}\left(B_K^2 - B_K\right) + 2k \geq TW(M^k, K) \geq \frac{1}{2}\left(B_K^2 - B_K\right).$$

Let us compare our estimate with that obtained by Takens and White. For an immersed oriented surface of genus g we estimate the diameters by the number $2g^2 + 5g + 3$, while the Takens–White estimate (for embeddings) gives $2g^2 + 3g + 3$. For example, for an immersion of two-dimensional torus in an Euclidean space our formula guarantees at least 10 diameters counted with their multiplicities, while the Takens–White estimate (for embeddings) guarantees no more than 8. For two-dimensional torus our estimate is sharp and reached in the class of embeddings of torus in three-dimensional space. For $M = S^k \times S^n$ we guarantee $2(n+k)+6$ diameters, and this estimate is sharp. For instance, for $S^7 \times S^7$ our formula gives 34 diameters, while the Takens–White estimate provides only 20.

The generalization of the Chekanov theorem makes it possible to extend the estimates to a wider class of hypersurfaces in $\mathbf{R}^n$, namely, wave fronts obtained from a hypersurface M in $\mathbf{R}^n$ by means of a deformation without (dangerous) selfcontacts.

In Sec. 1 the generalization of the Chekanov theorem is proved. In Sec. 2 the number of self-intersection points of the projection in T^*M of a Legendrian submanifold (of the class considered) of the space J^1M is estimated. In Sec. 3 the number of diameters of an immersed submanifold is estimated. In Sec. 4 the estimates obtained in Sec. 3 are extended to wave fronts. In Sec. 5 the accuracy of the estimates in Sec. 3 and Sec. 4 is discussed.

In this paper the results announced in [14, 15] are proved.

1. GENERALIZATION OF THE CHEKANOV THEOREM. CRITICAL POINTS OF QUASIFUNCTIONS

1.1. Formulation of the Basic Theorem. Let $p : E \to M$ be a bundle. The generic function h on E determines a Legendrian manifold in the space J^1M of 1-jets of the functions on a manifold M. The manifold of critical points along the fiber can naturally be mapped into J^1M (the differential h "along the base" which is well-posed in the critical point of the restriction of the function h along the fiber and the value of the function h are related to the critical point). The pairs obtained form the Legendrian manifold $\Lambda \subset J^1M$.

Definition 1.1. A Legendrian manifold homotopic to Λ in J^1M in the class of Legendrian embeddings is said to be an *E-quasifunction.*

Definition 1.2. *Critical points* of the Legendrian manifold Λ in J^1M are such points of Λ whose images belong to the zero section under the natural mapping $\rho_M : J^1M \to T^*M$. Critical points correspond to the points of the front (graph of the E-quasifunction) at which the tangent plane is parallel to $M \times 0$. We will call a critical point *nondegenerate* if $\rho_M(\Lambda)$ is transversal to the zero section at the image of the critical point. We now formulate the basic theorem.

Theorem 1.1. *Suppose Λ is a many-valued E-quasifunction on a closed manifold M and the bundle fiber $E \to M$ is compact. Then the number of critical points of Λ is at least*

(a) $\sum\limits_{i=0}^{\dim E} (b_i(E) + 2q_i(E))$, *where* $b_i(E) = \dim H_i(E, \mathbf{R})$ *and* $q_i(M)$ *is the minimum number of generators of the group* $\operatorname{Tors} H_i(E, \mathbf{Z})$, *if the critical points of* Λ *are not degenerate;*

(b) $\sum b_i(E, F)$, F *is the field, if the critical points of* Λ *are not degenerate;*

(c) $\operatorname{cl}(E, A) + 1$. *Here* $\operatorname{cl}(E, A)$ *is the cohomology length of the manifold* E *with coefficients in a commutative ring* A (see [4]).

Global properties of manifolds and symplectomorphisms given by a generating family were studied, for instance, in [10, 18].

1.2. Necessary Notation. We denote the canonical mapping $T^*B \to B$ by π_B and the projection $J^1B \to T^*B$ (forgetting the value) by ρ_B.

Suppose $p : E \to M$ is a smooth bundle with the fiber W, $E_0 \subset T^*E$ is a subbundle of the bundle $\pi_E : T^*E \to E$ formed by covectors whose restriction to the tangent space of the fiber is equal to zero, and π_ρ is the projection of E_0 in E. We denote the natural mapping of the bundle E_0 over T^*M ($E_0 \times \mathbf{R}$ over J^1M) by p_0 (p_1).

The described mappings can be naturally unified in the commutative diagram

$$\begin{tikzcd}
 & T^*E \times \mathbf{R} = J^1E \arrow[rr, "\rho_E"] & & T^*E \arrow[dr, "\pi_E"] & \\
E_0 \times \mathbf{R} \arrow[ur] \arrow[rr, "\rho_{E_0}"] \arrow[d, "p_1"] & & E_0 \arrow[ur] \arrow[rr, "\pi_\rho"] \arrow[d, "p_0"] & & E \arrow[d, "p"] \\
T^*M \times \mathbf{R} = J^1M \arrow[rr, "\rho_M"] & & T^*M \arrow[rr, "\pi_M"] & & M
\end{tikzcd}$$

1.3. Definition of the Generating Family. Suppose $p : E \to M$ is a bundle and $F : E \to \mathbf{R}$ is a smooth function such that the intersection of j^1F and $E_0 \times \mathbf{R}$ is transversal. Then $\Lambda = p_1 (j^1F \cap (E_0 \times \mathbf{R}))$ is an (immersed) Legendrian manifold. When the transversality condition is satisfied, F is called a *generating family* of the Legendrian manifold Λ. We then say that a generating family $\widetilde{F} : \widetilde{E} \to \mathbf{R}$ is a *stabilization* of a generating family $F : E \to \mathbf{R}$, if $\widetilde{E} = E \times \mathbf{R}^N$ and $\widetilde{F} = F + Q$, where Q is a nondegenerate quadratic form on $\mathbf{R}^N$. A generating family $\widetilde{F} : E \times \mathbf{R}^N \to \mathbf{R}$ will be called *quadratic at infinity* if it is the sum of a quadratic form on each fiber $\mathbf{R}^N$ and a function with bounded differential. We will sometimes denote the generating family F by $F(x, q)$, where q is a point of M and x is a point of the fiber of the bundle p.

1.4. Proof of Theorem 1.1. A function defined on the space of a vector bundle will be called *quadratic at infinity* if this function is the sum of a nondegenerate quadratic form along the fiber and a function with bounded differential. Then Theorem 1.1 directly follows from the following assertions.

Theorem 1.2. *Suppose* M *is a closed manifold and* $p : E \to M$ *is a bundle with a compact fiber. Then an* E*-quasifunction* Λ *on* M *can be defined by the generating family* $F : E \times \mathbf{R}^{N(\Lambda)} \to \mathbf{R}$ *quadratic at infinity.*

Theorem 1.3. *Suppose V is a vector bundle over a closed manifold E and $f : V \to \mathbf{R}$ is a function quadratic at infinity. Then the number of the critical points of the function f is at least*

(a) $\sum\limits_{i=0}^{\dim E} (b_i(E) + 2q_i(M))$ *if all the critical points are Morse;*

(b) $\sum b_i(E, F)$ *if all the critical points are Morse and F is a field of coefficients;*

(c) $\mathrm{cl}(E, A) + 1$.

Theorem 1.3 can be deduced from the well-known result obtained by Conley and Zehnder [11] (in [11] this theorem was proved for $E = T^n$, but the proof is valid in the general case). Theorem 1.2 can conveniently be proved in the following formulation.

