ISSN 1990-4789, Journal of Applied and Industrial Mathematics, 2009, Vol. 3, No. 1, pp. 1–4. © Pleiades Publishing, Ltd., 2009.
Original Russian Text © V.E. Alekseev, D.S. Malyshev, 2008, published in Diskretnyi Analiz i Issledovanie Operatsii, 2008, Vol. 15, No. 1, pp. 3–10.

Planar Graph Classes with the Independent Set Problem Solvable in Polynomial Time

V. E. Alekseev* and D. S. Malyshev

Department of Computational Mathematics and Cybernetics, State University of Nizhnii Novgorod, pr. Gagarina 23, Nizhnii Novgorod, 603950 Russia

Received September 20, 2007; in final form, January 29, 2008

Abstract—The polynomial solvability of the independent set problem is proved for an infinite family of subsets of the class of planar graphs.

DOI: 10.1134/S1990478909010013

INTRODUCTION

We will call a class X of graphs IS-*easy* if there is an algorithm solving the independent set problem for graphs in X in polynomial time. If the independent set problem remains NP-complete for graphs in X then X is called IS-*hard*. It is known that the class of all planar graphs is IS-hard. The goal of this article is to prove that some subsets of this class are IS-easy.

The direction for distinguishing IS-easy classes is suggested in [2, 3], where the complexity was studied of the independent set problem for some hereditary classes of graphs; these are the classes closed under removal of vertices. Each hereditary (and necessarily hereditary) class X is determined by a certain set of forbidden induced subgraphs Y. In this case we write $X = \mathrm{Free}(Y)$. If in addition Y is finite then X is called *finitely defined*.

The concept of *boundary class* of graphs is introduced in [3] as an inclusion minimal class presenting the intersection of a descreasing sequence of IS-hard classes. The following is proved in [3]:

Theorem 1. *A finitely defined class is* IS-*hard if and only if it includes an boundary class.*

Therefore, the knowledge of all boundary classes allows us to completely characterize the finitely defined IS-hard classes. It is also proved in [3] (assuming P≠NP) that some particular class is a boundary class. This is the class T of all graphs whose every connected component is a tree with at most three leaves. It remains unknown as yet whether there exist other boundary classes. This question is equivalent to that of the existence of a graph $G \in T$ such that $\mathrm{Free}(G)$ is IS-hard [3]. The difficulty of this problem is illustrated by the fact that the complexity of the independent set problem for $\mathrm{Free}(P_5)$ is presently unknown. At the same time, if instead of the whole set of hereditary classes we consider a part of it, we can hope for an irrefragable answer to this question. For instance, it is proved in [3] that T is a unique boundary class in the family of strongly hereditary classes, i.e., the classes of graphs closed under removal of vertices and edges.

In this article, we consider the hereditary subclasses of the class Planar of planar graphs. A class X is *finitely defined* with respect to Planar if $X = \mathrm{Planar} \cap \mathrm{Free}(Y)$, where Y is a finite set of graphs. We can also define the concept of a relative boundary class: a hereditary class Y of graphs is called *boundary relative to* a class X if there exists a sequence $Y_1 \supseteq Y_2 \supseteq \cdots$ of IS-hard subsets of X such that $\bigcap_{i=1}^{\infty} Y_i = Y$ and Y is a minimum class with this property. Proofs of an analog of Theorem 1 for relative boundary classes, as well as the fact that T is an boundary class relative to Planar, repeat almost verbatim the corresponding proofs in [3]. Even though we could not as yet determine whether other boundary classes relative to Planar exist, there is more noticeable progress toward this goal than in the case of (absolutely) boundary classes.

*E-mail: ave@uic.nnov.ru

Let $T_{i,j,k}$ be a tree with three leaves at distance i, j, and k from the (unique) degree 3 vertex. If $i = 0$ then this is P_{j+k}. Consider the classes

$$\text{Planar}(i, j, k) = \text{Planar} \cap \text{Free}(T_{i,j,k}).$$

The boundary classes relative to Planar distinct from T exist if and only if some of the classes Planar(i, j, k) are IS-hard. Let us verify that Planar$(1, 2, k)$ is an IS-easy class for every k.

Throughout this article we use the following notation: the set of vertices adjacent to a vertex a is $N(a)$; the set of vertices at distance 2 from a is $N_2(a)$; the set of vertices adjacent to both a and b is $N(a, b)$; and the distance in a graph between x and y is $d(x, y)$.

