ON THE EQUATIONS DEFINING AFFINE ALGEBRAIC GROUPS

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ABSTRACT. For the coordinate algebras of connected affine algebraic groups, we explore the problem of finding a presentation by generators and relations canonically determined by the group structure.

1. INTRODUCTION

Connected algebraic groups constitute a remarkable class of irreducible quasiprojective algebraic varieties. It contains the subclasses of abelian varieties and affine algebraic groups. These subclasses are basic: by Chevalley's theorem, every connected algebraic group G has a unique connected normal affine algebraic subgroup L such that G/L is an abelian variety; whence the variety G is an L-torsor over the abelian variety G/L. The varieties from these subclasses can be embedded in many ways as closed subvarieties in, respectively, projective and affine spaces. A natural question then arises as to whether there are distinguished embeddings and equations of their images, which are canonically determined by the group structure. For abelian varieties, this is the existence problem for canonically defined bases in linear systems and that of presentating homogeneous coordinate rings of ample invertible sheafs by generators and relations. These problems were explored and solved by D. Mumford [Mu 1966]. For affine algebraic groups, it is the existence problem of the canonically defined presentations of the coordinate algebras of such groups by generators and relations. We explore this problem in the present paper.

We fix as the base field an algebraically closed field k of arbitrary characteristic. In this paper, as in [Bor 1991], "variety" means "algebraic variety" in the sense of Serre [Se 1955, Subsect. 34]; every variety is taken over k.

Let G be a connected affine algebraic group and let $R_u(G)$ be its unipotent radical. In view of [Gr 1958, Props. 1, 2], [Ro 1956, Thm. 10], the underlying variety of G is isomorphic to the product of that of $G/R_u(G)$ and $R_u(G)$, and the latter is isomorphic to an affine space. Therefore, the problem under consideration is reduced to the case of reductive groups. Given this, henceforth G stands for a connected *reductive* algebraic group.

The simplest case of SL_2 is the guiding example. Take the polynomial k-algebra $k[x_1, x_2, x_3, x_4]$ in four variables x_i. The usual presentation of $k[\mathrm{SL}_2]$

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is given by the surjective homomorphism

$$\mu\colon k[x_1,x_2,x_3,x_4]\to k[\mathrm{SL}_2],\quad \mu(x_i)\left(\begin{bmatrix}a_1 & a_2\\ a_3 & a_4\end{bmatrix}\right)=a_i, \tag{1}$$

whose kernel is the ideal $(x_1x_4-x_2x_3-1)$. After rewriting, this presentation can be interpreted in terms of the group structure of SL_2 as follows.

We have $k[x_1,x_2,x_3,x_4]=k[x_1,x_3]\otimes_k k[x_2,x_4]$ and the restriction of μ to the subalgebra $k[x_1,x_3]$ (respectively, $k[x_2,x_4]$) is an isomorphism with the subalgebra $\mathcal{S}^+$ (respectively, $\mathcal{S}^-$) of $k[\mathrm{SL}_2]$ consisting of all regular functions invariant with respect to the subgroup U^+ (respectively, U^-) of all unipotent upper (respectively, lower) triangular matrices acting by right translations. Hence (1) yields the following presentation of $k[\mathrm{SL}_2]$ by generators and relations:

$$\begin{aligned} k[\mathrm{SL}_2] &\cong (\mathcal{S}^+\otimes_k\mathcal{S}^-)/\mathcal{I},\\ \mathcal{S}^+ &= k[\mu(x_1),\mu(x_3)]\cong k[x_1,x_3],\\ \mathcal{S}^- &= k[\mu(x_2),\mu(x_4)]\cong k[x_2,x_4],\\ \mathcal{I} &= \big(\mu(x_1)\otimes\mu(x_4)-\mu(x_2)\otimes\mu(x_3)-1\big). \end{aligned} \tag{2}$$

The subgroups U^+, U^- are opposite maximal unipotent subgroups of SL_2. The subalgebras $\mathcal{S}^+$, $\mathcal{S}^-$ are stable with respect to SL_2 acting by left translations, and $f:=\mu(x_1)\otimes\mu(x_4)-\mu(x_2)\otimes\mu(x_3)-1$ is the unique element of $(\mathcal{S}^+\otimes_k\mathcal{S}^-)^{\mathrm{SL}_2}$ determined by the conditions $f(e,e)=1$, $k[f]=(\mathcal{S}^+\otimes_k\mathcal{S}^-)^{\mathrm{SL}_2}$.

We show that there is an analogue of (2) for every connected reductive algebraic group G. Namely, we endow $k[G]$ with the G-module structure determined by left translations and fix in G a pair of opposite Borel subgroups B^+ and B^-. Let $U^\pm$ be the unipotent radical of $B^\pm$. Consider the G-stable subalgebras

$$\begin{aligned} \mathcal{S}^+ &:= \{f\in k[G]\mid f(gu)=f(g) \text{ for all } g\in G, u\in U^+\},\\ \mathcal{S}^- &:= \{f\in k[G]\mid f(gu)=f(g) \text{ for all } g\in G, u\in U^-\} \end{aligned} \tag{3}$$

of $k[G]$ and the natural multiplication homomorphism of k-algebras

$$\mu\colon \mathcal{S}^+\otimes_k\mathcal{S}^-\to k[G],\quad f_1\otimes f_2\mapsto f_1f_2. \tag{4}$$

For $k=\mathbb{C}$, the following conjectures were put forward in [FT 1992]:

Conjectures (D. E. Flath and J. Towber, 1992)**.**

(S) *The homomorphism μ is surjective.*
(K) *The ideal* $\ker\mu$ *in* $\mathcal{S}^+\otimes_k\mathcal{S}^-$ *is generated by* $(\ker\mu)^G$.

If these conjectures are true, then the problem under consideration is reduced to the following:

(a) find the canonically defined generators of the k-algebra $(\ker\mu)^G$,
(b) find the canonically defined presentations of $\mathcal{S}^\pm$ by generators and relations.

In [FT 1992], Conjectures (S) and (K) were proved for $k = \mathbb{C}$ and $G = \mathrm{SL}_n$, GL_n, SO_n, Sp_n by means of lengthy direct computations of some Laplace decompositions, minors, and algebraic identities between them. In Theorems 3 and 6 below we prove Conjectures (S) and (K) in full generality, with no restrictions on k and G.

In Theorems 7, 9 below we describe $\ker \mu$ as a vector space over k. In Theorem 10, we solve the above part (a) of the problem, finding the canonically defined generators of the k-algebra $(\ker \mu)^G$. We call them SL_2*-type relations* of the sought-for canonical presentation of $k[G]$ because for $G = \mathrm{SL}_2$ the element $\mu(x_1)\otimes\mu(x_4)-\mu(x_2)\otimes\mu(x_3)-1$ is just such a generator of $\mathcal{I}$ (see (2)). All of them are inhomogeneous of degree 2. If G is semisimple, they are indexed by the elements of the Hilbert basis $\mathscr{H}$ of the monoid of dominant weights of G. Note that the cardinality $|\mathscr{H}|$ of $\mathscr{H}$ is at least $\operatorname{rank} G$ with equality for simply connected G, but in the general case it may be much bigger. For instance, if $G = \mathrm{PGL}_r$, then $|\mathscr{H}| \geqslant p(r)+\varphi(r)-1$, where p and φ are, respectively, the classical partition function and the Euler function (see [Po 2011, Example 3.15]). Note that the problem of determining a full set of generators of the ideal $\ker \mu$ was formulated in [Fl 1994, Sect. 4] and, for $k = \mathbb{C}$, $G = \mathrm{SL}_n$, GL_n, SO_n, Sp_n, solved in [FT 1992] by lengthy direct computations.

For a semisimple group G whose monoid of dominant weights is freely generared (i.e., with $|\mathscr{H}| = \operatorname{rank} G$), a solution to the above part (b) of the problem in characteristic 0 was obtained (but not published) by B. Kostant; his proof appeared in [LT 1979, Thm. 1.1]. In arbitrary characteristic, such a solution is given by Theorems 1, 2, 11 below, which are heavily based on the main results of [RR 1985] and [KR 1987]. All relations in this case are homogeneous of degree 2. We call them *Plücker-type relations* of the sought-for canonical presentation of $k[G]$ because the k-algebra $\mathcal{S}^{\pm}$ for $G = \mathrm{SL}_n$ is the coordinate algebra of the affine multicone over the flag variety, and if $\operatorname{char} k = 0$, these relations are generated by the classical Plücker-type relations, obtained by Hodge [Ho 1942], [Ho 1943], that determine this multicone (see below Section 6). The set of these relations is a union of finite-dimensional vector spaces canonically determined by the group structure of G; these spaces are indexed by the elements of $\mathscr{H}\times\mathscr{H}$ and different spaces have zero intersection (see Theorem 11). Thus in this case, we obtain a canonical presentation of $k[G]$, in which all relations are quadratic and divided into two families: homogeneous relations of Plücker type and inhomogeneous relations of SL_2-type. As a parallel, we recall that any abelian variety is canonically presented as an intersection of quadrics in a projective space given by the Riemann equations; see [Ke 1989], [LB 1992].