Theorem 1.4. *Suppose $\Lambda_0 \subset J^1M$ is a Legendrian manifold defined by a generating family $F : E \to \mathbf{R}$ and the fiber of the bundle E is compact. Let $\{G^t\}_{t\in[0,1]}$ be a smooth family of contactomorphisms of the space J^1M, $G^0 = \mathrm{id}$. Then $\Lambda_t = G^t(\Lambda_0)$ can be defined by the generating family $\widetilde{F_t} : E \times \mathbf{R}^N \to \mathbf{R}$ quadratic at infinity.*

Proof. Suppose k_t is the contact Hamiltonian of the field $v_t = \left.\frac{\partial G^\tau}{\partial \tau}\right|_{t=\tau}$. Let us consider $K_t = p_1^* k_t : E_0 \times \mathbf{R} \to \mathbf{R}$ and extend K_t to a function $\widetilde{K_t} : J^1E \to \mathbf{R}$. Then $E_0 \times \mathbf{R}$ is invariant with respect to the flow $\tilde{G}^t$ given by the contact Hamiltonian $\widetilde{K_t}$,

$$p_1(\widetilde{G_t}(j^1F)) \cap (E_0 \times \mathbf{R}) = G^t(\Lambda_0) = \Lambda_t\,, \tag{$*$}$$

and $\tilde{G}^t(j^1F)$ intersects $E_0 \times \mathbf{R}$ transversally. The function $\widetilde{K_t}$ can be improved so that it becomes finite, and relation $(*)$ and the transversality condition are satisfied.

We can apply the Chekanov theorem ([8], Theorem 3.1) to the manifold j^1F and the flow $\tilde{G}^t$ determined by the contact Hamiltonian $\widetilde{K_t}$. The function $\widetilde{K_t}$ should be improved since the Chekanov theorem is valid for flows with compact support.

Consequently, the manifold $\tilde{G}^t(j^1F)$ is defined by a generating family $F_t : E \times \mathbf{R}^{N(t)} \to \mathbf{R}$ quadratic at infinity. Accordingly, $G^t(\Lambda_0)$ is also defined by the generating family F_t'. The proof is completed.

Remark 1.1. We can choose the number $N(t)$ to be independent of $t \in [0, 1]$.

A Legendrian manifold $\Lambda_1 \subset J^1M$ will be called a *good E-quasifunction* on M if Λ_1 is homotopic in the class of Legendrian embeddings to a Legendrian manifold Λ defined by a generating family $F : E \times \mathbf{R}^N \to \mathbf{R}$ quadratic at infinity.

Remark 1.2. Theorems 1.1 and 1.2 can be extended to good E-quasifunctions.

Example 1.1. Let us consider a Legendrian manifold Λ of the space of 1-jets of the functions on a circle whose projections in T^*S^1 and J^0S^1 are shown in Fig. 1. This Legendrian manifold can readily be defined by a generating family with a two-dimensional torus as the total space. All the Legendrian manifolds homotopic to this manifold in the class of Legendrian embeddings have at least four critical points with multiplicities. In the process of homotopy it is sufficient to have a single selfintersection and we can obtain a Legendrian manifold Λ_1 homotopic (in the class of Legendrian embeddings) to a manifold Λ_2 without critical points (projections of Λ_2 are shown in Fig. 2).

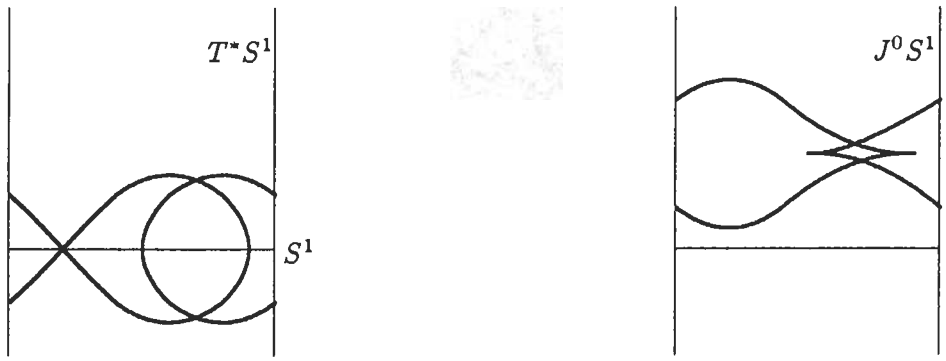

Fig. 1. Projections of $S^1 \times S^1$-quasifunction

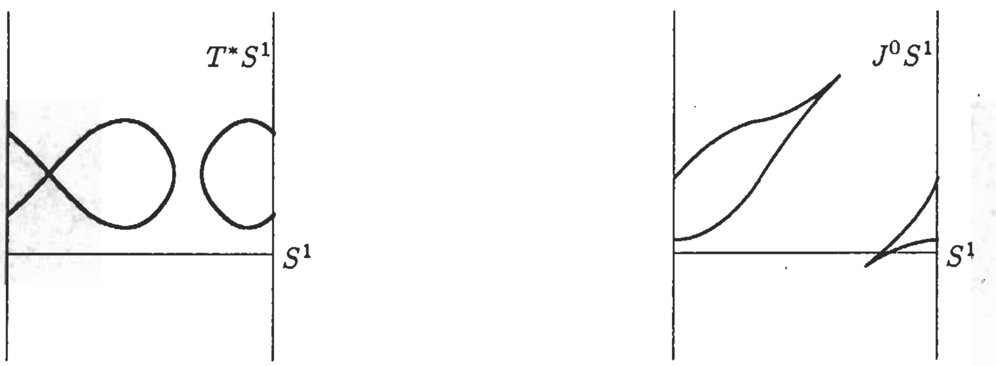

Fig. 2. Projections of a nonquasifunction

2. LAGRANGIAN SELFINTERSECTIONS

Suppose Λ is a Legendrian submanifold of J^1M, then $\rho_M(\Lambda)$ is an immersed Lagrangian submanifold of T^*M. If Λ is defined by a generating family $F : E \to \mathbf{R}$ (E is a bundle over M), then the number of selfintersection points $\rho_M(\Lambda)$ can be estimated in terms of the Betti numbers of Λ and E.

We need the following modification of Theorem 1.3.

Theorem 2.1. *Suppose V is the total space of a vector bundle over a closed manifold M, $g : V \to \mathbf{R}$ is a function quadratic at infinity, L is a Bott manifold of the function g lying at the level $g = C$, and the other critical points of g are Morse points. Then the number of Morse critical points of g is at least $|\sum b_i(M, \mathbf{Z}_2) - \sum b_i(L, \mathbf{Z}_2)|$.*

Proof. Without loss of generality, we can assume that all the critical values of g are different. Let us consider the function

$$\alpha(t) = \dim H_*(\{g \leq C + \varepsilon + t\}, \{g \leq C - t\}, \mathbf{Z}_2)$$

for a sufficiently small ε at $t > 0$.

From the Morse theory (see, e.g., [5]) it follows that $\alpha(t)$ is changed by ± 1 only when $C + \varepsilon + t$ or $C - t$ pass through critical values. Consequently, the number of the critical points is no less than $|\alpha(t_1) - \alpha(t_0)|$, where positive numbers t_1 and t_0 are sufficiently large and sufficiently small, respectively. The bundle V can be decomposed into the sum of the bundles V_+ and V_-, and the function g can be represented as the sum of a function on V_+ (a fiberwise positively defined quadratic form), a function on V_- (a fiberwise negatively defined quadratic form), and a function on V with bounded differential. From the Morse theory it follows that $\alpha(t_1)$ is equal to the dimension of the space of $\mathbf{Z}_2$-homologies of the Thom space of the bundle V_-. From the Thom isomorphism we obtain that $\alpha(t_1) = \sum b_i(M, \mathbf{Z}_2)$. From the Bott theorem (see [9]) it follows that $\alpha(t_0) = \sum b_i(L, \mathbf{Z}_2)$. The theorem is proved.

