Hyperholomorphic connections on coherent sheaves and stability

Misha Verbitsky,[1]

verbit@thelema.dnttm.ru, verbit@mccme.ru

Abstract

Let M be a hyperkähler manifold, and F a reflexive sheaf on M. Assume that F (outside of singularities) admits a connection ∇ with a curvature Θ which is invariant under the standard $SU(2)$-action on 2-forms. If Θ is square-integrable, such sheaf is called **hyperholomorphic**. Hyperholomorphic sheaves were studied at great length in [V3]. Such sheaves are stable and their singular sets are hyperkähler subvarieties in M. In the present paper, we study sheaves admitting a connection with $SU(2)$-invariant curvature which is not necessary L^2-integrable. We show that such sheaves are polystable.

Contents

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1 Introduction

Yang-Mills theory of holomorphic vector bundles is one of the most spectacular successes of modern algebraic geometry. Developed by Narasimhan-Seshadri, Kobayashi, Hitchin, Donaldson, Uhlenbeck-Yau and others, this theory proved to be very fruitful in the study of stability and the modular properties of holomorphic vector bundles. Bogomolov-Miyaoka-Yau inequality and Uhlenbeck-Yau theorem were used by Carlos Simpson in his groundbreaking works on variations of Hodge structures and flat bundles ([Sim]). Later, it was shown ([V1], [KV]) that Yang-Mills approach is also useful in hyperkähler geometry and lends itself to an extensive study of stable bundles, their modular and twistor properties.

From algebraic point of view, a coherent sheaf is much more natural kind of object than a holomorphic vector bundle. This precipitates the extreme importance of Bando-Siu theory [BS] which extends Yang-Mills geometry to coherent sheaves.

In the present paper, we study the ramifications of Bando-Siu theory, for hyperkähler manifolds.

1.1 Yang-Mills geometry and Bando-Siu theorem

Let M be a compact Kähler manifold, and ω its Kähler form. Consider the standard Hodge operator L on differential forms which multiplies a form by ω. Let Λ be the Hermitian adjoint operator.

Let B be a Hermitian holomorphic vector bundle, and

$$\Theta \in \Lambda^{1,1}(M, \operatorname{End}(B))$$

its curvature, considered as a (1,1)-form on M with coefficients in $\operatorname{End}(B)$. By definition, $\Lambda\Theta$ is a smooth section of $\operatorname{End}(B)$. The bundle B is called **Yang-Mills**, or **Hermitian-Einstein**, if $\Lambda\Theta$ is constant times the unit section of $\operatorname{End}(B)$.

For a definition of stability, see Subsection 3.2. Throughout this paper, stability is understood in the sense of Mumford-Takemoto.

Let B be a holomorphic bundle which cannot be decomposed onto a direct sum of non-trivial holomorphic bundles (such bundles are called **indecomposable**). By Uhlenbeck-Yau theorem, B admits a Yang-Mills connection if and only if B is stable; if exists, such a connection is unique ([UY]). This result allows one to deal with the moduli of stable holomorphic vector bundles in efficient and straightforward manner.

S. Bando and Y.-T. Siu ([BS]) extended the results of Uhlenbeck-Yau to coherent sheaves.

Definition 1.1: [BS] Let F be a coherent sheaf on M and ∇ a Hermitian connection on F defined outside of its singularities. Denote by Θ the curvature of ∇. Then ∇ is called **admissible** if the following holds

(i) $\Lambda\Theta \in \operatorname{End}(F)$ is uniformly bounded

(ii) $|\Theta|^2$ is integrable on M.

Any torsion-free coherent sheaf admits an admissible connection. An admissible connection can be extended over the place where F is smooth. Moreover, if a bundle B on $M\backslash Z$, $\operatorname{codim}_{\mathbb{C}} Z \geqslant 2$ is equipped with an admissible connection, then B can be extended to a coherent sheaf on M.

Therefore, the notion of a coherent sheaf can be adequately replaced by the notion of an admissible Hermitian holomorphic bundle on $M\backslash Z$, $\operatorname{codim}_{\mathbb{C}} Z \geqslant 2$.

A version of Uhlenbeck-Yau theorem exists for coherent sheaves (Theorem 4.8); given a torsion-free coherent sheaf F, F admits an admissible Yang-Mills connection ∇ if and only if F is polystable.

The following conjecture deals with Yang-Mills connections which are *not* admissible.

Conjecture 1.2: Let M be a compact Kähler manifold, F a torsion-free coherent sheaf on M with singularities in codimension at least 3, ∇ a Hermitian connection on F defined outside of its singularities, and Θ its curvature. Assume that $\Lambda(\Theta) = 0$. Then F can be extended to a stable sheaf on M.

This conjecture is motivated by the following heuristic argument.

Denote by ω the Kähler form on M, and let $n := \dim_{\mathbb{C}} M$. By Hodge-Riemann relations,

$$\operatorname{Tr}(\Theta \wedge \Theta) \wedge \omega^{n-2} = c|\Theta|^2 \operatorname{Vol}(M) \tag{1.1}$$

where c is a positive rational constant (this equality is true pointwise, assuming that $\Lambda\Theta = 0$). This equality is used in [Sim] to deduce the Bogomolov-Miyaoka-Yau inequality from the Uhlenbeck-Yau theorem.

By Gauss-Bonnet formula, the cohomology class of $\operatorname{Tr}(\Theta \wedge \Theta)$ can be expressed via $c_1(F)$, $c_2(F)$:

$$\frac{\sqrt{-1}}{2\pi^2} \operatorname{Tr}(\Theta \wedge \Theta) = 2c_2(F) - \frac{n-1}{n} c_1(F).$$

Therefore, the integral

$$\int_M \operatorname{Tr}(\Theta \wedge \Theta) \wedge \omega^{n-2} \tag{1.2}$$

"must have" cohomological meaning (we write "must have" to indicate here the element of speculation).

If indeed the integral (1.2) is expressed via cohomology, it is finite, and by (1.1) the curvature Θ is square-integrable.

In this paper, we study Conjecture 1.2 when M is a hyperkähler manifold, and ∇ is a hyperholomorphic connection (see Definition 3.2 for a definition and further discussion of the notion of hyperholomorphic bundle).

1.2 Hyperkähler and hypercomplex manifolds

A hypercomplex manifold is a manifold equipped with an action of quaternion algebra in its tangent bundle TM, such that for any quaternion $L, L^2 = -1$, the corresponding operator on TM defines an integrable structure on M. If, in addition, M is Riemannian, and (M, L) is Kähler for any quaternion $L, L^2 = -1$, then M is called **hyperkähler**.

A hyperkähler manifold is equipped with a natural action of the group $SU(2)$ on TM. By multiplicativity, we may extend this action to all tensor powers of TM. In particular, $SU(2)$ acts on the space of differential forms on M.

This action bears a deep geometric meaning encompassing the Hodge decomposition on M (see Lemma 2.6 and its proof). Moreover, the group $SU(2)$ preserves the Laplace operator, and henceforth acts on the cohomology of M (see e.g. [V0]).

Let η be an $SU(2)$-invariant 2-form on M. An elementary linear-algebraic calculation implies that $\Lambda\eta = 0$ (Lemma 3.8). Given a Hermitian vector bundle B with $SU(2)$-invariant curvature

$$\Theta \in \Lambda^2(M)_{SU(2)-inv} \otimes \operatorname{End}(B),$$

we find that $\Lambda\Theta = 0$, and therefore B is Yang-Mills.

Such bundles are called **hyperholomorphic**. The theory of hyperholomorphic bundles, developed in [V1], turns out to be quite useful in hyperkähler geometry, by the following reasons.

(i) For an arbitrary holomorphic vector bundle, a hyperholomorphic connection is Yang-Mills, and therefore unique. (1.3)

(ii) An $SU(2)$-invariant form is of the Hodge type $(1,1)$ with respect to any complex structure $L \in \mathbb{H}$, $L^2 = -1$ induced by the quaternionic action on M (Lemma 2.6). By Newlander-Nirenberg integrability theorem (Theorem 3.1), a hyperholomorphic bundle is holomorphic with respect to $I, J, K \in \mathbb{H}$. The converse is also true (Definition 3.2).

(iii) The moduli of hyperholomorphic bundles are hyperkähler (possibly singular) varieties. A normalization of such variety is smooth and hyperkähler ([V4]).

(iv) Let $L \in \mathbb{H}$, $L^2 = -1$ be a complex structure induced by the quaternionic action. Consider a stable holomorphic bundle on the Kähler manifold (M, L). Then B admits a hyperholomorphic connection if and only if the Chern classes $c_1(B)$, $c_2(B)$ are $SU(2)$-invariant (Theorem 3.9).

(v) Moreover, if $L \in \mathbb{H}$, $L^2 = -1$ is generic, and B is a stable holomorphic bundle on (M, L), then B is hyperholomorphic.

Using the results of Bando-Siu, we can extend the notion of hyperholomorphic connection to coherent sheaves ([V3]).

Definition 1.3: Let M be a hyperkähler manifold, and F a reflexive[1] coherent sheaf on the Kähler manifold (M, I). Consider an admissible (in the sense of Definition 1.1) Hermitian connection ∇ on F. The ∇ is called **admissible hyperholomorphic**, if its curvature is $SU(2)$-invariant. A stable reflexive sheaf is called **stable hyperholomorphic** if it admits an admissible hyperholomorphic connection.

The statements (i)-(ii) and (iv)-(v) of (1.3) hold true for hyperholomorphic sheaves. In addition to this, a hyperholomorphic sheaf with isolated singularities can be desingularized with a single blow-up ([V3]).

In examples, one often obtains coherent sheaves with Hermitian structure outside of singularities. For instance, a direct image of a Hermitian vector bundle is a complex of sheaves with cohomology equipped with the natural (Weil-Peterson) metrics. If we work in hyperkähler geometry, the corresponding Hermitian connection is quite often hyperholomorphic outside of singularities ([BBR]). However, the admissibility condition is rather tricky. In fact, we were unable to show in full generality that a sheaf with a connection and an $SU(2)$-invariant curvature is admissible.

