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Strongly Elliptic Second-Order Systems with Boundary Conditions on a Nonclosed Lipschitz Surface*

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Abstract. We consider boundary value problems and transmission problems for strongly elliptic second-order systems with boundary conditions on a compact nonclosed Lipschitz surface S with Lipschitz boundary. The main goal is to find conditions for the unique solvability of these problems in the spaces H^s, the simplest L_2-spaces of the Sobolev type, with the use of potential type operators on S. We also discuss, first, the regularity of solutions in somewhat more general Bessel potential spaces and Besov spaces and, second, the spectral properties of problems with spectral parameter in the transmission conditions on S, including the asymptotics of the eigenvalues.

Key words: Strong ellipticity, Lipschitz domain, nonclosed boundary, potential type operators, Bessel potential spaces, Besov spaces, regularity of solutions, spectral transmission problems, spectral asymptotics.

1. Statements of the Problems. For simplicity, we consider a strongly elliptic second-order system on the n-dimensional torus $\mathbb{T} = \mathbb{T}^n$ with 2π-periodic coordinates x_j $(j = 1, \dots, n)$ outside an $(n-1)$-dimensional Lipschitz surface S with $(n-2)$-dimensional Lipschitz boundary ∂S $(n \geqslant 2)$. More precisely, we assume that S is a part of a closed Lipschitz surface Γ dividing the torus into two domains $\Omega^\pm$ and that ∂S divides Γ into two domains $S = S_1$ and S_2. There is an obvious ambiguity in the choice of the additional part S_2 of the boundary Γ. The system is given on the entire torus and is written in the divergence form

$$Lu := -\sum \partial_j a_{j,k}(x)\partial_k u(x) + \sum b_j(x)\partial_j u(x) + c(x)u(x) = f(x). \tag{1.1}$$

Here the coefficients are $m \times m$-matrices with complex entries, and u is a column m-vector; $a_{j,k} \in C^1(\mathbb{T})$, $b_j \in C^{0,1}(\mathbb{T})$ (i.e., the b_j are Lipschitz), and $c \in L_\infty(\mathbb{T})$; $\partial_k = \partial/\partial x_k$. The sides of Γ facing $\Omega^\pm$ will be denoted by $\Gamma^\pm$. In a similar way, we define $S^\pm$. The boundary conditions will be posed on $S^\pm$.

By H^s we denote the L_2-spaces of Bessel potentials; for $s \geqslant 0$, these are the Sobolev–Slobodetskii L_2-spaces. All notation and assumptions are the same as in [3] except that the boundary surface itself has now a boundary. Some information on the spaces H^s, H^s_p, and B^s_p used in the present paper can be found in [3]; it is reproduced in somewhat extended form in [4]. One could also find it in [5]. (All the author's papers included in the bibliography can be found on his Internet page `http://www.agranovich.nm.ru`.)

The strong ellipticity is the uniform positive definiteness of the real part of the principal symbol, the matrix $a(x,\xi) = \sum a_{j,k}(x)\xi_j\xi_k)$ with real ξ, $|\xi| = 1$. In addition, we assume that the real part of the inner product $(cu, u)_{\mathbb{T}}$ is sufficiently large, so that the form

$$\Phi_{\mathbb{T}}(u,v) = \int_{\mathbb{T}} \Big[\sum a_{j,k}(x)\partial_k u(x) \cdot \partial_j \overline{v}(x) + \sum b_j(x)u(x) \cdot \overline{v}(x) + c(x)u(x) \cdot \overline{v}(x)\Big]\,dx \tag{1.2}$$

corresponding to the system is coercive on the space $H^1(\mathbb{T})$ in the sense that the (strengthened) Gårding inequality $C \operatorname{Re} \Phi_{\mathbb{T}}(u,u) \geqslant \|u\|^2_{H^1(\mathbb{T})}$ holds. As a consequence, the equation $Lu = f$ is uniquely solvable in $H^1(\mathbb{T})$ for $f \in H^{-1}(\mathbb{T})$ by the Lax–Milgram lemma on weak solutions of the

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abstract equation $Lu = f$, where L is the bounded operator defined by formula (1.4) below. We state this lemma in a form convenient to us (cf., e.g., [21]).

Lemma 1.1. *Let H be a Hilbert space, let H^* be the dual space of H with respect to the form (f,v), $v \in H$, $f \in H^*$, and let an $f \in H^*$ be given. Assume that a continuous sesquilinear form $\Phi(u,v)$ on H satisfies the inequality*

$$C \operatorname{Re} \Phi(u,u) \geqslant \|u\|_H^2. \tag{1.3}$$

Then there exists a unique element $u \in H$ such that

$$\Phi(u,v) = (f,v) \tag{1.4}$$

for all $v \in H$, and the operator $L^{-1}\colon f \mapsto u$ is bounded.

Furthermore, the forms $\Phi_{\Omega^\pm}(u,v)$ similar to the form $\Phi_{\mathbb{T}}(u,v)$ are coercive on the spaces $\widetilde{H}^1(\Omega^\pm)$, by which we mean that the inequalities $C \operatorname{Re} \Phi_{\Omega^\pm}(u,u) \geqslant \|u\|^2_{\widetilde{H}^1(\Omega^\pm)}$ hold. (Here $\widetilde{H}^s(\Omega^\pm)$ is the subspace of $H^s(\mathbb{T})$ formed by the elements supported in $\overline{\Omega^\pm}$.) This ensures the unique solvability of the Dirichlet problems in $\Omega^\pm$. We additionally assume that the forms $\Phi_{\Omega^\pm}$ are coercive on $H^1(\Omega^\pm)$; sufficient conditions for this are known. This ensures the unique solvability of the Neumann problems in $\Omega^\pm$. In particular, one can consider generalized inhomogeneous anisotropic elasticity systems (e.g., see [25]) and the Beltrami–Laplace equation with lower-order terms.