For an arbitrary reductive group G, let $\tau\colon \widehat{G}\to G$ be the universal covering. Then $\widehat{G} = Z\times C$, where Z is a torus, C is a simply connected semisimple group, $G = \widetilde{G}/\ker\tau$, and $\ker\tau$ is a finite central subgroup. The algebra $\mathcal{S}^{\pm}$ for $\widehat{G}$ is then the tensor product of $k[Z]$ and the algebra $\mathcal{S}^{\pm}$ for C. Since the presentation of $k[Z]$ is clear, and that of $\mathcal{S}^{\pm}$ for C are given by Theorems 1, 2, 11, the above part (b) of the problem is reduced to finding a presentation for the invariant algebra of the finite abelian group $\ker\tau$.

As an illustration, in the last Section 6 we consider the example of $G = \mathrm{SL}_n$, $\mathrm{char}\, k = 0$ and describe explicitly how the ingredients of our construction and the canonical presentation of $k[G]$ look like in this case.

The preprints [Po 1995], [Po 2000] of these results in characteristic 0 have been disseminated long ago. The validity of the results in arbitrary characteristic was announced in [Po 2000]. The author is pleased by the arisen occasion to finally present the complete proofs.

Notation and conventions.

Below we use freely the standard notation and conventions of [Bor 1991], [Ja 1987], [PV 1994], and [Sh 2013]. In particular, the algebra of functions regular on a variety X is denoted by $k[X]$, the field of rational functions on an irreducible X is denoted by $k(X)$, and the local ring of X at a point x is denoted by $\mathcal{O}_{x,X}$. For a morphism $\varphi\colon X \to Y$ of varieties, $\varphi^*\colon k[Y] \to k[X]$ denotes its comorphism.

All topological terms refer to the Zariski topology; the closure of Z in X is denoted by $\overline{Z}$ (each time it is clear from the context what is X).

The fixed point set of an action of a group P on a set S is denoted by S^P. Every action $\alpha\colon H \times X \to X$ of an algebraic group H on a variety X is always assumed to be regular (the latter means that α is a morphism). For every $h \in H$, $x \in X$, we write $g \cdot x$ in place of $\alpha(g, x)$. The H-orbit and the H-stabilizer of x are denoted respectively by $H \cdot x$ and H_x. Every homomorphism of algebraic groups is assumed to be algebraic.

The additively written group of characters (i.e., homomorphisms to the multiplicative group of k) of an algebraic group H is denoted by $\mathrm{X}(H)$. The value of a character $\lambda \in \mathrm{X}(H)$ at an element $h \in H$ is denoted by h^{λ}. Given a kH-module M, its weight space with weight $\lambda \in X(H)$ is denoted by M_{λ}.

We fix in G the maximal torus

$$T := B^+ \cap B^-$$

and identify $\mathrm{X}(B^{\pm})$ with $\mathrm{X}(T)$ by means of the restriction isomorphisms $\mathrm{X}(B^{\pm}) \to \mathrm{X}(T)$, $\lambda \mapsto \lambda|_T$.

By $\mathrm{X}(T)_+$ we denote the monoid of dominant weights of T determined by B^+. Below the highest weight of every simple G-module is assumed to be the highest weight with respect to T and B^+.

We denote by w_0 be the longest element of the Weyl group of T and fix in the normalizer of T a representative $\dot{w}_0$ of w_0. We then have $\dot{w}_0 B^{\pm} \dot{w}_0^{-1} = B^{\mp}$ and $\dot{w}_0 U^{\pm} \dot{w}_0^{-1} = U^{\mp}$. For every $\lambda \in \mathrm{X}(T)_+$, we put $\lambda^* := -w_0(\lambda) \in \mathrm{X}(T)_+$.

The set of all nonnegative rational numbers is denoted by $\mathbb{Q}_{\geqslant 0}$ and we put $\mathbb{N} := \mathbb{Z} \cap \mathbb{Q}_{\geqslant 0}$.

If $m \in \mathbb{Z}$, $m > 0$, we put $[m] := \{a \in \mathbb{Z} \mid 1 \leqslant a \leqslant m\}$.

For $d \in \mathbb{N}$, we denote by $[m]_d$ the set of all increasing sequences of d elements of $[m]$ (if $d \notin [m]$, then $[m]_d = \varnothing$).

2. Proof of Conjecture (S)

For every $\lambda \in \mathrm{X}(T)$, the spaces

$$\begin{aligned} \mathcal{S}^+(\lambda) &:= \{f \in \mathcal{S}^+ \mid f(gt) = t^{\lambda} f(g) \text{ for all } g \in G, t \in T\}, \\ \mathcal{S}^-(\lambda) &:= \{f \in \mathcal{S}^- \mid f(gt) = t^{w_0(\lambda)} f(g) \text{ for all } g \in G, t \in T\}. \end{aligned} \tag{5}$$

are the finite-dimensional (see, e.g., [Ja 1987, I.5.12.c)]) G-submodules of the G-modules $\mathcal{S}^+$ and $\mathcal{S}^-$ respectively. Since $\mathcal{S}^-(\lambda)$ is the right translation of $\mathcal{S}^+(\lambda)$ by $\dot{w}_0$, these G-submodules are isomorphic. In the notation of [Ja 1987, II.2.2], we have

$$\mathcal{S}^-(\lambda) = H^0(\lambda^*), \tag{6}$$

so by (6) and [Ja 1987, II.2.6, 2.2, 2.3], the following properties hold:

$$\left.\begin{array}{l} \text{(i) } \mathcal{S}^{\pm}(\lambda) \neq 0 \iff \lambda \in \mathrm{X}(T)_+; \\ \text{(ii) } \mathrm{soc}_G \mathcal{S}^{\pm}(\lambda) \text{ is a simple } G\text{-module with the highest weight } \lambda^*. \end{array}\right\} \tag{7}$$

If $\operatorname{char} k = 0$, then the G-module $\mathcal{S}^+(\lambda)$ is semisimple and hence $\mathcal{S}^+(\lambda) = \mathrm{soc}_G \mathcal{S}^+(\lambda)$ by (7)(ii). If $\operatorname{char} k > 0$, then, in general, this equality does not hold. From (3), (5), and (7)(i) we infer that

$$\begin{aligned} \mathcal{S}^+ &= \bigoplus_{\lambda \in \mathrm{X}(T)_+} \mathcal{S}^+(\lambda), \quad \mathcal{S}^+(\lambda)\mathcal{S}^+(\mu) \subseteq \mathcal{S}^+(\lambda+\mu), \\ \mathcal{S}^- &= \bigoplus_{\lambda \in \mathrm{X}(T)_+} \mathcal{S}^-(\lambda), \quad \mathcal{S}^-(\lambda)\mathcal{S}^-(\mu) \subseteq \mathcal{S}^-(\lambda+\mu), \end{aligned} \tag{8}$$

i.e., (8) are the $\mathrm{X}(T)_+$-gradings of the algebras $\mathcal{S}^+$ and $\mathcal{S}^-$. They are obtained from each other by the right translation by $\dot{w}_0$.

Theorem 1. *The linear span of $\mathcal{S}^{\pm}(\lambda)\mathcal{S}^{\pm}(\mu)$ over k is $\mathcal{S}^{\pm}(\lambda+\mu)$.*

Proof. This statement is the main result of [RR 1985]. Note that the difficulty lies in the case of positive characteristic: since $\mathcal{S}^{\pm}$ is an integral domain, if $\operatorname{char} k = 0$, then the claim immediately follows from (7)(i) and the inclusions in (8) because then $\mathcal{S}^{\pm}(\lambda+\mu)$ is a simple G-module. □

Theorem 2.

(i) *If $\mathscr{G}$ is a generating set of the semigroup $\mathrm{X}(T)_+$, then the k-algebra $\mathcal{S}^{\pm}$ is generated by the subspace $\bigoplus_{\lambda \in \mathscr{G}} \mathcal{S}^{\pm}(\lambda)$.*
(ii) *The k-algebras $\mathcal{S}^+$ and $\mathcal{S}^-$ are finitely generated.*

Proof. Part (i) follows from (8) and Theorem 1. Being the intersection of the lattice $\mathrm{X}(T)$ with a convex cone in $\mathrm{X}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by finitely many vectors, the semigroup $\mathrm{X}(T)_+$ is finitely generated. This, (i), and the inequality $\dim_k \mathcal{S}^{\pm}(\lambda) < \infty$ imply (ii). □

Now we are ready to turn to the proof of Conjecture (S).

Theorem 3. *The homomorphism μ is surjective.*

Our proof of Theorem 3 is based on two general results. The first is the following well-known surjectivity criterion:

Lemma 1. *The following properties of a morphism $\varphi\colon X \to Y$ of affine algebraic varieties are equivalent:*

(a) *φ is a closed embedding;*
(b) *$\varphi^*\colon k[Y] \to k[X]$ is surjective.*

Proof. See, e.g., [St 1974, 1.5]. □

The second is the closedness criterion for orbits of connected solvable affine algebraic groups that generalizes Rosenlicht's classical theorem on closedness of orbits of unipotent groups [Ro 1961, Thm. 2].