Let us consider a bundle $E^2 \to M$ induced by the diagonal embedding M in $M \times M$ from the bundle $E \times E \to M \times M$. The bundle fiber E^2 is the pullback of the bundle fiber E.

Theorem 2.2. *Suppose $\Lambda \subset J^1M$ is an E-quasifunction and $E \to M$ is a bundle with a compact fiber. Then the number of selfintersection points (counted with the multiplicities) of the projection $\rho_M(\Lambda)$ is at least $\frac{1}{2}\left|\sum b_i(E^2, \mathbf{Z}_2) - \sum b_i(\Lambda, \mathbf{Z}_2)\right|$.*

Proof. As follows from Theorem 1.2, Λ can be defined by the generating family $F\colon E\times\mathbf{R}^N\to\mathbf{R}$ quadratic at infinity. We consider the bundle $\widetilde{E} \to M$ induced by the diagonal embedding of M in $M \times M$ from the bundle $(E \times \mathbf{R}^N) \times (E \times \mathbf{R}^N) \to M \times M$. The fiber $\widetilde{E}$ is the pullback of the bundle fiber E multiplied by $\mathbf{R}^N$. On $\widetilde{E}$ we define the function $\widetilde{F}(x, y, q) = F(x, q) - F(y, q)$, where x and y are points of the bundle fiber $E \times \mathbf{R}^N \to M$, $q \in M$. The selfintersection points of the projection are in one-to-one correspondence with the critical points of the function $\widetilde{F} : \widetilde{E} \to \mathbf{R}$ with a positive critical value.

Lemma 2.1. *Suppose $F_1(x, q)$ and $F_2(y, q)$ are generating families of a Legendrian manifold $\Lambda \subset J^1M$. Then critical points of the function $F_1(x, q) - F_2(y, q)$ with zero critical value form a Bott manifold diffeomorphic to Λ.*

Proof. The assertion that the manifold of critical points with zero critical value is diffeomorphic to the manifold Λ is obvious. The assertion that this manifold is a Bott one is local. We will prove this assertion. Suppose $g_1(x, q)$ is a generating family of a germ of the Legendrian manifold Λ. The generating family $g_2(x, z, q) = g_1(x, q) + Q(z)$ will be called the stabilization of g if Q is a nondegenerate quadratic form. Two generating families $g_1(x, q)$ and $g_2(x, q)$ of the germ of the Legendrian manifold Λ are fiber equivalent if $g_1(x, q) = g_2(h(x, q), q)$, $(x, q) \to (h(x, q), q)$ is a diffeomorphism. Two generating families will be called stably fiber equivalent if these families become equivalent after stabilization. The assertion that the generating families F_1 and F_2 are of the Bott type remains valid when F_1 and F_2 are replaced by stably fiber equivalent generating families. According to [3], if g_1 and g_2 are generating families of the same germ of a Legendrian manifold, then g_1 is stably fibered equivalently to g_2. As F_1 or F_2 we take the following generating family. Suppose that at a considered point y the dimension of the kernel of the projection $\rho_M(\Lambda)$ in M is equal to k. Then there exist such canonical coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ in T^*M that the kernel of the projection coincides with $\left\langle \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_k} \right\rangle$ and the manifold $\rho_M(\Lambda)$ at the point $\rho_M(y)$ is tangent to $\left\langle \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_k}, \frac{\partial}{\partial p_{k+1}}, \ldots, \frac{\partial}{\partial p_n} \right\rangle$. Then the germ of Λ can be defined by the generating family $\sum_{i=1}^{k} x_i y_i + S(x_1, \ldots, x_k, q_{k+1}, \ldots, q_n)$, where $S = C + o(|x, q|^2)$. For this generating family we can readily show that the manifold is of the Bott type. The lemma is proved.

Let us continue the proof of Theorem 2.2. The points with zero critical value form a Bott manifold which is diffeomorphic to Λ according to Lemma 2.1. In the general position all the critical points of $\widetilde{F}$ with nonzero critical value are Morse points. The number of the selfintersection points is half the number of the Morse critical points. By virtue of Theorem 2.1, we obtain the estimate required. The proof is complete.

From Theorem 2.2 the following corollary is valid.

Corollary 2.1. *Let $E = M \times W$ be a trivial bundle and Λ an E-quasifunction. Then the number of selfintersection points of the projection Λ in T^*M (in the general position) is at least* $\frac{1}{2}[\sum b_i(M)\,(\sum b_i(W))^2 - \sum b_i(\Lambda)]$.

Example 2.1. Let us consider a Legendrian manifold Λ of the space of 1-jets of the functions on a circle given as an example in the previous section. The projections of all the Legendrian knots from a component containing Λ in T^*S^1 have at least three selfintersection points counted with their multiplicities. In the process of homotopy it is sufficient to have a single selfintersection and we can obtain a Legendrian manifold Λ_1 homotopic (in the class of Legendrian embeddings) to a manifold Λ_2 which has a single selfintersection point of the projection in T^*S^1 (see Fig. 2).

3. MINORATION OF THE NUMBER OF DIAMETERS OF MANIFOLDS IMMERSED IN $\mathbf{R}^N$

Suppose f is an immersion of a manifold M^n in $\mathbf{R}^{n+k}$. A segment connecting two different points $f(x)$ and $f(y)$ of the immersion and perpendicular to the tangent planes at these points is said to be the *diameter* of an immersed manifold $f(M^n)$. We formulate the principal result.

Theorem 3.1. *Suppose M^n is a closed manifold of dimension n and $B = \sum dim H_*(M, \mathbf{Z}_2)$. Then for a generic immersion the number of diameters of M^n in $\mathbf{R}^{n+k}$ is at least $\frac{1}{2}(B^2 + (n-1)B)$.*

In all particular examples our estimate is no worse than that in [16]. We give some corollaries of Theorem 3.1.

Corollary 3.1. *For generic immersion of a manifold M in $\mathbf{R}^n$ the number D of diameters is at least*

(a) $M = S^n$, $D \geq n + 1$;

(b) $M = T^n$, $D \geq 2^{2n-1} + (n-1)2^{n-1}$;

(c) $M = S_g^2$ *(oriented surface of genus g)*, $D \geq 2g^2 + 5g + 3$;

(d) $M = \mathbf{R}P^n$, $D \geq n^2 + n$.

The proof is the substitution of Theorem 3.1 to the formula.

Remark 3.1. The condition of genericity in Theorem 3.1 and below means that all the critical points of functions considered are Morse and (self)intersections are transversal. We can show that this is actually the condition of the general position (cf. [16]).

Before proving Theorem 3.1 we formulate certain convenient assertions.

Consider a function $F : S^{n+k-1} \times M \times M \to \mathbf{R}$, $F(x, \xi_1, \xi_2) = \langle x, f(\xi_1) - f(\xi_2)\rangle$. Here S^{n+k-1} is the sphere of radius 1 in $\mathbf{R}^{n+k}$ with its center at the origin, and $\langle \cdot, \cdot \rangle$ is a scalar product in $\mathbf{R}^{n+k}$.

Lemma 3.1. *Suppose f is a generic immersion and the point (x', ξ_1', ξ_2') is critical for the function $F = \langle x, f(\xi_1) - f(\xi_2)\rangle$. If $f(\xi_1') \neq f(\xi_2')$, then the segment $[f(\xi_1'), f(\xi_2')]$ is a diameter and $F(x', \xi_1', \xi_2') \neq 0$.*

Proof. Differentiating F with respect to $\xi_i (i = 1, 2)$ at the point (x', ξ_1', ξ_2'), we obtain $x' \perp f_*(T_{\xi_i'} M^n)$. Consequently, $f(\xi_1') \neq f(\xi_2')$ since the selfintersection is transversal. On the other hand, differentiating with respect to x, we obtain that x' is proportional to $f(\xi_1') - f(\xi_2')$. Hence, $[f(\xi_1'), f(\xi_2')]$ is a diameter. Since x' is proportional to $f(\xi_1') - f(\xi_2')$, we have $F(x', \xi_1', \xi_2') \neq 0$.