Let $\Omega_0 = \mathbb{T} \setminus \overline{S}$. Note that this domain is not Lipschitz. We need to define the space $H^1(\Omega_0)$. It would be inadequate to define it as the space of restrictions of functions in $H^1(\mathbb{T})$ to Ω_0, because, with this definition, the traces on $S^\pm$ of functions $u \in H^1(\Omega_0)$ would always coincide. We define $H^1(\Omega_0)$ as the space of functions $u \in L_2(\Omega_0)$ whose restrictions to $\Omega^\pm$ belong to H^1 and the traces $\gamma^\pm u = u^\pm$ on $\Gamma^\pm$ (they belong to $H^{1/2}(\Gamma)$) coincide on S_2; furthermore,

$$\|u\|^2_{H^1(\Omega_0)} = \|u\|^2_{H^1(\Omega^+)} + \|u\|^2_{H^1(\Omega^-)}. \tag{1.5}$$

It follows that the jump $[u] = u^- - u^+$ of u on S belongs to $\widetilde{H}^{1/2}(S)$. (By $\widetilde{H}^s(S)$ we denote the subspace of elements of $H^s(\Gamma)$ supported in $\overline{S}$.)

Remark. This definition of the space $H^1(\Omega_0)$ is equivalent to the following standard definition. This is the space of functions u that belong to L_2 in the domain Ω_0 along with their first derivatives $\partial_j u$ in the sense of distributions, and

$$\|u\|^2_{H^1(\Omega_0)} = \|u\|^2_{L_2(\Omega_0)} + \sum \|\partial_j u\|^2_{L_2(\Omega_0)}. \tag{1.6}$$

The equivalence follows from Nečas' integration by parts formula for H^1 functions in a Lipschitz domain [24, c. 121]: if we write it out in $\Omega^\pm$ for a function u belonging to $H^1(\Omega_0)$ in the sense of the first definition and a test function in $C_0^\infty(\Omega_0)$, then, after the addition of these formulas, the terms on S_2 cancel out.

Consequently, the space $H^1(\Omega_0)$ is independent of the choice of the surface S_2.

Since the system $Lu = f$ on the torus is uniquely solvable, we shall assume that $f = 0$ in Ω_0.

The solutions of boundary value problems for the system $Lu = 0$ are sought in $H^1(\Omega_0)$. (We consider only the simplest spaces until Section 6.) In particular, if φ is an arbitrary function in $C_0^\infty(\Omega_0)$, then $(u, \widetilde{L}\varphi)_{\Omega_0} = 0$. Here and below, $\widetilde{L}$ is the formal adjoint of the operator L.

We consider the following problems.

1°. The Dirichlet problem for the system $Lu = 0$ in Ω_0 with the Dirichlet conditions on $S^\pm$:

$$u^\pm = g^\pm \text{ on } S^\pm, \tag{1.7}$$

where $g^\pm \in H^{1/2}(S)$ and $[g] = g^- - g^+ \in \widetilde{H}^{1/2}(S)$.

2°. The Neumann problem for the same system in Ω with the Neumann conditions

$$T^\pm u = h^\pm \text{ on } S^\pm, \tag{1.8}$$

where $T^\pm u$ is the conormal derivative, $h^\pm \in H^{-1/2}(S)$, and $[h] \in \widetilde{H}^{-1/2}(S)$.

Let us comment on the latter setting. Recall that the conormal derivative $T^{\pm}u$ of a function u in $H^1(\Omega^{\pm})$ is in the general case *defined* via u and $Lu = f$ by means of Green's formula

$$(f, v)_{\Omega^{\pm}} = \Phi_{\Omega^{\pm}}(u, v) \mp (T^{\pm}u, v^{\pm})_{\Gamma}. \tag{1.9}$$

See [21, p. 117]. Here v is an arbitrary test function in $H^1(\Omega^{\pm})$, $f \in \widetilde{H}^{-1}(\Omega^{\pm})$, and the respective duality is used in the forms on the left- and right-hand sides. The conormal derivative belongs to $H^{-1/2}(\Gamma)$. Hence it belongs to $H^{-1/2}(S)$, and the same is true for the jump $[Tu] = T^-u - T^+u$. However, the following statement holds.

Proposition 1.2. *Let u be a solution of system* (1.1) *in Ω_0 belonging to $H^1(\Omega_0)$. Then $[Tu] \in \widetilde{H}^{-1/2}(S)$; i.e.,* $\mathrm{supp}[Tu] \subset \overline{S}$.

Proof. Let v be any function in $C_0^{\infty}(\Omega_0)$, and let Γ_0 be a neighborhood of its support lying inside Ω_0 and having a smooth boundary. Integrating by parts, we obtain $\Phi_{\Gamma_0}(u, v) = 0$. Hence

$$0 = \Phi_{\Omega_+}(u, v) + \Phi_{\Omega_-}(u, v) = (T^+u, v)_{S_2} - (T^-u, v)_{S_2},$$

and we see that $[Tu]_{S_2} = 0$. □ □

A surface S with boundary serves as a model of a nonclosed screen in acoustics and electrodynamics and of a crack in elasticity theory. In these cases, similar problems were considered by Stephan and Costabel–Stephan (in particular, see [28]–[30] and [12]). They considered elasticity problems for the Lamé system (the case of an isotropic medium). The $g^{\pm}$, as well as the $h^{\pm}$, were originally assumed to coincide. Generalizations to anisotropic media in three-dimensional domains were obtained in particular by Duduchava–Natroshvili–Shargorodskii [15] and Duduchava–Wendland [16]. See also [19] and the bibliography in these papers. In all these papers, the surface S and its boundary were assumed to be sufficiently smooth, and the Wiener–Hopf method was applied with the use of pseudodifferential operators in the form proposed by Eskin [17]. This permitted not only studying the regularity of the solution but also analyzing its asymptotic behavior near the boundary of S. These papers suggest that it is possible to generalize the simpler results on the unique solvability (or the Fredholm property) to arbitrary strongly elliptic second-order systems in Lipschitz domains without smoothness assumptions, pseudodifferential operators, and the Wiener–Hopf method. Just this possibility is realized in the present paper. We use the same technique of reduction of problems to equivalent equations on S. The author touched upon these problems in [6], but the generality is insufficient there. Clearly, the asymptotics of solutions could hardly be reached without the Wiener–Hopf method, but we make some progress in the problem on the regularity of solutions, which is useful also for spectral problems whose statement is given below (problems 5° and 6°).