Theorem 4. *Let a connected solvable affine algebraic group S act on an affine algebraic variety Z. Let x be a point of Z. Consider the orbit morphism $\tau : S \to Z$, $s \mapsto s \cdot z$. Then the following properties are equivalent:*

(a) *the orbit $S \cdot z$ is closed in Z;*
(b) *the semigroup $\{\lambda \in \mathrm{X}(S) \mid$ the function $S \to k$, $s \mapsto s^{\lambda}$ lies in $\tau^*(k[Z])\}$ is a group.*

Proof. This is proved in [Po 1989, Thm. 4] □

Remark 1. Since $\mathrm{X}(S)$ in Theorem 4 is a finitely generated free abelian group, it can be naturally regarded as a lattice in $\mathrm{X}(S) \otimes_{\mathbb{Z}} \mathbb{Q}$. Hence the following general criterion is applicable for verifying condition (b).

Let M be a nonempty subset of a finite dimensional vector space V over $\mathbb{Q}$. Let $\mathbb{Q}_{\geqslant 0}M$, conv$M$, and $\mathbb{Q}M$ be, respectively, the convex cone generated by M, the convex hull of M, and the linear span of M in V. Then the following properties are equivalent (see [Po 1989, p. 386]):

(i) 0 is an interior point of conv$M$,
(ii) $\mathbb{Q}_{\geqslant 0}M = \mathbb{Q}M$.

If M is a subsemigroup of V, then (i) and (ii) are equivalent to

(iii) M is a group.

Proof of Theorem 3.

1. We consider the action of G on its underlying algebraic variety by left translations. By Theorem 2, there is an irreducible affine algebraic variety X endowed with an action of G and a G-equivariant dominant morphism

$$\alpha\colon G \to X \ \text{ such that } \alpha^* \text{ is an isomorpshism } k[X] \xrightarrow{\cong} \mathcal{S}^+. \tag{9}$$

Let $x := \alpha(e)$. Since α is G-equivariant, we have

$$\alpha(g) = g \cdot x \ \text{ for every } g \in G, \tag{10}$$

and since α is dominant, the orbit $G \cdot x$ is open and dense in X. Consider the canonical projection $\pi\colon G \to G/U^+$. It is the geometric quotient for the action of U^+ on G by right translations. Therefore, (3) yields the isomorphism

$$\pi^*\colon k[G/U^+] \xrightarrow{\cong} \mathcal{S}^+ \tag{11}$$

and, since α is constant on the fibers π, there exists a G-equivariant morphism $\iota\colon G/U^+ \to X$ such that

$$\alpha = \iota \circ \pi. \tag{12}$$

From (12) we infer that the image of ι is $G \cdot x$. Since the group U^+ is unipotent, the algebraic variety G/U^+ is quasiaffine (see [Ro 1961, Thm. 3]). Therefore, $k(G/U^+)$ is the field of fractions of $k[G/U^+]$. On the other hand, $k(X)$ is the field of fractions of $k[X]$ inasmuch as X is affine. Using that (12) and isomorphisms (9), (11) yield the isomorphism $\iota^*\colon k[X] \xrightarrow{\cong} k[G/U^+]$, we conclude that ι is a birational isomorphism. Therefore, for a point z in general position in $G \cdot x$ the fiber $\iota^{-1}(z)$ is a single point. Being G-equivariant, ι is then injective. Finally, since G is smooth, $k[G]$ is integrally closed; therefore, $\mathcal{S}^{\pm}$ is integrally closed as well in view of (3) (see, e.g., [PV 1994, Thm. 3.16]). Thus X is normal, and hence by Zariski's Main Theorem, $\iota\colon G/U^+ \to G \cdot x$ is an isomorphism. Using that π is separable (see, e.g., [Bor 1991, II.6.5]), from this we infer that the following properties hold:

(i$_1$) $G_x = U^+$;
(ii$_1$) $G \to G \cdot x$, $g \mapsto \alpha(g) = g \cdot x$, is a separable morphism.

2. Let $y := \dot{w}_0 \cdot x$. Consider the G-equivariant morphism

$$\beta\colon G \to X, \quad g \mapsto g \cdot y. \tag{13}$$

From (10), (13), (i$_1$), and (ii$_1$) we infer that the following properties hold:

(i$_2$) $G_y = U^-$;
(ii$_2$) $G \to G \cdot y$, $g \mapsto \beta(g) = g \cdot y$, is a separable morphism;
(iii$_2$) β^* is an isomorphism $k[X] \xrightarrow{\cong} \mathcal{S}^-$.

3. Now consider the G-equivariant morphism

$$\gamma := \alpha \times \beta\colon G \to X \times X, \quad g \mapsto g \cdot z, \ \text{ where } z := (x, y). \tag{14}$$

From (14) and (i$_1$), (i$_2$) we obtain

$$G_z = G_x \cap G_y = U^+ \cap U^- = \{e\}, \tag{15}$$

hence γ is injective. We claim that γ is a closed embedding, i.e.,

(a) $G \to G \cdot z$, $g \mapsto g \cdot z$ is an isomorphism;
(b) $G \cdot z$ is closed in $X \times X$.

If this claim is proved, then the proof of Theorem 3 is completed as follows. Consider the isomorphism

$$k[X] \otimes_k k[X] \to k[X \times X], \quad f \otimes h \mapsto fh. \tag{16}$$

Then (4), (9), (iii$_3$), (14), (16) imply that μ is the composition of the homomorphisms

$$\mathcal{S}^+ \otimes_k \mathcal{S}^- \xrightarrow[\cong]{(\alpha^*)^{-1}\otimes(\beta^*)^{-1}} k[X] \otimes_k k[X] \xrightarrow[\cong]{(16)} k[X \times X] \xrightarrow{\gamma^*} k[G]. \tag{17}$$

Hence the surjectivity of μ is equivalent to the surjectivity of γ^*. By Lemma 1, the latter is equivalent to the property that γ is a closed embedding, i.e., that properties (a) and (b) hold.

Thus the proof of Theorem 3 is reduced to proving properties (a) and (b).

4. First, we shall prove property (a). Since γ is injective, this is reduced to proving separability of γ. In turn, in view of (14), the latter is reduced to proving that $\ker(d\gamma)_e$ is contained in $\operatorname{Lie} G_z$, i.e., that $\ker(d\gamma)_e = \{0\}$ because of (15) (see [Bor 1991, II.6.7]). Using loc.cit., from (10), (13), (i$_1$), (ii$_1$), (i$_2$), (ii$_2$) we infer that $\ker(d\alpha)_e \subseteq \operatorname{Lie} U^+$, $\ker(d\beta)_e \subseteq \operatorname{Lie} U^-$. In view of (14), we then have $\ker(d\gamma)_e = \ker(d\alpha)_e \cap \ker(d\beta)_e \subseteq \operatorname{Lie} U^+ \cap \operatorname{Lie} U^- = \{0\}$. This proves property (a).

5. Now we shall prove property (b). Actually, we shall prove the stronger property that the orbit $B^+ \cdot z$ is closed in $X \times X$: since the algebraic variety G/B^+ is complete, this stronger property implies property (b) (see [St 1974, Sect. 2.13, Lemma 2]). Using that B^+ is connected solvable, to this end we shall apply Theorem 4.

Namely, consider the morphism $\tau\colon B^+ \to X \times X$, $b \mapsto b \cdot z$ and the following subsemigroup M in $\mathrm{X}(B^+)$:

$$M := \{\lambda \in \mathrm{X}(B^+) \mid \text{ the function } B^+ \to k,\ b \mapsto b^\lambda \text{ lies in } \tau^*(k[X \times X]).\}$$

We identify $\mathrm{X}(B^+)$ with the lattice in $L := \mathrm{X}(B^+) \otimes_{\mathbb{Z}} \mathbb{Q}$. In view of Theorem 4 and Remark 1, the orbit $B^+ \cdot z$ is closed if and only if

$$\mathbb{Q}_{\geqslant 0} M = \mathbb{Q} M. \tag{18}$$

Given this, the problem is reduced to proving that property (18) holds. This is done below.

6. Since $\tau = \gamma|_{B^+}$, the algebra $\tau^*(k[X \times X])$ is the image of the homomorphism $\gamma^*(k[X \times X]) \to k[B^+]$, $f \mapsto f|_{B^+}$. From (17) we see that $\gamma^*(k[X \times X])$ contains $\mathcal{S}^+$ and $\mathcal{S}^-$. Hence the restrictions of $\mathcal{S}^+$ and $\mathcal{S}^+$ to B^+ lie in $\tau^*(k[X \times X])$. We shall exhibit some characters of B^+ lying in these restrictions.