1. PLANAR GRAPHS WITHOUT $T_{2,2,2}$

It is unknown whether Planar$(2, 2, 2)$ is IS-easy or IS-hard. In this section, we only prove that the independent set problem for it is polynomially equivalent to the same problem for the graphs of this class with degree bounded by some constant. We then use this fact for proving the main result of the article.

At the first stage, our principal algorithmic tool will be the compressions described in [1]. A *compression* is a mapping of the vertex set of a graph to itself which, while not an automorphism, is such that every pair of distinct nonadjacent vertices goes into distinct nonadjacent vertices. Therefore, a compression transforms a graph into an induced subgraph of it; in addition, it obviously preserves the independence number. Here we will use only the first and second order the compressions; i.e., compressions that fix all but one or two vertices.

A first order compression φ is written as $\binom{a}{b}$; this means that $\varphi(a) = b$, and the remaining vertices are fixed. This transformation is indeed a compression if and only if a and b are adjacent, and every vertex adjacent to b and distinct from a is adjacent to a. In other words,

$$N(b) - \{a\} \subseteq N(a) - \{b\}.$$

The transformation $\binom{a\ b}{c\ d}$, which means $\varphi(a) = c$, $\varphi(b) = d$, and the remaining vertices are fixed, is a second order compression if:

(i) $c \neq d$;

(ii) the graph has the edges (a, c) and (b, d), and has no edges (a, b) and (c, d);

(iii) except for a and b, every vertex adjacent to c is adjacent to a as well;

(iv) except for a and b, every vertex adjacent to d is adjacent to b as well.

A graph is called *incompressible* if it admits neither first nor second order compressions. An incompressible subgraph of a graph G resulting from G by a sequence of first and second order compressions is called a 2-*base* of G. A graph can have several 2-bases (for instance, C_4 has two bases), but they are isomorphic [1]. It is obvious that a 2-base of a graph can be found in polynomial time.

Lemma 1. *Take an incompressible graph G in* Planar$(2, 2, 2)$ *and its vertices a and b with $d(a, b) = 2$. Then $|N(a, b)| \leqslant 12$.*

Proof. Call a vertex $x \in N(a, b)$ *exceptional* if every vertex adjacent to x, except for a and b, belongs to $N(a, b)$, and a-*pure* (or b-*pure*) if there is a vertex adjacent to x but not adjacent to a (to b).

Observe that every vertex in $N(a, b)$ belongs to one of these three categories. Estimate the number of vertices of each type.

If x and y are two nonadjacent exceptional vertices then the mapping $\binom{a\ b}{x\ y}$ is a compression. Hence, the exceptional vertices form a full subgraph. It cannot contain more than three vertices since otherwise a subgraph K_5 will be formed. However, if there is exactly three exceptional vertices then $N(a, b)$ has no other vertices since otherwise a subgraph homeomorphic to K_5 will be formed. Consequently, either there is at most two exceptional vertices or $|N(a, b)| = 3$.

Suppose that G includes six a-pure vertices. Consider a plane embedding of G; it determines a cyclic ordering of the edges incident to each vertex and, consequently, of the adjacent vertices. Let $x_1, \dots, x_6$ be the a-pure vertices lying in this cyclic order with respect to a. For $i = 1, \dots, 6$, take some vertex y_i adjacent to x_i and not adjacent to a. These y_i need not be distinct, for instance, $y_1 = y_2$ is possible, although y_1, y_3, and y_5 are pairwise distinct and not adjacent to each other. However, then the set $\{a, x_1, x_3, x_5, y_1, y_3, y_5\}$ induces a subgraph $T_{2,2,2}$. Therefore, $N(a, b)$ contains at most five a-pure and at most five b-pure vertices, and so, at most 12 vertices overall. The proof of Lemma 1 is complete. ☐

Lemma 2. *In each incompressible graph in* Planar(2, 2, 2)*, the degree of every vertex is at most* 120.

Proof. Pick a vertex a of degree d in some incompressible graph $G \in$ Planar(2, 2, 2). Consider the bipartite subgraph H formed by the vertex sets $A = N(a)$ and $B = N_2(a)$ and all edges of G joining A with B. Let π be the cardinality of a maximum matching, and β, the cardinality of a minimum vertex covering of H. The König theorem yields $\pi = \beta$.

