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# On AGT relations with surface operator insertion and a stationary limit of beta-ensembles

A. Marshakov [a,b], A. Mironov [a,b,*], A. Morozov [b]

[a] Theory Department, Lebedev Physics Institute, Russia
[b] ITEP, Moscow, Russia

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ABSTRACT

We present a summary of what is currently known about of the AGT relations for conformal blocks with the additional insertion of the simplest degenerate operator, and a special choice of the corresponding intermediate dimension, in which the conformal blocks satisfy hypergeometric-type differential equations in the position of the degenerate operator. Special attention is devoted to the representation of the conformal block through using the beta-ensemble resolvents and to its asymptotics in the limit of large dimensions (both external and intermediate) taken asymmetrically in terms of the deformation epsilon-parameters. The next-to-leading term in the asymptotics defines the generating differential in the Bohr–Sommerfeld representation of the one-parameter deformed Seiberg–Witten prepotentials, (whose full two-parameter deformation leads to Nekrasov functions). This generating differential is also shown to be the one-parameter version of the single-point resolvent for the corresponding beta-ensemble, and its periods in the perturbative limit of the gauge theory are expressed through the ratios of the Harish–Chandra function. The Schrödinger/Baxter equations, considered earlier in this context, directly follow from the differential equations for the degenerate conformal block. This approach provides a powerful method for the evaluation of the single-deformed prepotentials, and even for the Seiberg–Witten prepotentials themselves. We primarily concentrate on the representative case of the insertion into the four-point block on a sphere and the one-point block on a torus.



## 1. Introduction

The AGT (Alday–Gaiotto–Tachikawa) conjecture [1] establishes explicit relations between the basic formulas in several principal branches of modern theory and thus naturally attracts an increasing amount of attention [1–23]. The main objects of investigation are various conformal blocks [24], and the statement claim is that they can be also represented

(1) as matrix-model [25] and/or beta-ensemble [26] partition functions in the Dijkgraaf–Vafa (DV) phase [27–29],
(2) as LMNS (Losev–Moore–Nekrasov–Satashvili) integrals [30],
(3) as combinations of the Nekrasov functions [31] (i.e., as a generalization of hypergeometric series expansions, [5]), and
(4) as exponentials of the deformed or "quantized" [11] SW (Seiberg–Witten) prepotentials [32], described in terms of integrable systems [33–35], and so on.

The AGT relations reflect a duality pattern [36], associated with the twisted compactification of the non-Lagrangian superconformal 6$d$ theory [37] for a $M5$-brane on a two-dimensional Riemann surface with boundaries, giving rise to a

* Corresponding author.
*E-mail addresses:* mars@itep.ru, mars@lpi.ru (A. Marshakov), mironov@itep.ru, mironov@lpi.ru (A. Mironov), morozov@itep.ru (A. Morozov).

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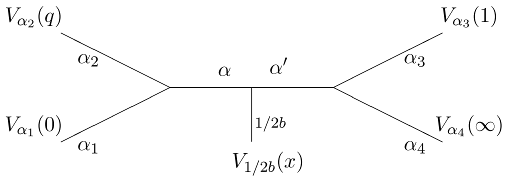


**Fig. 1.** Here, $z_{1,2,3,4} = (0, q, 1, \infty)$ and $q \ll x \ll 1$. In conformal theory, the structure constant for the degenerate primary vanishes unless $\alpha' = \alpha \pm 1/2b$ [24]. In the free-field representation of [15,17] for $\alpha' \neq \alpha \pm 1/2b$, there are additional screening insertions in the matrix-model ($\beta$-ensemble) representation, with *open* integration contours stretching from 0 to $x$. As explained in Section 3.2.2, such insertions violate differential equations naively following from Eq. (22) for the degenerate field, (2). Therefore, in this paper we consider only the case of $\alpha' = \alpha \pm 1/2b$ in this paper. The relation to the $a$-parameter in Yang–Mills theory is $\alpha = a + \epsilon/2$. In the limit of $\epsilon_2 \to 0$, the difference $1/2b = \frac{1}{2}\sqrt{-\frac{\epsilon_2}{\epsilon_1}}$ between $\alpha$ and $\alpha'$ gets becomes negligible, and $a$ in the corresponding Nekrasov function $F(\epsilon_1)$ at this limit can be considered as to be related to either $\alpha$ or $\alpha'$, thus restoring the symmetry of the diagram in application to the AGT relation.

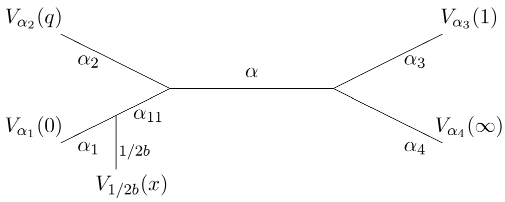


**Fig. 2.** The topology of the tree diagram implies a certain ordering of the pairings in the definition of the conformal block. From each OPE, only the contribution of one particular Verma module is picked up; therefore, the associativity of the OPE is restored only after sums are taken, summing over the intermediate dimensions. This diagram corresponds to an ordering that is different from that in Fig. 1: $x \gg q \gg 1$. Here, the intermediate dimension is $\alpha_{11} = \alpha_1 \pm 1/2b$. The two diagrams are connected by a duality transformation.

four-dimensional $\mathcal{N} = 2$ supersymmetric Yang–Mills theory, which can be further compactified down to 3, 2, 1, and 0 dimensions.

In this paper, we review the state of the art about these relations in the particular case of the four-point spherical conformal block with the additional insertion of the simplest degenerate primary field shown in Fig. 1,

$$B_5(x|z_i) = \left\langle V_{1/2b}(x) \prod_{i=1}^{4} V_{\alpha_i}(z_i) \right\rangle \tag{1}$$

as well as for the one-point conformal block on a torus, and their degenerate limits. For the spherical case, we consider only this type of diagram; all others (e.g., those in Fig. 2) can in principle be obtained from those in Fig. 1 by duality transformations (though we shall not consider this issue in this paper).

In the language of $4d$ SYM theory, such an insertion describes a "surface operator", produced by $M2$-brane, which lies entirely in the four-dimensional space–time and is located at a point $x$ on the Riemann surface. Within the CFT framework, this conformal block and the associated five-point correlation function are the standards objects of interest [24,38], since for a special choice of intermediate dimension they satisfy the hypergeometric-type differential equations in the $x$-variable for a special choice of the intermediate dimension, which do *not* hold for generic conformal blocks. This topic has been already addressed in relation with the AGT conjecture in [39,7,16,18,19].

In this paper, we describe the arrows in the following diagram:

$$B(x|z)$$

$$\swarrow^{(22)} \qquad\qquad \searrow^{(1)}$$

$$\left(b^2\partial_x^2 - \sum_i \frac{\partial_i}{x-z_i} - \sum_i \frac{\Delta_i}{(x-z_i)^2}\right) B(x|z) = 0 \qquad\qquad B(x|z) = \langle\langle \det(x-M) \rangle\rangle$$

$$\downarrow^{(4)} \qquad\qquad \downarrow^{(76)}$$

$$\log B(x|z) = \sum_k \frac{1}{k!} \left\langle\left\langle \Big(\mathrm{Tr}\ \log(x-M)\Big)^k \right\rangle\right\rangle_{conn}$$

$$\downarrow^{(83)}$$

$$\log B(x|z) = \frac{F(\epsilon_1)}{\epsilon_1\epsilon_2} + \frac{S(x;\epsilon_1)}{\epsilon_1} + O(\epsilon_2) \qquad\qquad \begin{cases} a = \oint_A dS(x;\epsilon_1) \\ \frac{\partial F(\epsilon_1)}{\partial a} = \oint_B dS(x;\epsilon_1) \end{cases}$$

$$\downarrow$$

$$\log B(x|z) = \frac{F_{SW}}{\hbar^2} + \frac{S_{SW}(x)}{\hbar} + O(\hbar^0) \qquad\qquad \begin{cases} a = \oint_A dS_{SW}(x) \\ \frac{\partial F_{SW}}{\partial a} = \oint_B dS_{SW}(x) \end{cases}$$

Hereafter, $\langle\cdots\rangle$ denote the CFT correlators, with $\langle\cdots\rangle_{free}$ stressing that this is the conformal theory of free massless fields, whereas $\langle\langle\cdots\rangle\rangle$ denote the $\beta$-ensemble averages, with the subscript *conn* referring to the connected correlators.

The right column deals with the matrix-model (beta-ensemble) representation, where the parameters of the conformal block define the shape of the potential, the number of integrations (DV phase) and the spectral complex curve. In this approach, $B_5(x|z)$ can be expressed [7,18,20] in terms of the exact resolvents of [40], which can be recursively constructed for any given spectral surface.