However, the following assertion is sufficient for most purposes.

Theorem 1.4: Let M be a compact hyperkähler manifold, I an induced complex structure, and F a reflexive sheaf on (M, I) which cannot be decomposed onto a direct sum of non-trivial coherent sheaves.[2] Assume that F is equipped with a Hermitian connection ∇ defined outside of the singular set of F, and the curvature of ∇ is $SU(2)$-invariant. Then F is stable.

Proof: This is Theorem 4.16. ■

1.3 Contents

This paper has the following structure.

[1] A torsion-free coherent sheaf is called **reflexive** if the natural monomorphism

$$F \longrightarrow F^{**} := \operatorname{Hom}(\operatorname{Hom}(F, \mathcal{O}), \mathcal{O})$$

is an isomorphism. A sheaf F^{**} is always reflexive. The natural functor $F \longrightarrow F^{**}$ is called **the reflexization**. For more details on reflexive sheaves, see Subsection 4.1 and [OSS].

[2] Such sheaves are called indecomposable.

- The present Introduction is independent from the rest of this paper.

- In Sections 2-3 we give preliminary definitions and state basic results about the geometry of hyperkähler manifolds and stable bundles. We follow [Bes], [UY] and [V1].

- In Subsections 4.1-4.2, we give an exposition of the theory of Bando-Siu and its applications to the hyperkähler geometry. We also give a definition of reflexive sheaves and list some of their properties.

- In Subsection 4.3, we state the main conjecture motivating our research (Conjecture 4.18). It is conjectured that, on any hyperkähler manifold, a hyperholomorphic connection on a reflexive sheaf (defined everywhere outside of singularities) has square-integrable curvature.

  We also state our main result (Theorem 4.16) which was explained earlier in this Introduction (Theorem 1.4).

- In Section 5 we work with positive (p,p)-forms and their singularities. We state an important lemma of Sibony, motivating Conjecture 4.18. Let η be a positive closed (p,p)-form with singularities in codimension at least $p+1$. Then η is L^1-integrable. This is used to prove Conjecture 4.18 in case of a sheaf with isolated singularities.

  As an intermediate result, we obtain the following proposition, which is quite useful in itself (Lemma 5.4). Let B be a hyperholomorphic bundle on a hyperkähler manifold M, $\dim_{\mathbb{H}} M = n > 1$, Θ its curvature. Denote by ω_I the Kähler form of (M, I). Consider the closed 4-form

$$r_2 := \frac{\sqrt{-1}}{2\pi^2} \operatorname{Tr}(\Theta \wedge \Theta)$$

representing (by Gauss-Bonnet) the cohomology class $2c_2(B) - \frac{n-1}{n} c_1(B)^2$. Then the $(2n-1, 2n-1)$-form $r_2 \wedge \omega_I^{2n-3}$ is positive.

- In Section 6 we study the first Chern class of a reflexive sheaf F admitting a hyperholomorphic connection outside of singularities. We show that $c_1(F)$ is $SU(2)$-invariant.

- In Section 7, we study the singularities of positive forms on a hyperkähler manifold. Consider a closed 2-form η which is smooth on $M \backslash Z$, where $codim_{\mathbb{R}} Z \geqslant 6$. Since $H^2(M\backslash Z) = H^2(M)$, we may consider the cohomology class $[\eta]$ as an element in $H^2(M)$. We define the degree $\deg_I$ of $[\eta]$ as follows

$$\deg_I(\eta) := \int_M [\eta] \wedge \omega_I^{n-1},$$

where ω_I is the Kähler form of (M, I).

Assume that η is a sum of a positive form η_+ and an $SU(2)$-invariant form. We show that $\deg_I[\eta] \geqslant 0$, and if $\deg_I[\eta] = 0$, then $\eta_+ = 0$.

• In Section 8, we prove L^1-integrability of a ∂_K-closed form $\eta_K^{2,0} \in \Lambda_K^{2,0}(M\backslash Z)$, where $I(\eta_K^{2,0}) = \overline{\eta}_K^{2,0}$, assuming that $\operatorname{Re} \eta(z, \overline{z})$ is non-negative for all $z \in T^{1,0}(M, I)$. This is essentially a hyperkähler version of two classical results from complex analysis - Sibony's lemma and Skoda-El Mir theorem. This result is used in Section 7 to show that certain closed forms with singularities represent cohomology classes of positive degree.

In the earlier versions of this paper this result was proven by a straightforward argument based on slicing, in the same way as one proves the L^1-integrability of a positive closed (p, p) form with singularities in codim $> 2p$ ([Sib]). To use slicing, one needs to approximate a hyperkähler manifold by a flat one, which leads to complicated estimates. Now these difficulties are avoided. In the latest version (starting from 2008), a coordinate-free approach to Sibony's lemma was used, based on the recent advances in the theory of ω^q-plurisubharmonic functions ([V7], [V8]).

• In the last section (Section 9), we use the results of Section 7 (the positivity of a degree of a closed 2-form η which is a sum of a positive and an $SU(2)$-invariant form) to prove our main result. Given a sheaf F admitting a connection with $SU(2)$-invariant curvature, we show that F is a direct sum of stable sheaves. This is done in the same way as one proves that a Yang-Mills bundle is polystable. We use the standard inequality between the curvature of a bundle and a sub-bundle, which is proven via the second fundamental form of a sub-bundle ([D1], [GH]).

2 Hyperkähler manifolds

This Section contains a compression of the basic and best known results and definitions from hyperkähler geometry, found, for instance, in [Bes], [Bea] and [V1].

Definition 2.1: ([Bes]) A **hyperkähler manifold** is a Riemannian manifold M endowed with three complex structures I, J and K, such that the following holds.

(i) the metric on M is Kähler with respect to these complex structures and

(ii) I, J and K, considered as endomorphisms of a real tangent bundle, satisfy the relation $I \circ J = -J \circ I = K$.

The notion of a hyperkähler manifold was introduced by E. Calabi ([Ca]).

Clearly, a hyperkähler manifold has a natural action of the quaternion algebra $\mathbb{H}$ in its real tangent bundle TM. Therefore its complex dimension is even. For each quaternion $L \in \mathbb{H}$, $L^2 = -1$, the corresponding automorphism of TM is an almost complex structure. It is easy to check that this almost complex structure is integrable ([Bes]).

Definition 2.2: Let M be a hyperkähler or hypercomplex manifold, and L a quaternion satisfying $L^2 = -1$. The corresponding complex structure on M is called **an induced complex structure**. The M, considered as a Kähler manifold, is denoted by (M, L). In this case, the hyperkähler structure is called **compatible with the complex structure** L.

Definition 2.3: Let M be a complex manifold and Θ a closed holomorphic 2-form over M such that $\Theta^n = \Theta \wedge \Theta \wedge ...$, is a nowhere degenerate section of a canonical class of M ($2n = dim_{\mathbb{C}}(M)$). Then M is called **holomorphically symplectic**.

Let M be a hyperkähler manifold; denote the Riemannian form on M by $\langle\cdot,\cdot\rangle$. Let the form $\omega_I := \langle I(\cdot), \cdot\rangle$ be the usual Kähler form which is closed and parallel (with respect to the Levi-Civita connection). Analogously defined forms ω_J and ω_K are also closed and parallel.

A simple linear algebraic consideration ([Bes]) shows that the form

$$\Theta := \omega_J + \sqrt{-1}\omega_K \tag{2.1}$$

is of type $(2,0)$ and, being closed, this form is also holomorphic. Also, the form Θ is nowhere degenerate, as another linear algebraic argument shows.

It is called **the canonical holomorphic symplectic form of a manifold M**. Thus, for each hyperkähler manifold M, and an induced complex structure L, the underlying complex manifold (M, L) is holomorphically symplectic. The converse assertion is also true:

Theorem 2.4: ([Bea], [Bes]) Let M be a compact holomorphically symplectic Kähler manifold with the holomorphic symplectic form Θ, a Kähler class $[\omega] \in H^{1,1}(M)$ and a complex structure I. Let $n = \dim_{\mathbb{C}} M$. Assume that $\int_M \omega^n = \int_M (Re\Theta)^n$. Then there is a unique hyperkähler structure $(I, J, K, (\cdot,\cdot))$ over M such that the cohomology class of the symplectic form $\omega_I = (\cdot, I\cdot)$ is equal to $[\omega]$ and the canonical symplectic form $\omega_J + \sqrt{-1}\,\omega_K$ is equal to Θ.

Theorem 2.4 follows from the conjecture of Calabi, proven by S.-T. Yau ([Y]). ∎

Let M be a hyperkähler manifold. We identify the group $SU(2)$ with the group of unitary quaternions. This gives a canonical action of $SU(2)$ on the tangent bundle, and all its tensor powers. In particular, we obtain a natural action of $SU(2)$ on the bundle of differential forms.

The following lemma is clear.

Lemma 2.5: The action of $SU(2)$ on differential forms commutes with the Laplacian.

Proof: This is Proposition 1.1 of [V0]. ∎

Thus, for compact M, we may speak of the natural action of $SU(2)$ in cohomology.

Further in this article, we use the following statement.

Lemma 2.6: Let ω be a differential form over a hyperkähler or hypercomplex manifold M. The form ω is $SU(2)$-invariant if and only if it is of Hodge type (p, p) with respect to all induced complex structures on M.