We also consider the following problems for the system $Lu = 0$ in Ω_0.

3°. The problem with the conditions

$$[u] = g, \quad [Tu] = h \text{ on } S. \tag{1.10}$$

Here $g \in \widetilde{H}^{1/2}(S)$ and $h \in \widetilde{H}^{-1/2}(S)$.

4°. The mixed problem with the conditions

$$u^+ = g \text{ on } S^+, \quad T^-u = h \text{ on } S^-. \tag{1.11}$$

Here again $g \in H^{1/2}(S)$ and $h \in H^{-1/2}(S)$. When considering this problem, we follow Duduchava–Natroshvili [14], who considered the anisotropic elasticity system, but we again do not use the smoothness assumptions and the Wiener–Hopf method. Here we assume that system (1.1) is formally self-adjoint. Compare with [21, p. 231–234], where general mixed problems in a Lipschitz domain with closed boundary are considered and equivalent equations on the boundary are obtained in the case of a formally self-adjoint system.

The literature on mixed problems is extremely wide, and problem 4° is a nonstandard version of these problems. In [5], we consider general (standard) mixed problems for strongly elliptic second-order systems in a Lipschitz domain with closed boundary and obtain equivalent equations on the boundary without the formal self-adjointness assumption. One can see there certain analogies with the present paper.

5°. The first spectral problem

$$[u] = 0, \quad u^{\pm} = -\lambda [Tu] \text{ on } S. \tag{1.12}$$

6°. The second spectral problem

$$[Tu] = 0, \quad [u] = -\lambda T^{\pm} u \text{ on } S. \tag{1.13}$$

Similar spectral problems in the case of a closed Lipschitz boundary were considered in [3] and [4]. Their original statement in the case of the Helmholtz equation is due to the Moscow physicist B. Z. Katsenelenbaum and his collaborators N. N. Voitovich and A. N. Sivov, see [31] or [8].

2. Potential Type Operators and Problem 3°**.** The inverse L^{-1} of the operator L on the torus is an integral operator; this is the so-called Newtonian potential. Its kernel $\mathscr{E}(x, y)$ is the fundamental solution for L. Recall that simple and double layer potentials are defined on functions given on Γ (with a little extra regularity) by the formulas

$$\mathscr{A}\psi(x) = \int_{\Gamma} \mathscr{E}(x, y)\psi(y)\, dS_y, \tag{2.1}$$

$$\mathscr{B}\varphi(x) = \int_{\Gamma} (\widetilde{T}_y^{+} \mathscr{E}^*(x, y))^* \varphi(y)\, dS_y. \tag{2.2}$$

Here $\widetilde{T}^{+}(\,\cdot\,)$ is the conormal derivative for the operator $\widetilde{L}$ formally adjoint to L (see [21]). These operators were studied by elementary tools in [21]; see also [3]. For these operators, we need a number of statements verified there. In particular, they can be found in [3]. Let us list them.

1. The operator $\mathscr{A}$ is extended to an operator that acts boundedly from $H^{-1/2}(\Gamma)$ into $H^1(\mathbb{T})$ and hence into $H^1(\Omega^{\pm})$. The function $u = \mathscr{A}\psi$ with $\psi \in H^{-1/2}(\Gamma)$ is a solution of the system $Lu = 0$ in $\Omega^{\pm}$. The trace $A\psi = \gamma^{\pm}\mathscr{A}\psi$ of this function on Γ is a bounded operator from $H^{-1/2}(\Gamma)$ into $H^{1/2}(\Gamma)$. Under the assumption that the forms $\Phi_{\Omega^{\pm}}$ are coercive on $\widetilde{H}^1(\Omega^{\pm})$, the operator A is invertible.

2. The operator $\mathscr{B}$ acts boundedly from $H^{1/2}(\Gamma)$ into $H^1(\Omega^{\pm})$. The function $u = \mathscr{B}\varphi$ with $\varphi \in H^{1/2}(\Gamma)$ is a solution of the system $Lu = 0$ in $\Omega^{\pm}$.

3. As in [21], we set

$$B = \tfrac{1}{2}(\gamma^{+}\mathscr{B} + \gamma^{-}\mathscr{B}) \quad \text{and} \quad \widehat{B} = \tfrac{1}{2}(T^{+}\mathscr{A} + T^{-}\mathscr{A}). \tag{2.3}$$

The first of these operators is the direct value of the double layer potential; it is bounded in $H^{1/2}(\Gamma)$. The second operator is bounded in $H^{-1/2}(\Gamma)$ and is equal to $\widetilde{B}^*$. This means that it is the adjoint to the direct value of the double layer potential for $\widetilde{L}$ with respect to the extension of the standard inner product in $L_2(\Gamma)$ to the direct product $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$. Under our assumption on the coerciveness of the forms $\Phi_{\Omega^{\pm}}$ on $H^1(\Omega^{\pm})$, the operators $\frac{1}{2}I \pm B$ and $\frac{1}{2}I \pm \widehat{B}$ are invertible.