The left column makes use of the CFT equation for the null-vector

$$\left(b^2L_{-1}^2 - L_{-2}\right)V_{1/2b} = 0 \tag{2}$$

or for the degenerate primary field $V_{1/2b}(x)$. This equation induces an equation for the conformal block $B_5(x|z)$ *only* provided when the new intermediate dimension takes a *special* value: such that the $\alpha$-parameters of the two lines, attached to $V_{1/2b}(x)$ (see Fig. 1) satisfy

$$\alpha - \alpha' = \pm 1/2b. \tag{3}$$

See Section 3.2.2 for details. With this selection rule, $B_5(x|z)$ satisfies the second-order differential equation, which actually has a typical shape of a non-stationary Schrödinger equation (cf. with [39]; in fact, it takes literally the form of the non-stationary Schrödinger equation only in the specific limit of large dimensions, which corresponds to the pure gauge theory), while the four-point conformal block with the degenerate field satisfies a stationary Schrödinger equation.

An important application of the $V_{1/2b}$ insertions into conformal blocks is that they describe the $\epsilon_2 \to 0$ limit of the Nekrasov functions, which we address as *stationary* for the reasons to be discussed below. This limit is technically non-trivial and very interesting, since it corresponds to a quantization [10,11] of the classical integrable systems, associated with the supersymmetric gauge theories through the standard dictionary of [34].[1] In general, the SW representation of the conformal block,

$$\begin{aligned} \frac{\partial \log B_4(z)}{\partial a_I} &= b^2 \oint_{B_I} \rho_1(x), \\ a_I &= \oint_{A_I} \rho_1(x) \end{aligned} \tag{4}$$

involves the exact one-point resolvent $\rho_1$ of the corresponding beta-ensemble (Dotsenko–Fateev matrix model [15]), which is a rather complicated quantity. However, in the $\epsilon_2 \to 0$ limit, things are simplified, in this limit because the multi-trace correlators in the beta-ensemble are factorized and the resolvent acquires a new representation

$$\begin{aligned} \rho_1(x) &= \left\langle\!\!\left\langle \mathrm{tr}\, \frac{1}{x-M} \right\rangle\!\!\right\rangle_{conn} \overset{\epsilon_2=0}{=} \frac{\left\langle\!\left\langle \left(\mathrm{tr}\,\frac{1}{x-M}\right) \det(x-M) \right\rangle\!\right\rangle_{conn}}{\langle\langle \det(x-M)\rangle\rangle_{conn}} \\ &= \frac{\partial}{\partial x} \log \langle\langle \det(x-M)\rangle\rangle_{conn}. \end{aligned} \tag{5}$$

At the same time, the beta-ensemble average of the determinant $\langle\langle \det(x-M)\rangle\rangle$ is generated, (even for both $\epsilon_{1,2} \neq 0$), by the insertion of an additional operator $V_{1/2b}(x)$ into the conformal block (for more precise formulas, see Section 3.3, (76)–(77)):

$$\langle\langle \det(x-M)\rangle\rangle = \left\langle V_{1/2b}(x)\cdots\right\rangle \sim B_5(x|z). \tag{6}$$

Therefore, one obtains a much simpler and very transparent SW representation of the free energy in the $\epsilon_2 = 0$ limit:

(i) insert an additional degenerate field $V_{1/2b}(x)$, i.e., substitute the original $B_4(z)$ by $B_5(x|z)$,
(ii) consider its asymptotics at small $\epsilon_2$,:

$$B_5(x|z) = \exp\left(-\frac{1}{\epsilon_1\epsilon_2}F(\epsilon_1) + \frac{1}{\epsilon_1}S(x;\epsilon_1) + O(\epsilon_2)\right) \tag{7}$$

then $\mathrm{d}S(x;\epsilon_1) \equiv \mathrm{d}_x S(x;\epsilon_1)$ is the generating Seiberg–Witten differential for $F(\epsilon_1) \equiv \lim_{\epsilon_2\to 0}\log B_4(z)$, which is the Nekrasov function in the $\epsilon_2 \to 0$ by the AGT relation

$$\begin{aligned} a_I &= \oint_{A_I} \mathrm{d}S(x;\epsilon_1), \\ \frac{\partial F(\epsilon_1)}{\partial a_I} &= b^2 \oint_{B_I} \mathrm{d}S(x;\epsilon_1) \end{aligned}$$

[1] Under this quantization, e.g., the spectral curve converts into a Baxter equation. For the further generalization of integrability, provided by the double deformation with both $\epsilon_1, \epsilon_2 \neq 0$, see Conclusion.

since $B_5(x|z)$ satisfies a second-order Schrödinger-like differential equation in $x$, the contour integrals can be considered as the Bohr–Sommerfeld periods, describing the monodromy of the "wave function" $\psi(x) = B_5(x|z)/B_4(z)$.

The standard dictionary of [34] relates the supersymmetric Yang–Mills theories having different matter contents with different classical integrable systems, and $dS_{SW}$ are associated classical short-action forms $\vec{p}d\vec{q}$, restricted to spectral curves, while $F(\epsilon_1)$ arises in this context as the Yang–Yang (YY) function [10], generating the TBA-like equations of the corresponding quantum integrable system [41]. Moreover, in the perturbative limit of gauge theory, the Bohr–Sommerfeld periods of $dS(x;\epsilon_1)$ are given by the logarithm of the ratios of the Harish–Chandra functions corresponding to the integrable theory (i.e., is related to the $S$-matrix). The four-point spherical conformal block captures the family of $SU(2)$ SYM systems with $N_f \le 2N_c = 4$ fundamental supermultiplets. The case of a single adjoint supermultiplet is described by a parallel theory of the one-point toric conformal block (also with additional insertion of $V_{1/2b}(x)$).

We provide more details about this construction in Section 4. In particular, an important role is played by the transparent asymmetry between $\epsilon_1$ and $\epsilon_2$, both in the Dotsenko–Fateev representation [15] of the conformal blocks, where only one screening $V_b$ is involved, and in the choice of the degenerate field $V_{1/2b}(x)$, which is used for insertions.

## 2. $B(x|z)$ in CFT

In this section, we describe the standard facts from $2d$ conformal field theory about the correlators and conformal blocks on sphere and torus with the degenerated field inserted [24,42] and fix the notation that is used throughout the text.

### 2.1. Degenerate primary

The Verma module $R_\Delta$ generated over the Virasoro highest weight $V_\Delta = |\Delta\rangle$, $L_nV_\Delta = 0$ for $n > 0$ and $L_0V_\Delta = \Delta V_\Delta$, consists of the linear combinations of the basis vectors $L_{-Y}V_\Delta$. Here, $Y$ denotes an arbitrary Young diagram, $Y = \{k_1 \ge k_2 \ge \cdots \ge k_l > 0\}$, $L_{-Y} \equiv L_{-k_1}, \ldots, L_{-k_l}$ and $L_Y = L_{k_l}, \ldots, L_{k_1}$. The Verma module $R_\Delta$ is considered degenerate if it contains another highest weight vector $\tilde{V} = \sum_Y \tilde{C}_Y L_{-Y}V_\Delta \ne V_\Delta$, inside, satisfying $L_n\tilde{V} = 0$ for $n > 0$. In this case, $\tilde{V}$ has a vanishing norm.

At the first level, $R_\Delta$ is degenerate only if $\Delta = 0$. If at the second level, $\tilde{V} = \left(\xi L_{-1}^2 - L_{-2}\right)V_\Delta$, and there are two non-trivial conditions: $L_1\tilde{V} = 0$ and $L_2\tilde{V} = 0$. They imply, respectively, that

$$\xi = \frac{3}{2(2\Delta+1)} \tag{8}$$

and

$$8\Delta + c = 12\xi\Delta \tag{9}$$

or, together

$$\Delta = \frac{5 - c \pm \sqrt{(c-1)(c-25)}}{16}. \tag{10}$$

Parameterizing the central charge and dimension as

$$\begin{aligned} c &= 1 - 6Q^2 = 1 - 6\left(b - \frac{1}{b}\right)^2, \\ \Delta &= \alpha(\alpha - Q) = \alpha\left(\alpha - b + \frac{1}{b}\right) \end{aligned} \tag{11}$$

we obtain four solutions:

$$\boxed{\begin{cases} \alpha = \frac{1}{2b} \\ \xi = b^2, \end{cases}} \quad \boxed{\begin{cases} \alpha = -\frac{b}{2} \\ \xi = \frac{1}{b^2}, \end{cases}} \quad \boxed{\begin{cases} \alpha = \frac{3b}{2} - \frac{1}{b} \\ \xi = b^2, \end{cases}} \quad \boxed{\begin{cases} \alpha = b - \frac{3}{2b} \\ \xi = \frac{1}{b^2}. \end{cases}} \tag{12}$$

In what follows, we work with the first of these four solutions (boxed), so that the original highest weight primary $V_{1/2b}$ of the degenerate Verma module satisfies

$$\tilde{V} = \left(b^2L_{-1}^2 - L_{-2}\right)V_{1/2b} = 0 \tag{13}$$

and has dimension

$$\Delta_{1/2b} = -\frac{1}{2} + \frac{3}{4b^2}. \tag{14}$$

One can impose this constraint on all correlators with insertions of the primary $V_{1/2b}$, and the degeneracy of Verma module implies that this constraint is a self-consistent requirement. The conformal Ward identities imply that such correlators satisfy peculiar differential equations, (see Section 2.3). In the free-field realization of CFT, this constraint is imposed almost automatically, (see Section 3.1), and this result is also easily seen from the DF/multi-Penner $\beta$-ensemble representation of the corresponding conformal blocks shown below in Section 3.2.2.