Proof: Let I be an induced complex structure, and $\rho_I : U(1) \longrightarrow SU(2)$ the corresponding embedding, induced by the map $\mathbb{R} = \mathfrak{u}(1) \longrightarrow \mathfrak{su}(2)$, $1 \longrightarrow I$. The Hodge decomposition on $\Lambda^*(M)$ coincides with the weight decomposition of the $U(1)$-action ρ_I. An $SU(2)$-invariant form is also invariant with respect to ρ_I, and therefore, has Hodge type (p,p). Conversely, if a η is invariant with respect to ρ_I, for all induced complex structures I, then η is invariant with respect to the Lie group G generated by these $U(1)$-subgroups of $SU(2)$. A trivial linear-algebraic argument ensures that G is the whole $SU(2)$. This proves Lemma 2.6. ■

3 Hyperkähler manifolds and stable bundles

3.1 Hyperholomorphic connections

Let B be a holomorphic vector bundle over a complex manifold M, ∇ a connection in B and $\Theta \in \Lambda^2(M)\otimes End(B)$ be its curvature. This connection is called **compatible with a holomorphic structure** if $\nabla_X(\zeta) = 0$ for any holomorphic section ζ and any antiholomorphic tangent vector field $X \in T^{0,1}(M)$. If there exists a holomorphic structure compatible with the given Hermitian connection then this connection is called **integrable**.

One can define a **Hodge decomposition** in the space of differential forms with coefficients in any complex bundle, in particular, $End(B)$.

Theorem 3.1: Let ∇ be a Hermitian connection in a complex vector bundle B over a complex manifold. Then ∇ is integrable if and only if $\Theta \in \Lambda^{1,1}(M, \mathrm{End}(B))$, where $\Lambda^{1,1}(M, \mathrm{End}(B))$ denotes the forms of Hodge type (1,1). Also, the holomorphic structure compatible with ∇ is unique.

Proof: This is Proposition 4.17 of [Kob], Chapter I. ■

This proposition is a version of Newlander-Nirenberg theorem. For vector bundles, it was proven by M. Atiyah and R. Bott.

Definition 3.2: [V1] Let B be a Hermitian vector bundle with a connection ∇ over a hyperkähler manifold M. Then ∇ is called **hyperholomorphic** if the curvature of ∇ is $SU(2)$-invariant.

Example 3.3: (Examples of hyperholomorphic bundles)

(i) Let M be a hyperkähler manifold, and TM be its tangent bundle equipped with the Levi–Civita connection ∇. Consider a complex structure on TM induced from the quaternion action. Then ∇ is a Hermitian connection which is integrable with respect to each induced complex structure, and hence, is hyperholomorphic.

(ii) For B a hyperholomorphic bundle, all its tensor powers are hyperholomorphic.

(iii) Thus, the bundles of differential forms on a hyperkähler manifold are also hyperholomorphic.

3.2 Hyperholomorphic bundles and Yang-Mills connections.

Definition 3.4: Let F be a coherent sheaf over an n-dimensional compact Kähler manifold M. We define **the degree** $\deg(F)$ (sometimes the degree is also denoted by $\deg c_1(F)$) as

$$\deg(F) = \int_M \frac{c_1(F) \wedge \omega^{n-1}}{vol(M)}$$

and $\operatorname{slope}(F)$ as

$$\operatorname{slope}(F) = \frac{1}{\operatorname{rank}(F)} \cdot \deg(F).$$

The number $\operatorname{slope}(F)$ depends only on a cohomology class of $c_1(F)$.

Let F be a torsion-free coherent sheaf on M and $F' \subset F$ its proper subsheaf. Then F' is called **destabilizing subsheaf** if $\operatorname{slope}(F') \geqslant \operatorname{slope}(F)$

A coherent sheaf F is called **stable** [1] if it has no destabilizing subsheaves. A coherent sheaf F is called **polystable** if it is a direct sum of stable sheaves of the same slope.

Let M be a Kähler manifold with a Kähler form ω. For differential forms with coefficients in any vector bundle there is a Hodge operator $L : \eta \longrightarrow \omega \wedge \eta$. There is also a fiberwise-adjoint Hodge operator Λ (see [GH]).

Definition 3.5: Let B be a holomorphic bundle over a Kähler manifold M with a holomorphic Hermitian connection ∇ and a curvature $\Theta \in \Lambda^{1,1} \otimes End(B)$. The Hermitian metric on B and the connection ∇ defined by this metric are called **Yang-Mills** if

$$\Lambda(\Theta) = constant \cdot \operatorname{Id}\Big|_B,$$

[1] In the sense of Mumford-Takemoto

where Λ is a Hodge operator and $\operatorname{Id}\Big|_B$ is the identity endomorphism which is a section of $End(B)$.

A holomorphic bundle is called **indecomposable** if it cannot be decomposed onto a direct sum of two or more holomorphic bundles.

The following fundamental theorem provides examples of Yang-Mills bundles.

Theorem 3.6: (Uhlenbeck-Yau) Let B be an indecomposable holomorphic bundle over a compact Kähler manifold. Then B admits a Hermitian Yang-Mills connection if and only if it is stable. Moreover, the Yang-Mills connection is unique, if it exists.

Proof: [UY]. ∎

Proposition 3.7: Let M be a hyperkähler manifold, L an induced complex structure and B be a complex vector bundle over (M, L). Then every hyperholomorphic connection ∇ in B is Yang-Mills and satisfies $\Lambda(\Theta) = 0$ where Θ is a curvature of ∇.

Proof: We use the definition of a hyperholomorphic connection as one with $SU(2)$-invariant curvature. Then Proposition 3.7 follows from the

Lemma 3.8: Let $\Theta \in \Lambda^2(M)$ be a $SU(2)$-invariant differential 2-form on M. Then $\Lambda_L(\Theta) = 0$ for each induced complex structure L.[2]

Proof: This is Lemma 2.1 of [V1]. ∎

Let M be a compact hyperkähler manifold, I an induced complex structure. For any stable holomorphic bundle on (M, I) there exists a unique Hermitian Yang-Mills connection which, for some bundles, turns out to be hyperholomorphic. It is possible to tell exactly when this happens.

Theorem 3.9: Let B be a stable holomorphic bundle over (M, I), where M is a hyperkähler manifold and I is an induced complex structure over M. Then B admits a compatible hyperholomorphic connection if and only if the first two Chern classes $c_1(B)$ and $c_2(B)$ are $SU(2)$-invariant.[3]

Proof: This is Theorem 2.5 of [V1]. ∎

[2] By Λ_L we understand the Hodge operator Λ adjoint to the multiplication by the Kähler form associated with the complex structure L.

[3] We use Lemma 2.5 to speak of action of $SU(2)$ in cohomology of M.

4 Hyperholomorphic sheaves

In [BS], S. Bando and Y.-T. Siu developed machinery allowing one to apply the methods of Yang-Mills theory to torsion-free coherent sheaves. In [V3], their work was applied to generalise the results of [V1] (see Section 3) to coherent sheaves. The first two subsections of this Section are a compilation of the results and definitions of [BS] and [V3].

4.1 Stable sheaves and Yang-Mills connections

In this subsection, we repeat the basic definitions and results from [BS] and [OSS].

Definition 4.1: Let X be a complex manifold, and F a coherent sheaf on X. Consider the sheaf $F^* := \mathcal{H}om_{\mathcal{O}_X}(F, \mathcal{O}_X)$. There is a natural functorial map $\rho_F : F \longrightarrow F^{**}$. The sheaf F^{**} is called **a reflexive hull**, or **reflexization** of F. The sheaf F is called **reflexive** if the map $\rho_F : F \longrightarrow F^{**}$ is an isomorphism.

Remark 4.2: For all coherent sheaves F, the map $\rho_{F^*} : F^* \longrightarrow F^{***}$ is an isomorphism ([OSS], Ch. II, the proof of Lemma 1.1.12). Therefore, a reflexive hull of a sheaf is always reflexive. Moreover, a reflexive hull can be obtained by restricting to a non-singular set of F subset and taking the pushforward ([OSS], Ch. II, Lemma 1.1.12). More generally, a reflexive sheaf is isomorphic to a pushforward of its restriction to an open set $M\backslash Z$, for any complex analytic subset $Z \subset M$ of codimension at least 2.

Lemma 4.3: Let X be a complex manifold, F a coherent sheaf on X, Z a closed analytic subvariety, $\operatorname{codim} Z \geqslant 2$, and $j : (X\backslash Z) \hookrightarrow X$ the natural embedding. Assume that the pullback j^*F is reflexive on $(X\backslash Z)$. Then the pushforward j_*j^*F is also reflexive.

Proof: This is [OSS], Ch. II, Lemma 1.1.12. ■

Lemma 4.4: Let F be a reflexive sheaf on M, and X its singular set. Then $\operatorname{codim}_M X \geqslant 3$

Proof: This is [OSS], Ch. II, 1.1.10. ■

Claim 4.5: Let X be a Kähler manifold, and F a torsion-free coherent sheaf over X. Then F (semi)stable if and only if F^{**} is (semi)stable.

Proof: This is [OSS], Ch. II, Lemma 1.2.4. ■

The admissible Hermitian metrics, introduced by Bando and Siu in [BS], play the role of the ordinary Hermitian metrics for vector bundles.

Let X be a Kähler manifold. In Hodge theory, one considers the operator $\Lambda : \Lambda^{p,q}(X) \longrightarrow \Lambda^{p-1,q-1}(X)$ acting on differential forms on X, which is adjoint to the multiplication by the Kähler form. This operator is also defined on differential forms with coefficients in a bundle. Consider a curvature Θ of a bundle B as a 2-form with coefficients in $\operatorname{End}(B)$. Then $\Lambda\Theta$ is a section of $\operatorname{End}(B)$.

Definition 4.6: Let X be a Kähler manifold, and F a reflexive coherent sheaf over X. Let $U \subset X$ be the set of all points at which F is locally trivial. By definition, the restriction $F\Big|_U$ of F to U is a bundle. An **admissible metric** on F is a Hermitian metric h on the bundle $F\Big|_U$ which satisfies the following assumptions

(i) the curvature Θ of (F, h) is square integrable, and

(ii) the corresponding section $\Lambda\Theta \in \operatorname{End}(F\Big|_U)$ is uniformly bounded.

Definition 4.7: Let X be a Kähler manifold, F a reflexive sheaf over X, and h an admissible metric on F. Consider the corresponding Hermitian connection ∇ on $F\Big|_U$. The metric h and the Hermitian connection ∇ are called **Yang-Mills** if its curvature satisfies

$$\Lambda\Theta \in \operatorname{End}(F\Big|_U) = c \cdot \mathsf{id}$$

where c is a constant and id the unit section $\mathsf{id} \in \operatorname{End}(F\Big|_U)$.