Thus, we construct a mapping of the set of critical points of F into diameters. We can readily verify that this is the mapping surjective and that each diameter corresponds to exactly four critical points of F (in the case of generic immersion). Namely, the following assertion is valid.

Assertion 3.1. *Suppose that $[f(\xi_1'), f(\xi_2')]$ is a diameter of the manifold $f(M^n)$. Then $\left(\pm\frac{f(\xi_1')-f(\xi_2')}{\|f(\xi_1')-f(\xi_2')\|}, \xi_1', \xi_2'\right)$ and $\left(\pm\frac{f(\xi_1')-f(\xi_2')}{\|f(\xi_1')-f(\xi_2')\|}, \xi_2', \xi_1'\right)$ are critical points of the function $F = \langle x, f(\xi_1) - f(\xi_2)\rangle$.*

The following computation of homologies is the key result for establishing the estimate of Theorem 3.1.

Lemma 3.2. *Suppose M is a closed manifold and l is an involution on $S^N \times M \times M$, $l(x, y, z) = (-x, z, y)$. Then the sum of the Betti numbers of the factor space $S^N \times M \times M$ with respect to the action l is equal to $B^2 + NB$.*

Proof. Suppose (α^i) is a cell division of M, σ^0, $\widehat{\sigma}^0$, σ^1, $\widehat{\sigma}^1, \ldots, \sigma^N$, $\widehat{\sigma}^N$ is the standard cell division of S^N invariant with respect to the antipodal involution, and σ^i, $\widehat{\sigma}^i$ are the cells of dimension i. Then $S^N \times M \times M$ is decomposed into the cells $\sigma^k \times \alpha^i \times \alpha^j$ and $\widehat{\sigma}^k \times \alpha^i \times \alpha^j$. The involution l transfers the cell $\sigma^k \times \alpha^i \times \alpha^j$ to the cell $\widehat{\sigma}^k \times \alpha^j \times \alpha^i$. This cell division $S^N \times M \times M$ induces a cell division in $S^N \times M \times M/_l$ since it is invariant with respect to the involution l. We denote the corresponding complex of cell chains (with $\mathbf{Z}_2$-coefficients) of the space $S^N \times M \times M/_l$ by $C_*(S^N \times M \times M/_l)$.

Suppose ∂ is a boundary operator in $C_*(M)$, ∂_1 is a boundary operator in $C_*(M \times M)$, and ∂_2 is a boundary operator in $C_*(S^N \times M \times M/_l)$. We identify (formally) the cells $S^N \times M \times M/_l$ with the cells $\sigma^k \times \alpha^i \times \alpha^j$. Then $\partial_2(\sigma^k \times \alpha^i \times \alpha^j) = \sigma^k(\partial_1(\alpha^i \times \alpha^j)) + \sigma^{k-1}(\alpha^i \times \alpha^j + \alpha^j \times \alpha^i)$ (we assume that $\sigma^{-1} = 0$).

Let us consider the case $\partial = 0$. Here the complex of the cell chains $C_*(S^N \times M \times M/_l, \mathbf{Z}_2)$ is graded by the dimension of the cell in the sphere S^N, since from $\partial = 0$ there follows $\partial_1 = 0$. We denote homologies with respect to this graduation by H_k. We now calculate the dimension of H_k. In this graduation the complex $C_*(S^N \times M \times M/_l)$ is as follows: for any k, $0 \leq k \leq N$, $C_k(S^N \times M \times M/_l, \mathbf{Z}_2)$ is isomorphic to $C_*(M \times M, \mathbf{Z}_2)$, and the boundary operator is the mapping of symmetrization s, $s(\alpha^i \otimes \alpha^j) = \alpha^i \otimes \alpha^j + \alpha^j \otimes \alpha^i$. We have $\dim \ker s = \frac{1}{2}(B^2 + B)$, $\dim \operatorname{im} s = \frac{1}{2}(B^2 - B)$. $H_N = \ker s$, consequently $\dim H_N = \frac{1}{2}(B^2 + B)$. $H_k = \ker s/_{\operatorname{im} s}$ for $0 < k < N$, consequently $\dim H_k = B$ for $0 < k < N$. $H_0 = C_*(M, \mathbf{Z}_2) \otimes C_*(M, \mathbf{Z})/_{\mathrm{im}\, s}$, consequently $\dim H_0 = \frac{1}{2}(B^2 + B)$. (From $\partial = 0$ it follows that $\dim C_*(M, \mathbf{Z}_2) \otimes C_*(M, \mathbf{Z}_2) \simeq H_*(M, \mathbf{Z}_2) \otimes H_*(M, \mathbf{Z}_2)$.)

We now consider the case $\partial \neq 0$. We reduce it to the case $\partial = 0$. Using the given cell division of M, we construct a cell space M' such that the complexes $C_*(S^N \times M \times M/_l, \mathbf{Z}_2)$ and $C_*(S^N \times M' \times M'/_l, \mathbf{Z}_2)$ are isomorphic and $\dim H_*(S^N \times M' \times M'/_l, \mathbf{Z}_2) = B^2 + NB$. This suggests that if $\partial \neq 0$, the lemma is valid. Any complex with the coefficients in a field can be decomposed into the sum of a (trivial) complex of homologies and two-term exact complexes.

We now consider the space M'' which is the union of spheres (wedge product) in which the number of the spheres of dimension k is equal to $\dim H_k(M, \mathbf{Z}_2)$. M' is the wedge of M'' and the disks corresponding to short exact complexes. Then the spaces $S^N \times M' \times M'/_l$ and $S^N \times M'' \times M''/_l$ are homotopically equivalent since $S^N \times M'' \times M''/_l$ is the strong deformation retract of $(S^N \times M' \times M')/_l$, consequently, their homologies are identical. The case $S^N \times M'' \times M''/_l$ is considered for $\partial = 0$. The lemma is proved.

Lemma 3.3. *Suppose $p : E \to B$ is a bundle with the fiber $\mathbf{R}P^k$ (S^k). Then $\sum b_i(E, \mathbf{Z}_2) \leq (k+1) \sum b_i(B, \mathbf{Z}_2)$ ($\sum b_i(E, \mathbf{Z}_2) \leq 2 \sum b_i(B, \mathbf{Z}_2)$).*

Proof. Let us consider the case $k > 0$. Here the bundle with the fiber $\mathbf{R}P^k$ (S^k) is $\mathbf{Z}_2$-homologically simple (see [6]), since the homologies of fiber in any dimension are, at most, one-dimensional. Consequently, the spectral sequence (with the coefficients from $\mathbf{Z}_2$) calculates the homologies of the total space E. The dimension of the term E_2 of this spectral sequence is equal to $(k+1) \sum b_i(B, \mathbf{Z}_2)$ $(2 \sum b_i(B, \mathbf{Z}_2))$. The dimension of homologies of the total space E is not greater than the dimension of the term E_2 of the spectral sequence.

For $k = 0$ (the fiber S^0) the Smith theory (see [7]) gives the estimate required. The proof is completed.

Proof of Theorem 3.1. The function $F(x, \xi_1, \xi_2)$ is invariant with respect to the action of the involution on $S^{n+k-1} \times M^n \times M^n$, $(x, \xi_1, \xi_2) \to (-x, \xi_2, \xi_1)$. The involution acts without fixed points, consequently, the quotient set of this involution is a smooth manifold (we denote this manifold as $S^{n+k-1} \times M^n \times M^n/_{\mathbf{Z}_2}$). On the quotient the function F determines a smooth function $\widetilde{F}$.