### 2.2. Conformal Ward identities

The spherical correlators of primaries satisfy the simple chain of conformal Ward identities [24]:

$$\left\langle T(z)\prod_i V_{\alpha_i}(z_i)\right\rangle = \left(\sum_i \frac{1}{z-z_i}\partial_i + \sum_i \frac{\Delta_i}{(z-z_i)^2}\right)\left\langle \prod_i V_{\alpha_i}(z_i)\right\rangle \tag{15}$$

and the similar ones for multiple insertions of the stress tensor $T(z)$. In fact, three of the derivatives $\partial_i = \partial/\partial z_i$ in (15) can be always eliminated with the help of the projective $SL(2)$-invariance for the spherical correlators

$$\begin{aligned}
0 &= \left\langle \left(L_{-1}\prod_i V_{\alpha_i}(z_i)\right)\right\rangle = \sum_i \partial_i \left\langle \left(\prod_i V_{\alpha_i}(z_i)\right)\right\rangle, \\
0 &= \left\langle L_0 \prod_i V_{\alpha_i}(z_i)\right\rangle = \sum_i (z_i\partial_i + \Delta_i)\left\langle \prod_i V_{\alpha_i}(z_i)\right\rangle, \\
0 &= \left\langle L_1 \prod_i V_{\alpha_i}(z_i)\right\rangle = \sum_i (z_i^2\partial_i + 2z_i\Delta_i)\left\langle \prod_i V_{\alpha_i}(z_i)\right\rangle.
\end{aligned} \tag{16}$$

Eqs. (15)–(16) (and similar equations w.r.t. the variables $\bar{z}_i$) for the spherical correlation function holds for any conformal block $B_I(\{z_i\})$, with an arbitrary choice of the points $\{z_i\}$ and intermediate dimensions, which appear in the channel decomposition of the correlator

$$\left\langle \prod_i V_{\alpha_i}(z_i)\right\rangle = \sum \mathcal{C}_{I\bar{J}} B_I(\{z_i\})\bar{B}_J(\{z_i\}) \tag{17}$$

i.e., for non-vanishing $\mathcal{C}_{I\bar{J}}$, where $I$ and $\bar{J}$ are the corresponding holomorphic and anti-holomorphic multi-indices.

In particular, the generic four-point correlator that solves Eq. (16) can be presented in the form

$$\left\langle \prod_{i=1}^4 V_{\alpha_i}(z_i)\right\rangle = z_{13}^{-2\Delta_1} z_{23}^{\Delta_1+\Delta_4-\Delta_2-\Delta_3} z_{34}^{\Delta_1+\Delta_2-\Delta_3-\Delta_4} z_{24}^{\Delta_3-\Delta_1-\Delta_2-\Delta_4} \times (\bar{z}\text{ part}) \times G(x,\bar{x}) \tag{18}$$

where $z_{ij} \equiv z_i - z_j$ and $G$ is the function of only the double ratios $x = \frac{z_{12}z_{34}}{z_{13}z_{24}}$ and similarly for $\bar{x}$. This fact allows one to choose the fields located at $z_1 = 0$, $z_2 = x$, $z_3 = 1$ and $z_4 = \infty$; the four-point conformal block in formula (17) acquires the form

$$\begin{aligned}
B_\Delta(x) &\equiv B_\Delta^{(12;34)}(x) = x^{\Delta-\Delta_1-\Delta_2}\sum_{n>0} B_{\Delta,n}x^n \\
&= x^{\Delta-\Delta_1-\Delta_2}\left(1 + \frac{(\Delta+\Delta_1-\Delta_2)(\Delta+\Delta_3-\Delta_4)}{2\Delta}x + \cdots\right).
\end{aligned} \tag{19}$$

### 2.3. Equation for the conformal block

For our purposes in this paper, we distinguish one of the primaries, $V_{\alpha_0}(x)$ at some point $z_0 = x$, with the dimension $\Delta_0 = \Delta(\alpha_0)$, which will later be made degenerate at the second level. Integrating (15) over $z$ with the weight $(z-x)^{-1}$, one obtains

$$\left\langle L_{-2}V_{\alpha_0}(x)\prod_i V_{\alpha_i}(z_i)\right\rangle = \left(\sum_i \frac{1}{x-z_i}\partial_i + \sum_i \frac{\Delta_i}{(x-z_i)^2}\right)\left\langle V_{\alpha_0}(x)\prod_i V_{\alpha_i}(z_i)\right\rangle \tag{20}$$

and, similarly,

$$\left\langle L_{-1}^2 V_{\alpha_0}(x)\prod_i V_{\alpha_i}(z_i)\right\rangle = \partial_x^2 \left\langle V_{\alpha_0}(x)\prod_i V_{\alpha_i}(z_i)\right\rangle. \tag{21}$$

Choosing $\alpha_0 = \frac{1}{2b}$ and making use of (13), one obtains that

$$\left(b^2\partial_x^2 - \sum_i \frac{1}{x-z_i}\partial_i - \sum_i \frac{\Delta_i}{(x-z_i)^2}\right)\left\langle V_{1/2b}(x)\prod_i V_{\alpha_i}(z_i)\right\rangle = 0. \tag{22}$$

Now we apply this equation to the conformal block and realize that it fixes a specific intermediate dimensions in the conformal block.

*2.3.1. Four-point conformal block with the degenerate field*

Eq. (16) are sufficient to reduce (22) to a *single*-variable differential equation in the case of only *three* variables $z_{1,2,3}$: the three Eqs. (16) allow us to the expression of all the three derivatives $\partial_i$. Substituting these expressions back into (22), one obtains [24]:

$$\left\{b^2\partial_x^2 + \sum_{i=1}^3 \frac{1}{x-z_i}\partial_x + \frac{3x-z_1-z_2-z_3}{(x-z_1)(x-z_2)(x-z_3)}\Delta_{1/2b} + \frac{(z_1-z_2)(z_3-z_1)\Delta_1}{(x-z_1)^2(x-z_2)(x-z_3)}\right.$$
$$\left. + \frac{(z_1-z_2)(z_2-z_3)\Delta_2}{(x-z_1)(x-z_2)^2(x-z_3)} + \frac{(z_2-z_3)(z_3-z_1)\Delta_3}{(x-z_1)(x-z_2)(x-z_3)^2}\right\} B_4(x|z_1,z_2,z_3) = 0. \tag{23}$$

If $z_{1,2,3}$ are placed at 0, 1, $\infty$, then this equation simplifies to

$$\left\{b^2x(x-1)\partial_x^2 + (2x-1)\partial_x + \Delta_{1/2b} + \frac{\Delta_1}{x} - \frac{\Delta_2}{x-1} - \Delta_3\right\} B_4(x|0,1,\infty) = 0. \tag{24}$$

Conjugation with a factor $x^\alpha(1-x)^\beta$ with specially adjusted $\alpha$ and $\beta$ converts this equation into an ordinary hypergeometric equation with the solution

$$\begin{aligned} &B_4(x|0,1,\infty) = x^{\alpha_1/b}(1-x)^{\alpha_2/b}F(A,B;C;x)\\ &A = \frac{1}{2b^2} + \frac{\alpha_1}{b} + \frac{\alpha_2}{b} - \frac{\alpha_3}{b}\\ &B = \frac{1}{b}\sum_{i=1}^3 \alpha_i + 2\Delta_{1/2b}, \qquad C = \frac{1}{b^2} + \frac{2\alpha_1}{b}. \end{aligned} \tag{25}$$

Eqs. (24), (25) are consistent with the generic formula (19) *only* if the dimensions $\Delta_1$ and $\Delta$ are related by the fusion rule[2]

$$\begin{aligned} &\alpha = \alpha_1 \pm \frac{1}{2b}\\ &\Delta_1 = \alpha_1\left(\alpha_1 - b + \frac{1}{b}\right), \qquad \Delta = \Delta_\alpha = \alpha\left(\alpha - b + \frac{1}{b}\right) \end{aligned} \tag{26}$$

where the two choices of the sign correspond to the two linearly independent solutions of (24), and, in the case of the sign "minus" in (26), one has to choose in (25) instead of $F(A,B;C;x)$, the other solution to the hypergeometric equation $x^{1-C}F(A-C+1, B-C+1; 2-C; x)$.