4. One has

$$T^{\pm}\mathscr{A} = \pm\tfrac{1}{2}I + \widehat{B}, \quad \gamma^{\pm}\mathscr{B} = \mp\tfrac{1}{2}I + B, \quad T^{+}\mathscr{B} = T^{-}\mathscr{B}. \tag{2.4}$$

The operator $H = -T^{\pm}\mathscr{B}$ is the so-called hypersingular operator; it acts boundedly from $H^{1/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$. Under our assumption on the coerciveness of the forms $\Phi_{\Omega^{\pm}}$ on $H^1(\Omega^{\pm})$, the operator H is invertible. The operator H^{-1} is connected with A by the formulas

$$H^{-1} = (\tfrac{1}{4}I - B^2)^{-1}A = A(\tfrac{1}{4}I - \widehat{B}^2)^{-1}. \tag{2.5}$$

5. The formula

$$u = \mathscr{B}\varphi - \mathscr{A}\psi, \quad \text{where } \varphi = [u],\ \ \psi = [Tu], \tag{2.6}$$

represents a solution of the system $Lu = 0$ in $\Omega^{\pm}$ via the jumps on Γ. Here $u \in H^1(\Omega^{\pm})$, $\varphi \in H^{1/2}(\Gamma)$, and $\psi \in H^{-1/2}(\Gamma)$.

We shall augment this list where necessary.

Now let us proceed to the case of a surface S with boundary. In this case, formula (2.6) can be written out with the jumps on S:

$$u = \mathscr{B}[u]_S - \mathscr{A}[Tu]_S. \tag{2.7}$$

This function is a solution of the system $Lu = 0$ outside $\overline{S}$ belonging to $H^1(\Omega_0)$, because it belongs to $H^1(\Omega^\pm)$ and $u^+ = u^-$ on S_2. It is easily seen that the following statement holds.

Proposition 2.1. *Problem* 3° *is uniquely solvable, and the solution is given by* (2.7).

3. Dirichlet and Neumann Problems. Let us introduce the operators

$$A_S\psi = (A\psi)|_S, \quad B_S\varphi = (B\varphi)|_S \qquad (\psi \in \widetilde{H}^{-1/2}(S),\ \varphi \in \widetilde{H}^{1/2}(S)). \tag{3.1}$$

Obviously, A_S is a bounded operator from $\widetilde{H}^{-1/2}(S)$ into $H^{1/2}(S)$, and B_S is a bounded operator from $\widetilde{H}^{1/2}(S)$ into $H^{1/2}(S)$.

Consider the Dirichlet problem. In (2.7), we pass onto two sides of S, add the resulting relations and divide by 2, thus obtaining

$$g = B_S\varphi - A_S\psi, \quad \text{where } g = \tfrac{1}{2}(g^+ + g^-),\ \varphi = g^- - g^+. \tag{3.2}$$

This is an equation for ψ, and it similar to those used in [28]–[30], [12], [15], and [16].

Proposition 3.1. *The operator* A_S *satisfies the Gårding type inequality*

$$\|\psi\|^2_{\widetilde{H}^{-1/2}(S)} \leqslant C_1 \operatorname{Re}(A_S\psi, \psi)_S, \qquad \psi \in \widetilde{H}^{-1/2}(S), \tag{3.3}$$

and hence is invertible by the Lax–Milgram lemma.

Here, on the right-hand side, we use the duality of the spaces $H^{1/2}(S)$ and $\widetilde{H}^{-1/2}(S)$ with respect to the extension of the standard inner product in $L_2(S)$ to the direct product of these spaces. Inequality (3.3) follows from the similar inequality

$$\|\psi\|^2_{H^{-1/2}(\Gamma)} \leqslant C_1 \operatorname{Re}(A\psi, \psi)_\Gamma, \qquad \psi \in H^{-1/2}(\Gamma), \tag{3.4}$$

for the operator A, which implies the invertibility of A by the same lemma. The derivation of this inequality is omitted in [3], where the similar inequality for the operators $N_\pm$ (Neumann-to-Dirichlet operators) is verified. To make the exposition self-contained, we verify inequality (3.4).

For $u = \mathscr{A}\psi$, Green's formulas (1.9) in $\Omega^\pm$, together with the relation $\psi = -[T\mathscr{A}\psi]$ (see the first relation in (2.4)), imply the formula

$$\Phi_{\Omega^+}(u, u) + \Phi_{\Omega^-}(u, u) = (\psi, A\psi)_\Gamma.$$

From the (strengthened) Gårding inequalities in $\Omega^\pm$, we obtain

$$\|u\|^2_{H^1(\Omega^+)} + \|u\|^2_{H^1(\Omega^-)} \leqslant C_2 \operatorname{Re}(\psi, A\psi)_\Gamma = C_2 \operatorname{Re}(A\psi, \psi)_\Gamma.$$

Since our Neumann problems in $\Omega^\pm$ are uniquely solvable and since the a priori estimates for their solutions in the case of the homogeneous system are two-sided, we have

$$\|T^\pm u\|_{H^{-1/2}(\Gamma)} \leqslant C_3 \|u\|_{H^1(\Omega^\pm)}.$$

Hence

$$\|\psi\|^2_{H^{-1/2}(\Gamma)} = \|[T\mathscr{A}\psi]\|^2_{H^{-1/2}(\Gamma)} \leqslant C_1 \operatorname{Re}(A\psi, \psi)_\Gamma.$$

This proves (3.4).

Theorem 3.2. *The Dirichlet problem in the setting* 1° *has exactly one solution in* $H^1(\Omega_0)$.

Indeed, the jump $[Tu]_S$ can be found from the Dirichlet data with the use of Eq. (3.2), and then the solution is constructed by formula (2.7). For the zero Dirichlet data, this solution is zero.

Let us proceed to the Neumann problem. Here the situation is similar. Let us introduce the operator

$$H_S\varphi = (H\varphi)|_S \qquad (\varphi \in \widetilde{H}^{1/2}(S)). \tag{3.5}$$

This is a bounded operator from $\widetilde{H}^{1/2}(S)$ into $H^{-1/2}(S)$. We compute the conormal derivatives of both sides of (2.7) on both sides of S, add, and divide by 2, thus obtaining

$$h = -H_S\varphi - \widehat{B}_S\psi, \quad \text{where} \quad h = \tfrac{1}{2}(h^+ + h^-), \quad \psi = h^- - h^+. \tag{3.6}$$

This is an equation for φ, and again it is similar to the equations used in the same papers [28]–[30], [12], [15], and [16].