Lemma 3.4. *For a generic immersion the critical points of the function $\widetilde{F}$ (function F) with zero critical value form a Bott manifold diffeomorphic to the projectivization (spherization) of the normal bundle of a manifold.*

Remark 3.2. Here, the genericity condition is the transversality of selfintersection.

Proof of Lemma 3.4. From Lemma 3.3 it follows that the critical points with zero critical value of the function F are contained in $S^{n+k-1} \times \Delta \subset S^{n+k-1} \times M^n \times M^n$ (Δ is the diagonal in $M^n \times M^n$). These points form a Bott manifold diffeomorphic to the spherization of the normal bundle. A point (x, ξ, ξ) is critical if and only if $x \perp f_*(T_\xi(M^n))$. The fact that this manifold is a Bott one can be verified in the local coordinates. The lemma is proved.

We now continue the proof of the theorem.

From 3.3 and 3.4, it follows that the number of diameters is not less than half the number of critical points of the function $\widetilde{F}$ with nonzero critical value. The critical points of the function $\widetilde{F}$ with zero critical value form a Bott manifold diffeomorphic to $P\nu(M^n)$, which is the projectivization of the normal bundle (see Lemma 3.4). Thus, from the Morse theory [4] it follows that for a Bott function on a closed manifold the sum of the Betti numbers (with $\mathbf{Z}_2$-coefficients) of all Bott manifolds is no less than the sum of the Betti numbers (with $\mathbf{Z}_2$-coefficients) of the manifold. Accordingly,

$$2D + \sum b_i(P\nu(M^n), \mathbf{Z}_2) \geq \sum b_i \left(S^{n+k-1} \times M^n \times M^n /_{\mathbf{Z}_2}, \mathbf{Z}_2 \right),$$

where D is the number of diameters.

We set $B = \sum b_i(M^n, \mathbf{Z}_2)$. From Lemma 3.2 it follows that

$$\sum b_i \left(S^{n+k-1} \times M^n \times M^n /_{\mathbf{Z}_2}, \mathbf{Z}_2 \right) = B^2 + (n+k-1)B.$$

Lemma 3.3 implies that $\sum b_i(P\nu(M^n), \mathbf{Z}_2) \leq kB$. Thus, $D \geq \frac{1}{2}(B^2 + (k-1)B)$, which proves the theorem.

4. DIAMETERS OF WAVE FRONTS

The aim of this section is to extend the domain of application of the estimates obtained in Sec. 3. Our estimates of the number of diameters of smooth hypersurfaces are conserved under the deformation of the hypersurfaces in the wave front class. For this it is necessary to require that the type of the corresponding Legendrian knot do not change under deformation.

We recall some standard facts of contact geometry.

Suppose B is a smooth manifold. A (cooriented) hyperplane in the tangent space at a given point is called a (cooriented) contact element of the manifold B applied at this point. All the (cooriented) contact elements of B form the space PT^*B (ST^*B) of a projectivized cotangent bundle (spherization of the cotangent bundle). The space PT^*B (ST^*B) has a contact structure defined canonically (see [2]).

An immersed submanifold X with transversal selfintersections in B defines a Legendrian submanifold $P(X) \subset PT^*B$ ($S(X) \subset ST^*B$) representing a set of (cooriented) contact elements tangent to X (see [2]).

A (cooriented) wave front in B is the projection of a Legendrian submanifold PT^*B (ST^*B) in B. For a generic Legendrian submanifold in PT^*B (ST^*B) its (cooriented) wave front in B is a singular stratified (cooriented) hypersurface which at any point has a (cooriented) tangent plane. A generic Legendrian submanifold in PT^*B (ST^*B) can be uniquely restored from its wave front in B. We will identify a Legendrian submanifold PT^*B (ST^*B) with its wave front.

Consider a closed immersed submanifold L with transversal selfintersections in the Euclidean space $\mathbf{R}^{n+1}$.

Definition 4.1. A wave front Σ is called a *Chekanov wave front of type L* if Σ is a wave front of a Legendrian submanifold W of the space $PT^*\mathbf{R}^{n+1}$, where W is Legendrian isotopic to the manifold $P(L) \subset PT^*\mathbf{R}^{n+1}$.

A segment connecting two different points of the wave front and perpendicular to it at its endpoints is said to be a diameter of the wave front in $\mathbf{R}^{n+1}$.

Theorem 4.1. *The number of diameters of a Chekanov wave front of type L^k, counted with the multiplicities, is at least $\frac{1}{2}(B^2 - B) + \frac{kB}{2}$.*

Before proving Theorem 4.1 we formulate the following assertion (see [1]).

Proposition 4.1. *The space $ST^*\mathbf{R}^{n+1}$ is contactomorphic to J^1S^n.*

Let us give an idea of the proof. We identify S^n with the standard unit sphere $||x|| = 1$ in $\mathbf{R}^{n+1}$ and the cotangent vector to S^n with a vector perpendicular to x (using the metric). Then to a point (u, p, q) from $J^1S^n = \mathbf{R} \times T^*S^n$ we relate a (cooriented) contact element parallel to x at the point $uq + p$. We obtain a mapping from J^1S^n onto $ST^*\mathbf{R}^{n+1}$. We can verify that this is a contactomorphism. In what follows, using the contactomorphism, we will identify J^1S^n with $ST^*\mathbf{R}^{n+1}$ when necessary.

Lemma 4.1. *Suppose X is an immersed submanifold with transversal selfintersections in $\mathbf{R}^{n+1}$. Then $S(X)$ can be defined by the generating family $F : S^n \times X \to \mathbf{R}$, $F(q, x) = \langle q, x\rangle$.*

Proof. This is an assertion of the support function theory. In the Lagrangian case it can be found in [3].

Lemma 4.1 and Proposition 4.1 give many natural examples of Legendrian submanifolds in J^1S^n defined by generating families.

Proof of Theorem 4.1. A Chekanov wave front $\Sigma \subset \mathbf{R}^{n+1}$ of type L^k is the projection in $\mathbf{R}^{n+1}$ of a Legendrian submanifold Λ of the space $PT^*\mathbf{R}^{n+1}$ which is Legendrian isotopic to $P(L^k) \subset PT^*\mathbf{R}^{n+1}$. Consider the natural mapping of "forgeting the coorientation of a contact element" of the space $ST^*\mathbf{R}^{n+1}$ in the space $PT^*\mathbf{R}^{n+1}$. For this mapping the preimage of the manifold Λ is a Legendrian submanifold $\tilde{\Lambda}$ of space $ST^*\mathbf{R}^{n+1}$ which is Legendrian isotopic to $S(L^k) \subset ST^*\mathbf{R}^{n+1}$. According to Theorem 1.2 and Lemma 4.1, the Legendrian manifold $\tilde{\Lambda} \subset ST^*\mathbf{R}^{n+1} = J^1S^n$ can be given by the generating family $F(q, \xi, z)$ $(q \in S^n, \xi \in L, z \in \mathbf{R}^N)$ which is quadratic at infinity.

For a generic Chekanov wave front $\Sigma \subset \mathbf{R}^{n+1}$ of type L^k the number of diameters Σ is equal just to a quarter of the number of critical points of the function

$$F_1 : S^n \times L \times \mathbf{R}^N \times L \times \mathbf{R}^N \to \mathbf{R},$$
$$F_1(q, \xi_1, z_1, \xi_2, z_2) = F(q, \xi_1, z_1) + F(-q, \xi_2, z_2),$$
$$q \in S^n, \quad \xi_i \in L, \quad z_i \in \mathbf{R}^N, \quad i = 1, 2,$$

with nonzero critical value.