One can easily check directly that the conformal block from the r.h.s. of (28)

$$\begin{aligned} B_{\Delta_\alpha}^{(1,1/2b;34)}(x) &= x^{\Delta_\alpha-\Delta_1-\Delta_{1/2b}}\left(1 + \frac{(\Delta_\alpha+\Delta_{1/2b}-\Delta_1)(\Delta_\alpha+\Delta_3-\Delta_4)}{2\Delta_\alpha}x + \cdots\right)\\ &\underset{(26)}{=} B_4(x|0,1,\infty) \end{aligned} \tag{27}$$

which solves (24). Formula (17) now acquires the form

$$\begin{aligned} \left\langle V_1(0)V_{1/2b}(x)V_3(1)V_4(\infty)\right\rangle &= \sum_\Delta C_{1,1/2b}^\Delta C_{34}^\Delta |B_\Delta^{(1,1/2b;34)}(x)|^2\\ &= \sum_{\alpha=\alpha_1\pm\frac{1}{2b}} C_{1,1/2b}^{\Delta_\alpha}C_{34}^{\Delta_\alpha}|B_{\Delta_\alpha}^{(1,1/2b;34)}(x)|^2 \end{aligned} \tag{28}$$

since *only* for the choice (26), the structure constant $C_{1,1/2b}^{\Delta_\alpha}$ is non-vanishing [24]. Here, we obtained this fact indirectly by solving the equation for the correlator. We shall derive this fact straightforwardly using the $\beta$-ensemble representation for the conformal blocks in the next section.

2 We specially stress this point here: in the original normalizations of [24], when the structure constants are not absorbed into the definition of conformal block, the generic formula (19) *by no means simplifies*, or, e.g., as is stated in Appendix of [21], gives rise to a vanishing result. Expression (19) simplifies only under the conditions (26), and all other conformal blocks, though being non-vanishing themselves, do not give any contribution to the correlator due to the vanishing nature of the structure constants, (see (28)).

*2.3.2. Five-point conformal block with the degenerate field*

When there are *four* variables $z_{1,2,3,4}$, one can use (16) to eliminate three out of the four derivatives $\partial_i$:

$$\begin{aligned}&\left\{b^2\partial_x^2+\frac{3x^2-2x(z_1+z_2+z_3)+z_1z_2+z_2z_3+z_3z_1}{(x-z_1)(x-z_2)(x-z_3)}\partial_x+\frac{(z_1-z_4)(z_2-z_4)(z_3-z_4)}{(x-z_1)(x-z_2)(x-z_3)(x-z_4)}\partial_4\right.\\&+\frac{(z_1-z_2)(z_2-z_3)(z_3-z_1)}{(x-z_1)(x-z_2)(x-z_3)}\left(\frac{\Delta_1}{(x-z_1)(z_2-z_3)}+\frac{\Delta_2}{(x-z_2)(z_3-z_1)}+\frac{\Delta_3}{(x-z_3)(z_1-z_2)}\right)\\&-\frac{\left(3z_4^2-2z_4(z_1+z_2+z_3)+z_1z_2+z_2z_3+z_3z_1\right)x-\left(2z_4^3-(z_1+z_2+z_3)z_4^2+z_1z_2z_3\right)}{(x-z_1)(x-z_2)(x-z_3)(x-z_4)^2}\Delta_4\\&\left.+\frac{3x-z_1-z_2-z_3}{(x-z_1)(x-z_2)(x-z_3)}\Delta_{1/2b}\right\}B_5(x|z_1,z_2,z_3,z_4)=0.\end{aligned}\tag{29}$$

If $z_{1,2,3}$ are placed at $0,1,\infty$, this equation for $B(x|0,1,\infty,q)\equiv B(x|q)$ simplifies to

$$\begin{aligned}&\left\{b^2x(x-1)\partial_x^2+(2x-1)\partial_x-\frac{q(q-1)}{x-q}\partial_q+\Delta_{1/2b}+\frac{\Delta_1}{x}-\frac{\Delta_2}{x-1}-\Delta_3+\frac{q^2-(2q-1)x}{(x-q)^2}\Delta_4\right\}\\&B_5(x|q)=0\end{aligned}\tag{30}$$

$x$ and $x-1$ in the denominators can again be eliminated by conjugation. The resulting equation can be represented as the one being on an elliptic curve (torus) with coordinate $x-q$ and ramification point $q^{-1}$ [38]. The double pole $(x-q)^2$ then becomes a Weierstrass function.

*2.3.3. Toric block with one z-variable*

Instead of (15) and (16), a toric correlator satisfies a pair of equations: the conformal Ward identity [42] (we normalize the correlators so that the toric partition function is $Z(\tau,\bar\tau)=\langle 1\rangle$)

$$\left\langle T(z)\prod_i V_{\alpha_i}(z_i)\right\rangle=2\pi\mathrm{i}\frac{\partial}{\partial\tau}\left\langle\prod_i V_{\alpha_i}(z_i)\right\rangle+\sum_i\Big((\zeta_*(z-z_i|\tau)+2\eta_1z)\partial_i+\Delta_i\wp_*(z-z_i|\tau)\Big)\left\langle\prod_i V_{\alpha_i}(z_i)\right\rangle\tag{31}$$

with

$$\begin{aligned}&\zeta_*(z|\tau)\equiv\partial_z\log\theta_*(z|\tau)=\zeta(z|\tau)-2\eta_1z,\qquad \wp_*(z|\tau)\equiv-\partial_z\zeta_*(z|\tau)=\wp(z|\tau)+2\eta_1\\&\eta_1=\zeta\left(\frac12|\tau\right)=-2\pi\mathrm{i}\partial_\tau\log\eta\left(\mathrm{e}^{\mathrm{i}\pi\tau}\right),\qquad \eta(\tau)=\mathrm{e}^{\mathrm{i}\pi\tau/12}\prod_{n>0}\left(1-\mathrm{e}^{2\mathrm{i}n\pi\tau}\right)\end{aligned}\tag{32}$$

and the torus counterpart of (16), is similar to

$$0=\left\langle L_{-1}\Big(\prod_i V_{\alpha_i}(z_i)\Big)\right\rangle=\sum_i\partial_i\left\langle\prod_i V_{\alpha_i}(z_i)\right\rangle.\tag{33}$$

Note that (33) ensures the correctness of the double periodicity in $z$ of Eq. (31), while for the periodicity in $\{z_i\}$-variables, the presence of the $\tau$-derivative is extremely important.

As a corollary of (13) and (31), one obtains a torus counterpart of Eq. (22): the correlator with the degenerate field insertion now satisfies

$$\begin{aligned}&\left(-2\pi\mathrm{i}\frac{\partial}{\mathrm{d}\tau}+b^2\partial_x^2-\sum_j\left(\zeta_*(x-z_j|\tau)\partial_j+\Delta_j\wp_*(x-z_j)\right)\right)\left\langle V_{1/2b}(x)\prod_i V_{\alpha_i}(z_i)\right\rangle\\&=2\eta_1\Delta_{1/2b}\left\langle V_{1/2b}(x)\prod_i V_{\alpha_i}(z_i)\right\rangle.\end{aligned}\tag{34}$$

In the particular case of a single $z$-variable (to be put at $z=0$), we obtain:

$$\left(-2\pi\mathrm{i}\frac{\partial}{\partial\tau}+b^2\partial_x^2+\zeta_*(x|\tau)\partial_x-\Delta_\alpha\wp_*(x)\right)\left\langle V_{1/2b}(x)V_\alpha(0)\right\rangle=2\eta_1\Delta_{1/2b}\left\langle V_{1/2b}(x)V_\alpha(0)\right\rangle\tag{35}$$

or, after multiplication by $\eta^A\theta_*(x)^{-1/2b^2}$ with $\frac{A}{2}=\Delta_\alpha+\frac{1}{b^2}-1$, this equation turns into (cf., e.g., with [16])

$$\left(2\pi\mathrm{i}\frac{\partial}{\partial\tau}-b^2\partial_x^2+\left(\Delta_\alpha+\frac{1}{4b^2}-\frac12\right)\wp(x)\right)\cdot\left(\eta^{-A}\theta_*(x)^{1/2b^2}\left\langle V_{1/2b}(x)V_\alpha(0)\right\rangle\right)=0\tag{36}$$

and the same equation is satisfied by any toric two-point conformal block, arising in the decomposition of the correlator

$$\left\langle V_{1/2b}(x)V_\alpha(0)\right\rangle = \sum_{\Delta,\pm} C^{\Delta_\pm}_{\Delta_\alpha\Delta}\left|B^{\pm}_{\Delta,\Delta_\alpha}(x|\tau)\right|^2 \tag{37}$$

with

$$\Delta_\pm = \Delta_{\alpha_\pm}, \quad \alpha_\pm = \alpha \pm \frac{1}{2b}. \tag{38}$$