First consider the restriction of $\mathcal{S}^+(\lambda)$ to B^+ for $\lambda \in \mathrm{X}(T)_+$. Note that $\mathcal{S}^+(\lambda)$ contains a function f such that $f(e) \neq 0$. Indeed, in view of (7)(i) and Borel's fixed point theorem, $\mathcal{S}^+(\lambda)$ contains a B^--stable line ℓ. The group B^- acts on ℓ by means of a character $\nu \in \mathrm{X}(B^-)$. Take a nonzero function $f \in \ell$. For every $b \in B^-$, $u \in U^+$, we then have $f(b^{-1}u) = b^\nu f(u) \overset{(3)}{=} b^\nu f(e)$; whence $f(e) \neq 0$ because B^-U^+ is dense in G. This proves the existence of f. Multiplying f by $1/f(e)$, we may assume that $f(e) = 1$. Then for every $b \in B^+$, we deduce from (3), (5) that $f(b) = b^\lambda f(e) = b^\lambda$, i.e., $f|_{B^+}$ is the character $B^+ \to k$, $b \mapsto b^\lambda$. This proves the inclusion

$$\mathrm{X}(B^+)_+ \subseteq M. \tag{19}$$

Now consider the restriction of $\mathcal{S}^-(\lambda)$ to B^+ for $\lambda \in \mathrm{X}(T)_+$. In view of (7)(ii), there is a B^+-stable line ℓ in $\mathcal{S}^-(\lambda)$, on which B^+ acts by the character $\lambda^* \in \mathrm{X}(B^+)$. Take a nonzero function $f \in \ell$. We may assume that $f(e) = 1$: this is proved as above with $\nu = \lambda$, replacing B^- by B^+, and U^+ by U^-. For every $b \in B^+$, we then have $f(b^{-1}) = b^{\lambda^*}$, i.e., $f|_{B^+}$ is the character $B^+ \to k$, $b \mapsto b^{-\lambda^*} = b^{w_0(\lambda)}$. This proves the inclusion

$$-\mathrm{X}(B^+)_+ \subseteq M. \tag{20}$$

Since $\mathbb{Q}_{\geqslant 0}(\mathrm{X}(B^+)_+) - \mathbb{Q}_{\geqslant 0}(\mathrm{X}(B^+)_+) = L$, the inclusions (19), (20) imply the equality $\mathbb{Q}_{\geqslant 0}M = L$; whence a fortiori the equality (18) holds. This completes the proof of Theorem 3. □

3. Proof of Conjecture (K)

We now intend to describe the ideal $\ker\mu$ in $\mathcal{S}^+ \otimes_k \mathcal{S}^-$. This is done in Sections 3 and 4 in several steps: first in Theorem 6 we prove that $\ker\mu$ is generated by $(\ker\mu)^G$, then in Theorem 7 we describe $\ker\mu$ as a vector space, and finally in Theorem 10 we find a standard finite generating set of $\ker\mu$.

The first step is based on the following general statement:

Theorem 5. *Let Z be an affine algebraic variety endowed with an action of a reductive algebraic group H. Let $a \in Z$ be a point such that the orbit morphism*

$$\varphi\colon H \to Z, \quad h \mapsto h \cdot a$$

is a closed embedding. Then the ideal $\ker\varphi^*$ in $k[Z]$ is generated by $(\ker\varphi^*)^H$.

For the proof of Theorem 5 we need the following

Lemma 2. *Let $\psi\colon Y \to Z$ be a morphism of irreducible affine algebraic varieties and let $z \in \psi(Y)$ be a smooth point of Z. Assume that for each point $y \in \psi^{-1}(z)$ the following hold:*

(i) *y is a smooth point of Y;*
(ii) *the differential $d_y\psi$ is surjective.*

Then the ideal $\{f \in k[Y] \mid f|_{\psi^{-1}(z)} = 0\}$ of $k[Y]$ is generated by $\psi^*(\mathfrak{m})$, where $\mathfrak{m} := \{h \in k[Z] \mid h(z) = 0\}$.

Proof. Given a nonzero function $f \in k[Y]$, below we denote by Y_f the principal open subset $\{y \in Y \mid f(y) \neq 0\}$ of Y; it is affine and $k[Y_f] = k[Y]_f$.

1. Let $s_1, \dots, s_d$ be a system of generators of the ideal $\mathfrak{m}$ of $k[Z]$. Put $t_i := \psi^*(s_i)$. Then we have

$$\{y \in Y \mid t_1(y) = \cdots = t_d(y) = 0\} = \psi^{-1}(z). \tag{21}$$

We claim that, for every point $a \in Y$, there is a function $h_a \in k[Y]$ such that the principal open subset $U = Y_{h_a}$ is a neighborhood of a and $I_U := \{f \in k[U] \mid f|_{\psi^{-1}(z)\cap U} = 0\}$ is the ideal of $k[U]$ generated by $t_1|_U, \ldots, t_d|_U$.

Proving this, we consider two cases.

First, consider the case where $a \notin \psi^{-1}(z)$. Then any principal open neighborhood of a not intersecting $\psi^{-1}(z)$ may be taken as U because in this case $I_U = k[U]$ and, in view of (21) and Hilbert's Nullstellensatz, $k[U] = k[U]t_1|_U + \cdots + k[U]t_d|_U$.

Second, consider the case where $a \in \psi^{-1}(z)$. Let $n = \dim Y, m = \dim Z$. Since a and z are the smooth points, the assumption (ii) yields the equality

$$\dim \ker d_a\psi = n - m. \tag{22}$$

The functions $s_1, \ldots, s_d$ generate the maximal ideal of $\mathcal{O}_{z,Z}$. Therefore, renumbering them if necessary, we may (and shall) assume that $s_1, \ldots, s_m$ is a system of local parameters of Z at z, i.e., $\bigcap_{i=1}^m \ker d_z s_i = \{0\}$. Since $d_a t_i = d_a\psi \circ d_z s_i$, we then infer from (ii) that $\bigcap_{i=1}^m \ker d_a t_i = \ker d_a\psi$. In view of (22), the latter equality implies the existence of functions $f_1, \ldots, f_{n-m} \in \mathcal{O}_{a,Y}$ such that $t_1, \ldots, t_m, f_1, \ldots, f_{n-m}$ is a system of local parameters of Y at a. Let

$$F := \{y \in Y \mid t_1(y) = \cdots = t_m(y) = 0\}. \tag{23}$$

By [Sh 2013, Chap. II, §3, Sect. 2, Thm. 4], there is a principal open neighborhood U of a such that $F \cap U$ is an irreducible smooth $(n-m)$-dimensional closed subvariety of U whose ideal in $k[U]$ is generated by $t_1|_U, \ldots, t_m|_U$. On the other hand, (21) and (23) yield $\psi^{-1}(z) \subseteq F$ and, by the fiber dimension theorem, every irreducible component of $\psi^{-1}(z)$ has dimension $\geqslant n - m$. Hence $U \cap F = \psi^{-1}(z) \cap U$. This and (21) prove the claim.

2. Using this claim, the proof of Lemma 2 is completed as follows. Since $Y = \bigcup_{a \in Y} Y_{h_a}$ and Y is quasi-compact, there exists a finite set of points $a_1, \ldots, a_r \in Y$ such that

$$Y = \bigcup_{i=1}^r Y_{h_i}, \quad \text{where } h_i := h_{a_i}. \tag{24}$$

Now, let $f \in k[Y]$ be a function such that $f|_{\psi^{-1}(z)} = 0$. Then, in view of the definition of h_a, for every $i = 1, \ldots, r$, we have

$$fh_i^{b_i} = c_{i,1}t_1 + \cdots + c_{i,d}t_d \quad \text{for some } c_{i,j} \in k[Y] \text{ and } b_i \in \mathbb{N}. \tag{25}$$

From (24) and Hilbert's Nullstellensatz we infer that there are functions $q_1, \ldots, q_r \in k[Y]$ such that

$$1 = q_1h_1^{b_1} + \cdots + q_sh_s^{b_r}. \tag{26}$$

From (25) and (26) we then deduce that

$$f = \left(\sum_{i=1}^r q_ic_{i,1}\right)t_1 + \cdots + \left(\sum_{i=1}^r q_ic_{i,d}\right)t_d \in k[Y]t_1 + \cdots + k[Y]t_d.$$

This completes the proof. □

Proof of Theorem 5. There is a closed equivariant embedding of Z in an affine space on which H operates linearly (see [Ro 1961, Lemma 2], [PV 1994, Thm. 1.5]). Hence we may (and shall) assume that Z is an irreducible smooth affine algebraic variety.

Since G is reductive, $k[Z]^G$ is a finitely generated k-algebra (see, e.g., [MF 1982, Thm. A.1.0] and the references therein). Denote by $Z/\!\!/H$ the affine algebraic variety $\operatorname{Specm}(k[Z]^G)$ and by $\pi\colon Z \to Z/\!\!/H$ the morphism corresponding to the inclusion homomorphism $k[Z]^G \hookrightarrow k[Z]$.

The condition on the point a implies that its H-stabilizer is trivial,

$$H_a = \{e\}. \tag{27}$$

Hence $H \cdot a$ is a closed H-orbit of maximal dimension. Taking into account that in every fiber of π there is a unique closed orbit lying in the closure of every orbit contained in this fiber (see [MF 1982, Cors. 1.2, A.1.0]), from this we deduce the equality

$$\pi^{-1}(\pi(a)) = H \cdot a. \tag{28}$$

Since the group $\{e\}$ is linearly reductive, from (27) and the separability of φ we infer by [BR 1985, Prop. 7.6] that there is a smooth affine subvariety S of the H-variety Z, which is an étale slice at $a \in S$. In view of (27), this means the following:

(i) $\pi|_S\colon S \to Z/\!\!/H$ and $\psi\colon H \times S \to Z$, $(h, s) \mapsto h \cdot s$ are the étale morphisms;

(ii) the diagram

$$\begin{tikzcd} H \times S \arrow[r, "\psi"] \arrow[d, "\mathrm{pr}_2"'] & Z \arrow[d, "\pi"] \\ S \arrow[r, "\pi|_S"] & Z/\!\!/H \end{tikzcd}$$

is a Cartesian square, i.e., it is commutative and the morphism $H \times S \to S \times_{Z/\!\!/H} Z$ determined by ψ and pr_2 is an isomorphism.

