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Packet of Gravity Surface Waves at High Reynolds Numbers

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Abstract—Within the framework of the Lagrangian approach a method for describing a wave packet on the surface of an infinitely deep, viscous fluid is developed. The case, in which the inverse Reynolds number is of the order of the wave steepness squared is analyzed. The expressions for fluid particle trajectories are determined, accurate to the third power of the steepness. The conditions, under which the packet envelope evolution is described by the nonlinear Schrödinger equation with a dissipative term linear in the amplitude, are determined. The rule, in accordance with which the term of this type can be correctly added in the evolutionary equation of an arbitrary order is formulated.

Keywords: wave packet, viscosity, Lagrangian coordinates, nonlinear Schrödinger equation.

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The evolution of the packet of potential gravity surface waves is governed by the nonlinear Schrödinger equation in the third order of the perturbation theory [1] and by the Dysthe equation in the fourth approximation. To allow for viscosity it is conventional—purely phenomenologically, that is, from general considerations,—to add a linear-in-amplitude term in these evolutionary equations [3, 4]. The conditions, at which the introduction of an additional term into these equations may be correct, are not discussed. The value of the coefficient of the amplitude represents an arbitrary parameter and its physical meaning is not explained. In this connection, the question arises about the validation of the dissipative equations employed.

The main problem in describing waves in a viscous fluid is connected with the boundary condition on the free surface. In the solution of the linear approximation, in which the boundary condition is transferred onto the horizontal surface, a new spatial scale, namely, the boundary layer thickness, arises. It depends on the wave frequency and for actually observable gravity waves is not greater than a few centimeters [5]. In this layer the vortical component of the wave field decreases exponentially. For this reason, in the quadratic and higher approximations the boundary condition can be transferred onto the horizontal level only if the wave amplitude is of the same order or smaller than the boundary layer thickness. Otherwise, the formulation of the surface boundary conditions on the undisturbed level seems inadequate (see [6, §3.4]). Because of this, in the Eulerian description of weakly-decaying quasistationary waves an orthogonal curvilinear coordinate system with the free surface as a coordinate line is used [6, 7]. However, the situation becomes fundamentally simpler, if the Lagrangian coordinates are used [8–12]. In this case. the free surface shape is known, although the form itself of the equations of fluid motion becomes considerably more intricate.

In this study, on the basis of the Lagrangian approach the problem of the packet of gravity surface waves in an infinitely deep, viscous fluid is considered. It is assumed that the inverse Reynolds number based on the high-frequency filling parameters is of the order of the wave steepness squared. As a result, the fluid flow is potential. However, starting from the third approximation, the boundary conditions on the free surface involve viscous terms.

On the free surface of a viscous fluid two conditions must be fulfilled, namely, that of continuity for the vertical and horizontal components of the momentum flux. The nonlinear Schrödinger equation with a dissipative linear term is obtained from the former condition, whereas the latter condition cannot be satisfied.

This is due to the fact that near the free surface viscosity cannot be disregarded in principle. The non-vanishing of the horizontal component of the momentum flux means that near the free surface the mean flow determined in the process of the problem solution is actually different from that given by the potential solution. However, the flow itself does not enter in the evolutionary equation for the packet amplitude. For this reason, it can be said that under approximations adopted the nonlinear Schrödinger equation with linear dissipation is fairly validated. From the process of solution construction there follows the rule of correct introduction of the dissipative linear term into a evolutionary equation of order n: the inverse Reynolds number must be of the order of the wave steepness to the power $n - 1$.

1. FORMULATION OF THE PROBLEM

The complete system of equations of the dynamics of an incompressible viscous fluid in the Lagrange variables for two-dimensional plane-parallel flow in the Earth's gravity field is as follows [13, 14]

$$[X, Y] = \frac{D(X, Y)}{D(a, b)} = 1,$$
$$X_{tt} = -\rho^{-1}[p, Y] + \nu\{[X, [X, X_t]] + [Y, [Y, X_t]]\}, \tag{1.1}$$
$$Y_{tt} = -g - \rho^{-1}[X, p] + \nu\{[X, [X, Y_t]] + [Y, [Y, Y_t]]\}.$$

Here, $X(a, b, t)$ and $Y(a, b, t)$ are the coordinates of fluid particle locations, a and b are its Lagrangian coordinates, ρ is the density, p is the pressure, g is the gravity acceleration (the y axis is directed vertically upwards), and ν is kinematic viscosity. The brackets mean the operation of calculating the Jacobian with respect to the variables a and b. The first equation of system (1.1) is the continuity equation and reflects the fact that at the initial moment of time the fluid particle coordinates are a and b.