*2.3.4. The "non-conformal" limit*

When the four external dimensions $\Delta(\alpha_i)$ become large (while the intermediate dimension $\Delta$ is kept finite), one can make the double ratio $q = \frac{(z_2-z_1)(z_4-z_3)}{(z_3-z_1)(z_4-z_2)}$ small, so that the dimensional transmutation takes place, and a new finite parameter $\Lambda^4 = q\sqrt{\Delta_1\Delta_2\Delta_3\Delta_4}$ emerges instead of $q$ and four $\Delta_i$. This limit corresponds to the pure $\mathcal{N} = 2SU(2)$ SYM theory and thus is referred to as the "non-conformal limit" in the AGT literature, (see [6]). On the CFT side, this limit is associated with a peculiar coherent state

$$|\Delta, \Lambda\rangle = \sum_Y \Lambda^{2|Y|}Q_\Delta^{-1}\Big([1^{|Y|}], Y\Big)L_{-Y}|\Delta\rangle \tag{39}$$

so that the four-point conformal block turns into

$$B_\Delta^{12;34}(q) \to \langle\Delta, \Lambda|\Delta, \Lambda\rangle = \sum_{n\geq 0}\Lambda^{4n}Q^{-1}([1^n], [1^n]) \tag{40}$$

where the sum goes over the single-row Young diagrams $Y = [1^n]$, and $Q(Y, Y') = \langle\Delta|L_Y L_{-Y'}|\Delta\rangle$ is the block-diagonal Shapovalov form for the Virasoro algebra. The same result can of course be obtained from a similar limit of the one-point toric conformal block $B_{\Delta,\Delta_\alpha}(\tau)$, which corresponds on the SYM side to obtaining the pure gauge theory from the infinite-mass limit $e^{\pi i\tau}\Delta_\alpha = \Lambda^2$ (being fixed when $\Delta_\alpha \to \infty$ and $\tau \to +i\infty$) of the $\mathcal{N} = 2^*$ theory with an adjoint supermultiplet:

$$B_{\Delta,\Delta_\alpha}(\tau) \to \langle\Delta, \Lambda|\Delta, \Lambda\rangle = \sum_n \Lambda^{4n}Q^{-1}([1^n], [1^n]). \tag{41}$$

In this paper, we are interested in the conformal block with additional insertion of the degenerate primary $V_{1/2b}(x)$. There are three possibilities for obtaining the equation for this conformal block. First of all, one can obtain the equation directly by insertion of the degenerate primary into the matrix element (41):

$$\mathcal{B}_5(x|z_1, z_2, z_3, z_4) \to \left\langle\Delta, \Lambda|V_{1/2b}(x)|\Delta, \Lambda\right\rangle. \tag{42}$$

This approach was performed in [9]. The two other possibilities are those which we discussed above: one can take the limit of infinite masses in Eq. (30) for the five-point conformal block $B_5$, or consider a similar limit for the toric conformal block (37),

$$B^{\pm}_{\Delta,\Delta_\alpha}(x|\tau) \to \left\langle\Delta, \Lambda|V_{1/2b}(x)|\Delta, \Lambda\right\rangle. \tag{43}$$

All the three methods definitely lead to the same equation. For instance, in the latter case, Eq. (36) is substituted by its periodic Toda-chain (sine-Gordon) analogue. Indeed, in the peculiar Inozemtsev limit [43], the Weierstrass function $\wp(x|\tau)$ turns into a hyperbolic cosine. To see this result, rewrite first the Weierstrass function as an expansion in inverse sines:

$$\wp(x) = \sum_{m,n\in\mathbb{Z}}\frac{1}{(x+m+n\tau)^2} - C(\tau) = \sum_{n\in\mathbb{Z}}\frac{\pi^2}{\sin^2\pi(x+n\tau)} - C(\tau) \tag{44}$$

(where the factor $\pi$ emerges in the argument of sin due to periodicity under $x \to x+i\pi$, while the factor $\pi$ in the numerator is present since $\frac{\pi}{\sin\pi x} \sim \frac{1}{x}$), and

$$C(\tau) = \frac{1}{3} + 2\sum_{n\geq 1}\frac{\pi^2}{\sin^2\pi(n\tau)}. \tag{45}$$

Next, put set $x = i\xi - \tau/2$. In the Inozemtsev limit, there are two terms, surviving from this sum in the leading order in the $e^{2\pi i\tau}$-expansion:

$$\frac{\pi^2}{\sin^2\pi x} \longrightarrow -4\pi^2 e^{i\pi\tau}e^{-2\pi\xi} \tag{46}$$

and

$$\frac{\pi^2}{\sin^2\pi(x+\tau)} \longrightarrow -4\pi^2 e^{i\pi\tau} e^{+2\pi\xi}. \tag{47}$$

Thus,

$$\wp(x) \to -8\pi^2 e^{\pi i\tau}\cosh 2\pi\xi. \tag{48}$$

Of course, $\partial_x = -i\partial_\xi$, and the Calogero–Schrödinger equation (36) finally turns into (under the rescaling of $2\pi\xi \to \xi$)

$$\left(b^2\partial_\xi^2 - 2\Lambda^2\cosh\xi + \frac{1}{4}\frac{\partial}{\partial\log\Lambda}\right)\left\langle\Delta,\Lambda|V_{1/2b}(\xi)|\Delta,\Lambda\right\rangle = 0. \tag{49}$$

This formula coincides with [9, (A.13)] up to some trivial rescalings of the conformal block.

## 3. $B(x|z)$ in free-field/$\beta$-ensemble realizations

### 3.1. Free fields [44,45]

The chiral free-field propagator is given by

$$\langle\phi(z)\phi(0)\rangle = -2\log z. \tag{50}$$

For the exponential primary fields

$$V_\alpha =: e^{i\alpha\phi} : \tag{51}$$

one can write

$$\begin{aligned}\prod_j V_{\alpha_j}(z_j) &= \prod_j : e^{i\alpha_j(z_j)} := \prod_{i<j}(z_i-z_j)^{2\alpha_i\alpha_j} : e^{\sum_j \alpha_j\phi(z_j)} : \\ &= \prod_{i<j}(z_i-z_j)^{2\alpha_i\alpha_j} : \prod_j V_{\alpha_j}(z_j) :\end{aligned} \tag{52}$$

The holomorphic stress tensor

$$T = -\frac{1}{4}(\partial\phi)^2 + \frac{iQ}{2}\partial^2\phi \tag{53}$$

obviously satisfies

$$T(z)T(0) = \frac{c}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0) + O(z) \tag{54}$$

and

$$T(z)V_\alpha(0) = \sum_k \frac{1}{z^{k+2}}L_kV_\alpha(0) = \frac{\Delta_\alpha}{z^2}V_\alpha(0) + \frac{1}{z}\partial V_\alpha(0) + :\left(-\frac{1}{4}(\partial\phi)^2 + i\left(\alpha+\frac{Q}{2}\right)\partial^2\phi\right)V_\alpha(0): + O(z) \tag{55}$$

with the central charge and dimension exactly given by (11). The screening currents with unit dimension are $V_b$ and $V_{-1/b}$, since $\Delta_b = \Delta_{-1/b} = 1$.

The null-vector condition implies that

$$\begin{aligned}\left(b^2L_{-1}^2 - L_{-2}\right)V_\alpha &= b^2\partial^2V_\alpha - :\left(-\frac{1}{4}(\phi)^2 + i\left(\alpha+\frac{Q}{2}\right)\partial^2\phi\right)V_\alpha : \\ &= :\left(\left(\alpha^2b^2-\frac{1}{4}\right)(\partial\phi)^2 + i\left(\alpha b^2 - \alpha - \frac{Q}{2}\right)\partial^2\phi\right)V_\alpha :\end{aligned} \tag{56}$$

and the r.h.s. vanishes for $\alpha = \frac{1}{2b}$ $\left(\text{and } Q = b - \frac{1}{b}\right)$. In what follows, we shall omit the normal-ordering signs for the free-field operators, when their presence is obvious.

### 3.2. $B(x|z)$ in the $\beta$-ensemble representation

#### 3.2.1. Conformal block in the free-field representation

In the free-field realization, the arbitrary generic conformal block on a sphere is given by

$$B_I(\{z_i\}) = \left\langle \prod_i e^{i\alpha_i\phi(z_i)} \prod_{\gamma_I}\left(\int e^{ib\phi(u)}du\right)^{N_{\gamma_I}} \right\rangle_{free} \tag{57}$$

where angular brackets imply the correlator in the theory of the $2d$ chiral field (50) according to the rule, (52). The number of screening insertions $N_{\gamma_I}$ and the choice of the integration contours $\gamma_I$ themselves depend on the particular choice $I$ of the conformal block.