Proposition 3.3. *The operator H_S satisfies the Gårding type inequality*

$$\|\varphi\|^2_{\widetilde{H}^{1/2}(S)} \leqslant C_4 \operatorname{Re}(H_S\varphi, \varphi)_S, \qquad \varphi \in \widetilde{H}^{1/2}(S), \tag{3.7}$$

and hence is invertible by the Lax–Milgram lemma.

Here the duality of the spaces $H^{-1/2}(S)$ and $\widetilde{H}^{1/2}(S)$ is used on the right-hand side. Inequality (3.7) follows from the similar inequality

$$\|\varphi\|^2_{H^{1/2}(\Gamma)} \leqslant C_4 \operatorname{Re}(H\varphi, \varphi)_\Gamma, \qquad \varphi \in H^{1/2}(\Gamma), \tag{3.8}$$

for the operator H. To prove the latter inequality, we set $u = \mathscr{B}\varphi$ and derive the relation

$$\Phi_{\Omega^+}(u, u) + \Phi_{\Omega^-}(u, u) = (H\varphi, \varphi)_\Gamma$$

from Green's formulas. Then we proceed as in the previous case, using the fact that the estimates for the solutions of the Dirichlet problem are two-sided.

Inequalities (3.4), (3.3), (3.8), and (3.7) express the well-known idea on the "strong ellipticity" of potential type boundary operators corresponding to strongly elliptic systems. See the hints and the literature in [20].

Now we obtain the following theorem similar to Theorem 3.2.

Theorem 3.4. *The Neumann problem in the setting* 2° *has exactly one solution in* $H^1(\Omega_0)$.

Note that in [5] we consider operators N_S and D_S similar to A_S and H_S.

4. Mixed Problem. To construct its solution u, we seek the jumps

$$\varphi = [u]_S \in \widetilde{H}^{1/2}(S), \quad \psi = [Tu]_S \in \widetilde{H}^{-1/2}(S). \tag{4.1}$$

By computing the conormal derivative of the function (2.7) on S^- and the boundary value of this function on S^+, we obtain the equations

$$\begin{aligned} -H_S\varphi - (-\tfrac{1}{2}I + \widehat{B}_S)\psi &= h, \\ (-\tfrac{1}{2}I + B_S)\varphi - A_S\psi &= g. \end{aligned} \tag{4.2}$$

Here the order of the equations is not accidental: we follow [14] and [21]. The operator

$$\mathscr{T} = \begin{pmatrix} H_S & -\tfrac{1}{2}I + \widehat{B}_S \\ \tfrac{1}{2}I - B_S & A_S \end{pmatrix} \tag{4.3}$$

acts boundedly from the space

$$\widetilde{\mathscr{H}} = \widetilde{H}^{1/2}(S) \times \widetilde{H}^{-1/2}(S) \tag{4.4}$$

into the space

$$\mathscr{H} = H^{-1/2}(S) \times H^{1/2}(S). \tag{4.5}$$

These two spaces are dual with respect to the extension of the inner product $(u_1, v_1)_S + (u_2, v_2)_S$ to their direct product. For the column $U = (\varphi, \psi)'$, we have

$$(\mathscr{T}U, U) = (H_S\varphi, \varphi)_\Gamma + ((-\tfrac{1}{2}I + \widehat{B}_S)\psi, \varphi)_\Gamma + ((\tfrac{1}{2}I - B_S)\varphi, \psi)_\Gamma + (A_S\psi, \psi)_\Gamma. \tag{4.6}$$

Assume that L is formally self-adjoint. This assumption includes the condition

$$\sum b_j \nu_j = 0 \ \text{ on } \ S; \tag{4.7}$$

see [4]. Then $\widehat{B} = B^*$, and one sees from (4.6) that

$$\operatorname{Re}(\mathscr{T}U, U) = (H_S\varphi, \varphi)_\Gamma + (A_S\psi, \psi)_\Gamma; \tag{4.8}$$

the other terms cancel out. (The symbol Re on the right-hand side is not needed here.) We use Propositions 3.1 and 3.3 and find that the equation $\mathscr{T}U = F$, where F is the column $(h, g)'$, is uniquely solvable by the Lax–Milgram lemma. This proves the following theorem.

Theorem 4.1. *If the operator L is formally self-adjoint, then problem 4° is uniquely solvable.*

One can consider the case in which only the principal part of L is formally self-adjoint, but we do not dwell on this.

5. Spectral Problems. The following claim holds for *eigenfunctions.*

Proposition 5.1. *Problems 5° and 6° are equivalent to the equations*

$$A_S\psi = \lambda\psi, \quad \textit{where } \psi = [Tu]_S,\ u = -\mathscr{A}\psi, \tag{5.1}$$

and

$$H_S^{-1}\varphi = \lambda\varphi, \quad \textit{where } \varphi = [u]_S,\ u = \mathscr{B}\varphi, \tag{5.2}$$

respectively.

The verification is similar to that made in [3] for the case of a closed boundary.

Here we point out that the invertible operators $A_S\colon \widetilde{H}^{-1/2}(S) \to H^{1/2}(S)$ and $H_S\colon \widetilde{H}^{1/2}(S) \to H^{-1/2}(S)$ act from a wider space into a narrower space and from a narrower space into a wider space, respectively. In particular, the space $L_2(S)$ is an intermediate space. Hence the spectral problems make sense.

In what follows, we present spectral results similar to those given in [4] for the case of a closed boundary Γ, and so we omit some details.