One of the main results of [BS] is the following analogue of the Uhlenbeck-Yau theorem (Theorem 3.6).

Theorem 4.8: Let M be a compact Kähler manifold, and F a coherent sheaf without torsion. Then F admits an admissible Yang–Mills metric if and only if F is polystable. Moreover, if F is stable, then this metric is unique, up to a constant multiplier.

Proof: [BS], Theorem 3. ∎

Remark 4.9: Clearly, the connection, corresponding to a metric on F, does not change when the metric is multiplied by a scalar. The Yang–Mills metric on a polystable sheaf is unique up to a component-wise multiplication by scalar multipliers. Thus, the Yang–Mills connection of Theorem 4.8 is unique.

4.2 Stable hyperholomorphic sheaves over hyperkähler manifolds

Let M be a compact hyperkähler manifold, I an induced complex structure, F a torsion-free coherent sheaf over (M, I) and F^{**} its reflexization. Recall that the cohomology of M are equipped with a natural $SU(2)$-action (Lemma 2.5). The motivation for the following definition is Theorem 3.9 and Theorem 4.8.

Definition 4.10: Assume that the first two Chern classes of the sheaves F, F^{**} are $SU(2)$-invariant. Then F is called **stable hyperholomorphic** if F is stable. If F is a direct sum of stable hyperholomorphic sheaves, F is called **polystable hyperholomorphic**,

Remark 4.11: The slope of a hyperholomorphic sheaf is zero, because a degree of an $SU(2)$-invariant second cohomology class is zero (Lemma 3.8).

Let M be a hyperkähler manifold, I an induced complex structure, and F a torsion-free sheaf over (M, I). Consider the natural $SU(2)$-action in the bundle $\Lambda^i(M, B)$ of the differential i-forms with coefficients in a vector bundle B. Let $\Lambda^i_{inv}(M, B) \subset \Lambda^i(M, B)$ be the bundle of $SU(2)$-invariant i-forms.

Definition 4.12: Let $Z \subset (M, I)$ be a complex subvariety of codimension at least 2, and F a reflexive sheaf on (M, I), such that $F\Big|_{M\backslash Z}$ is a bundle. Consider an admissible metric h on $F\Big|_{M\backslash Z}$, and let ∇ be the associated connection. Then ∇ is called **admissible hyperholomorphic** if its curvature

$$\Theta_\nabla = \nabla^2 \in \Lambda^2 \left(M, \operatorname{End}\left(F\Big|_{M\backslash Z}\right)\right)$$

is $SU(2)$-invariant, i. e. belongs to $\Lambda^2_{inv}\left(M, \operatorname{End}\left(F\Big|_{M\backslash Z}\right)\right)$.

Remark 4.13: This is the same definition as Definition 1.3.

Theorem 4.14: Let M be a compact hyperkähler manifold, I an induced complex structure and F a reflexive sheaf on (M, I). Then F admits a hyperholomorphic connection if and only if F is polystable hyperholomorphic.

Proof: This is [V3], Theorem 3.19. ■

4.3 Weakly hyperholomorphic sheaves

Definition 4.15: Let M be a hyperkähler manifold, I an induced complex structure, and F a torsion-free coherent sheaf on M. Assume that outside of a closed complex analytic set $Z \subset (M, I)$, $\operatorname{codim}_{\mathbb{C}} Z \geqslant 3$, the sheaf F is smooth and equipped with a connection ∇. Assume, moreover, that the curvature of ∇ is $SU(2)$-invariant. Then F is called **weakly hyperholomorphic**.

The main result of this paper is the following theorem.

Theorem 4.16: Let M be a compact hyperkähler manifold, I an induced complex structure, and F a reflexive sheaf on (M, I). Assume that F is weakly hyperholomorphic. Then F is polystable.
Proof: See Section 9. ■

Remark 4.17: From Theorem 4.16 it follows that all stable summands F_i of F are hyperholomorphic (see Remark 9.3).

By Theorem 4.14, F admits a unique admissible hyperholomorphic Yang-Mills connection ∇_1. However, we do not know whether $\nabla = \nabla_1$ or not.

Conjecture 4.18: Under assumptions of Theorem 4.16, the connection ∇ is admissible.

Clearly, Theorem 4.16 is implied by Theorem 4.14 and Conjecture 4.18.

Example 4.19: Let B be a hyperholomorphic bundle on a product $M_1 \times M_2$ of two hyperkaehler manifolds, and $M_1 \times M_2 \xrightarrow{\pi} M_1$ the projection map. From the usual twistor argument (see e. g. [KV] or [BBR]) it follows that a derived direct image $R^i\pi_*B$ admits a hyperholomorphic connection outside of its singularities. If, in addition, M_1 is generic in its deformation class, all its subvarieties have even codimension. In particular, $R^i\pi_*B$ is smooth outside of codimension 2, and its reflexization $(R^i\pi_*B)^{**}$ is weakly hyperholomorphic. The stability of direct images of stable bundles is an important question which is partially solved by Theorem 4.16.

5 Positive forms and hyperholomorphic connections

To justify Conjecture 4.18, we prove it for sheaves with isolated singularities.

5.1 Singularities of positive closed forms

Definition 5.1: Let M be a complex manifold, and η a real-valued (p,p)-form on M. Then η is called **positive** if for any p-tuple of vector fields

$$\alpha_1, ..., \alpha_1 \in \Lambda^{1,0}(M),$$

we have

$$(\sqrt{-1}\,)^p \eta(\alpha_1, \overline{\alpha}_1, \alpha_2, \overline{\alpha}_2, ...) \geqslant 0.$$

For an excellent exposition of the theory of positive forms and currents, see [D1]. Further on, we shall need the following important lemma.

Lemma 5.2: Let M be a Kähler manifold, $Z \subset M$ a closed complex subvariety, $\operatorname{codim} Z > p$, and η a closed positive (p,p)-form on $M\backslash Z$. Then η is locally L^1-integrable.

Proof: [Sib]. ■

5.2 Weakly holomorphic sheaves with isolated singularities

The following proposition is not used anywhere in this paper. We include it to justify Conjecture 4.18, and, ultimately - to support Theorem 4.16 with a simple and convincing argument, albeit valid only in a special case.

Proposition 5.3: Let M be a compact hyperkähler manifold, I an induced complex structure, F a torsion-free coherent sheaf on (M,I) with isolated singularities, and $Z \subset (M,I)$ a finite set containing the singular points of F. Consider a hyperholomorphic connection ∇ on $F\Big|_{M\backslash Z}$. Then ∇ is admissible, in the sense of Definition 4.6.[1]

Proof: Consider the curvature as a form $\Theta \in \Lambda^{1,1}(M) \otimes \mathfrak{su}(B)$, where $\mathfrak{su}(B)$ denotes the Lie algebra of traceless skew-Hermitian endomorphisms of B Let $r_2 \in \Lambda^{2,2}(M)$ be the form

$$r_2 := \frac{\sqrt{-1}}{2\pi^2} \operatorname{Tr}(\Theta \wedge \Theta)$$

representing, by Gauss-Bonnet formula, the cohomology class $2c_2(B) - \frac{n-1}{n}c_1(B)^2$. Let ω be the Kähler form of (M, I), and $\operatorname{Vol}(M)$ is volume form. By the Hodge-Riemann relations, for $\Lambda\Theta = 0$, we have

$$\operatorname{Tr}(\Theta \wedge \Theta) \wedge \omega^{n-2} = (4(n^2 - n))^{-1} \|\Theta\|^2 \operatorname{Vol}(M)$$

(see, e.g. [BS] or [V3]). Therefore, Θ is square-integrable if and only if $r_2 \wedge \omega^{n-1}$ lies in $L^1(M)$.

We use the following fundamental lemma; we shall complete the proof of Proposition 5.3 at the end of this section.

Lemma 5.4: Let M, $\dim_{\mathbb{C}} M = n$, $n > 2$ be a hyperkähler manifold, (B, ∇) a holomorphic bundle with a hyperholomorphic connection, Θ its curvature, and $r_2 := \frac{\sqrt{-1}}{2\pi^2} \operatorname{Tr}(\Theta \wedge \Theta)$ be the corresponding 4-form, representing (by Gauss-Bonnet) $2c_2(B) - \frac{n-1}{n}c_1(B)^2$. Consider the $(n-1, n-1)$-form $r_2 \wedge \omega^{n-3}$. Then $r_2 \wedge \omega^{n-3}$ is positive.[2]

Proof: The statement of Lemma 5.4 is essentially linear-algebraic. Let $x \in M$ be a point, and $z_1, ...z_n, \overline{z}_1, ...\overline{z}_n$ a standard basis in the complexified cotangent space $T_x^*M \otimes \mathbb{C}$, such that the Kähler form is written as

$$\omega\Big|_x = \sqrt{-1} \sum_i z_i \wedge \overline{z}_i.$$

Denote by $w_{ij} \in \Lambda^{n-1,n-1}(M)$ the form

$$w_{ij} := z_1 \wedge ...\check{z}_i... \wedge z_n \wedge \overline{z}_1 \wedge ...\check{\overline{z}}_j... \wedge \overline{z}_n$$

[1] Since ∇ is Yang-Mills, it is admissible if and only if its curvature is square-integrable.

[2] Since the statement of Lemma 5.4 is local, it is also true for an admissible hyperholomorphic connection on a reflexive sheaf.

where $z_1 \wedge ... \wedge \check{z}_i \wedge ... \wedge z_n$ denotes a product of all z_k *except* z_i.