For example, in the case of four-point conformal block, one can write

$$\begin{aligned} B_{\Delta}^{(12;34)}(q) &= \left\langle \mathrm{e}^{\mathrm{i}\alpha_1\phi(0)}\mathrm{e}^{\mathrm{i}\alpha_2\phi(q)}\mathrm{e}^{\mathrm{i}\alpha_3\phi(1)}\mathrm{e}^{\mathrm{i}\alpha_4\phi(\infty)} \left(\int_0^q \mathrm{e}^{\mathrm{i}b\phi(u)}\mathrm{d}u\right)^{N_x} \left(\int_0^1 \mathrm{e}^{\mathrm{i}b\phi(v)}\mathrm{d}v\right)^{N_1} \right\rangle_{free} \\ &= q^{2\alpha_1\alpha_2}(1-q)^{2\alpha_2\alpha_3} \int \prod_{a<a'} (U_a - U_{a'})^{2b^2} \prod_a U_a^{2\alpha_1 b}(1-U_a)^{2\alpha_3 b}(q-U_a)^{2\alpha_2 b}\mathrm{d}U_a \end{aligned} \tag{58}$$

where $\{U_a\} = \{\{u\}, \{v\}\}$, $a = 1, \ldots, N_x + N_1$, with

$$\begin{aligned} N_x &= \frac{1}{b}(\alpha - \alpha_1 - \alpha_2) \\ N_1 &= \frac{1}{b}(Q - \alpha - \alpha_3 - \alpha_4) \end{aligned} \tag{59}$$

being the number of contours stretched between $u = 0, q$ and $v = 0, 1$, respectively.

### 3.2.2. *Four-point conformal block in the $\beta$-ensemble representation*

The purpose of this section is to show that the differential equation (22) survives for the conformal block only provided that no screening contour terminates at the position of the degenerate operator.

If one of the fields is degenerate at the second level, say $V_{1/2b}(x)$, formula (58) gives the solution to the second-order null-vector Eq. (24) only for $N_x = 0$, i.e.,

$$\begin{aligned} B_4(x|0,1,\infty) &= \left\langle \mathrm{e}^{\frac{\mathrm{i}}{2b}\phi(x)}\mathrm{e}^{\mathrm{i}\alpha_1\phi(0)}\mathrm{e}^{\mathrm{i}\alpha_2\phi(1)}\mathrm{e}^{\mathrm{i}\alpha_3\phi(\infty)} \left(\int_0^1 \mathrm{e}^{\mathrm{i}b\phi(v)}\mathrm{d}v\right)^{N_1} \right\rangle_{free} \\ &= x^{\alpha_1/b}(1-x)^{\alpha_2/b} \int \prod_{a<a'} (v_a - v_{a'})^{2b^2} \prod_a v_a^{2\alpha_1 b}(1-v_a)^{2\alpha_2 b}(x-v_a)\mathrm{d}v_a \\ N_1 &= \frac{1}{b}(Q - \alpha - \alpha_3 - \alpha_4). \end{aligned} \tag{60}$$

Eq. (24) automatically follows for the r.h.s. of (60) when applying (56) for the field $V_{1/2b}(x)$. Consider the free-field correlator with one degenerate field and some number of screenings (with yet unspecified contours) inserted, then

$$\left\langle \left\{ (b^2 L_{-1}^2 - L_{-2})\mathrm{e}^{\frac{\mathrm{i}}{2b}\phi(x)} \right\} \mathrm{e}^{\mathrm{i}\alpha_1\phi(0)}\mathrm{e}^{\mathrm{i}\alpha_2\phi(1)}\mathrm{e}^{\mathrm{i}\alpha_3\phi(\infty)} \prod_I \left(\int_{\gamma_I} \mathrm{e}^{\mathrm{i}b\phi(u)}\mathrm{d}u\right)^{N_{\gamma_I}} \right\rangle_{free} = 0 \tag{61}$$

is obviously true, due to Eq. (56) at $\alpha = \frac{1}{2b}$. As for an arbitrary conformal theory above, one can write

$$\begin{aligned} & b^2\partial_x^2 \left\langle \mathrm{e}^{\frac{\mathrm{i}}{2b}\phi(x)}\mathrm{e}^{\mathrm{i}\alpha_1\phi(0)}\mathrm{e}^{\mathrm{i}\alpha_2\phi(1)}\mathrm{e}^{\mathrm{i}\alpha_3\phi(\infty)} \prod_I \left(\int_{\gamma_I} \mathrm{e}^{\mathrm{i}b\phi(u)}\mathrm{d}u\right)^{N_{\gamma_I}} \right\rangle_{free} \\ &= b^2 \left\langle L_{-1}^2 \mathrm{e}^{\frac{\mathrm{i}}{2b}\phi(x)}\mathrm{e}^{\mathrm{i}\alpha_1\phi(0)}\mathrm{e}^{\mathrm{i}\alpha_2\phi(1)}\mathrm{e}^{\mathrm{i}\alpha_3\phi(\infty)} \prod_I \left(\int_{\gamma_I} \mathrm{e}^{\mathrm{i}b\phi(u)}\mathrm{d}u\right)^{N_{\gamma_I}} \right\rangle_{free} \\ &= \left\langle L_{-2} \mathrm{e}^{\frac{\mathrm{i}}{2b}\phi(x)}\mathrm{e}^{\mathrm{i}\alpha_1\phi(0)}\mathrm{e}^{\mathrm{i}\alpha_2\phi(1)}\mathrm{e}^{\mathrm{i}\alpha_3\phi(\infty)} \prod_I \left(\int_{\gamma_I} \mathrm{e}^{\mathrm{i}b\phi(u)}\mathrm{d}u\right)^{N_{\gamma_I}} \right\rangle_{free} \\ &= \oint_x \frac{\mathrm{d}z}{z-x} \left\langle \left(T(z)\mathrm{e}^{\frac{\mathrm{i}}{2b}\phi(x)}\right) \mathrm{e}^{\mathrm{i}\alpha_1\phi(0)}\mathrm{e}^{\mathrm{i}\alpha_2\phi(1)}\mathrm{e}^{\mathrm{i}\alpha_3\phi(\infty)} \prod_I \left(\int_{\gamma_I} \mathrm{e}^{\mathrm{i}b\phi(u)}\mathrm{d}u\right)^{N_{\gamma_I}} \right\rangle_{free} \\ &= -\sum_{w=0,1,\infty} \oint_x \frac{\mathrm{d}z}{z-x} \left\langle \mathrm{e}^{\frac{\mathrm{i}}{2b}\phi(x)} \left( T(z)\mathrm{e}^{\mathrm{i}\alpha_1\phi(0)}\mathrm{e}^{\mathrm{i}\alpha_2\phi(1)}\mathrm{e}^{\mathrm{i}\alpha_3\phi(\infty)} \prod_I \left(\int_{\gamma_I} \mathrm{e}^{\mathrm{i}b\phi(u)}\mathrm{d}u\right)^{N_{\gamma_I}} \right) \right\rangle_{free}. \end{aligned} \tag{62}$$

The r.h.s. of this equation obviously results in the same expression as in the r.h.s. of (20), further giving rise to (24), (25) exactly as in Section 2.3.1, but now directly for the particular conformal block, written in the form of the free-field correlator (60). One has to use here only the commutativity of the stress-energy tensor with the screening operator

$[T(z), \int V_b(u)\mathrm{d}u] = 0$, following from the fact that the singular part of the corresponding OPE (55)

$$T(z)V_b(u) = \frac{\partial}{\partial u}\left(\frac{V_b(u)}{z-u}\right) + \cdots \tag{63}$$

is the total derivative.

This argument should be applied, however, with extra care in the case of non-closed contours, (as in (58) and (60)). Usually (see, e.g., [25]) the desired result is achieved by the analytic continuation of the result of free-field computation from the parameter values of parameters, ensuring automatic vanishing of this result at the end-point of the contour. However, this process is not always possible, and we shall see immediately that in our case the argument is applicable only for $N_x = 0$, i.e., when there is no integration of the screening current with the end-point at $u = x$.