Consider the operator A_S. The case of a formally self-adjoint system (1.1) is most favorable. In this case $(A_S\psi_1, \psi_2)_S$ is an inner product in $\widetilde{H}^{-1/2}(S)$, with respect to which A_S is a self-adjoint operator with positive discrete spectrum. The eigenfunctions form an orthonormal basis, and it remains an orthogonal basis in $H^{1/2}(S)$ with respect to the inner product $(A_S^{-1}\varphi_1, \varphi_2)_S$. This result can be carried on to the intermediate spaces $H^s(S) = \widetilde{H}^s(S)$, $|s| < 1/2$. The eigenvalues λ_j of A_S arranged in nonincreasing order, taking multiplicities into account, satisfy the estimate $\lambda_j \leqslant Cj^{-1/(n-1)}$. Moreover, an asymptotic formula for the eigenvalues is true. We indicate a way to obtain it in Section 7.

If only the principal part of L is formally self-adjoint, then A_S is a weak perturbation of the self-adjoint operator A_S^0 constructed from the system $L_0u = 0$ with "adjusted" lower-order terms. The eigenvalues, starting from some number, are contained in an arbitrarily narrow sector with bisector $\mathbb{R}_+$. The system of root functions is complete in $\widetilde{H}^{-1/2}(S)$ and $H^{1/2}(S)$ and is a basis for the Abel–Lidskii summability method of order $n-1+\varepsilon$ with parentheses in these spaces, where $\varepsilon > 0$ can be arbitrarily small. The completeness result can be carried on to the intermediate spaces.

Finally, in the most general case the spectrum is discrete, and the estimate $s_j \leqslant C_1 j^{-1/(n-1)}$ for the s-numbers is preserved. The completeness and the summability by the Abel–Lidskii method are preserved if the opening angle of the sector with bisector $\mathbb{R}_+$ containing all values of the forms $\Phi_{\Omega^\pm}(u, u)$ is less than $\pi/(n-1)$. The eigenvalues lie in this sector. To prove this, one derives the following optimal estimate for the resolvent of the operator A_S^{-1} in the complementary sector: if

$$(A_S^{-1} - \lambda I)\varphi = \psi \tag{5.3}$$

(note that the letters φ and ψ had a different meaning in the preceding text), then

$$\|\varphi\|_{H^{1/2}(S)} + |\lambda|\|\varphi\|_{\widetilde{H}^{-1/2}(S)} \leqslant C_2\|\psi\|_{\widetilde{H}^{-1/2}(S)}. \tag{5.4}$$

See the hints in [3, Section 6]. Again, the completeness result carries on to intermediate spaces.

The spectral properties of the operator H_S^{-1} are similar to those just presented. If L is formally self-adjoint, then we equip $H^{-1/2}(S)$ with the inner product $(H_S^{-1}\psi, \psi)_S$.

6. Regularity of Solutions.

6.1. As before, we assume that the forms $\Phi_{\Omega^\pm}$ are coercive on $H^1(\Omega^\pm)$.

Now, instead of H^σ, we need the more general spaces H_p^σ of Bessel potentials and the Besov spaces B_p^σ, $1 < p < \infty$; for $p = 2$, these spaces coincide with H^σ (e.g., see [3] or [4]). First, assume that the boundary is closed (does not itself have a boundary). Then the solution of the homogeneous system $Lu = 0$ in $\Omega^\pm$ can be sought in the spaces $H_p^{1/2+s+1/p}(\Omega^\pm)$ or $B_p^{1/2+s+1/p}(\Omega^\pm)$, the Dirichlet data are given in $B_p^{1/2+s}(\Gamma)$, and the Neumann data are given in $B_p^{-1/2+s}(\Gamma)$, $|s| < 1/2$. (For other s, the Dirichlet data on the Lipschitz surface Γ or S become meaningless in general.) Set $t = 1/p$. As in the author's previous papers, the admissible points (s, t) form the square

$$Q = \{(s,t) : |s| < 1/2,\ 0 < t < 1\}. \tag{6.1}$$

The Dirichlet and Neumann problems, which are uniquely solvable at the center $(0, 1/2)$ of the square, remain to be uniquely solvable at least for $|s| < \varepsilon$ and $|t - 1/2| < \delta$ with sufficiently small positive ε and δ by Shneiberg's theorem on the extrapolation of invertibility for operators in interpolation scales [27]. No additional assumptions about the system are needed in this case. Furthermore, probably for smaller ε and δ, one can extend the assertions on the boundedness and invertibility of the operators

$$A\colon B_p^{-1/2+s}(\Gamma) \to B_p^{1/2+s}(\Gamma) \quad \text{and} \quad H\colon B_p^{1/2+s}(\Gamma) \to B_p^{-1/2+s}(\Gamma).$$

The operators $\frac{1}{2}I \pm B$ in $B_p^{1/2+s}(\Gamma)$ and $\frac{1}{2}I \pm \widehat{B}$ in $B_p^{-1/2+s}(\Gamma)$ are bounded and invertible as well. One can also generalize formula (2.6). All of this is explained in [3] and [4].

Passing to the case of a nonclosed boundary, first of all it is necessary to define the space $H_p^{1/2+s+1/p}(\Omega_0)$. We define it in the same way as in the already considered case of $s = 0$ and $p = 2$: this space consists of the functions $u \in L_p(\Omega_0)$ that belong to $H_p^{1/2+s+1/p}$ in $\Omega^\pm$ and whose traces on Γ (which belong to $B_p^{1/2+s}(\Gamma)$) coincide on S_2, so that the jump $[u]$ belongs to $\widetilde{B}_p^{1/2+s}(S)$. The space $B_p^{1/2+s+1/p}(\Omega_0)$ is defined in a similar way. The remark in Section 1 extends at least to the spaces $H_p^1(\Omega)$.

Now we can consider the solutions of the system $Lu = 0$ in $H_p^{1/2+s+1/p}(\Omega_0)$ or $B_p^{1/2+s+1/p}(\Omega_0)$. In the Dirichlet problem,

$$g^\pm \in B_p^{1/2+s}(S) \quad \text{and} \quad [g] \in \widetilde{B}_p^{1/2+s}(S). \tag{6.2}$$

In the Neumann problem,

$$h^\pm \in B_p^{-1/2+s}(S) \quad \text{and} \quad [h] \in \widetilde{B}_p^{-1/2+s}(S), \tag{6.3}$$

because Proposition 1.2 can be generalized as follows.