Write $r_2 \wedge \omega^{n-3}$ in this basis as

$$r_2 \wedge \omega^{n-3} = \sum_{i,j} B_{ij} w_{ij}, \quad B_{ij} \in \mathbb{R}$$

To prove Lemma 5.4, we need to show that

$$B_{ii} \geqslant 0, \ i = 1, ...n. \tag{5.1}$$

Indeed, positivity of a real $(n-1, n-1)$-form ν is equivalent to the positivity of a product $\sqrt{-1}\, \nu \wedge z \wedge \overline{z}$, for each $z \in \Lambda^{1,0}(M)$. Choosing the basis $z_1, ..., z_n \in \Lambda^{1,0}(M)$ in such a way that $z_i = z$, we find that whenever $B_{1,1} \geqslant 0$ for each orthonormal basis $z_1, ..., z_n$, the form $r_2 \wedge \omega^{n-3}$ is positive.

Write $\Theta\Big|_x$ as

$$\Theta\Big|_x = \sum_{i,j} z_i \wedge \overline{z}_j A_{ij}$$

where $A_{ij} \in \mathfrak{su}\left(B\Big|_x\right)$. An easy calculation implies

$$B_{ii} = -(n-3)! \sum_{k,l} \operatorname{Tr}(A_{kl}^2) + (n-3)! \sum_{k,l} \operatorname{Tr}(A_{kk} A_{ll}) \quad k \neq l \neq i \tag{5.2}$$

(the sum is performed over all $k, l = 1, ...n$, satisfying $k \neq l \neq i$).

The first summand of the right hand side of (5.2) is non-negative, because $A_{kl} \in \mathfrak{su}\left(B\Big|_x\right)$, and the Killing form on $\mathfrak{su}\left(B\Big|_x\right)$ is negative definite.

To prove (5.1), it remains to show that the second summand of the right hand side of (5.2) is non-negative:

$$C_{ii} := \sum_{k,l} \operatorname{Tr}(A_{kk} A_{ll}) \geqslant 0 \quad (k \neq l \neq i). \tag{5.3}$$

Clearly,

$$C_{ii} = \sum_{k \neq l} \operatorname{Tr}(A_{kk} A_{ll}) - 2 \left(\sum_{k=1,...,i-1,i+1,...n} \operatorname{Tr}(A_{kk} A_{ii}) \right). \tag{5.4}$$

Since $\Lambda(\Theta) = 0$, we have

$$\sum_{k=1}^{n} A_{kk} = 0. \tag{5.5}$$

Therefore,

$$\sum_{k=1,...,i-1,i+1,...n} \operatorname{Tr}(A_{kk}A_{ii}) = -\operatorname{Tr}(A_{ii}^2). \tag{5.6}$$

Plugging (5.6) into (5.4), we obtain that (5.5) gives

$$\begin{aligned} C_{ii} = \sum_{k\neq l} \operatorname{Tr}(A_{kk}A_{ll}) + 2\operatorname{Tr}(A_{ii}^2) &= -\sum_{k} \operatorname{Tr}(A_{kk}^2) + 2\operatorname{Tr}(A_{ii}^2) \\ &= -\left(\sum_{k=1,...,i-1,i+1,...n} \operatorname{Tr}(A_{kk}^2)\right) + \operatorname{Tr}(A_{ii}^2) \end{aligned} \tag{5.7}$$

The formula (5.7) was obtained using only the Yang-Mills property of the connection; it is true for all Kähler manifolds. Now recall that ∇ is hyperholomorphic. Renumbering the basis $z_1, ...z_n$, we may assume that the number i is odd. Fix the standard quaternion triple I, J, K. This fixes a choice of a holomorphic symplectic form (2.1). Changing $z_1, ..z_{i-1}, z_{i+1}, ...z_n$ if necessary, we may also assume that the holomorphic symplectic form is written as

$$\Omega = z_1 \wedge z_2 + ... + z_i \wedge z_{i+1} \wedge ... + z_{n-1} \wedge z_n$$

Let $J \in SU(2)$ be an operator of $SU(2)$ given by $J \in \mathbb{H}$. An easy calculation insures that J maps the 2-form $z_i \wedge \overline{z}_i$ to $-z_i \wedge \overline{z}_i$. Since Θ is $SU(2)$-invariant, we obtain that $A_{ii} = -A_{i+1,i+1}$. Plugging this into (5.7), we find

$$C_{ii} = -\sum_{k\neq i,i+1} \operatorname{Tr}(A_{kk}^2)$$

Since the Killing form is negative definite, the number C_{ii} is non-negative. This proves Lemma 5.4. ■

Return to the proof of Proposition 5.3. We have shown that $r_2 \wedge \omega^{n-3}$ is a positive $(n-1, n-1)$-form; this form is also closed, and smooth outside of a complex analytic subset of codimension n. By Lemma 5.2, such form is L^1-integrable. We proved Proposition 5.3. ■

6 $SU(2)$-invariance of the Chern class

6.1 The Dolbeault spectral sequence and the Hodge filtration

Further on in this section, we shall need the following proposition

Proposition 6.1: Let X be a complex manifold, and $Z \subset X$ a real analytic subvariety admitting a stratification by smooth real analytic subvarieties of even dimension. Assume that $\operatorname{codim}_{\mathbb{R}} Z \geqslant 2m$, $m \geqslant 2$. Let $U = X \backslash Z$, and let B be a holomorphic bundle on X. Consider the natural map of holomorphic cohomology $\varphi: H^n(X,B) \longrightarrow H^n(U,B)$. Then φ is an isomorphism, for $n \leqslant m-2$.

Remark 6.2: For $n=0$, Proposition 6.1 becomes the well known Hartogs theorem. For Z complex analytic, Proposition 6.1 is also well known ([Sch]). Further on, we shall use Proposition 6.1 when X is a hyperkähler manifold with an induced complex structure J, and $Z \subset M$ a complex analytic subvariety of (M,I).

The proof of Proposition 6.1.

Using the Meyer-Vietoris exact sequence, we find that it suffices to prove Proposition 6.1 when X is an open ball and the bundle B is trivial. Using induction by $\dim Z$, we may also assume that Z is smooth (otherwise, prove Proposition 6.1 for $X = X \backslash Sing(Z)$, and then apply Proposition 6.1 to the pair $(X,\ Sing(Z))$.

Shrinking X further and applying Meyer-Vietoris, we may assume that there exists a smooth holomorphic map $f: X \longrightarrow Y$ inducing a real analytic isomorphism $f: Z \xrightarrow{\sim} Y$. Shrinking X again, we assume that $X = Y \times B$, where B is an open ball in $\mathbb{C}^r$, $r \geqslant m$, and $f: X \longrightarrow Y$ is the standard projection map. Denote by $g: X \longrightarrow B$ the other standard projection map.

Consider the Künneth decomposition of the differential forms

$$\Lambda^{0,a}(U) = \bigoplus_{b+c=a} f^*\Lambda^{0,b}(Y) \otimes_{C^\infty(X\backslash Z)} g^*\Lambda^{0,c}(B) \tag{6.1}$$

Decomposing the Dolbeault complex of U in accordance with (6.1), we obtain

$$\overline{\partial} = \overline{\partial}_B + \overline{\partial}_Y, \tag{6.2}$$

where $\overline{\partial}_B$, $\overline{\partial}_Y$ are the Dolbeault differentials on $\Lambda^{0,*}(B)$, $\Lambda^{0,*}(Y)$. Consider the bicomplex spectral sequence, associated with the bicomplex (6.1) and the decomposition (6.2). The cohomology of the complex $(f^*\Lambda^{0,*}(B), \overline{\partial}_B)$ form a C^∞-bundle on Y, with the fibers in $y \in Y$ identified with $H^q(\mathcal{O}_{f^{-1}(y)})$. Therefore, the E_1-term of this spectral sequence, $H^*(\Lambda^{0,*}(U), \overline{\partial}_B)$ can be identified with the space of global sections of a graded C^∞-bundle $R^{p,q}$ on

Y,

$$R^{p,q}\Big|_y = H^q(\mathcal{O}_{f^{-1}(y)}) \otimes_{\mathbb{C}} \Lambda^{0,*}(T_yY).$$

By construction, the fibers $f^{-1}(y)$ are isomorphic to an open ball without a point: $B^r\backslash pt$. The cohomology of the structure sheaf on $B^r\backslash pt$ are well known; in particular, we have an isomorphism $H^i(\mathcal{O}_{f^{-1}(y)}) \cong H^i(B)$, $i \leqslant r-2$ ([Sch]). Therefore, the natural functorial morphism from cohomology of X to the cohomology of U induces an isomorphism

$$E_1^{p,q}(X) \cong E_1^{p,q}(U), \tag{6.3}$$

for $q \leqslant r-2$. This spectral sequence converges to $H^*(U, \mathcal{O}_U)$. Therefore, (6.3) implies an isomorphism $H^n(\mathcal{O}_X) \cong H^n(\mathcal{O}_U)$, for $n \leqslant m-2$. We have proved Proposition 6.1. ■

Corollary 6.3: In assumptions of Proposition 6.1, consider the natural map $H^{2n}(X) \xrightarrow{\rho} H^{2n}(U)$. Since $n \leqslant m-2$, $\operatorname{codim}_{\mathbb{R}} Z > 2n$, and ρ is an isomorphism. Let $F^0(X) \subset F^1(X) \subset ... \subset F^{2n}(X) = H^{2n}(X)$, $F^0(U) \subset F^1(U) \subset ... \subset F^{2n}(U) = H^{2n}(U)$ be the Hodge filtration on the cohomology of X and U. Assume that X is compact and Kähler. Then ρ induces an isomorphism

$$\rho: \ F^i(X) \longrightarrow F^i(U)$$

for $i \leqslant n$.