The system of equations (1.1) must be supplemented with boundary conditions. In the case of free gravity waves on an infinitely deep water these are the impermeability condition on the bottom ($Y_t = 0$ at $b = -\infty$) and the absence of viscous stresses from the free surface

$$T_{ik}n_k = -p_0 n_i, \quad \mathbf{n}\{n_x, n_y\} = \mathbf{n}\left\{-\frac{Y_a}{\sqrt{X_a^2 + Y_a^2}}, \frac{X_a}{\sqrt{X_a^2 + Y_a^2}}\right\}, \quad b = 0,$$
$$T_{xx} = -p + 2\nu\rho[X_t, Y], \quad T_{yy} = -p - 2\nu\rho[Y_t, X], \quad T_{xy} = \nu\rho\big([Y_t, y] - [X_t, X]\big). \tag{1.2}$$

Here, $\mathbf{T}_{ik}$ is the viscous stress tensor, p_0 is the constant external pressure, and $\mathbf{n}$ is the outward normal to the free surface.

We will introduce the complex coordinates $W = X + iY$ and $\overline{W} = X - iY$, where the bar denotes a complex-conjugate quantity. Then system (1.1) takes the form:

$$[\overline{W}, W] = 2i, \quad W_{tt} = -ig + i\rho^{-1}[p, W] + \frac{\nu}{2}\{[W, [\overline{W}, W_t]] + [\overline{W}, [W, W_t]]\}. \tag{1.3}$$

In studying wave motions of fluids it is convenient to go over to the modified Lagrangian coordinates $q = a + \sigma t$, b, t [10], where the quantity σ is constant. This is equivalent to going over into a reference frame in motion at a velocity σ. As becomes clear in what follows, it coincides with the phase velocity of the high-frequency packet "filling". In the new coordinates system (1.3) can be written as follows:

$$[\overline{W}, W] = \frac{D(\overline{W}, W)}{D(q, b)} = 2i,$$
$$W_{tt} + 2\sigma W_{tq} + \sigma^2 W_{qq} = \frac{i[p, W]}{\rho} - ig \tag{1.4}$$
$$+ \frac{\nu}{2}\{[W, [\overline{W}, W_t + \sigma W_q]] + [\overline{W}, [W, W_t + \sigma W_q]]\}.$$

Here, the brackets mean the operation of taking the Jacobian with respect to the variables q and b.

We will now non-dimensionalize this system by introducing new variables

$$W = LW_n, \quad q = Lq_n, \quad t = (L/\sigma)t_n, \quad p = \rho\sigma^2 p_n, \tag{1.5}$$

where L is a scale length (in what follows, the subscripts are omitted). Then the system of equations (1.4) takes the form:

$$\begin{gathered}[\overline{W},\, W] = \frac{D(\overline{W},\, W)}{D(q,\, b)} = 2i, \\ W_{tt} + 2W_{tq} + W_{qq} = -i\frac{gL}{\sigma^2} + i[p,\, W] \\ + \frac{1}{2R_*}\big\{[W,\, [\overline{W},\, W_t + W_q]] + [\overline{W},\, [W,\, W_t + W_q]]\big\}.\end{gathered} \tag{1.6}$$

The equation of motion contains two dimensionless parameters, namely, the Reynolds number $R_* = \sigma L/\nu$ and the quantity gL/σ^2.

In the formulas for the boundary conditions (1.2) we will go over to the complex coordinates of a fluid particle trajectory; then, with account for Eq. (1.5) we obtain

$$\begin{gathered}T_{xx} = -p - \frac{i}{R_*}\operatorname{Im}\big([W_t + W_q,\, W] - [W_t + W_q,\, \overline{W}]\big), \\ T_{yy} = -p + \frac{i}{R_*}\operatorname{Im}\big([W_t + W_q,\, W] + [W_t + W_q,\, \overline{W}]\big), \\ T_{xy} = -\frac{1}{R_*}\operatorname{Re}\big[W_t + W_q,\, W\big].\end{gathered} \tag{1.7}$$

We will consider the behavior of a weakly-nonlinear wave train with slowly varying parameters (amplitude, frequency, and wavenumber). For this purpose, we will use the method of expansion of a derivative (multiscale method [15]).