Proposition 6.1. *Let u be a solution of the system $Lu = 0$ in Ω_0 belonging to $H_p^{1/2+s+1/p}(\Omega_0)$. Then $[Tu] \in \widetilde{B}_p^{-1/2+s}(S)$.*

The proof is similar to that of Proposition 1.2 (with L_2 replaced by L_p).

Next, we obtain the boundedness of the operators

$$A_S\colon \widetilde{B}_p^{-1/2+s}(S) \to B_p^{1/2+s}(S) \quad \text{and} \quad H_S\colon \widetilde{B}_p^{1/2+s}(S) \to B_p^{-1/2+s}(S), \tag{6.4}$$

as well as of the operators

$$\tfrac{1}{2}I \pm B_S\colon \widetilde{B}_p^{1/2+s}(S) \to B_p^{1/2+s}(S) \quad \text{and} \quad \tfrac{1}{2}I \pm \widehat{B}_S\colon \widetilde{B}_p^{-1/2+s}(S) \to B_p^{-1/2+s}(S) \tag{6.5}$$

for $|s| < \varepsilon$ and $|t - 1/2| < \delta$ with sufficiently small ε and δ. Now note that the families of spaces used here are interpolation families for the complex interpolation method with respect to each of the indices. For the families $\{B_p^{1/2+s}(S)\}$ and $\{B_p^{-1/2+s}(S)\}$, this follows from the fact that the transition from spaces on Γ to the corresponding spaces on S is a retraction. (The inverse passage with the use of the extension operator is a coretraction.) As for the families $\{\widetilde{B}_p^{1/2+s}(S)\}$ and

$\{\widetilde{B}_p^{-1/2+s}(S)\}$, one can use the duality theorem for the complex interpolation method, because these spaces are dual to $B_{p'}^{-1/2-s}(S)$ and $B_{p'}^{1/2-s}(S)$, respectively, and reflexive. See, e.g., [9, Sections 4.5 and 6.4]. Since our operators are invertible at the center of the square, we can apply Shneiberg's theorem and conclude that these operators are invertible for our (s,t), possibly with smaller ε and δ. Thus, the following theorem holds.

Theorem 6.2. *The operators* (6.4) *and* (6.5) *are invertible for* $|s| < \varepsilon$ *and* $|t-1/2| < \delta$ *with sufficiently small* ε *and* δ.

The same is true for the operator (4.3) if $L = \widetilde{L}$. Formula (2.7) can be generalized as well. This leads to the following main result.

Theorem 6.3. *There exist* $\varepsilon > 0$ *and* $\delta > 0$ *such that for any* $g^\pm \in B_p^{1/2+s}(S)$ *and* $h^\pm \in B_p^{-1/2+s}(S)$, $|s| < \varepsilon$, $|t-1/2| < \delta$, *the Dirichlet and Neumann problems, respectively, have unique solutions in* $H_p^{1/2+s+1/p}(\Omega_0)$. *Furthermore, problems* 3° *and* 4° *are uniquely solvable in the same spaces. The spaces* H *in these statements can be replaced by the similar spaces* B.

As a consequence, the regularity of solutions increases—within the indicated limits—with increasing regularity of $g^\pm$ and $h^\pm$.

6.2. Let us proceed to spectral problems.

Theorem 6.4. *The root functions of the operator* A_S *belong to the union of the spaces* $B_p^{1/2+s}(S)$, $|s| < \varepsilon$, $|t-1/2| < \delta$. *The spectrum is independent of* (s,t). *In the spaces* $B_p^{1/2+s}(S)$ *and* $\widetilde{B}_p^{-1/2+s}(S)$, *the assertions on the basis property are preserved for* $p=2$ *provided that* L *is formally self-adjoint*; *in all other cases*, *the assertions on the completeness and the summability of the series by the Abel–Lidskii method are preserved. The assertions on the basis property and completeness extend to the spaces intermediate with respect to* s.

Similar results hold for the operator H_S^{-1} *with the spaces* $\widetilde{B}_p^{-1/2+s}(S)$ *and* $B_p^{1/2+s}(S)$ *replaced by* $B_p^{-1/2+s}(S)$ *and* $\widetilde{B}_p^{1/2+s}(S)$, *respectively.*

We give some clarifying remarks. The smoothness of the eigenfunctions of A_S is seen from the equation $A_S\psi = \lambda\psi$. Similarly, one obtains the smoothness of the associated functions. Using the (dense) embeddings of our spaces, we verify, as in [1], that the assertions on the completeness and the independence of the spectrum from (s,t) are preserved. The interval of values of s for which the basis property is obtained in the case of a formally self-adjoint operator L becomes somewhat wider. The summability of spectral expansions by the Abel–Lidskii method is derived with the use of an abstract theorem in [2] from the resolvent estimate

$$\|\varphi\|_{B_p^{1/2+s}(S)} + |\lambda|\|\varphi\|_{\widetilde{B}_p^{-1/2+s}(S)} \leqslant C\|\psi\|_{\widetilde{B}_p^{-1/2+s}(S)}, \tag{6.6}$$

which generalizes (5.4) and can be obtained by interpolation theory techniques again with the use of Shneiberg's theorem. Namely, one first obtains an estimate, uniform with respect to the parameter, of the first term on the left-hand side and then uses Eq. (5.3) to estimate the second term.

The completeness is additionally obtained in all spaces corresponding to the points of the square Q, in which the spaces where the completeness has been established are densely embedded. The embeddings are dense automatically.

7. Spectral Asymptotics. Here we obtain asymptotic formulas for the eigenvalues of A_S and, under additional assumptions, for the eigenvalues of H_S^{-1}. See Theorem 7.1 below. Now we assume that the operator L is formally self-adjoint. First, we present the results for operators in a domain with closed boundary; cf. [4].