Proof: Consider the E_2-term of the Dolbeault spectral sequence. Since $E_2^{p,q}(X) = H^q(\Omega^p(X))$ and $E_2^{p,q}(U) = H^q(\Omega^p(U))$, the restriction map $E_2^{2n-i,i}(X) \longrightarrow E_2^{2n-i,i}(U)$ is an isomorphism for $i \leqslant n$ (Proposition 6.1). By definition, F^r is a union of all elements of $\oplus_{i \leqslant r} E_2^{2n-i,i}$ which survive under the higher differentials, up to the images of these differentials. On the other hand, X is Kähler and compact, hence the Dolbeault spectral sequence of X degenerates in E^2. Therefore, the map $\rho: \ F^i(X) \longrightarrow F^i(U)$ is surjective. It is injective because $H^{2n}(X) \xrightarrow{\rho} H^{2n}(U)$ is an isomorphism. ■

Corollary 6.4: Let X be a compact Kähler manifold, $Z \subset X$ a real analytic subvariety admitting a stratification by smooth real analytic subvarieties of even dimension, $\operatorname{codim}_{\mathbb{R}} Z \geqslant 6$, and $U := X\backslash Z$. Given a closed $(1,1)$-form η on U, the corresponding cohomology class $[\eta] \in H^2(U) = H^2(X)$ has Hodge type $(1,1)$.

Proof: Since the η is a $(1,1)$-form, the cohomology class $[\eta] \in H^2(U)$ belongs to the $F^1(U)$-term of the Hodge filtration. By Corollary 6.3,

$$[\eta] \in F^1(X) = H^{2,0}(X) \oplus H^{1,1}(X). \tag{6.4}$$

Replacing the complex structure I on X by $-I$, we obtain $H^{2,0}(X, I) = H^{0,2}(X, -I)$. Applying the same argument to the cohomology class $[\eta]$ on $(X, -I)$, we obtain

$$[\eta] \in F^1(X, -I) = H^{1,1}(X) \oplus H^{0,2}(X). \tag{6.5}$$

Comparing (6.4) and (6.5), we obtain Corollary 6.4. ■

6.2 Closed $SU(2)$-invariant forms

The main result of this section is the following theorem.

Theorem 6.5: Let M be a compact hyperkähler manifold, I an induced complex structure and F a weakly hyperholomorphic sheaf on (M, I). Then the Chern class $c_1(F)$ is $SU(2)$-invariant.

Proof: Let $Z \subset (M, I)$ be the singular set of F. By Lemma 4.4, $\operatorname{codim}_{\mathbb{C}} Z \geqslant 3$. Consider the form $\eta := \operatorname{Tr}\Theta$ on $U := M \backslash Z$, where Θ is the curvature of $F\Big|_U$. Since $\operatorname{codim}_{\mathbb{C}} Z \geqslant 3$, we have $H^2(M) = H^2(U)$. Clearly, the cohomology class $[\eta] \in H^2(U) = H^2(M)$ is equal to $c_1(F)$.

Let L be an arbitrary induced complex structure on M. Applying Corollary 6.4 to the Kähler manifold (M, L) and a closed $(1,1)$-form η on $M \backslash Z$, we find that the cohomology class $[\eta] = c_1(F)$ is of type $(1,1)$ with respect to L. By Lemma 2.6, $c_1(F)$ is $SU(2)$-invariant. ■

7 Positivity and hyperkähler geometry

Let M be a compact hyperkähler manifold, $\dim_{\mathbb{R}} M = 2n$, I an induced complex structure, and ω_I the corresponding Kähler form (Section 2). Given $\eta \in H^2(M)$, we define

$$\deg_I(\eta) := \int_M \eta \wedge \omega_I^{n-1}.$$

A 2-form is called **pure of weight 2** if it is a sum of forms which lie in 3-dimensional irreducible $SU(2)$-subrepresentations of $\Lambda^2(M)$.

The aim of this section is the following theorem.

Theorem 7.1: In the above assumptions, let $Z \subset (M, I)$, $\operatorname{codim}_{\mathbb{C}} Z \geqslant 3$ be a closed complex subvariety, and $H^2(M) \longrightarrow H^2(M\backslash Z)$ the induced isomorphism. Consider a closed (1,1)-form η on $(M\backslash Z, I)$, and let $[\eta]$ be its cohomology class in $H^2(M\backslash Z) = H^2(M)$. Assume that η admits a decomposition $\eta = \eta_0 + \eta_+$, where η_0 is $SU(2)$-invariant, and η_+ is positive and pure of weight 2. Then $\deg_I([\eta]) \geqslant 0$, and the equality is reached if and only if $\eta_+ = 0$.

Remark 7.2: The "if" part of the last statement is an immediate consequence of Lemma 3.8, Lemma 2.6 and Corollary 6.4.

The proof of Theorem 7.1 takes the rest of this Section.

Lemma 7.3: Let M be a hyperkähler manifold, I, J, K the standard triple of induced complex structures, and η a $(1,1)$-form on (M, I). Consider the decomposition $\eta = \eta_+ + \eta_0$, where η_0 is $SU(2)$-invariant, and η_+ is pure of weight 2. Consider the Hodge decomposition

$$\eta = \eta_K^{2,0} + \eta_K^{1,1} + \eta_K^{0,2} \tag{7.1}$$

associated with K. Then

(i)
$$\eta_K^{1,1} = \eta_0, \quad \text{and} \quad \eta_K^{2,0} + \eta_K^{0,2} = \eta_+. \tag{7.2}$$

(ii) The correspondence $\eta_+ \longrightarrow \eta_K^{2,0}$ induces an isomorphism between the bundle $\Lambda_K^{2,0}(M)$ and the bundle $\Lambda_{I,+}^{1,1}(M)$ of (1,1)-forms of weight 2.

(iii) Moreover, this identification maps the real structure $\eta_K^{2,0} \longrightarrow I(\overline{\eta}_K^{2,0})$ to the standard real structure on $\Lambda_{I,+}^{1,1}(M)$.

Proof: Let K act on $\Lambda^*(M)$ multiplicatively as follows

$$K(x_1 \wedge x_2 \wedge ...) := K(x_1) \wedge K(x_2) \wedge ...$$

Denote the action of I in the same way. Since the eigenvalues of K on $\Lambda^1(M)$ are $\pm\sqrt{-1}$, the operator K has eigenvalues ± 1 on $\Lambda^2(M)$: it acts on $\Lambda_K^{1,1}(M)$ as 1, and on $\Lambda_K^{2,0}(M) \oplus \Lambda_K^{0,2}(M)$ as -1:

$$K\Big|_{\Lambda_K^{1,1}(M)} = 1, \quad K\Big|_{\Lambda_K^{2,0}(M)\oplus\Lambda_K^{0,2}(M)} = -1 \tag{7.3}$$

The central element $c \in SU(2)$ acts trivially on $\Lambda^2(M)$; therefore, I and K commute on $\Lambda^2(M)$. We obtain that K preserves the fixed space of I:

$$K(\Lambda^{1,1}_I(M)) = \Lambda^{1,1}_I(M).$$

If a 2-form η is fixed by I and K, it is also fixed by $K \circ I = J$. By (7.3), this means that η is of type $(1,1)$ with respect to I, J and K. A simple linear-algebraic argument implies that η is of type $(1,1)$ with respect to all induced complex structures. By Lemma 2.6, this implies that η is $SU(2)$-invariant. We proved the first equation of (7.2).

Now, if η is pure of weight 2, K acts on η as on the Kähler form ω_I; it is easy to check, then, that $K(\eta) = -\eta$. This implies that $\eta \in \Lambda^{2,0}_K(M) \oplus \Lambda^{0,2}_K(M)$. Conversely, if η belongs to $\Lambda^{2,0}_K(M)$ or $\Lambda^{0,2}_K(M)$, it has weight ± 2 with respect to the Cartan algebra element, corresponding to $\sqrt{-1}\,K \in \mathfrak{su}(2) \subset \mathbb{H}$; therefore, η is pure of weight 2. This proves (7.2). This identifies the bundles $\Lambda^{2,0}_K(M)$, $\Lambda^{0,2}_K(M)$ and the bundle $\Lambda^{1,1}_{I,+}(M)$ of $(1,1)$-forms of weight 2. This identification is compatible with the real structure on forms, and this gives the last assertion of Lemma 7.3. ■

Definition 7.4: Let M be a hyperkähler manifold, I, J, K the standard triple of induced complex structures, and $\rho \in \Lambda^{2,0}_K(M)$ a (2,0)-form on (M, K) satisfying $I(\rho) = \overline{\rho}$. Consider the real part $\mathsf{Re}(\rho)$ of ρ. Then

$$2I(\mathsf{Re}(\rho)) = I(\rho) + I(\overline{\rho}) = \rho + \overline{\rho} = 2\,\mathsf{Re}(\rho).$$

Therefore, $\mathsf{Re}(\rho)$ lies inside $\Lambda^{1,1}_I(M, \mathbb{R})$. We say that ρ is **K-positive** if the form $Re(\rho)$ is positive on (M, I).

Theorem 7.1 is an immediate corollary of the following Proposition, which is a hyperkähler version of Lemma 5.2.

Proposition 7.5: Let M be a hyperkähler manifold, I, J, K the standard triple of induced complex structures, $Z \subset (M, I)$ a compact complex subvariety, $\operatorname{codim}_{\mathbb{C}} Z \geqslant 3$, and $\rho \in \Lambda^{2,0}_K(M)$ a ∂_K-closed $(2,0)$-form on $(M \backslash Z, K)$, which satisfies $\rho = I(\overline{\rho})$. Assume that ρ is K-positive. Then (i) ρ is locally L^1-integrable on M, and (ii) ∂_K-closed as a current on M.

We prove Proposition 7.5 in Section 8. Let us show how to deduce Theorem 7.1 from Proposition 7.5.

Let $\Omega_K := \omega_I + \sqrt{-1}\,\omega_J$ be the holomorphic symplectic form on (M, K), and $n := \dim_{\mathbb{H}} M$. Consider the $(4n-2)$-form

$$E := \Omega_K^{n-1} \wedge \overline{\Omega}_K^n.$$

Further on, we shall need the following lemma.

Lemma 7.6: Let $[\eta] \in H_I^{1,1}(M)$ be a cohomology class, on a compact hyperkähler manifold (M, I, J, K), $\dim_{\mathbb{R}} M = 4n$. Then

$$\lambda_n \deg_I[\eta] = \mathsf{Re}\left(\int_M [\eta]_K^{2,0} \wedge E\right), \tag{7.4}$$

where λ_n is a positive constant, depending on n only, $[\eta]_K^{2,0}$ is a $(2,0)$-part of $[\eta]$ with respect to K, and E the $(4n-2)$-form constructed above.