From (i) and (ii) we deduce that $\pi(a)$ is a smooth point of $Z/\!\!/H$ and the differentials $d_{(e,a)}\psi$, $d_a(\pi|_S)$ are isomorphisms. Since $d_{(e,a)}\mathrm{pr}_2$ is clearly surjective, (ii) then implies that $d_a\pi$ is surjective, too.

Now, in view of (28) and transitivity of the action of H on $H \cdot a$, we conclude that $d_z\pi$ is surjective for every point $z \in \pi^{-1}(\pi(a))$. In view of Lemma 2, this implies the claim of Theorem 5. $\square$

Theorem 6. *The ideal* $\ker \mu$ *in* $\mathcal{S}^+ \otimes_k \mathcal{S}^-$ *is generated by* $(\ker \mu)^G$.

Proof. In the proof of Theorem 3 we have shown that

— the homomorphism μ is the composition of the homomorphisms (17);

— the morphism γ is a closed embedding.

In view of these facts, Theorem 6 is equivalent to the claim that the ideal $\ker \gamma^*$ in $k[X \times X]$ is generated by $(\ker \gamma^*)^G$. This claim follows from Theorem 5. $\square$

4. Structure of $(\ker \mu)^G$

We shall use the following lemma for describing $(\ker \mu)^G$ as a vector space.

Lemma 3.

$$\dim(\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\nu))^G = \begin{cases} 1 & \text{if } \nu = \lambda^*, \\ 0 & \text{if } \nu \neq \lambda^* \end{cases} \quad \textit{for every } \lambda, \nu \in \mathrm{X}(T)_+, \tag{29}$$

$$(\mathcal{S}^+ \otimes_k \mathcal{S}^-)^G = \bigoplus_{\lambda \in \mathrm{X}(T)_+} (\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G. \tag{30}$$

Proof. In view of (8), the equality (30) follows from (29). To prove (29), we note that
$$(\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\nu))^G \cong \mathrm{Hom}_G(\mathcal{S}^+(\lambda)^*, \mathcal{S}^-(\nu))$$
and, in view of (6), the G-module $\mathcal{S}^+(\lambda)^*$ is the universal highest weight module of weight λ (the Weyl module); in particular, for each G-module M, there is an isomorphism

$$\mathrm{Hom}_G(\mathcal{S}^+(\lambda)^*, M)) \xrightarrow{\cong} (M^{U^+})_\lambda, \tag{31}$$

where the right-hand side of (31) is the weight space of T (see [Ja 1987, II.2.13, Lemma]). Since $\mathcal{S}^-(\nu)^{U^+}$ is a line on which B^+ acts by means of ν^* (see [Ja 1987, II.2.2, Prop.]), this proves (29). $\square$

We identify $k[G] \otimes_k k[G]$ with $k[G \times G]$ by the isomorphism

$$k[G] \otimes_k k[G] \to k[G \times G], \quad f_1 \otimes f_2 \mapsto ((a, b) \mapsto f_1(a) f_2(b)). \tag{32}$$

Thus $\mathcal{S}^+ \otimes_k \mathcal{S}^-$ is regarded as a subalgebra of $k[G \times G]$, and (4), (32) yield the equality

$$f(a, a) = \mu(f)(a) \text{ for every } f \in \mathcal{S}^+ \otimes_k \mathcal{S}^- \text{ and } a \in G. \tag{33}$$

Theorem 7.

(i) *If $f \in (\mathcal{S}^+ \otimes_k \mathcal{S}^-)^G$, then $f - f(e, e) \in (\ker \mu)^G$.*

(ii) *Every $h \in (\ker \mu)^G$ can be uniquely written in the form*

$$h = \sum (h_\lambda - h_\lambda(e, e)), \quad h_\lambda \in (\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G, \tag{34}$$

the sum is taken over a finite set of nonzero elements $\lambda \in \mathrm{X}(T)_+$.

Proof. (i) Since μ is G-equivariant, its restriction to $(\mathcal{S}^+ \otimes_k \mathcal{S}^-)^G$ is a homomorphism to $k[G]^G = k$. Hence $\mu(f)$ is a constant. In view of (33), this implies (i).

(ii) If (34) holds, then the decomposition (30) implies that h_λ is the natural projection of h to $(\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G$ determined by this decomposition; whence the uniqueness of (34). To prove the existence, let h_λ be the aforementioned projection of h to $(\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G$. Then $h = \sum_{\lambda \in F} h_\lambda$ for a finite set $F \subset \mathrm{X}(T)_+$. Hence $0 = \mu(h) = \sum_{\lambda \in F} \mu(h_\lambda)$. As above, $\mu(h_\lambda) = h_\lambda(e, e)$; this implies the equality (34), where the sum is taken over all $\lambda \in F$. Since h_0 is a constant, we may assume that F does not contain 0. This proves (ii). $\square$

In the next lemma, for brevity, we put (cf. [Ja 1987])

$$\begin{gathered} V(\lambda) := \mathcal{S}^-(\lambda)^* \cong \mathcal{S}^+(\lambda)^*, \quad L(\lambda) := V(\lambda)/\mathrm{rad}_G V(\lambda), \\ \pi_\lambda\colon V(\lambda) \to L(\lambda) \text{ is the canonical projection.} \end{gathered} \tag{35}$$

The G-module $V(\lambda)$ (hence $L(\lambda)$ as well) is generated by a B^+-stable line of weight λ (see [Ja 1987, II, Sect. 2.13, Lemma]); whence $V(\lambda)$ is also generated by a B^--stable line of weight $-\lambda^*$.

Also, for the G-modules P and Q, we denote by $\mathscr{B}(P\times Q)$ the G-module of all bilinear maps $P\times Q \to k$; we then have the isomorphism of G-modules

$$P^* \otimes_k Q^* \xrightarrow{\cong} \mathscr{B}(P\times Q), \quad f\otimes h \mapsto fh. \tag{36}$$

Lemma 4. *For all elements $\lambda, \nu \in \mathrm{X}(T)_+$, the following hold*:

(a) $\dim \mathscr{B}(V(\lambda)\times V(\nu))^G = \begin{cases} 1 & \text{if } \nu = \lambda^*, \\ 0 & \text{if } \nu \neq \lambda^*. \end{cases}$

(b) $\dim \mathscr{B}(L(\lambda)\times L(\nu))^G = \begin{cases} 1 & \text{if } \nu = \lambda^*, \\ 0 & \text{if } \nu \neq \lambda^*. \end{cases}$

(c) *Every nonzero element $\theta \in \mathscr{B}(L(\lambda)\times L(\lambda^*))^G$ is a nondegenerate pairing $L(\lambda)\times L(\lambda^*) \to k$.*

(d) *If $l^+ \in L(\lambda)$, $l^- \in L(\lambda^*)$ are the nonzero semi-invariants of, respectively, B^+ and B^-, then $\theta(l^+, l^-) \neq 0$ for θ from* (c). *For every nonzero element $\epsilon \in k$, there exists a unique θ such that $\theta(l^+, l^-) = \epsilon$.*

(e) *Every element $\vartheta \in \mathscr{B}(V(\lambda)\times V(\lambda^*))^G$ vanishes on $\ker \pi_\lambda \times \ker \pi_{\lambda^*}$. If $\vartheta \neq 0$, then ϑ is a nondegenerate pairing $V(\lambda)\times V(\lambda^*) \to k$.*

(f) *Let $v^+ \in V(\lambda)$ and $v^- \in V(\lambda^*)$ be, respectively, the nonzero B^+- and B^--semi-invariants of weights λ and $-\lambda$ that generate the G-modules $V(\lambda)$ and $V(\lambda^*)$. Then $\vartheta(v^+, v^-) \neq 0$ for every nonzero element $\vartheta \in \mathscr{B}(V(\lambda)\times V(\lambda^*))^G$.*

Proof. G-modules, Part (a) follows from (29), (36), (35). Part (b) is proved similarly, using that $L(\lambda)$ is a simple G-module with highest weight λ (see [Ja 1987, II.2.4]). The simplicity of $L(\lambda)$ implies (c) because the left and right kernels of θ are G-stable.

Proving (d), take a basis $\{p_1, \ldots, p_s\}$ of $L(\lambda)$ such that $p_1 = l^+$ and every p_i is a weight vector of T. Let $\{p_1^*, \ldots, p_s^*\}$ be the basis of $L(\lambda^*)$ dual to $\{p_1, \ldots, p_s\}$ with respect to θ. Let $L(\lambda)'$ be the linear span over k of all p_i's with $i > 1$. Then $L(\lambda)'$ is B^--stable, and, for every element $u \in U^-$, we have $u\cdot p_1 = p_1 + p'$, where $p' \in L(\lambda)'$ (see, e.g., [St 1974, Sect. 3.3, Prop. 2 and p. 84]). Then, for every elements $\alpha_1, \ldots, \alpha_s \in k$, we have

$$\begin{aligned} (u\cdot p_1^*)\Big(\sum_{i=1}^{s} \alpha_i p_i\Big) &= p_1^*\Big(\sum_{i=1}^{s} \alpha_i (u^{-1}\cdot p_i)\Big) \\ &= p_1^*(\alpha_1 p_1 + \text{an element of } L(\lambda)') \\ &= \alpha_1 = p_1^*\Big(\sum_{i=1}^{s} \alpha_i p_i\Big); \end{aligned}$$

whence $u\cdot p_1^*=p_1^*$. Therefore, $l^-=\lambda p_1^*$ for a nonzero $\lambda\in k$, hence $\theta(l^+,l^-)=\lambda\neq 0$. This and (b) prove (d).