According to Lemma 2.1, the manifold of the critical points with zero critical value is a Bott manifold since the generating family $-F(-x, \xi, z) = G(x, \xi, z)$ defines the same Legendrian submanifold as the generating family $F(x, \xi, z)$.

The function F_1 determines the function $\widetilde{F}_1$ on the factor $S^n \times L \times \mathbf{R}^N \times L \times \mathbf{R}^N$ according to the action of the involution $l_1 : l_1(q, \xi_1, z_1, \xi_2, z_2) = (-q, \xi_2, z_2, \xi_1, z_1)$. Critical points of $\widetilde{F}_1$ form a Bott manifold diffeomorphic to L. $S^n \times L \times \mathbf{R}^N \times L \times \mathbf{R}^N/_{l_1}$ is the vector bundle over $S^n \times L \times L/_l$ and the function $\widetilde{F}_1$ is quadratic at infinity.

Twice the number $2D$ of the diameters is equal to the number of the critical points with nonzero critical value. From Theorem 2.1 it follows that in the generic case

$$D \geq \frac{1}{2}\left(\sum b_i\left(S^n \times L \times L/_l, \mathbf{Z}_2\right) - \sum b_i(L, \mathbf{Z}_2)\right).$$

Using Lemma 3.2, we obtain the estimate required.

4.1. Passing and Counterpassing Diameters of Cooriented Wave Fronts. Consider a closed submanifold L with transversal selfintersections in the Euclidean space $\mathbf{R}^{n+1}$.

Definition 4.2. A cooriented wave front Σ is called a *cooriented Chekanov wave front of type* L if Σ is a cooriented wave front of a Legendrian submanifold which is Legendrian isotopic to $S(L) \subset ST^*\mathbf{R}^{n+1}$.

A diameter of a cooriented wave front with the coorientations at the endpoints of the same (opposite) directions will be called a *passing* (*counterpassing*) diameter.

Theorem 4.2. *The number of passing (resp. counterpassing) diameters of a cooriented Chekanov wave front of type L, counted with the multiplicities, is no less than $B^2 - B$ (resp. $B^2 + nB$).*

Proof. A cooriented Chekanov wave front $\Sigma \subset \mathbf{R}^{n+1}$ of type L is the projection of a Legendrian submanifold $\Lambda \subset ST^*\mathbf{R}^{n+1}$ which is Legendrian isotopic to $S(L) \subset ST^*\mathbf{R}^{n+1}$. Passing diameters of Σ are in one-to-one correspondence (for a generic wave front Σ) with the selfintersection points of Lagrangian manifold $\rho_{S^n}(\Lambda) \subset T^*S^n$. Legendrian manifold Λ is an $S^n \times L$-quasifunction. Consequently, we can use the estimate obtained in Corollary 2.1. Thus, the number of passing diameters of a cooriented Chekanov wave front of type L is at least $\frac{1}{2}(2B^2 - \sum b_i(S(L))$. Applying Lemma 3.3 to the bundle $S(L) \to L$, we obtain the required estimate of the number of passing diameters.

The Legendrian manifold Λ is an $S^n \times L$-quasifunction. Consequently, according to Theorem 1.2, Λ is given by generating family $F : S^n \times L \times \mathbf{R}^N \to \mathbf{R}$ quadratic at infinity. The number of counterpassing diameters (for a generic wave front Σ) is twice as large as that of critical points of the function F_1 defined in the proof of Theorem 4.1. Generally, all the critical points are nondegenerate (this is the distinction from Theorem 4.1) and their number can be estimated by using Morse theory and Lemma 3.2.

Remark 4.1. Cooriented Chekanov wave fronts of type L are cooriented wave fronts obtained from an equidistant L by homotopy in the wave front class without dangerous selfcontacts.

Remark 4.2. In [13], Ferrand independently considered a similar problem and obtained Theorem 4.2 in the particular case of L being a point.

Example 4.1. Consider a circle embedded in $\mathbf{R}^3$. Its naturally cooriented equidistant is a cooriented Chekanov wave front of type S^1. According to Theorem 4.2, this front has at least two passing and eight counterpassing diameters. This estimate is sharp, 10 diameters of a two-dimensional torus are shown in Fig. 3. Note that the image of the cooriented Chekanov wave front of type S^1 under the Gaussian mapping is the whole sphere S^2, although the degree of the mapping equals zero.

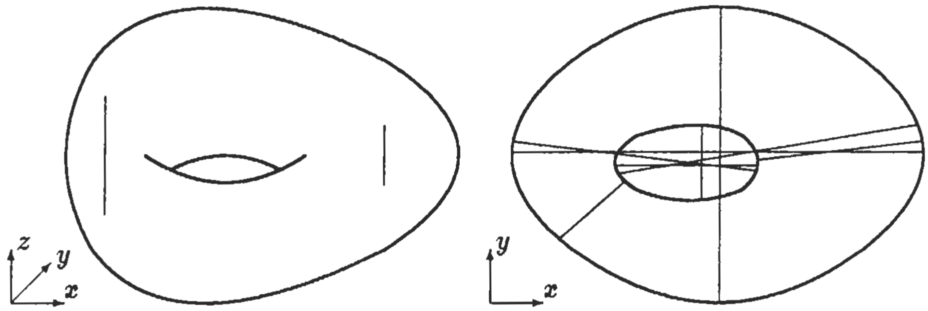

Fig. 3. The ten diameters of a torus: two of them are vertical, the other eight are in the plane section

5. SHARPNESS OF THE ESTIMATES OF THE NUMBER OF WAVE FRONT DIAMETERS

The aim of this section is to discuss the estimates obtained in Secs 4 and 3.

The estimate in Sec. 4 is sharp for the n-dimensional sphere S^n realized by an ellipsoid with different semiaxes in $(n+1)$-dimensional space.

Let us consider the Euclidean space $\mathbf{R}^{n+k+1}$ and the standard sphere S^n of unit radius in $\mathbf{R}^{n+1} \subset \mathbf{R}^{n+k+1}$. Suppose L is the set of those points of $\mathbf{R}^{n+k+1}$, the distance from which to S^n is equal to $\frac{1}{4}$. Let L be arbitrarily cooriented.

Assertion 5.1. *The manifold L is diffeomorphic to $S^n \times S^k$ and we can slightly perturb L and reduce it to the general position so that the number of diameters will be equal to $2(n+k)+6$. Then the numbers of the passing and counterpassing diameters are equal to 2 and $2(n+k)+4$, respectively.*

In order to prove Assertion 5.1 we need the following simple lemma.

Lemma 5.1. *Let M be a smooth manifold and $F_t : M \times [0,1] \to \mathbf{R}$ be a smooth function on $M \times [0,1]$ such that*

(a) *a compact submanifold N of the manifold M is the Bott manifold of the function $F_0 : M \to \mathbf{R}$;*

(b) *the restriction of the function $\left.\frac{\partial F_t}{\partial t}\right|_{t=0} : M \to \mathbf{R}$ on the submanifold N is a function with nondegenerate critical points whose number is equal to S.*

Then there exists a neighborhood of the submanifold N in the manifold M such that for fairly small positive ε all the critical points of the function F_ε in this neighborhood are nondegenerate and their number is equal to S.

Proof of Assertion 5.1. In fact, L is diffeomorphic to $S^n \times S^k$ since the normal bundle to S^n is trivial. Consider the function $f = \|\xi_1 - \xi_2\|^2$ on $L \times L \setminus \Delta$ (Δ is the diagonal). We describe its critical points. The critical points of the function f form Bott manifolds which are the manifolds of oriented diameters.

The manifold of oriented diameters of length $\frac{1}{2}$ denoted by $D_{\frac{1}{2}}$ is diffeomorphic to $S^n \times S^k$.