Proof: By construction, $[\eta]_K^{2,0} \wedge E = [\eta] \wedge E$. Also, $\deg_I[\eta] = \int_M [\eta] \wedge \omega_I^{2n-1}$. Since $[\eta] \in H_I^{1,1}(M)$, we have

$$[\eta] \wedge E = [\eta] \wedge E_I^{2n-1,2n-1},$$

where $E_I^{2n-1,2n-1}$ is $(2n-1, 2n-1)$-part of E, taken with respect to I. Then (7.4) is implied by the relation

$$E_I^{2n-1,2n-1} = \lambda_n^{-1} \omega_I^{2n-1} \tag{7.5}$$

This relation is proven by direct calculation (see e.g. [V2, Section 3], where the whole algebra generated by ω_I, ω_J and ω_K is explicitly calculated). ∎

Given a ∂_K-exact 2-form $\mu = \partial_K \mu'$, we have

$$d(E \wedge \mu') = E \wedge \mu.$$

Therefore, the number

$$E(\rho) := \int_M \rho \wedge E$$

depends only on the ∂_K-Dolbeault cohomology class of $\rho \in \Lambda^{2,0}(M)$.

Return to the assumptions of Theorem 7.1. The form $\eta_K^{2,0}$ by construction satisfies assumptions of Proposition 7.5. Therefore, $\eta_K^{2,0}$ is locally L^1-integrable. We shall interpret $\deg_I[\eta]$ as integral $\lambda_n^{-1} \int_M \eta_K^{2,0} \wedge E$, using the following homological argument.

Lemma 7.7: Let M be a compact Kähler manifold, $Z \subset M$ a real analytic subvariety of codimension at least 6. Then the natural restriction map $H^2(\mathcal{O}_M) \longrightarrow H^2(\mathcal{O}_{M\backslash Z})$ is injective.

Proof There is a natural map φ from the p-th de Rham cohomology of a complex manifold M to its p-th holomorphic cohomology $H^p(M, \mathcal{O}_M)$: given a closed p-form η, the $(0,p)$-part of η is $\overline{\partial}$-closed and represents a class in $H^p(M, \mathcal{O}_M)$. Consider the commutative diagram

$$\begin{array}{ccc} H^2_{DR}(M) & \xrightarrow{j} & H^2_{DR}(M\backslash Z) \\ \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow{\scriptstyle\varphi_Z} \\ H^2(\mathcal{O}_M) & \xrightarrow{j_0} & H^2(\mathcal{O}_{M\backslash Z}) \end{array} \tag{7.6}$$

(here H^*_{DR} denotes the de Rham cohomology group). By definition, $\ker\varphi$, $\ker\varphi_Z$ is the $F^1H^2_{DR}$-part of $H^2_{DR}(M)$, $H^2_{DR}(M\backslash Z)$, where F^i denotes the Hodge filtration on cohomology. The Dolbeault spectral sequence gives the E_2-term corresponding to $F^1H^2_{DR}(V)$ for any complex manifold V as follows:

$$0 \longrightarrow H^0(\Omega^2(V)) \longrightarrow E_2(F^1H^2_{DR}(V)) \longrightarrow H^1(\Omega^1(V)) \longrightarrow 0.$$

Using the functoriality of Dolbeault spectral sequence, we obtain the following diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to & H^0(\Omega^2(M)) & \to & E_2(F^1H^2_{DR}(M)) & \to & H^1(\Omega^1(M)) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & H^0(\Omega^2(M\backslash Z)) & \to & E_2(F^1H^2_{DR}(M\backslash Z)) & \to & H^1(\Omega^1(M\backslash Z)) & \to 0 \end{array} \tag{7.7}$$

Using Proposition 6.1, we obtain that the vertical arrows of (7.7) are isomorphisms. Therefore, the $E_2(F^1H^2_{DR})$-terms for M and $M\backslash Z$ are isomorphic. Since the Dolbeault spectral sequence for M degenerates in E_2, and $E_2(F^1H^2_{DR}(M)) = E_2(F^1H^2_{DR}(M\backslash Z))$, all differentials d_i, $i > 2$ for the Dolbeault spectral sequence of $M\backslash Z$ vanish on $F^1H^2_{DR}(M\backslash Z)$. Also, for $E_2(F^qH^p_{DR}(M))$ $(p, q \leqslant 1)$, the natural restriction map

$$E_2(F^qH^p_{DR}(M)) \longrightarrow E_2(F^qH^p_{DR}(M\backslash Z))$$

is an isomorphism, as follows from Proposition 6.1, hence the differentials d_i, $i > 2$ vanish on the terms $E_2(F^qH^p_{DR}(M\backslash Z))$ $(p, q \leqslant 1)$ as well. This implies that the Dolbeault spectral sequence degenerates in $E_2(F^1 H^2_{DR}(M\backslash Z))$, and the bottom row of (7.7) gives an exact sequence

$$0 \longrightarrow H^0(\Omega^2(M\backslash Z)) \longrightarrow F^1 H^2_{DR}(M\backslash Z) \longrightarrow H^1(\Omega^1(M\backslash Z)) \longrightarrow 0.$$

Applying (7.7) again, we obtain that the restriction map induces an isomorphism

$$F^1 H^2_{DR}(M\backslash Z) \cong F^1 H^2_{DR}(M).$$

In terms of (7.6) this is interpreted as an isomorphism $\ker\varphi = \ker\varphi_Z$. The left arrow of (7.6) is surjective because M is Kähler. An elementary diagram chasing using surjectivity of φ and $\ker\varphi = \ker\varphi_Z$ implies that j_0 is indeed injective. We proved Lemma 7.7. ■

Return now to the proof of Theorem 7.1. Consider $[\eta]$ as an element of $H^2_{DR}(M\backslash Z)$. Then $\varphi_Z([\eta]) = [\eta_K^{0,2}]_{\overline{\partial}}$, where $\eta_K^{0,2}$ denotes the $\eta_K^{0,2}$-component of η, and $[\cdot]_{\overline{\partial}}$ its Dolbeault class in $H^2(\mathcal{O}_{M\backslash Z})$. Now, $[\eta_K^{0,2}]_{\overline{\partial}}$ belongs to the image of $j_0(H^2(\mathcal{O}_M))$, because $\eta_K^{0,2} = \overline{\eta_K^{2,0}}$ is L^1-integrable, and the cohomology of currents are equal to cohomology of forms. Lemma 7.7 implies now that

$$\varphi([\eta]) = \varphi_Z([\eta]) = [\eta_K^{0,2}]_{\overline{\partial}}.$$

Using (7.4) we may compute $\deg_I[\eta]$ as an integral

$$\deg_I[\eta] = \lambda_n^{-1}\int_M \eta_K^{2,0}\wedge E$$

(this makes sense, because $\eta_K^{2,0}$ is L^1-integrable). Then

$$\deg_I[\eta] = 2^{-(n-1)}\int_M \eta_K^{2,0}\wedge E = \int_M \eta_+\wedge\omega_I^{2n-1} \tag{7.8}$$

(the first equation holds by (7.5), and the second one is implied by $(\eta_K^{2,0})_I^{1,1} = \eta_+$, which is essentially a statement of Lemma 7.3).

Since η_+ is a positive 2-form, the integral (7.8) is non-negative, and positive unless $\eta_+ = 0$. We have reduced Theorem 7.1 to Proposition 7.5. ■

8 Positive $(2,0)$-forms on hyperkähler manifolds

The purpose of this Section is to prove Proposition 7.5. We deduce this result from the quaternionic version of the classical Sibony's Lemma and Skoda-El Mir Theorem ([E], [Sk], [Sib], [D2]), which is proven in [V8].

Theorem 8.1: Let (M, I, J, K, g) be a hyperkähler manifold, and $Z \subset (M, I)$ a closed complex subvariety. Consider a (2,0)-form η on $(M, I)\backslash Z$, which satisfies the following conditions

(a) $J\eta = \overline{\eta}$

(b) $\eta(x, J\overline{x}) \geqslant 0$, for any $x \in T^{1,0}(M, I)$.

(c) $\partial_I \eta = 0$.

Then

(i) (Sibony's Lemma) The form η is locally integrable around Z, if Z is compact and satisfies $\operatorname{codim}_{\mathbb{C}} Z \geqslant 3$.

(ii) (Skoda-El Mir theorem) Assume that η is locally integrable around Z. Consider the trivial extension $\widetilde{\eta}$ of η to M as a $(2,0)$-current on (M, I). Then $\partial_I \widetilde{\eta} = 0$.

Proof: See [V8], Theorem 1.2 and Theorem 1.3. ∎

Now we deduce Proposition 7.5 from Theorem 8.1.

We work in assumptions of Proposition 7.5. Let $\rho \in \Lambda^{2,0}(M\backslash Z, K)$ be a K-positive, ∂_K-closed form on $M\backslash Z$. Since $SU(2)$ acts on the quaternionic triples (I, J, K) transitively, there exists $g \in SU(2)$ mapping K to I and I to J, with $SU(2)$ acting on quaternions by conjugation. Let

$$\eta := g(\rho) \in \Lambda^{2,0}(M\backslash Z, I)$$

be the $(2,0)$-form on $(M\backslash Z, I)$ corresponding to ρ under the isomorphism induced by g.

Since $I(\rho) = \overline{\rho}$, η satisfies $J(\eta) = \overline{\eta}$. Also, K-positivity of ρ is equivalent to $\eta(x, I\overline{x}) \geqslant 0$, for any $x \in T^{1,0}(M, I)$. To apply Theorem 8.1 to η, it remains to show that $\partial_I \eta = 0$.