If a vertex $x \in A$ is not adjacent to any vertex of B then the mapping $\binom{a}{x}$ is a compression. Consequently, each vertex of A is adjacent to at least one vertex of B, and H contains d edges having no pairwise common vertices in A. By Lemma 1, every vertex of B is of degree at most 12 in H. Hence, it takes at least $d/12$ vertices to cover these d edges. Consequently, $\pi \geqslant d/12$.

Pick a matching in H with π edges and consider the graph H' obtained from H by contracting all edges in this matching. If $d > 120$ then H' contains at least 11 vertices. Since it is planar, it contains a vertex b of degree at most 5. Among the vertices not adjacent to b, there is a pair not adjacent to each other (otherwise a subgraph K_5 will be formed). Consequently, H' includes an independent set consisting of three vertices. Corresponding to these vertices, H contains three edges, which together with the incident vertices induce a subgraph $3K_2$. However, then these six vertices together with a induce in G a subgraph $T_{2,2,2}$. The proof of Lemma 2 is complete. □

It is proved in [5] that, given Δ and k, the class of all graphs in which the degrees of vertices are at most Δ and which include no induced subgraphs $T_{1,k,k}$ is IS-easy. By Lemma 2, this yields

Lemma 3. *The class* Planar(1, 2, 2) *is* IS-*easy.*

In the next section, we will prove a more general statement.

2. PLANAR GRAPHS WITHOUT $T_{1,2,k}$

Theorem 2. *The class* Planar(1, 2, k) *is* IS-*easy for every* k.

Proof. A clique in a graph G is called *separating* if the removal of all its vertices from G increases the number of connected components of G. It is proved in [3] that if there exists a polynomial-time algorithm which solves the independent set problem for connected graphs without separating cliques in a hereditary class X then X is IS-easy. Therefore, consider a connected graph $G \in$ Planar(1, 2, k) without separating cliques. If G includes no induced subgraph $T_{1,2,2}$ then we can apply to G a polynomial-time algorithm for Planar(1, 2, 2), which exists by Lemma 3. Let us show that if G includes no subgraph $T_{1,2,2}$ then the radius of G is at most $k + 2$.

Suppose that G includes a subgraph $T_{1,2,2}$ with the vertex set $S = \{a, b_1, b_2, b_3, c_1, c_2\}$ and edges (a, b_1), (a, b_2), (a, b_3), (b_1, c_1), and (b_2, c_2). Show that in G the distance from a to every other vertex is at most $k + 2$.

Take a vertex d outside of S. Among the shortest paths joining a and d choose a path through the greatest number of vertices of S. It is obvious that this number is at most three. Denote this path by $P = x_0, x_1, \ldots, x_t$, where $x_0 = a$ and $x_t = d$. Call an edge not lying in P and joining a vertex of this path to a vertex of S a *chord*. If (x_i, y) is a chord $(y \in S)$ then $i \leqslant 3$, for otherwise there would be a shorter path from a to d. Suppose that $t \geqslant k + 3$. Consider the various possibilities for the relation between P and the vertices of S, and show that, in all cases, the graph includes either an induced subgraph $T_{1,2,k}$ or a separating clique.

Case 1. The path P passes through three vertices of S. There are two equivalent possibilities. Consider one of them: $x_1 = b_1$ and $x_2 = c_1$.

1.1. There is no chords. Then S and $x_3, \ldots, x_k$ generate $T_{1,2,k}$.

1.2. There are some chords. The unique chord in this case must be the edge (x_3, c_2) since all other chords would provide a shorter path from a to d. If this chord is present then the vertices $c_2, x_1, x_2, \ldots, x_{k+3}$ induce $T_{1,2,k}$.

For brevity, henceforth we will simply list the vertices which must be removed in order for the remaining vertices of S to form together with some vertices of P (not necessarily all) a subgraph $T_{1,2,k}$. For instance, in the case in question, we must remove a, b_2, and b_3.

Case 2. The path P passes through two vertices of S, and $x_1 = b_1$. If chords are absent then we remove b_3. If there is a chord joining x_3 with either c_1 or c_2 then we have a path from a to d of the same length passing through three vertices of S. The presence of chords joining x_3 to either b_2 or b_3 would provide a shorter path from a to d. Consequently, there can be only chords incident to x_2.