The manifold of oriented diameters of length $\frac{3}{2}$ (both ends of the diameter lie on the standard sphere of radius $\frac{3}{4}$ in $\mathbf{R}^{n+1}$) is diffeomorphic to the sphere S^n. We denote this manifold by $D_{\frac{3}{2}}$.

The manifold of oriented diameters of length $\frac{5}{2}$ (both ends of the diameter lie on the standard sphere of radius $\frac{5}{4}$ in $\mathbf{R}^{n+1}$) is diffeomorphic to the sphere S^n. We denote this manifold by $D_{\frac{5}{2}}$.

Finally, two manifolds of oriented diameters of length 2 (the first end lies on the standard sphere of radius $\frac{3}{4}$ ($\frac{5}{4}$) in $\mathbf{R}^{n+1}$ and the second end on the standard sphere of radius $\frac{5}{4}$ ($\frac{3}{4}$) in $\mathbf{R}^{n+1}$) are diffeomorphic to the sphere S^n. We denote these manifolds by D_2 ($\tilde{D}_2$).

The function f has no other critical points. We begin to perturb L.

Step 1. Let $e_1, \ldots, e_{n+1}, v_1, \ldots, v_k$ be an orthonormal basis in $\mathbf{R}^{n+k+1}$, $e_i \in \mathbf{R}^{n+1}$ for any $i = 1, \ldots, n+1$. Consider the map $A : \mathbf{R}^{n+k+1} \to \mathbf{R}^{n+k+1}$ $A(\sum \alpha_i e_i + \sum \beta_j v_j) = \sum \alpha_i e_i + \sum \lambda_j \beta_j v_j$, where λ_j are different numbers close to unity. We will describe the Bott manifolds of the function $f = \|A(\xi_1) - A(\xi_2)\|^2$ on $L \times L \setminus \Delta$.

The manifolds $D_{\frac{3}{2}}$, $D_{\frac{5}{2}}$, D_2 и $\tilde{D}_2$ are the Bott ones and do not change. The manifold $D_{\frac{1}{2}}$ is split off in $k+1$ pairs of Bott manifolds of oriented diameters, each of which is diffeomorphic to the sphere S^n. The manifolds $\{(A(x + \frac{1}{4}v_j), A(x - \frac{1}{4}v_j)),\ x \in S^n \subset \mathbf{R}^{n+k+1}\}$ and $\{(A(x - \frac{1}{4}v_j), A(x + \frac{1}{4}v_j)),\ x \in S^n \subset \mathbf{R}^{n+k+1}\}$, $j = 1, \ldots, k$, of the Bott manifolds of oriented diameters of length $\frac{\lambda_j}{2}$ will be denoted by $d_{\frac{\lambda_j}{2}}$ and $\tilde{d}_{\frac{\lambda_j}{2}}$, respectively. The manifolds $\{(\frac{5}{4}x, \frac{3}{4}x),\ x \in S^n \subset \mathbf{R}^{n+k+1}\}$ and $\{(\frac{3}{4}x, \frac{5}{4}x),\ x \in S^n \subset \mathbf{R}^{n+k+1}\}$ of the manifolds of oriented diameters of length $\frac{1}{2}$ will be denoted by $d_{\frac{1}{2}}$ and $\tilde{d}_{\frac{1}{2}}$, respectively.

Thus, the critical points of the function $\|A(\xi_1) - A(\xi_2)\|^2$ form the Bott manifolds $D_{\frac{3}{2}}$, D_2, $\tilde{D}_2$, $D_{\frac{5}{2}}$, $d_{\frac{1}{2}}$, $\tilde{d}_{\frac{1}{2}}$, and $2k$ the manifolds $d_{\frac{\lambda_j}{2}}$, $\tilde{d}_{\frac{\lambda_j}{2}}$ $j = 1, \ldots, k$.

Step 2. Let us perturb $A(L)$. We would like to construct the vector field v in $\mathbf{R}^{n+k+1}$ and find out how many (oriented) diameters the manifold $g^t(A(L))$ will have for small positive t (here g^t is a transformation of the phase flow of the field v in a time $[0, t]$). The difficulty lies in the fact that up to now the ends of different diameters coincide.

Let us consider the function $f_t = \|g^t(A(\xi_1)) - g^t(A(\xi_2))\|^2$. In order to find out the number of the critical points of this function at small t we must investigate the function $\frac{\partial f_t}{\partial t}\Big|_{t=0}$ on the Bott manifolds of the function f_0. We can readily verify that

$$\frac{\partial f_t}{\partial t}\Big|_{t=0} (\xi_1, \xi_2) = 2\langle v(A(\xi_1)) - v(A(\xi_2)), A(\xi_1)) - A(\xi_2)\rangle. \qquad (**)$$

Construction of the vector field v. Let $(\alpha_1, \ldots, \alpha_{n+1}, \beta_1, \ldots, \beta_k)$ be coordinates in $\mathbf{R}^{n+k+1}$ with the above-mentioned basis e_i, v_j, and $S^n \subset \mathbf{R}^{n+1}$ is the standard sphere of radius 1 with center at the point 0.

Let v be a vector field in $\mathbf{R}^{n+k+1}$ such that

(a) on the spheres $\{x + \frac{\lambda_m}{4} v_m,\ x \in S^n\}$ this field is equal to $\dot{\alpha}_i = 0$, $\dot{\beta}_j = 0$, $\dot{\beta}_k = 1$, $i = 1, \ldots, n+1$, $j = 1, \ldots, k-1$;

(b) on the spheres $\{x - \frac{\lambda_m}{4} v_m,\ x \in S^n\}$ this field is equal to zero;

(c) on the sphere $\{\frac{5}{4}x,\ x \in S^n\}$ this field is equal to $\dot{\alpha}_i = \mu_i \alpha_i$, $\dot{\beta}_j = 0$, $i = 1, \ldots, n+1$, $j = 1, \ldots, k$;

(d) on the sphere $\{\frac{3}{4}x,\ x \in S^n\}$ this field is equal to $\dot{\alpha}_i = \mu_i \alpha_i + c_i$, $\dot{\beta}_j = 0$, $i = 1, \ldots, n+1$, $j = 1, \ldots, k$, $c_1^2 + \ldots + c_{n+1}^2 = 1$.

Lemma 5.2. *On the manifold $d_{\frac{\lambda_i}{2}}$ $(\tilde{d}_{\frac{\lambda_i}{2}})$ the function $\frac{\partial f_t}{\partial t}\big|_{t=0}$ has exactly two nondegenerate critical points, $i = 1, \ldots, k$.*

Proof. We identify each diameter of the manifold of oriented diameters $d_{\frac{\lambda_i}{2}}$ $(\tilde{d}_{\frac{\lambda_i}{2}})$ with the projection of its beginning in $\mathbf{R}^{n+1}$. For this map $d_{\frac{\lambda_i}{2}}$ $(\tilde{d}_{\frac{\lambda_i}{2}})$ is diffeomorphically mapped onto the sphere S^n and the function $\frac{\partial f_t}{\partial t}\big|_{t=0}$ will coincide with the function $2\alpha_{n+1}$. On the sphere the nonzero linear function has two nondegenerate critical points. The proof is completed.