7.1. The paper [7] gives an asymptotic formula for the eigenvalues of an integral operator on an "almost smooth" Lipschitz surface Γ, namely, for a negative-order pseudodifferential operator whose kernel is the restriction to $\Gamma \times \Gamma$ of the kernel of an elliptic pseudodifferential operator in $\mathbb{R}^n$ of order less by one. An almost smooth surface is defined there as a surface infinitely smooth outside a closed set of zero measure. One can consider a torus instead of $\mathbb{R}^n$.

Our operator A may just serve as an example, provided that the coefficients of L are infinitely smooth. If this is the case, then we take the Newtonian potential for the operator on the torus; its kernel is the fundamental solution for L. The assumption about the infinite smoothness of the coefficients of L can eventually be removed by approximating the given coefficients by infinitely smooth ones; hence we shall assume that the coefficients are infinitely smooth. The operator A is originally considered in $L_2(\Gamma)$.

To derive the asymptotics, one uses a special partition of the surface into small parts U_j ($j = 0, \dots, K$). The operator A can be written in the form $\sum A_{j,k}$, where $A_{j,k} = \theta_j A \theta_k \cdot$ and θ_j is the characteristic function of the part U_j. The asymptotics in the case of the operator $A_{j,j}$ corresponding to a smooth part U_j of Γ, $j > 0$, is known, e.g., from the results by Birman and Solomyak. The key point is an estimate of the s-numbers of the other operators $A_{j,k}$. It has the form

$$s_l(A_{j,k}) \leqslant C \min(\varepsilon_j, \varepsilon_k) l^{-1/(n-1)}, \tag{7.1}$$

where ε_j is estimated via the measure of the set U_j and tends to zero together with this measure. Here it is important that our operator A satisfies the original assumption on the kernel. The estimate (7.1) permits one to use the perturbation technique developed by Birman–Solomyak [10] and obtain the desired result. See [7, Section 4] for details. Clearly, the result can also be obtained for operators of the form $\theta A \theta \cdot$, where θ is the characteristic function of a part of the surface Γ. Just this is used in what follows.

The above assumption on the structure of the kernel does not hold for the operator H^{-1}. But estimates of the form (7.1) for its s-numbers can also be obtained if the operator can be written in two forms $T_1 A$ and $A T_2$, where T_1 and T_2 are bounded operators. It is seen from (2.5) that H^{-1} admits such representations. However, we need the boundedness and invertibility of the operators $\frac{1}{2} I \pm B$ and $\frac{1}{2} I \pm \widehat{B}$ in $L_2(\Gamma)$ rather than in the spaces indicated in Section 2. There are results of this kind in the literature, but they were obtained (on the basis of a totally different approach to problems in Lipschitz domains) not for all strongly elliptic systems. They are certainly true for the Beltrami–Laplace equation [23] and the Lamé system [13], to which lower-order terms can be added. In these cases, the asymptotics can also be obtained by this method for the operator H^{-1} on an almost smooth surface. In the other cases, one obtains the order-sharp estimate of the eigenvalues.

The fact that A and H^{-1} are treated as operators in $L_2(\Gamma)$ does not affect the eigenfunctions and eigenvalues.

The restriction that the surface Γ should be almost smooth was removed in the case of the operator A by Rozenblum and Tashchiyan [26].

7.2. Let us proceed to operators on a nonclosed surface. Let

$$A_S \psi = \lambda \psi. \tag{7.2}$$

Here originally $\psi \in \widetilde{H}^{-1/2}(S)$. But ψ, together with the left-hand side, belongs to $H^{1/2}(S)$. Hence ψ belongs to the intermediate spaces $\widetilde{H}^s(S) = H^s(S)$, $|s| < 1/2$, in particular, to $L_2(S)$. This permits one, using the extension of functions by zero, to rewrite Eq. (7.2) in the form

$$\theta A \theta \psi = \lambda \psi, \tag{7.3}$$

where θ is the characteristic function of S. Conversely, (7.2) follows from (7.3).

Let us present the definitive result. The eigenvalues $\lambda_j(A_S)$ of A_S are numbered in nonincreasing order, counting multiplicities.

Theorem 7.1. *One has*

$$\lambda_j(A_S) = C_{A_S} j^{-1/(n-1)} + o(j^{-1/(n-1)}), \tag{7.4}$$

where

$$C_{A_S}^{n-1} = (2\pi)^{-(n-1)} \iint_{T^* S} n_\alpha(x', \xi') \, dx' d\xi', \tag{7.5}$$

$\alpha(x',\xi')$ is the principal symbol of A, and $n_\alpha(x',\xi')$ is the number of eigenvalues of α greater than 1.

A similar result holds for the operator H_S^{-1} in the case of an almost smooth surface Γ, provided that L is either a scalar operator or a matrix operator with the Lamé operator in the principal part.

The cotangent bundle and the symbol on a Lipschitz surface are understood formally, but the cotangent spaces and the symbol are defined almost everywhere on S, and therefore formula (7.5) has a meaning both in the case of an almost smooth surface (see [7]) and in the general case (see [26]).

We do not dwell on the computation of the symbols of A and H^{-1} (in the smooth case); see the hints in [4].

In conclusion, we make three remarks.

1. We assume that both domains $\Omega^\pm$ lie on the torus. In principle, the torus can be replaced by a domain containing Γ with smooth boundary and, say, with the homogeneous Dirichlet condition on the boundary. Likewise, it is possible to consider a more general smooth manifold with smooth boundary or without boundary. It is also possible (and useful) to consider $\mathbb{R}^n$; then one should pose appropriate conditions at infinity in Ω^-, which, of course, affects the choice of spaces. See [11].

2. If one assumes that the forms $\Phi_{\Omega^\pm}$ are coercive but not in the strengthened sense, then, instead of the unique solvability of the problems and the invertibility of the operators, one obtains results on their Fredholm property with index zero. These generalizations do not require much effort but lead to complicated statements, and so we have avoided this.

3. This paper was extended (mixed problems and spectral asymptotics were added) in December of 2010.

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