Indeed, due to (63), the integrand in the correlator (62) contains a term of the form

$$\begin{aligned}\frac{\partial}{\partial u}\cdot\frac{1}{x-u}\left\langle e^{\frac{i}{2b}\phi(x)}e^{i\alpha_1\phi(0)}e^{i\alpha_2\phi(1)}e^{i\alpha_3\phi(\infty)}e^{ib\phi(u)}\right\rangle_{free} \overset{=}{(60)}\ & \frac{\partial}{\partial u}\left(\frac{1}{x-u}\cdot u^{2\alpha_1 b}(1-u)^{2\alpha_2 b}(x-u)\right)\\ =\ & \frac{\partial}{\partial u}\left(u^{2\alpha_1 b}(1-u)^{2\alpha_2 b}\right)\end{aligned} \tag{64}$$

i.e., the extra pole from (63) exactly cancels the zero at $x = u$, coming from the contraction of the degenerate field with the screening current

$$V_{1/2b}(x)\cdot V_b(u) = e^{\frac{i}{2b}\phi(x)}\cdot e^{ib\phi(u)} \overset{\sim}{(50)} (x-u). \tag{65}$$

The integral of (64) along the contour between $u = 0$ and $u = x$ is obviously non-vanishing, contrary to the integral between $u = 0$ and $u = 1$, which can be treated as vanishing, at least in the sense of analytic continuation. We therefore conclude that (58) satisfies the second-order differential equation in $x$-variable, if the field at the point $x$ is degenerate on second level *and* $N_x = 0$, i.e., exactly for (60).

### 3.2.3. *n-point conformal block in the $\beta$-ensemble representation*

For the five-point conformal block with $q \ll x \ll 1$, $q = \frac{(z_2-z_1)(z_3-z_4)}{(z_3-z_1)(z_2-z_4)}$, (see Fig. 1), one has [15,17]

$$\begin{aligned}B_5(x|q) = \Big\langle &: e^{i\alpha_1\phi(0)} :: e^{i\alpha_2\phi(q)} :: e^{\frac{i}{2b}\phi(x)} :: e^{i\alpha_3\phi(1)} :: e^{i\alpha_4\phi(\infty)} :\\ &\left(\int_0^q : e^{ib\phi(u)} : \mathrm{d}u\right)^{N_q}\left(\int_0^x : e^{ib\phi(v)} : \mathrm{d}v\right)^{N_x}\left(\int_0^1 : e^{ib\phi(w)} : \mathrm{d}w\right)^{N_1}\Big\rangle_{free}\end{aligned} \tag{66}$$

where

$$\begin{aligned}\alpha &= \alpha_1+\alpha_2+bN_q,\\ \tilde\alpha &= \alpha+\frac{1}{2b}+bN_x,\\ \alpha_4 &= b-\frac{3}{2b}-\alpha_1-\alpha_2-\alpha_3-b(N_q+N_x+N_1).\end{aligned} \tag{67}$$

Here, the angular brackets imply only the free-field computation (52), i.e.,

$$\begin{aligned}B_5(x|q) &\sim q^{2\alpha_1\alpha_2}(1-q)^{2\alpha_2\alpha_3}x^{\alpha_1/b}(1-x)^{\alpha_3/b}(q-x)^{\alpha_2/b}\\ &\quad\times\int\underline{\prod_i(x-U_i)}\prod_{i<j}(U_i-U_j)^{2b^2}\prod_i U_i^{2\alpha_1 b}(1-U_i)^{2\alpha_3 b}(q-U_i)^{2\alpha_2 b}\mathrm{d}U_i\\ &= e^{\frac{1}{2b^2}W(x)}B_4(q)\frac{\int\underline{\prod_i(x-U_i)}\prod_{i<j}(U_i-U_j)^{2b^2}\prod_i e^{W(U_i)}\mathrm{d}U_i}{\int\prod_{i<j}(U_i-U_j)^{2b^2}\prod_i e^{W(U_i)}\mathrm{d}U_i}\\ &= \exp\left(\frac{1}{2b^2}\int^x W'(\tilde x)\mathrm{d}\tilde x\right)B_4(q)\left\langle\!\left\langle\underline{\text{“}\det\text{”}(x-M)}\right\rangle\!\right\rangle\end{aligned} \tag{68}$$

where we used (58), and

$$\begin{aligned}&W(x) = 2b\sum_{i=1}^4\alpha_i\log(x-z_i)\\ &(i = 1,2,3,4,\ z_i = \{0,q,1,\infty\})\end{aligned} \tag{69}$$

is the logarithmic potential of the beta-ensemble with $\beta = b^2$, corresponding to (58). As usual, we symbolically denote the r.h.s. as a "matrix-model" average, as if $M$ were a "matrix" with eigenvalues $U_i$: $M = \mathrm{diag}(U_i) = \mathrm{diag}(\{u\}, \{v\}, \{w\})$, determinant "det"$(x - M) \equiv \prod_i (x - U_i)$ and integration measure $\mathrm{d}M \equiv \prod_{i<j}(U_i - U_j)^{2\beta} \prod_i \mathrm{d}U_i$. Double angular brackets denote the beta-ensemble average with specific integration contours, different for the three different constituents of the "eigenvalue set" $\{U_a\}$, as in Eq. (66).

As in the case of the four-point function (60), it is easy to check that this multiple integral indeed satisfies (22), but only for $N_x = 0$, which also implies the same fusion rule as in the case of the four-point conformal block.

Similarly, the $n$-point conformal block with $n-4$ degenerated operators is given for the comb-like diagram [15,17] by the free-field average, $q \ll x_1 \ll \cdots \ll x_{n-4} \ll 1$

$$\begin{aligned} B_n(x_a|q) = \Bigg\langle &: e^{i\alpha_1\phi(0)} :: e^{i\alpha_2\phi(q)} : \prod_a : e^{\frac{i}{2b}\phi(x_a)} :: e^{i\alpha_3\phi(1)} :: e^{i\alpha_4\phi(\infty)} : \\ &\left(\int_0^q : e^{ib\phi(u)} : \mathrm{d}u\right)^{N_q} \prod_a \left(\int_0^{x_a} : e^{ib\phi(v_a)} : \mathrm{d}v_a\right)^{N_{x_a}} \left(\int_0^1 : e^{ib\phi(w)} : \mathrm{d}w\right)^{N_1} \Bigg\rangle_{free} \end{aligned} \tag{70}$$

with

$$\begin{aligned} &\alpha = \alpha_1 + \alpha_2 + bN_q, \\ &\alpha^{(a)} = \alpha^{(a-1)} + \frac{1}{2b} + bN_{x_a}, \\ &\alpha_4 = b - \frac{n-2}{2b} - \alpha_1 - \alpha_2 - \alpha_3 - b\left(N_q + \sum_a N_{x_a} + N_1\right) \end{aligned} \tag{71}$$

where $\alpha^{(a)}$ refers to the intermediate channels.

The eigenvalue model average now looks like

$$\begin{aligned} B_n(x_a|q) &\sim q^{2\alpha_1\alpha_2}(1-q)^{2\alpha_2\alpha_3} \prod_a x_a^{\alpha_1/b}(1-x_a)^{\alpha_3/b}(q-x_a)^{\alpha_2/b} \prod_{a<b}(x_a - x_b)^{\frac{1}{b^2}} \\ &\quad \times \int \underline{\prod_a \prod_i (x_a - U_i)} \prod_{i<j}(U_i - U_j)^{2b^2} \prod_i U_i^{2\alpha_1 b}(1-U_i)^{2\alpha_3 b}(q-U_i)^{2\alpha_2 b} \mathrm{d}U_i \\ &= \exp\left(\sum_a \frac{1}{2b^2}\int^{x_a} W'(\tilde{x})\mathrm{d}\tilde{x}\right) B_4(q) \left\langle\!\!\left\langle \underline{\prod_a \text{“det”}(x_a - M)} \right\rangle\!\!\right\rangle. \end{aligned} \tag{72}$$

Again, this multiple integral indeed satisfies (22), but only for all $N_{x_a} = 0$, which also implies the same fusion rule as in the case of the four-point conformal block.