Lemma 5.3. *If the numbers μ_i are different, then the function $\frac{\partial f_t}{\partial t}\big|_{t=0}$ has exactly $2n+2$ nondegenerate critical points on the manifold $D_{\frac{5}{2}}$ $(D_{\frac{3}{2}})$.*

Proof. We will identify the point x of the sphere S^n with the point $(\frac{5}{4}x, -\frac{5}{4}x)$ of the manifold $D_{\frac{5}{2}}$ (with the point $(\frac{3}{4}x, -\frac{3}{4}x)$ of the manifold $D_{\frac{3}{2}}$, respectively). By virtue of relation (**) $\frac{\partial f_t}{\partial t}\big|_{t=0}(x) = 10\langle x, P_\mu(x)\rangle$ $(\frac{\partial f_t}{\partial t}\big|_{t=0}(x) = 6\langle x, P_\mu(x)\rangle$, respectively), where P_μ is an operator which transfers $(\alpha_1, \ldots, \alpha_{n+1})$ to $(\mu_1\alpha_1, \ldots, \mu_{n+1}\alpha_{n+1})$. Thus, in such an identification, on the manifold $D_{\frac{5}{2}}$ $(D_{\frac{3}{2}})$ the function $\frac{\partial f_t}{\partial t}\big|_{t=0}$ is a restriction of a quadratic form to the sphere. On the sphere S^n the quadratic form with different eigenvalues has exactly $(2n+2)$ nondegenerate critical points. The proof is completed.

Lemma 5.4. *If the numbers μ_i are fairly small, then the function $\frac{\partial f_t}{\partial t}\big|_{t=0}$ has exactly two nondegenerate critical points on each manifold D_2, $\tilde{D}_2$, $d_{\frac{1}{2}}$, and $\tilde{d}_{\frac{1}{2}}$.*

Proof. Let us consider the manifold $D_2 = \{(\frac{3}{4}x, \frac{3}{4}x, x \in S^n\}$. Denote a vector with the coordinates $(c_1, \ldots, c_{n+1}, 0, \ldots, 0)$ by C (here, $c_1, \ldots, c_{n+1}$ are the numbers involved in constructing the field v). According to relation (**), $\frac{\partial f_t}{\partial t}\big|_{t=0}(\frac{3}{4}x, \frac{5}{4}x) = 2\langle \frac{5}{4}x - \frac{3}{4}x, \frac{5}{4}P_\mu(x) - (\frac{3}{4}P_\mu(x) + C)\rangle = \langle x, \frac{1}{2}P_\mu(x) - C)\rangle$. Consequently, for small μ_i the function $\frac{\partial f_t}{\partial t}\big|_{t=0}$ differs only slightly from the restriction of the nonzero linear function to a sphere. The lemma is proved.

Thus, from Lemma 5.1 and Lemmas 5.2, 5.3, and 5.4 it follows that the manifold $g^t(A(L))$ (at small positive t) has exactly $4(n+k)+12$ oriented diameters. It is easy to see that four of them correspond to passing diameters, and the other $4(n+k)+8$ to counterpassing diameters. Assertion 5.1 is proved.

Our estimate of the number of diameters is sharp for an oriented surface of genus g embedded in three-dimensional space. Let us construct an example. Consider the three-dimensional Euclidean space $\mathbf{R}^3$ with coordinates (x, y, z). In the plane $\mathbf{R}^2$ with coordinates (x, y) we consider g nonintersecting circles lying inside a $(g+1)$th circle so that none of the first g circles lies inside another. Let the centers of these circles be different. We perturb each of these $(g+1)$ circles a little, transforming it in an ellipse. The curve obtained in $\mathbf{R}^2$ will be denoted L_g. L_g is the boundary of a disc with g holes $D_g \subset \mathbf{R}^2$. There is a function $f : \mathbf{R}^2 \to \mathbf{R}$ such that f is positive inside D_g and negative in $\mathbf{R}^2 \setminus D_g$. Thus, the function f is equal to zero on L_g. Moreover, we require that the function f have exactly $g+1$ Morse critical points each of which lies inside D_g. It is easy to show that a function satisfying these conditions does exist. Let ε be positive. Then the surface in $\mathbf{R}^3$ given by the equation $z^2 = \varepsilon f(x, y)$ is a smooth oriented embedded surface of genus g.

Assertion 5.2. *For a fairly small positive ε the surface $z^2 = \varepsilon f(x, y)$ has exactly $2g^2 + 5g + 3$ diameters. Exactly $g+1$ of them are parallel to the axis z and projected along the z axis to critical points of the function f, while the other $2g^2 + 4g + 2$ diameters are diameters of the curve L_g. For a fairly small positive ε the surface $z^2 = \varepsilon f(x, y)$ is a generic surface (with respect to the diameters).*

We do not prove Assertion 5.2 here.

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REFERENCES

1. Arnold, V.I., *Topological Invariants of Plane Curves and Caustics*, Providence (R.I.): Amer. Math. Soc., 1994. (*Univ. Lect. Ser.*, vol. 5).
2. Arnold, V.I. and Givental, A.B., Symplectic Geometry, *Itogi Nauki i Tekhniki. Sovremennye Problemy Matematiki. Fundamentalnye Napravleniya*, Moscow: VINITI, 1985, vol. 4, pp. 5–139. English transl. in: *Encyclopaedia of Math. Sc.*, Springer, vol. 4.
3. Arnold, V.I., Varchenko, A.N., and Gusein-Zade, S.M., *Osobennosti differentsiruemykh otobrazhenii* (Singularities of Differentiable Mappings), Moscow: Nauka, 1982, vol. 1.
4. Dubrovin, B.A., Novikov, S.P., and Fomenko A.T., *Sovremennaya geometriya* (Contemporary Geometry), Moscow: Nauka, 1984, vol. 2.
5. Milnor, J., *Morse Theory*, Princeton: Princeton Univ. Press, 1963.
6. Fomenko, A.T. and Fuks, D.B., *Kurs gomotopicheskoi topologii* (A Course of Homotopic Topology), Moscow: Nauka, 1989.
7. Viro, O.Ya. and Fuks, D.B., Homologies and Cohomologies, *Itogi Nauki i Tekhniki. Sovremennye Problemy Matematiki. Fundamentalnye Napravleniya*, Moscow: VINITI, 1988, vol. 24, pp. 123–240.
8. Chekanov, Yu.V., Critical Points of Quasifunctions and Generating Families of Legendrian Manifolds, *Funct. Anal. and Appl.*, 1996, vol. 30, pp. 118–128.
9. Bott, R., Lectures on Morse Theory, Old and New, *Bull. Amer. Math. Soc.* (N.S.), 1982, vol. 7, no. 2, pp. 331–358.
10. Chaperon, M., *On Generating Families. A Floer's Memorial Volume*, Basel etc.: Birkhäuser, 1996.

11. Conley, C. and Zehnder, E., The Birkhoff-Lewis Fixed Point Theorem and a Conjecture of V.I. Arnold, *Invent. Math.*, 1983, vol. 73, pp. 33–49.

12. Eliashberg, Y. and Gromov, M., Lagrangian Intersection and the Stable Morse Theory, *Preprint ETHZ*, May 1996.

13. Ferrand, E., On a Theorem of Chekanov, in: *Symplectic Singularities and Geometry of Gauge Fields*, Warszawa, 1997, pp. 39–48. (*Banach Center Publ.*, vol. 39).

14. Pushkar', P.E., Diameters of Immersed Manifolds and of Wave Fronts, *C. R. Acad. Sci. Paris*, Sér 1, 1997, vol. 326, pp. 201–205.

15. Pushkar', P.E., Generalization of the Chekanov Theorem: Diameters of Immersed Manifolds and Wave Fronts, *Preprint* FI-ST1997-011, Toronto: Fields Inst. Res. Math. Sci., June 1997.

16. Takens, F. and White, J., Morse Theory of Double Normals of Immersions, *Indiana Univ. Math. J.*, 1971, vol. 21, pp. 11–17.

17. Théret, D., *Thése Univ. Paris VII.*

18. Viterbo, C., Symplectic Topology as the Geometry of Generating Functions, *Ann. Math.*, 1992, vol. 292, pp. 685–710.

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