2.1. If the chord (x_2, c_2) is present then we remove b_2, b_3, and c_1.

2.2. If (x_2, c_2) is absent and (x_2, b_2) is present then we remove a, b_3, and c_1.

2.3. The chords (x_2, c_2) and (x_2, b_2) are absent, but (x_2, b_3) is present. If (x_2, c_1) is present then we remove b_1, b_2, and c_2; and if it is absent then we remove a, b_2, and c_2.

2.4. If (x_2, c_2), (x_2, b_2), and (x_2, b_3) are absent then we remove c_1.

Case 3. The path P passes through b_3, and there exist no paths of the same length from a to d passing through b_1 or b_2. If there is no chords then we remove c_1. No chord incident to x_3 can be present for the same reasons as in Case 2. Hence, only the chords incident to x_2 can be present. If a chord connects x_2 to either b_1 or b_2 then a path of the same length is formed passing through this vertex. It remains to consider the chords (x_2, c_1) and (x_2, c_2). If both are present then we remove a, b_2, and b_3; and if only (x_2, c_1) is present then we remove c_2.

Case 4. The path P avoids the vertices of S except for a. If there is no chords then we remove b_1 and c_1. For the same reasons as above, no chords incident to x_3 can be present, and as for x_2, only the chords to c_1 and c_2 can be present. If only one of them is present then we remove x_1, and if both then we remove x_1, a, b_2, and b_3. It remains to consider the case that there is only one chord from x_1.

4.1. The chord (x_1, b_1) is present, but (x_1, b_2) and (x_1, b_3) are not. If (x_1, c_2) is present then we remove a, b_3, and c_1; and if it is absent then we remove b_1 and c_1.

4.2. The chord (x_1, b_3) is present, but (x_1, b_1) and (x_1, b_2) are not. If (x_1, c_1) is present then we remove a, b_2, and c_2; and if it is absent then we remove b_3 and c_2.

4.3. The chords (x_1, b_1) and (x_1, b_2) are present, but (x_1, b_3) is not. If (x_1, c_1) is present then we remove b_1, b_2, and c_2; and if it is absent then we remove a, b_3, and c_2.

4.4. The chords (x_1, b_1) and (x_1, b_3) are present, but (x_1, b_2) is not. If (x_1, c_1) is present then we remove b_1, b_3, and c_2; and if it is absent then we remove a, b_2, and c_2.

4.5. The chords (x_1, b_1), (x_1, b_2), and (x_1, b_3) are present. In this case, the graph includes three triangles with the edge (a, x_1). In a plane embedding, each edge is incident to at most two faces. Consequently, in every plane embedding, at least one of these triangles does not bound a face. Then it is a separating clique.

Thus, the radius of G is at most $k + 2$. It is shown in [6] that every planar graph of radius r has treewidth at most $3r + 1$. Also it is known [4] that the independent set problem is solvable in polynomial time in the class of graphs of tree-width bounded by a constant. The proof of the theorem is complete. □

REFERENCES

1. V. E. Alekseev, “On Compressed Graphs,” Problemy Kibernet. **36**, 23–31 (1979).
2. V. E. Alekseev, “On Influence of Local Constraints on the Complexity of the Definition of Graph Independence Number,” in *Combinatoric and Algebraic Methods in Applied Mathematics* (Gor’kov. Gos. Univ., Gor’kii, 1983), pp. 3–13.
3. V. E. Alekseev, “On Easy and Hard Hereditary Classes of Graphs with Respect to the Independent Set Problem,” Discrete Appl. Math. **132** (1–3), 17–26 (2004).
4. H. L. Bodlaender, “Dynamic Programming on Graphs with Bounded Treewidth,” in *Lecture Notes in Computer Sciences*, Vol. 317: *Automata, Languages and Programming* (Springer, Berlin, 1988), pp. 105–118.
5. V. Lozin and M. Milanic, “A Polynomial Algorithm for Finding an Independent Set of Maximum Weight in Fork-Free Graphs,” in *Proceedings of the Eighteenth Annual ACM-SIAM Symposium on Discrete Algorithms*. (ACM Press, New York, 2007).
6. N. Robertson and P. Seymour, “Graph Minors. III. Planar Treewidth,” J. Combin. Theory, Ser. B, **36** (1), 49–64 (1984).