Consider the complex vector space

$$\mathcal{H} = \langle d, IdI, JdJ, KdK \rangle \subset \operatorname{End}(\Lambda^*(M)),$$

generated by the de Rham differential and its twists. Clearly, $\mathcal{H}$ is preserved by the natural action of $SU(2)$ on $\operatorname{End}(\Lambda^*(M))$. As a representation of

$SU(2)$, $\mathcal{H}$ has weight 1, that is, the standard generator $\sqrt{-1}\,I$ of the Cartan subalgebra of $SU(2)$ acts on $\mathcal{H}$ with eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$. Therefore, $\mathcal{H}$ is isomorphic to a sum of two irreducible, 2-dimensional representations of $SU(2)$. This implies that the space

$$\mathcal{H}_K^{1,0} := \{\delta \in \mathcal{H} \ \ | \ \ K(\delta) = \sqrt{-1}\,\delta\}$$

is 2-dimensional. This space is clearly generated by ∂_K and $I\overline{\partial}_K I$. For any $\delta \in \mathcal{H}_K^{1,0}$, $\delta(\rho) = 0$, because $\partial_K(\rho) = I\overline{\partial}_K I(\rho) = 0$. Since

$$g(\mathcal{H}_K^{1,0}) = \mathcal{H}_I^{1,0} := \{\delta \in \mathcal{H} \ \ | \ \ I(\delta) = \sqrt{-1}\,\delta\},$$

all elements of $\mathcal{H}_I^{1,0}$ vanish on $g(\rho) = \eta$. Therefore, $\partial_I \eta = 0$. We obtain that η satisfies conditions (a), (b), (c) of Theorem 8.1.

By Theorem 8.1 (i), η is locally integrable; this proves Proposition 7.5 (i). By Theorem 8.1 (ii), a trivial extension of η to M is ∂_I-closed as a current. Backtracking the above argument, we find that this is equivalent to ∂_K-closedness of the trivial extension of ρ. We proved Proposition 7.5 (ii).

9 Stability of weakly hyperholomorphic sheaves

In this section, we use Theorem 6.5 and Theorem 7.1 to prove Theorem 4.16.

Let M be a complex hyperkähler manifold, I an induced complex structure, and F a reflexive sheaf on (M, I). Assume that F is weakly hyperholomorphic.

We are going to show that F is polystable. On contrary, let $F' \subset F$ be a destabilizing subsheaf. Replacing F' by its reflexization if necessary, we may assume that F' is reflexive.

Let Z be the union of singular sets of F and F'. By Lemma 4.4, $\operatorname{codim}_{\mathbb{C}} Z \geqslant 3$. Denote by U the complement $U := M\backslash Z$. On U, both sheaves F and F' are bundles. Let $\Theta \in \Lambda^{1,1}(U) \otimes \operatorname{End}(F)$ be the curvature of F, and $\Theta' \in \Lambda^{1,1}(U) \otimes \operatorname{End}(F')$ the curvature of F'. Denote by $A \in \Lambda^{1,0}(U, \operatorname{Hom}(F/F', F'))$ the so-called *second fundamental form of a subbundle* F' ([GH]). The curvature of F' can be expressed through Θ and A as

$$\Theta' = \Theta\Big|_{F'} - A \wedge A^{\perp}, \tag{9.1}$$

where $A^{\perp} \in \Lambda^{1,0}(U, \operatorname{Hom}(F', F/F'))$ is the Hermitian adjoint of A, and the form

$$\Theta\Big|_{F'} \in \Lambda^{1,1}(U) \otimes \operatorname{End}(F')$$

is obtained from Θ by the orthogonal projection $\operatorname{End}(F) \longrightarrow \operatorname{End}(F')$.

The following claim is quite elementary.

Claim 9.1: Let F be a reflexive coherent sheaf over a complex manifold X, and $F' \subset F$ a reflexive subsheaf. Assume that F is equipped with a Hermitian structure outside of singularities. Assume, moreover, that the second fundamental from of $F' \subset F$ vanishes. Then

$$F \cong F' \oplus (F/F')^{**}, \tag{9.2}$$

where $(F/F')^{**} = \operatorname{Hom}(\operatorname{Hom}(F/F', \mathcal{O}_X), \mathcal{O}_X)$ denotes the reflexive hull of F/F'.

Proof: Let Z be the union of singular sets of F and F'. By Lemma 4.4, $\operatorname{codim}_{\mathbb{C}} Z \geqslant 3$. Consider the orthogonal decomposiion

$$F\Big|_{X\backslash Z} \cong F'\Big|_{X\backslash Z} \oplus (F/F')\Big|_{X\backslash Z}, \tag{9.3}$$

By the definition of the second fundamental form, the connection on F preserves the decomposition (9.3) if and only if this form vanishes (see [GH]). Denote by $j: (X\backslash Z) \hookrightarrow X$ the natural embedding. By [OSS], Ch. II, Lemma 1.1.12 (see Remark 4.2), we have $j_* j^* F = F$, $j_* j^* F' = F'$, $j_* j^*(F/F') = (F/F')^{**}$. Comparing this with (9.3), we obtain the decomposition (9.2). Claim 9.1 is proven. ∎

Return to the proof of Theorem 4.16. By Theorem 6.5, the cohomology class $c_1(F)$ is $SU(2)$-invariant. Therefore, by Lemma 3.8, $\deg c_1(F) = 0$. To show that F is polystable, we need to show that for any reflexive subsheaf $F' \subset F$, we have $\deg c_1(F') \leqslant 0$, and if the equality is reached, then the decomposition (9.2) holds.

By (9.1), we have

$$-\operatorname{Tr}\Theta' = -\operatorname{Tr}\left(\Theta\Big|_{F'}\right) + \operatorname{Tr}(A \wedge A^{\perp}) \tag{9.4}$$

This form represents $-c_1(F')$. The first summand of the right hand side of (9.4) is $SU(2)$-invariant. Indeed, the form Θ is by our assumptions $SU(2)$-invariant, and $\Theta\Big|_{F'}$ is obtained by orthogonal projection from $\Lambda^2(M) \otimes \operatorname{End}(F)$ to $\Lambda^2(M) \otimes \operatorname{End}(F')$, but this projection obviously commutes with $SU(2)$. The second summand of the right hand side of (9.4)is manifestly positive. We arrive in the situation which is close to that dealt in Theorem 7.1.

To apply Theorem 7.1, we need a closed 2-form η on U which is a sum of an $SU(2)$-invariant form and a positive form of weight 2; however, there is no reason why $\operatorname{Tr}(A \wedge A^{\perp})$ should be pure of weight 2. Therefore, to use Theorem 7.1, we need the following elementary linear-algebraic lemma.

Lemma 9.2: Let U be a hyperkähler manifold, I an induced complex structure on U and η a positive $(1,1)$-form on (U, I). Consider the decomposition

$$\eta = \eta_0 + \eta_+,$$

where η_0 is $SU(2)$-invariant, and η_+ pure of weight 2. Then η_+ is positive; moreover, $\eta_+ = 0 \Rightarrow \eta = 0$.

Proof: Let K be an induced complex structure satisfying $I \circ K = -K \circ I$. Consider the multiplicative action of K on $\Lambda^*(M)$ defined in Lemma 7.3. By Lemma 7.3, $\eta_+ = \frac{1}{2}(\eta - K(\eta))$. On the other hand, the cone of positive forms is generated by the 2-forms

$$\eta_z := \sqrt{-1}\, z \wedge \overline{z},$$

where $z \in \Lambda^{1,0}(M, I)$. Clearly,

$$-K(\eta_z) = \sqrt{-1}\, K(z) \wedge K(\overline{z}).$$

Since $K(\overline{z}) \in \Lambda^{1,0}(M, I)$, the form $-K(\eta_z)$ is positive.

We have shown that, $\eta_+ = \frac{1}{2}(\eta - K(\eta))$ is a sum of two positive forms; this form is positive, and it is non-zero unless $\eta = 0$. This proves Lemma 9.2.■

Return to the situation described by (9.4). We have shown that the 2-form $(-\operatorname{Tr}\Theta')$ is a sum of a positive form $\mu = \operatorname{Tr}\left(A \wedge A^{\perp}\right)$ and an $SU(2)$-invariant form. Decomposing μ onto $SU(2)$-invariant and pure weight 2 parts as in Lemma 9.2, we find that $-\operatorname{Tr}\Theta'$ is a sum of an $SU(2)$-invariant form and a positive form η_+ which is pure of weight 2 with respect to the $SU(2)$-action.

Now we can apply Theorem 7.1. We find that $\deg \operatorname{Tr}\Theta' \leqslant 0$, and the equality is reached only if $\eta_+ = 0$. If F' is destabilizing, we have $\deg \operatorname{Tr}\Theta' \geqslant 0$. By Lemma 9.2, this is equivalent to $\mu = 0$, or, what is the same, $A = 0$. Now, if $A = 0$, then F splits as in Claim 9.1.

We have shown that for any destabilizing reflexive subsheaf $F' \subset F$, the sheaf F splits as $F = F' \oplus (F/F')^{**}$. This means that F is polystable.

Remark 9.3: In assumptions of Theorem 4.16, one can easily see that all stable direct summands F_i of F are also hyperholomorphic, that is, have $SU(2)$-invariant c_1, c_2. Indeed, $c_1(F_i)$ can be computed using the curvature of ∇ as indicated above, and by (9.4) it is $SU(2)$-invariant. For any coherent sheaf A, let

$$D(A) := 2c_2(A) - \frac{\operatorname{rk} A - 1}{\operatorname{rk} A} c_1(A)^2$$

be the *discriminant* of A. For any stable coherent sheaf A with $SU(2)$-invariant first Chern class, $D(A)$ satisfies the inequality

$$\int_M \omega_J^{2n-2} \wedge D(A) \leqslant \int_M \omega_I^{2n-2} \wedge D(A) \tag{9.5}$$

([V3], Claim 3.21), and the equality is reached of and only A is hyperholomorphic. Since the discriminant is additive, and (9.5) is true for the direct sum of all F_i, it is true for each summand F_i.

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Misha Verbitsky
Laboratory of Algebraic Geometry, SU-HSE,
7 Vavilova Str., Moscow, Russia, 117312
verbit@mccme.ru