We will assume that the function W depends on the spatial $q_0 = q$, $q_1 = \varepsilon q$, $q_2 = \varepsilon^2 q$, and b and temporal $t_1 = \varepsilon t$ and $t_2 = \varepsilon^2 t$ variables (ε is a small parameter of the wave steepness) and can be represented in the form of the expansion

$$\begin{gathered}W(q_0,\, q_1,\, q_2,\, b,\, t_1,\, t_2) = q_0 + ib + \varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3 + O(\varepsilon^4), \\ w_j = w_j(q_0,\, q_1,\, q_2,\, b,\, t_1,\, t_2),\end{gathered} \tag{1.8}$$

while the pressure in the fluid is determined by the relation

$$\begin{gathered}p = p_0 - b + \varepsilon p_1 + \varepsilon^2 p_2 + \varepsilon^3 p_3 + O(\varepsilon^4), \\ p_j = p_j(q_0,\, q_1,\, q_2,\, b,\, t_1,\, t_2).\end{gathered} \tag{1.9}$$

The substitution of the first two terms determining the hydrostatic pressure in the second equation of system (1.6) makes it possible to obtain the value of the second dimensionless parameter: $gL/\sigma^2 = 1$. If it assumed that $L = k^{-1}$, while $\sigma = \omega/k$ is the phase velocity of the wave, then this relation coincides with the dispersion equation of waves on deep water.

In the general case, the problem of describing the packet evolution in a viscous fluid is exceptionally complicated. However, we will make a simplifying assumption. We will assume that the Reynolds number is fairly high, so that $R_*^{-1} = \alpha\varepsilon^2$, where α is a numerical coefficient of the order of unity. This relation is equivalent to the assumption that the carrying wave length $\lambda = (\nu/\alpha\varepsilon^2\sqrt{g})^{2/3}$. The chosen restriction on the Reynolds number means that the viscosity effects exhibit themselves only in the cubic approximation.

In any approximation, apart from the consideration of the wave component of the motion, the mean flows should also be determined. These can be conveniently determined using the Helmholtz equation, which follows from the equations of motion but does not involve the pressure. In the Lagrangian coordinates it takes the form:

$$\frac{\partial\Omega}{\partial t} = \nu\{[Y,\ [Y,\ \Omega]] + [X,\ [X,\ \Omega]]\}, \quad \Omega = [X_i,\ X] + [Y_i,\ Y]. \tag{1.10}$$

Here, Ω is the plane flow vorticity, while the brackets mean the operation of taking the Jacobian with respect to the Lagrangian variables a and b.

In the variables introduced above this equation can be rewritten as follows:

$$\mathrm{Re}\left\{\frac{\partial}{\partial t}\left[W_t + W_q,\ \overline{W}\right] + \frac{\partial}{\partial q}\left[W_t + W_q,\ \overline{W}\right] - \frac{1}{R_*}\left[W,\ [\overline{W},\ [W_t + W_q,\ \overline{W}]]\right]\right\} = 0. \tag{1.11}$$

In this equation all the quantities are dimensionless, while the Jacobians are now taken with respect to the variables q and b. In the complex representation the vorticity $\Omega = \mathrm{Re}\left[W_t + W_q,\ \overline{W}\right]$. It can also be represented in the form of a series in the small parameter of wave steepness: $\Omega = \sum_{n=1} \varepsilon^n \Omega_n$.

2. LINEAR APPROXIMATION

The calculations can be conveniently performed in accordance with the following procedure. First, the system of equations consisting of the continuity and Helmholtz equations is solved. Then from the known form of the complex coordinate W_n the pressure p_n is determined from the equation of motion. Thereupon all the expressions thus determined are substituted in the boundary conditions, whence evolutionary equations for the wave packet envelope are derived.