It follows from (35), (a), (b) that the embedding

$$\mathscr{B}(L(\lambda)\times L(\lambda^*))^G\to\mathscr{B}(V(\lambda)\times V(\lambda^*))^G,\quad \theta\mapsto\theta\circ(\pi_\lambda\times\pi_{\lambda^*})$$

is an isomorphism. Part (e) follows from this and (c).

Part (f) follows from (d) and (e), because $\pi_\lambda(v^+)$ and $\pi_{\lambda^*}(v^-)$ are, in view of (35), the nonzero semi-invariants of, respectively, B^+ and B^-. □

Lemma 5. *Let an algebraic group H act on a algebraic variety Z and let V be a finite-dimensional submodule of the H-module $k[Z]$. Then the morphism*

$$\varphi\colon Z\to V^*,\quad \varphi(a)(f)=f(a)\ \textit{for every}\ a\in Z, f\in V \tag{37}$$

has the following properties:

(i) *φ is H-equivariant;*
(ii) *the restriction of φ^* to $(V^*)^*$ is an isomorphism $(V^*)^*\to V$;*
(iii) *φ^* exercises an isomorphism between $k[\overline{\varphi(Z)}]$ and the subalgebra of $k[Z]$ generated by V.*

Proof. Part (i) is proved by direct verification.

Every function $f\in V$ determines an element $l_f\in(V^*)^*$ by the formula $l_f(s)=s(f)$, $s\in V^*$. It is immediate that $V\to(V^*)^*$, $f\mapsto l_f$ is a vector space isomorphism and that (37) implies $\varphi^*(l_f)=f$. This proves (ii).

Let $\iota\colon(V^*)^*\to k[\overline{\varphi(Z)}]$ be the restriction homomorphism. The k-algebra $k[\overline{\varphi(Z)}]$ is generated by $\iota((V^*)^*)$. Part (iii) now follows from the fact that φ^* exercises an embedding of $k[\overline{\varphi(Z)}]$ in $k[Z]$ and, in view of (ii), the image of $\iota((V^*)^*)$ under this embedding is V. □

Corollary 1. *In the notation of Lemma 5, let $V\neq\{0\}$ and let the orbit $H\cdot a$ be dense in Z. Then $\varphi(a)\neq 0$.*

We call the morphism (37) the *covariant determined by the submodule V*.

Lemma 6. *Let λ be an element of $\mathrm{X}(T)_+$ and let*

$$\varphi^+\colon G\to\mathcal{S}^+(\lambda)^*,\quad \varphi^-\colon G\to\mathcal{S}^-(\lambda^*)^*$$

be the covariants determined by the submodules $\mathcal{S}^+(\lambda)$ and $\mathcal{S}^-(\lambda^*)$ of the G-module $k[G]$. Then $v^+:=\varphi^+(e)$ and $v^-:=\varphi^-(e)$ are, respectively, the nonzero B^+- and B^--semiinvariants of weights λ and $-\lambda$.

Proof. First, we have $v^+\neq 0$, $v^-\neq 0$ by Corollary 1. Next, for every $f\in\mathcal{S}^+(\lambda)$, $b\in B^+$, we have

$$\begin{aligned}(b\cdot v^+)(f)&=\varphi^+(e)(b^{-1}\cdot f)\overset{(37)}{=\!=}(b^{-1}\cdot f)(e)\\&=f(b)\overset{(5)}{=}b^\lambda f(e)\overset{(37)}{=\!=}(b^\lambda v^+)(f);\end{aligned}$$

whence $b\cdot v^+=b^\lambda v^+$, i.e., v^+ is a nonzero B^+-semi-invariant of weight λ, as claimed. For v^- the proof is similar. □

Theorem 8. *The restriction of μ to $(\mathcal{S}^+(\lambda)\otimes_k\mathcal{S}^-(\lambda))^G$ for every $\lambda\in \mathrm{X}(T)_+$ is an isomorphism $(\mathcal{S}^+(\lambda)\otimes_k\mathcal{S}^-(\lambda^*))^G \xrightarrow{\cong} k[G]^G = k$.*

Proof. In view of (33) and Lemma 3, the proof is reduced to showing that there is a function $f\in(\mathcal{S}^+(\lambda)\otimes_k\mathcal{S}^-(\lambda^*))^G$ such that $f(e,e)\neq 0$.

Consider the covariants φ^+ and φ^- from Lemma 6 and the G-equivariant morphism

$$\varphi := \varphi^+\times\varphi^-\colon G\times G\to \mathcal{S}^+(\lambda)^*\times\mathcal{S}^-(\lambda^*)^*.$$

Lemma 4(a) and (35) imply that $\mathscr{B}(\mathcal{S}^+(\lambda)^*\times\mathcal{S}^-(\lambda^*)^*)^G$ contains a nonzero element ϑ. By Lemma 5, the function $f:=\vartheta\circ\varphi\colon G\times G\to k$ is contained in $(\mathcal{S}^+(\lambda)\otimes_k\mathcal{S}^-(\lambda^*))^G$. For this f, using Lemmas 6 and 4(f), we obtain

$$f(e,e)=\vartheta(\varphi(e,e))=\vartheta(\varphi^+(e),\varphi^-(e))\neq 0. \tag{38}$$

This completes the proof. □

Corollary 2. *For every element $\lambda\in\mathrm{X}(T)_+$, there exists a unique element*

$$s_\lambda\in(\mathcal{S}^+(\lambda)\otimes_k\mathcal{S}^-(\lambda^*))^G\subseteq k[G\times G]\ \ \textit{such that}\ s_\lambda(e,e)=1. \tag{39}$$

If $\{f_1,\dots,f_d\}$ and $\{h_1,\dots,h_d\}$ are the bases of $\mathcal{S}^+(\lambda)$ and $\mathcal{S}^-(\lambda^*)$ dual with respect to a nondegenerate G-invariant pairing $\mathcal{S}^+(\lambda)\times\mathcal{S}^-(\lambda^*)\to k$ (the latter exists by (36) and Lemma 4), then $\varepsilon:=\sum_{i=1}^d f_i(e)h_i(e)\neq 0$ and

$$s_\lambda=\varepsilon^{-1}\Big(\sum_{i=1}^d f_i\otimes h_i\Big).$$

Proof. First, note that if P, Q are the finite dimensional kG-modules, $\theta\in\mathscr{B}(P,Q)^G$ is a nondegenerate pairing $P\times Q\to k$, and $\{p_1,\dots,p_m\}$ and $\{q_1,\dots,q_m\}$ are the bases of P and Q dual with respect to θ, then $\sum_{i=1}^m p_i\otimes q_i$ is a nonzero element of $(P\otimes_k Q)^G$ (not depending on the choice of these bases). Indeed, θ determines the isomorphism of G-modules

$$\begin{aligned}&\phi\colon P\otimes_k Q\to \operatorname{Hom}(P,P),\\ &(\phi(p\otimes q))(p')=\theta(p',q)p,\ \text{where } p,p'\in P,\ q\in Q.\end{aligned} \tag{40}$$

From (40) we then obtain

$$\Big(\phi\Big(\sum_{i=1}^m p_i\otimes q_i\Big)\Big)(p_j)=\sum_{i=1}^m\theta(p_j,q_i)p_i=\sum_{i=1}^m\delta_{ij}p_i=p_j,$$

therefore, $\phi\big(\sum_{i=1}^m p_i\otimes q_i\big)=\mathrm{id}_P$; whence the claim.

For $P=\mathcal{S}^+(\lambda)$, $Q=\mathcal{S}^-(\lambda^*)$, it yields that $\sum_{i=1}^d f_i\otimes h_i$ is a nonzero element of $(\mathcal{S}^+(\lambda)\otimes_k\mathcal{S}^-(\lambda^*))^G$. Theorem 8 and (33) then complete the proof. □

Remark 2. For $\operatorname{char}k=0$, there is another characterization of s_λ. Namely, let $\mathscr{U}$ be the universal enveloping algebra of $\operatorname{Lie}G$. Every $\mathcal{S}^\pm(\lambda)$ is endowed with the natural $\mathscr{U}$-module structure. Let $\{x_1,\dots,x_n\}$ and $\{x_1^*,\dots,x_n^*\}$ be the bases of $\operatorname{Lie}G$ dual with respect to the Killing form Φ. Identify $\operatorname{Lie}T$ with its dual space by means of Φ. Let σ be the sum of all positive roots. For every $\lambda \in \mathrm{X}(T)_+$, put

$$c_\lambda := \Phi(\lambda+\sigma,\lambda)+\Phi(\lambda^*+\sigma,\lambda^*) \tag{41}$$

and consider on the space $\mathcal{S}^+(\lambda)\otimes_k \mathcal{S}^-(\lambda^*)$ the linear operator

$$\Delta := \sum_{i=1}^{n}(x_i\otimes x_i^* + x_i^*\otimes x_i). \tag{42}$$

Proposition 1. *The following properties of an element $t \in \mathcal{S}^+(\lambda)\otimes_k \mathcal{S}^-(\lambda^*)$ are equivalent:*

(i) $t = s_\lambda$;
(ii) $\Delta(t) = -c_\lambda t$ *and* $t(e,e) = 1$.