Substituting expansions (1.8) and (1.9) in the hydrodynamics equations (1.6) and (1.11) yields the systems of equations for the corresponding approximations. In the linear case the continuity and Helmholtz equations are as follows:

$$\mathrm{Im}(w_{1b} + iw_{1q_0}) = 0, \qquad \mathrm{Re}\,(w_{1b} + iw_{1q_0})_{q_0q_0} = -\Omega_{1q_0} = 0.$$

Their solution diminishing with the depth can be written as follows:

$$w_1 = A(q_l,\ t_l)\exp(b + iq_0) + \psi_1(q_l,\ b,\ t_l), \quad l \geq 1. \tag{2.1}$$

In this equation the first term describes the oscillatory motion of fluid particles and the second term is the drift flow. The wave amplitude A is a function of "slow" variables. The function ψ_1 is real; it depends on both the slow coordinates and the vertical Lagrangian coordinate.

The distinctive feature of the motion under consideration is that the drift flow ψ_1 arises already in the first approximation. Mathematically, this is due to the fact that in the linear approximation the solution is determined accurate to an arbitrary function dependent on the fast coordinates q_l and t_l and independent of the slow coordinate q_0. In studying the stationary potential wave on the fluid surface (Stokes wave) the solution is sought as a function of the coordinates q_0 and b only [10]. For this reason, the linear approximation does not include the drift flow but this flow appears in the quadratic approximation.

In the most general case it would be well to add a purely imaginary constant determining the mean fluid level to solution (2.2). It is convenient to let it be equal to zero, thus fixing this level at the horizon, $Y = 0$. The vorticity $\Omega_1 = 0$. The equation for p_1 is obtained from the second equation (1.6)

$$p_{1q_0} + ip_{1b} = iw_{1q_0} - w_{1q_0q_0}. \tag{2.2}$$

In accordance with solution (2.1), the right side of the above expression is zero; therefore, the quantity p_1 is constant. In the problem under consideration viscous forces start to play a role only in the cubic approximation; because of this, in the first two approximations it is the pressure constancy condition on the free surface (cf. Eqs. (1.2) and (1.7)) that is taken for the boundary condition. For the sake of simplicity, we can let $p_1|_{b=0} = 0$. Thus, from the first approximation it follows that $p_1 = 0$ for any arbitrary b.

Comparing expression (2.2) with Eq. (1.4) in the dimensional variables it can readily be seen that the vanishing of the right side of Eq. (2.2) is the consequence of the dispersion equation for the waves on deep water which was used before for writing the equations in the dimensionless variables.

3. QUADRATIC APPROXIMATION

In the second approximation the continuity and Helmholtz equations are as follows:

$$\begin{gathered}\mathrm{Im}\,(w_{2b} + iw_{2q_0} - w_{1q_0}\overline{w}_{1b} + iw_{1q_1}) = 0,\\ \mathrm{Re}\left(w_{2q_0b} + iw_{2q_0q_0} + iw_{1q_0t_1} + 2iw_{1q_1q_0} + w_{1q_1b} + w_{1t_1b} - [w_{1q_0}, \overline{w}]\right) = -\Omega_2 = 0.\end{gathered} \tag{3.1}$$

Here, the second equation was singly integrated with respect to q_0. Here and in what follows the Jacobian is taken with respect to the variables q_0 and b. Also in the second approximation the flow is potential ($\Omega_2 = 0$).

Substituting solution (2.2) in relations (3.1) we arrive at the following system of equations

$$\mathrm{Im}\,(w_{2b} + iw_{2q_0} - i(A\psi_{1b} - A_{q_1})\exp(b + iq_0) - i|A|^2\exp 2b + i\psi_{1q_1}) = 0,$$

$$\mathrm{Re}\,(w_{2q_0b} + iw_{2q_0q_0} - (A_{q_1} - A\psi_{1b})\exp(b + iq_0) + \psi_{1q_1b} + \psi_{1t_1b} + 2|a|^2\exp 2b) = 0.$$

Its solution can be conveniently written in the form:

$$w_2 = Q_2(q_l, b, t_l)\exp(b + iq_0) + \psi_2(q_l, b, t_l) + if_2(q_l, b, t_l), \quad l \geq 1, \tag{3.2}$$

where ψ_2 and f_2 are real functions of slow variables.

The conditions on the choice of the functions Q_2 and f_2 and the equation for the mean displacement ψ_1 are as follows:

$$Q_2 = i\big(A\psi_1 - bA_{q_1}\big), \quad f_{2b} = |A|^2\exp 2b - \psi_{1q_1}, \quad \psi_{1t_1b} + \psi_{1q_1b} = -2|A|^2\exp 2b. \tag{3.3}$$

The function ψ_2 can be arbitrary and in this approximation is not calculated. In the second approximation the oscillation amplitude is determined accurate to an arbitrary function independent of the coordinate b. We will let it zero, since otherwise the amplitude A could be overdetermined by including this function in it. The mean vertical particle displacement f_2 is determined from the known function of the drift flow in the first approximation ψ_1. In view of the fact that on the bottom there are no vertical displacements of fluid particles, for the function f_2 the following distribution in the depth is valid

$$f_2 = \frac{1}{2}|A|^2\exp 2b - \int\limits_{-\infty}^{b}\psi_{1q_1}\,db.$$

As $b \to 0$, the both terms of this formula vanish. On the free surface the function f_2 can be other than zero. This means that in the second approximation the mean level generally does not coincide with the surface $Y = 0$ and deviates from it by the quantity $f_2(0)$. Integrating the last equation (3.3) with respect to b we obtain

$$\psi_{1q_1} + \psi_{1t_1} = -|A|^2\exp 2b.$$

The integration constant is zero, since on the bottom the fluid is at rest. The minus sign means that both the mean flow and the wave move in the negative direction of the X axis. The function ψ_1 is determined from the known wave amplitude. In the second approximation the drift flow form (function ψ_2) should be determined from the third-approximation equations.

In the quadratic approximation Eq. (1.6) takes the form:

$$w_{2q_0q_0} - iw_{2q_0} + p_{2q_0} + ip_{2b} = iw_{1q_1} - 2w_{1q_0q_1} - 2w_{1q_0t_1}.$$

In view of Eqs. (3.2) and (3.3), we obtain

$$p_{2q_0} + ip_{2b} = -i(A_{q_1} + 2A_{t_1})\exp(b + iq_0) + i\psi_{1q_1}.$$

Here, the pressure constancy condition on the free surface is taken as the boundary condition for the above equation. In view of this, we obtain that the wave packet amplitude satisfies the equation

$$A_{t_1} + \frac{1}{2}A_{q_1} = 0. \tag{3.4}$$

The expression for the pressure takes the form:

$$p_2 = \int_0^b \psi_{1q_1}\, db.$$

From Eq. (3.4) it follows that the packet envelope travels in the negative direction of the X axis at a velocity half as large as the phase velocity. If a new coordinate $\xi_1 = q_1 - t_1/2$ is introduced, then the wave amplitude is a function of this variable, that is, $A = A(\xi_1, q_l, t_l)$, $l \geq 2$. The function ψ_1 and, therefore, the pressure p_2 do not possess this property.

We will emphasize that for the two first approximations the solutions are constructed as in the case of an inviscid fluid: the fluid flow is potential, while on the free surface the pressure constancy condition is satisfied. The viscosity effect manifests itself only in the third approximation.

4. CUBIC APPROXIMATION

The third-approximation equations are as follows:

$$\begin{gathered}
\operatorname{Im}\left(w_{3b} + iw_{3q_0} + iw_{1q_2} + iw_{2q_1} + w_{1b}\overline{w}_{1q_1} + [w_1, \overline{w}_2]\right) = 0, \\
-\Omega_3 = \operatorname{Re}\Big\{iw_{3q_0q_0} + w_{3q_0b} + i(w_{2q_0t_1} + 2w_{2q_0q_1} + w_{1q_0t_2} \\
+ 2w_{1q_0q_2} + w_{1q_1t_1} + w_{1q_1q_1}) + w_{2t_1b} + w_{2q_1b} + w_{1t_2b} + w_{1q_2b} \\
- [w_{1q_0}, \overline{w}_2] - [w_{2q_0}, \overline{w}_1] - [w_{1q_1}, \overline{w}_1] - [w_{1t_1}, \overline{w}_1] - \frac{D(w_{1q_0}, \overline{w}_1)}{D(q_1, b)}\Big\} = 0, \\
w_{3q_0q_0} - iw_{3q_0} + p_{3q_0} + ip_{3b} = iw_{1q_2} + iw_{2q_1} - p_{2q_1} + i[p_2, w_1] \\
- w_{1t_1t_1} - w_{1q_1q_1} - 2w_{1q_1t_1} - 2w_{1q_0t_2} - 2w_{2q_0t_1} - 2w_{1q_0q_2} - 2w_{q_0q_1}.
\end{gathered} \tag{4.1}$$