Proof. By [Bou 1975, Chap. VIII, §6, Sect. 4, Cor.], the Casimir element $\Omega := \sum_{i=1}^n x_i x_i^* \in \mathscr{U}$ acts on any simple $\mathscr{U}$-module with the highest weight γ as scalar multiplication by $\Phi(\gamma+\sigma,\gamma)$. Since $\Phi(\gamma+\sigma,\gamma)>0$ if $\gamma\neq 0$, the kernel of Ω in any finite dimensional $\mathscr{U}$-module V coincides with V^G. We apply this to $V=\mathcal{S}^+(\lambda)\otimes_k \mathcal{S}^-(\lambda^*)$. For any elements $f\in\mathcal{S}^+(\lambda)$, $h\in\mathcal{S}^-(\lambda^*)$, we deduce from (41), (42) the following:

$$\begin{aligned}\Omega(f\otimes h) &= \sum_{i=1}^{n}\big(x_ix_i^*(f)\otimes h + x_i^*(f)\otimes x_i(h)+x_i(f)\otimes x_i^*(h)+f\otimes x_ix_i^*(h)\big)\\ &= \Omega(f)\otimes h+f\otimes\Omega(h)+\Delta(f\otimes h) = c_\lambda(f\otimes h)+\Delta(f\otimes h).\end{aligned}$$

Now Corollary 2 and the aforesaid about $\ker\Omega$ complete the proof. □

Theorem 9. *Let $\lambda_1,\ldots,\lambda_m$ be a system of generators of the monoid $\mathrm{X}(T)_+$. Then $(\ker\mu)^G$ is the linear span over k of all monomials of the form*

$$(s_{\lambda_1}-1)^{d_1}\cdots(s_{\lambda_m}-1)^{d_m}, \text{ where } d_i\in\mathbb{N},\ d_1+\cdots+d_m>0,$$

where s_{λ_i} is defined in Corollary 2.

Proof. By Theorem 7(i), the linear span L referred to in Theorem 9 is contained in $(\ker\mu)^G$. In view of Theorem 7, to prove the converse inclusion $(\ker\mu)^G\subseteq L$, we have to show that, for every function

$$f\in(\mathcal{S}^+(\lambda)\otimes_k\mathcal{S}^-(\lambda^*))^G, \tag{43}$$

we have $f-f(e,e)\in L$. Since $\lambda_1,\ldots,\lambda_m$ is a system of generators of $\mathrm{X}(T)_+$, there are the integers $d_1,\ldots,d_m\in\mathbb{N}$ such that $\lambda=\sum_{i=1}^m d_i\lambda_i$. From (39) and (8) we then infer that $h:=\prod_{i=1}^m s_{\lambda_i}^{d_i}\in(\mathcal{S}^+(\lambda)\otimes_k\mathcal{S}^-(\lambda^*))^G$ and $h(e,e)=1$. This, (43), and (29) imply that $f=f(e,e)h$. Therefore,

$$f-f(e,e)=f(e,e)(h-1)=f(e,e)\Big(\prod_{i=1}^m((s_{\lambda_i}-1)+1)^{d_i}-1\Big). \tag{44}$$

The right-hand side of (44) clearly lies in L. This completes the proof. □

Theorem 10. *Let $\lambda_1, \ldots, \lambda_m$ be a system of generators of the monoid $\mathrm{X}(T)_+$. Then the ideal $\ker\mu$ in $\mathcal{S}^+ \otimes_k \mathcal{S}^-$ is generated by $s_{\lambda_1}-1, \ldots, s_{\lambda_m}-1$, where s_{λ_i} is defined in Corollary 2.*

Proof. This follows from Theorems 6 and 9. □

5. Presentation of $\mathcal{S}^{\pm}$

If the group G is semisimple, then the semigroup $\mathrm{X}(T)_+$ has no units other than 0. Hence the set $\mathscr{H}$ of all indecomposable elements of $\mathrm{X}(T)_+$ is finite,

$$\mathscr{H} = \{\lambda_1, \ldots, \lambda_d\}, \tag{45}$$

generates $\mathrm{X}(T)_+$, and every generating set of $\mathrm{X}(T)_+$ contains $\mathscr{H}$ (see, e.g., [Lo 2005, Lemma 3.4.3]). Note that $\mathscr{H}$, called the *Hilbert basis of* $\mathrm{X}(T)_+$, in general is not a free generating system of $\mathrm{X}(T)_+$ (i.e., it is not true that every element $\alpha \in \mathrm{X}(T)_+$ may be uniquely expressed in the form $\alpha = \sum_{i=1}^d c_i\lambda_i$, $c_i \in \mathbb{N}$). Namely, it is free if and only if $G = G_1 \times \cdots \times G_s$ where every G_i is either a simply connected simple algebraic group or isomorphic to SO_{n_i} for an odd n_i (see [St 1975, §3], [Ri 1979, Prop. 4.1], [Ri 1982, Prop. 13.3], [Po 2011, Remark 3.16]). In particular, if G is simply connected, then $\mathscr{H}$ coincides with the set of all fundamental weights and generates $\mathrm{X}(T)_+$ freely. Note that $\lambda_i^* \in \mathscr{H}$ for every i.

To understand presentation of $\mathcal{S}^{\pm}$, denote respectively by $\mathrm{Sym}\,\mathcal{S}^{\pm}(\lambda_i)$ and $\mathrm{Sym}^m\,\mathcal{S}^{\pm}(\lambda_i)$ the symmetric algebra and the mth symmetric power of $\mathcal{S}^{\pm}(\lambda_i)$. The naturally $\mathbb{N}^d$-graded free commutative k-algebra

$$\mathcal{F}^{\pm} := \mathrm{Sym}\,\mathcal{S}^{\pm}(\lambda_1) \otimes_k \cdots \otimes_k \mathrm{Sym}\,\mathcal{S}^{\pm}(\lambda_d) \tag{46}$$

may be viewed as the algebra of regular functions $k[L^{\pm}]$ on the vector space

$$L^{\pm} := \mathcal{S}^{\pm}(\lambda_1)^* \oplus \cdots \oplus \mathcal{S}^{\pm}(\lambda_d)^*.$$

Let e_i be the ith unit vector of $\mathbb{N}^d$ and let $\mathcal{F}^{\pm}_{p,q}$ be the homogeneous component of $\mathcal{F}^{\pm}$ of degree e_p+e_q. We have the natural isomorphisms of G-modules

$$\varphi^{\pm}_{p,q}\colon \mathcal{F}^{\pm}_{p,q} \xrightarrow{\cong} \mathcal{S}^{\pm}_{p,q} := \begin{cases} \mathcal{S}^{\pm}(\lambda_p) \otimes_k \mathcal{S}^{\pm}(\lambda_q) & \text{if } p \neq q, \\ \mathrm{Sym}^2\,\mathcal{S}^{\pm}(\lambda_p) & \text{if } p = q. \end{cases} \tag{47}$$

By Theorems 1, 2 the natural multiplication homomorphisms

$$\phi^{\pm}\colon \mathcal{F}^{\pm} \to \mathcal{S}^{\pm} \quad \text{and} \quad \psi^{\pm}_{p,q}\colon \mathcal{S}^{\pm}_{p,q} \to \mathcal{S}^{\pm}(\lambda_p + \lambda_q) \tag{48}$$

are surjective. Since $\mathcal{F}^{\pm}$ is a polynomial algebra, the surjectivity of $\phi^{\pm}$ reduces finding a presentation of $\mathcal{S}^{\pm}$ by generators and relations to describing $\ker\phi^{\pm}$. If $d = \dim T$, the following explicit description of $\ker\phi^{\pm}$ is available:

Theorem 11. *Let G be a connected semisimple group such that the Hilbert basis* (45) *freely generates the semigroup $\mathrm{X}(T)_+$. Then*

(i) *the ideal $\ker\phi^{\pm}$ of the $\mathbb{N}^d$-graded k-algebra $\mathcal{F}^{\pm}$ is homogeneous;*
(ii) *this ideal is generated by the union of all its homogeneous components of the total degree 2;*

(iii) *the set of these homogeneous components coincides with the set of all subspaces* $(\varphi^{\pm}_{p,q})^{-1}(\ker\psi^{\pm}_{p,q})$, $1 \leqslant p \leqslant q \leqslant d$.