Also in this approximation the fluid flow is potential ($\Omega_3 = 0$). This is attributable to the fact that we consider the high-Reynolds-number approximation and in the n-th approximation the viscous term in the Helmholtz equation is determined by the lower-order vorticities. If these are identically zeros, then $\Omega_n = 0$.

Physically, this means that vorticity diffusion into the fluid depth is neglected. Thus, the viscosity effect is realized only via the boundary conditions on the free surface.

Substituting the expressions for w_1 and w_2 in the two first equations of system (4.1) we obtain the following equations

$$\mathrm{Im}\Big\{ iw_{3q_0} + w_{3b} + 2(1+b)\overline{A}_{q_1}A\exp 2b + (\psi_{1q_2} + \psi_{2q_1}) + \exp(b+iq_0)$$

$$\times\Big[iA_{q_2} + bA_{q_1q_1} - (b\psi_1 + \psi_1)_b A_{q_1} - \Big(i\psi_2 + f_2 - \frac{1}{2}\psi_1^2\Big)_b A\Big]\Big\} = 0,$$

$$\mathrm{Re}\Big\{ iw_{3q_0q_0} + w_{3q_0b} + 2i(b+3)\overline{A}_{q_1}A\exp 2b + (\psi_{2t_1} + \psi_{2q_1} + \psi_{1t_2} + \psi_{1q_2})_b - \exp(b+iq_0)$$

$$\times\Big[A_{q_2} - ibA_{q_1q_1} + i(b\psi_1 + \psi_1)_b A_{q_1} + \Big(if_2 - \psi_2 - \frac{i\psi_1^2}{2}\Big)_b A + 4i|A|^2A\exp 2b\Big]\Big\} = 0.$$

From the second equation of this system it follows that

$$\psi_{2t_1} + \psi_{2q_1} + \psi_{1t_2} + \psi_{1q_2} = \frac{i}{4}(2b+5)(A_{q_1}\overline{A} - \overline{A}_{q_1}A)e^{2b}. \tag{4.2}$$

The expression for the third-approximation solution is as follows:

$$w_3 = Q_3\exp(b+iq_0) + \frac{1}{2}|A|^2\overline{A}\exp(3b - iq_0) + \psi_3 + if_3, \tag{4.3}$$

$$Q_3 = \Big(i\psi_2 + f_2 - \frac{\psi_1^2}{2}\Big)A + (1+b)\psi_1 A_{q_2} - ibA_{q_2} - \frac{b^2}{2}A_{q_1q_1} + |A|^2A\exp 2b,$$

$$f_{3b} = i(1+b)(\overline{A}_{q_1}A - A_{q_1}\overline{A}) - \psi_{1q_2} - \psi_{2q_1}. \tag{4.4}$$

The function ψ_1 is determined in considering the next approximation. The calculation of the mean flow is a separate problem. From Eqs. (4.2)–(4.4) it can be seen that it has both horizontal and vertical velocity components, namely, $f_3(-\infty) = 0$, $f_2(0)$, and $f_3(0) \neq 0$. The packet in motion entrains fluid particles in not only the horizontal motion but also in the vertical motion determined by the form of the envelope amplitude and the horizontal flows of the first and second approximations. In this case, along with the near-surface drift of fluid particles, there exists also a flow named the irrotational return flow [16]. However, our purpose is to derive the evolutionary equation for the wave packet envelope.