Proof. This is the main result of [KR 1987]. □

Remark 3. In characteristic 0 for the first time the proof of Theorem 11 was obtained (but not published) by B. Kostant; his proof appeared in [LT 1979, Thm. 1.1]. In this case, (47) and the surjectivity of $\psi^{\pm}_{p,q}$ yield that $\psi^{\pm}_{p,q}$ is the projection of $\mathcal{S}^{\pm}_{p,q}$ to the Cartan component of $\mathcal{S}^{\pm}_{p,q}$, and $\ker\psi^{\pm}_{p,q}$ is the unique G-stable direct complement to this component. The subspace $\ker\psi^{\pm}_{p,q}$ admits the following description using the notation of Remark 2 (loc. cit.). Let $\{x_1,\ldots,x_n\}$ and $\{x_1^*,\ldots,x_n^*\}$ be the dual bases of $\operatorname{Lie} G$ with respect to Φ. Then $\ker\psi^{+}_{p,q}$ is the image of the linear transformation $\big(\sum_{s=1}^{n}(x_s\otimes x_s^*+x_s^*\otimes x_s)\big)-2\Phi(\lambda_p^*,\lambda_q^*)\mathrm{id}$ of the vector space $\mathcal{S}^{\pm}_{p,q}$.

Summing up, if G is a connected semisimple group such that the Hilbert basis (45) freely generates the semigroup $\mathrm{X}(T)_+$, then the sought-for canonical presentation of $k[G]$ is given by the surjective homomorphism

$$\phi := \phi^+\otimes\phi^- : \mathcal{F} := \mathcal{F}^+\otimes_k\mathcal{F}^- \to k[G] \tag{49}$$

of the polynomial k-algebra $\mathcal{F}$ and the following generating system $\mathscr{R}$ of the ideal $\ker\phi$. Identify $\mathcal{F}^+$ and $\mathcal{F}^-$ with subalgebras of $\mathcal{F}$ in the natural way. Then $\mathscr{R}=\mathscr{R}_1\bigsqcup\mathscr{R}_2$, where

$$\mathscr{R}_1=\bigcup_{p,q}\Big(\big(\varphi^{+}_{p,q}\big)^{-1}\big(\ker\psi^{+}_{p,q}\big)\cup\big(\varphi^{-}_{p,q}\big)^{-1}\big(\ker\psi^{-}_{p,q}\big)\Big) \tag{50}$$

(see the definition of $\varphi^{\pm}_{p,q}$, $\psi^{\pm}_{p,q}$ in (47), (48)) and

$$\mathscr{R}_2=\{s_{\lambda_1},\ldots,s_{\lambda_d}\} \tag{51}$$

(see the definition of s_{λ_i} in Corollary 2). The elements of $\mathscr{R}_1$ (respectively, $\mathscr{R}_2$) are the Plücker-type (respectively, the SL_2-type) relations of the presentation.

The canonical presentation of $k[G]$ is redundant. To reduce the size of $\mathscr{R}_1$, we may replace every space $\ker\psi^{\pm}_{p,q}$ in (50) by a basis of this space. Finding such a basis falls within the framework of Standard Monomial Theory.

6. An example

As an illustration, here we explicitly describe the canonical presentation of $k[G]$ for $G=\mathrm{SL}_n$, $n\geqslant 2$, and $\operatorname{char} k=0$.

Let T be the maximal torus of diagonal matrices in G, and let B^+ (respectively, B^-) be the Borel subgroup of lower (respectively, upper) triangular matrices in G. Then

$$\mathscr{H}=\{\varpi_1,\ldots,\varpi_{n-1}\},\quad\text{where}$$
$$\varpi_d\colon T\to k,\quad \operatorname{diag}(a_1,\ldots,a_n)\mapsto a_{n-d+1}\cdots a_n.$$

Every pair $i_1, i_2 \in [n]$ determines the function

$$x_{i_1,i_2}\colon G \to k, \quad \begin{pmatrix} a_{1,1} & \ldots & a_{1,n} \\ \ldots & \ldots & \ldots \\ a_{n,1} & \ldots & a_{n,n} \end{pmatrix} \mapsto a_{i_1,i_2}. \tag{52}$$

The k-algebra generated by all functions (52) is $k[G]$.

For every $d \in [n-1]$ and every sequence $i_1, \ldots, i_d$ of d elements of $[n]$, put

$$f^-_{i_1,\ldots,i_d} := \det\begin{pmatrix} x_{i_1,1} & \ldots & x_{i_1,d} \\ \ldots & \ldots & \ldots \\ x_{i_d,1} & \ldots & x_{i_d,d} \end{pmatrix}, \quad f^+_{i_1,\ldots,i_d} := \det\begin{pmatrix} x_{i_1,n-d+1} & \ldots & x_{i_1,n} \\ \ldots & \ldots & \ldots \\ x_{i_d,n-d+1} & \ldots & x_{i_d,n} \end{pmatrix}.$$

For every fixed d, all functions $f^-_{i_1,\ldots,i_d}$ (respectively, $f^+_{i_1,\ldots,i_d}$) such that $i_1 < \cdots < i_d$ are linearly independent over k and their linear span over k is the simple G-module $\mathcal{S}^-(\varpi_d)$ (respectively, $\mathcal{S}^+(\varpi_d)$); see, e.g., [FT 1992, Prop. 3.2]. Therefore, denoting by $x^\pm_{i_1,\ldots,i_d}$ the element $f^\pm_{i_1,\ldots,i_d}$ of the k-algebra $\mathcal{F}^\pm$ defined by (46), we identify $\mathcal{F}^\pm$ with the polynomial k-algebra in variables $x^\pm_{i_1,\ldots,i_d}$, where d runs over $[n-1]$ and $i_1, \ldots, i_d$ runs over $[n]_d$. Correspondingly, the k-algebra $\mathcal{F}$ is identified with the polynomial k-algebra in the variables $x^-_{i_1,\ldots,i_d}$ and $x^+_{i_1,\ldots,i_d}$, the homomorphism (49) takes the form

$$\phi\colon \mathcal{F} \to k[G], \quad x^+_{i_1,\ldots,i_d} \mapsto f^+_{i_1,\ldots,i_d}, \;\; x^-_{i_1,\ldots,i_d} \mapsto f^-_{i_1,\ldots,i_d},$$

and $\phi^\pm = \phi|_{\mathcal{F}^\pm}$. Below the sets (50) and (51) are explicitly specified using this notation.

First, we will specify the Plücker-type relations. It is convenient to introduce the following elements of $\mathcal{F}^\pm$. Let $i_1, \ldots, i_d$ be a sequence of $d \in [n-1]$ elements of $[n]$, and let $j_1, \ldots, j_d$ be the nondecreasing sequence obtained from $i_1, \ldots, i_d$ by permutation. Then we put

$$x^\pm_{i_1,\ldots,i_d} = \begin{cases} \operatorname{sgn}(i_1, \ldots, i_d) x^\pm_{j_1,\ldots,j_d} & \text{if } i_p \neq i_q \text{ for all } p \neq q, \\ 0 & \text{otherwise.} \end{cases}$$

The k-algebra $\mathcal{S}^\pm$ is the coordinate algebra of the affine multicone over the flag variety, see [To 1979]. By the well-known classical Hodge's result [Ho 1942], [Ho 1943] (see also [To 1979, p. 434, Cor. 1]) the ideal $\ker \phi^\pm$ is generated by all elements of the form

$$\sum_{l=1}^{q+1} (-1)^l x^\pm_{i_1,\ldots,i_{p-1},j_l} x^\pm_{j_1,\ldots,\widehat{j_l},\ldots,j_{q+1}}, \tag{53}$$

where p and q run over $[n-1]$, $p \leqslant q$, and $i_1, \ldots, i_{p-1}$ and $j_1, \ldots, j_{q+1}$ run over $[n]_{p-1}$ and $[n]_{q+1}$ respectively. Since every element (53) is homogeneous of degree 2, this result together with Theorem 11 imply that, for every fixed $p, q \in [n-1]$, the set $\left(\varphi^\pm_{p,q}\right)^{-1}\left(\ker \psi^\pm_{p,q}\right)$ in (50) is the linear span of all elements (53), where $i_1, \ldots, i_{p-1}$ and $j_1, \ldots, j_{q+1}$ run over $[n]_{p-1}$ and $[n]_{q+1}$ respectively. This describes the Plücker-type relations (50).

Secondly, we will describe s_{ϖ_d}. If $\boldsymbol{i} \in [n]_{n-d}$ is a sequence $i_1, \ldots, i_{n-d}$, we put $x^{\pm}_{\boldsymbol{i}} := x^{\pm}_{i_1,\ldots,i_{n-d}}$ and denote by $\boldsymbol{i}^* \in [n]_d$ the unique sequence $j_1, \ldots, j_d$ whose intersection with $i_1, \ldots, i_{n-d}$ is empty. Let $\mathrm{sgn}(\boldsymbol{i}, \boldsymbol{i}^*)$ be the sign of the permutation $(i_1, \ldots, i_{n-d}, j_1, \ldots, j_d)$. Then by [FT 1992, Thm. 3.1(b)],

$$s_{\varpi_d} = \sum_{\boldsymbol{i} \in [n]_{n-d}} \mathrm{sgn}(\boldsymbol{i}, \boldsymbol{i}^*) x^{-}_{\boldsymbol{i}} x^{+}_{\boldsymbol{i}^*}.$$

This describes the SL_2-type relations (51).

A similar description of the presentation of $k[G]$ may be given for the classical groups G of several other types: for them, the Plücker-type (respectively, the SL_2-type) relations are obtained using [LT 1979], [LT 1985] (respectively, [FT 1992]).

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