Substituting the solutions for the three first approximations in the last equation of system (4.1) we obtain the equation for determining the pressure

$$p_{3q_0} + ip_{3b} = -\Big[2iA_{t_2} + 2|A|^2Ae^{2b} + \frac{1}{4}A_{\xi_1\xi_1}\Big]\exp(b+iq_0)$$

$$+ |A|^2\overline{A}\exp(3b - iq_0) + i(\psi_{1q_2} + \psi_{2q_1}).$$

Integrating this equation we obtain an expression for the pressure.

$$p_3 = 2\mathrm{Re}\Big[-A_{t_2} + \frac{1}{8}iA_{\xi_1\xi_1} + \frac{1}{2}i|A|^2Ae^{2b}\Big]\exp(b+iq_0)$$

$$+ \int_0^b (\psi_{1q_2} + \psi_{2q_1})\,db + p_3^*(q_1, t_1), \quad l \geqslant 1, \quad \xi_1 = q_1 - t_1/2. \tag{4.5}$$

The function p_3^* is determined from the boundary conditions

$$(T_{yl}n_l)_3 = -p_3 - 2\alpha \mathrm{Re}\, iw_{1q_0b} = 0, \quad b = 0, \tag{4.6}$$

$$(T_{xl}n_l)_3 = \alpha \mathrm{Re}\,(w_{1q_0b} - iw_{1q_0q_0}) = 0. \tag{4.7}$$

Equations (4.6) and (4.7) correspond to the conditions of the equality of the vertical and horizontal momentum fluxes to zero on the free boundary. The subscript "3" for the force components that act on the surface means that only the third-order terms are taken into account. Substituting expression for the pressure (4.5) in Eq. (4.6) yields the evolutionary equation for the wave packet amplitude envelope

$$iA_{t_2} + \frac{1}{2}|A|^2A + \frac{1}{8}A_{\xi_1\xi_1} + i\alpha A = 0. \tag{4.8}$$

The function $p_3^* \equiv 0$. Equation (4.8) is the nonlinear Schrödinger equation with the linear dissipative term. In the dimensional variables it takes the form:

$$iA_{t_2} + \frac{1}{2}\omega k^2|A|^2A + \frac{1}{8}\frac{\omega}{k^2}A_{\xi_1\xi_1} + i\alpha\omega A = 0. \tag{4.9}$$

It should be noted that it is the wave frequency that plays the role of the amplitude decrement.

Equation (4.9) can be written returning to the original variables t and $\xi = q - \sigma t$. We will introduce a new function $A^* = A/\varepsilon$; then the evolutionary equation is brought into the form:

$$iA_t^* + \frac{1}{2}\omega k^2|A^*|^2A^* + \frac{1}{8}\frac{\omega}{k^2}A_{\xi\xi}^* + i\frac{\alpha\omega}{R_*}A^* = 0. \tag{4.10}$$

An important feature of Eqs. (4.8)–(4.10) is that they do not include the drift flow ψ_1. This is the consequence of the fact that the case of fairly high Reynolds numbers is considered. If the inverse Reynolds number is of the order of the wave steepness or of the order of unity, then the evolutionary equation will include the mean flow.

We will now note a shortcoming of the solution constructed. It can easily be seen that the boundary condition for the tangential component of the momentum flux (4.7) is not fulfilled. This is due to the fact that viscosity is important only within a very narrow surface layer. Accordingly, the flow within this layer possesses large gradients and, generally speaking, the viscous terms in the equations of motion cannot be disregarded. Equations (4.3) to (4.5) describe a potential flow and are valid outside the near-surface boundary layer. Within the layer they should be refined with account for viscosity. Since the condition for the tangential component of the momentum flux is not fulfilled, the variations will primarily concern the mean flow. However, it does not enter in the evolutionary equation for the amplitude. On the basis of this it may be inferred that Eqs. (4.8) to (4.10) are a fairly good approximation for studying the wave packet damping at high Reynolds numbers.

The method of constructing the solution is so organized that the viscosity effect is taken into account only in the boundary condition for the normal component of the momentum flux. The linear dissipative term appears in the n-order evolutionary equation if it coincides with the least order of the nonzero viscous term. This means that the quantity $1/R_*$ must be of the order of ε^{n-1}. Thus, the greater the order of the evolutionary equation the greater the Reynolds number at which the use of the nonlinear Schrödinger equation with linear dissipation is validated.

Summary. The method of modified Lagrangian coordinates is developed for describing the packet of gravity surface waves at high Reynolds numbers. The conditions of applicability of the nonlinear Schrödinger equation with a linear-in-amplitude dissipative term are determined. The algorithm for constructing the solution tested for the three first approximations can be applied to an analysis of the higher-order flows.

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