### 3.3. Resolvent expansion of the conformal block

Applying the general identity

$$\begin{aligned} \log\langle\langle e^L \rangle\rangle &= \log\left(1 + \langle\langle L \rangle\rangle + \frac{1}{2}\langle\langle L^2 \rangle\rangle + \frac{1}{6}\langle\langle L^3 \rangle\rangle + \cdots\right) \\ &= \langle\langle L \rangle\rangle + \frac{1}{2}\left(\langle\langle L^2 \rangle\rangle - \langle\langle L \rangle\rangle^2\right) + \frac{1}{6}\left(\langle\langle L^3 \rangle\rangle - 3\langle\langle L^2 \rangle\rangle\langle\langle L \rangle\rangle + 2\langle\langle L \rangle\rangle^3\right) + \cdots \\ &= \sum_k \frac{1}{k!}\langle\langle L^k \rangle\rangle_{conn} \end{aligned} \tag{73}$$

to the r.h.s. of (68), one concludes that $\log B_5(x|z)$ can be represented as a sum of connected correlators. In this case, $e^L = \det(x - M)$ and

$$L = \mathrm{Tr}\,\log(x - M) \equiv \sum_i \log(x - U_i) = \sum_i \int^x \frac{\mathrm{d}\tilde{x}}{\tilde{x} - U_i} = \int^x \mathrm{Tr}\,\frac{\mathrm{d}\tilde{x}}{\tilde{x} - M}. \tag{74}$$

Thus,

$$\langle\langle L^k \rangle\rangle_{conn} = \int^x \cdots \int^x \left\langle\!\!\left\langle \mathrm{Tr}\,\frac{\mathrm{d}x_1}{x_1 - M} \cdots \mathrm{Tr}\,\frac{\mathrm{d}x_k}{x_k - M} \right\rangle\!\!\right\rangle_{conn} = \int^x \cdots \int^x \rho_k(x_1, \ldots, x_k) \tag{75}$$

is a $k$-fold integral of a $k$-fold connected multi-resolvent $\rho_k(x_1,\dots,x_k)$ for the beta-ensemble (68). One therefore obtains

$$\log\frac{B_5(x|q)}{B_4(q)}=\frac{1}{2b^2}\int^x W'(x')\mathrm{d}x'+\sum_k\frac{1}{k!}\int^{x\;\otimes k}\rho_k(x_1,\dots,x_k). \tag{76}$$

Similarly, for the $n$-point conformal block we get from (72),

$$\log\frac{B_n(x_a|q)}{B_4(q)}=\sum_a\frac{1}{2b^2}\int^{x_a}W'(\tilde{x})\mathrm{d}\tilde{x}+\sum_k\frac{1}{k!}\sum_{a_1,\dots,a_k=1}^{n-4}\int^{x_{a_1}}\cdots\int^{x_{a_k}}\rho_k(\tilde{x}_{a_1},\dots,\tilde{x}_{a_k}). \tag{77}$$

As explained in detail in [40], the multi-resolvents are poly-differentials on the spectral curve, recursively defined from the Virasoro-like constraints (the Ward identities for the matrix model or beta-ensemble). This construction (sometimes called "topological recursion") depends only on spectral curve with distinguished coordinates endowed with a generating differential. In the multi-Penner model with the potential $W(x)$, the spectral curve is $\beta^2y^2=W'(x)^2+\sum_i\frac{\beta c_i}{x-z_i}$ with the coefficients $c_i$ being the linear combinations of the $N$-variables with $\alpha$-dependent coefficients (for particular examples of multi-resolvents in this case, see [8,12–14,20]). From now on, by making a shift, we absorb the term $W'(x)$ into the definition of one-point $\rho_1(x)$ (which is very natural as well and within the framework of [40]).

Formulas (76) and (77) contain the *exact* multi-resolvents, including the contributions of all genera,

$$\rho_k=\sum_{p\geq 0}\hbar^{p-1}\rho^{(p|k)}. \tag{78}$$

They coincide with the expressions conjectured in [18,20] (and prove them) for all values of $\beta=b^2$, not only for $\beta=1$ and $Q=0$. In the general case, one just need only consider the beta-ensemble multi-resolvents instead of the matrix-model ones, exploited in [18,20].

*Comment. One has to be very careful with fixing the values of four external dimensions, corresponding to the beta-ensemble producing the resolvents. Indeed, for fixed $\alpha_1$, $\alpha_2$ and $\alpha_3$, the fourth charge $\alpha_4$ is determined by the size of the beta-ensemble (or numbers of the inserted screening operators inserted) and the number of degenerated fields inserted, (71). For instance, the value of $\alpha_4$ used in $B_5(x|q)$ differs by $1/2b$ from that in $B_4(z)$, i.e., the average in formula (68) is calculated with the beta-ensemble corresponding to $\alpha_4$ shifted by $1/2b$ as compared to that describing the Nekrasov function itself. However, in the planar limit, the difference disappears.*

As suggested in [15,17], being based on the matrix-model experience [28,46], the free energy can be viewed as a (double-deformed) prepotential with $\rho_1$ playing the role of the generating differential:

$$\begin{aligned}\frac{\partial\log B_4(q)}{\partial a_I}&=b^2\oint_{B_I}\rho_1(x),\\ a_I&=\oint_{A_I}\rho_1(x).\end{aligned} \tag{79}$$

This conjecture still remains to be proved. In Section 4, we consider a *weaker* form of this conjecture, in the limits of small $\epsilon_2$.

## 4. SW theory in the limit $\epsilon_2\to 0$

### 4.1. The limit of the conformal block

Now we are going to consider the limit of $\epsilon_2\to 0$. To do this, we restore the parameters $\epsilon_{1,2}$ of deformation of the Nekrasov functions in the conformal blocks by rescaling the charges $\alpha\to\alpha/g_s$ (i.e., the potential $W(x)\to W(x)/g_s$ (69)) with the string coupling $g_s^2\equiv-\epsilon_1\epsilon_2$. In the limit of small $\epsilon_2$ in the beta-ensemble with $\beta=b^2=-\epsilon_1/\epsilon_2$

$$B_4\sim\int\prod_i\mathrm{d}x_i\exp\left(\frac{1}{g_s}W(x_i)\right)\prod_{i<j}(x_i-x_j)^{2\beta} \tag{80}$$

the multi-resolvents behave as

$$\rho_k\sim g_s^{2k-2} \tag{81}$$

(as an illustration, we list in the Appendix the first few multi-resolvents in the simplest Gaussian case). This result means that when $\epsilon_2\to 0$ and therefore $g_s\to 0$, only the one-point resolvent survives in (76).

As we already noted above using the knowledge from matrix models [28,46], one can calculate the matrix-model partition function using the spectral curve, endowed with a generating differential. We expect the same claim to be correct for the beta-ensembles, and since the partition function is now given by $B_4(q)$, formula (79) should be valid, where, as usual, the integrals are taken over the $A$- and $B$-cycles of the spectral curve determined by the planar limit of the one-point resolvent.

Now taking the limit $\epsilon_2 \to 0$, using (76) and (81) and noting that, in this limit, $\log B_4$ behaves as $1/g_s^2$ [31] one finally obtains

$$B_5(x|q) = \exp\left(-\frac{1}{\epsilon_1\epsilon_2}F(\epsilon_1) + \frac{1}{\epsilon_1}S(x;\epsilon_1) + O(\epsilon_2)\right) \tag{82}$$

where $F(\epsilon_1)$ does not depend on $x$, while it can and does depend on $q$, $\mathrm{d}S(x;\epsilon_1) \equiv \epsilon_1\rho_1$ and

$$\begin{aligned} a &= \oint_A \mathrm{d}S(x;\epsilon_1), \\ \frac{\partial F(\epsilon_1)}{\partial a} &= \oint_B \mathrm{d}S(x;\epsilon_1) \end{aligned} \tag{83}$$

where we have rescaled $a \to a/\epsilon_1$ as compared with formula (79).

### 4.2. The Schrödinger equation for $S(x)$

Now one can obtain the Schrödinger equation for the ratio of the conformal blocks

$$\frac{B_5(x|q)}{B_4(q)} = \exp\left(\frac{S(x;\epsilon_1)}{\epsilon_1}\right) \equiv \psi(x). \tag{84}$$

To accomplish this task, consider the solution to the equation

$$\left(b^2\partial_x^2 + \frac{2x-1}{x(x-1)}\partial_x + \mathcal{O}\right)B_5(x|q) = 0 \tag{85}$$

where the operator

$$\begin{aligned} \mathcal{O} &= -\frac{q(q-1)}{x(x-1)(x-q)}\partial_q + \frac{1}{\epsilon_1\epsilon_2}\mathcal{V}(x|z) = -\frac{q(q-1)}{x(x-1)(x-q)}\partial_q \\ &\quad + \frac{1}{\epsilon_1\epsilon_2}\frac{1}{x(1-x)}\left[\Delta_{1/2b} + \frac{\Delta_1}{x} - \frac{\Delta_2}{x-1} - \Delta_3 + \frac{q^2-(2q-1)x}{(x-q)^2}\Delta_4\right] \end{aligned} \tag{86}$$

acts only on the $q$-variable. Then, in the leading order in $\epsilon_2^{-1}$, one has [19]

$$\left(\frac{\partial S}{\partial x}\right)^2 + \epsilon_1\frac{\partial^2 S}{\partial x^2} = \frac{q(q-1)}{x(x-1)(x-q)}\partial_q F(\epsilon_1) + \mathcal{V}(x|q) \tag{87}$$

or

$$\left(-\epsilon_1^2\partial_x^2 + \mathcal{V}(x)\right)\psi(x) = \frac{(q-1)E}{x(x-1)(x-q)}\psi(x) \tag{88}$$

where $E$ is an $x$-independent quantity

$$E = \frac{\partial F(\epsilon_1)}{\partial \log q}. \tag{89}$$

Note that the limit $\epsilon_2 \to 0$ in (85) is quite unusual: in such a limit, $b = -\epsilon_1/\epsilon_2 \to \infty$, unlike the naive semiclassical limit of the Schrödinger equation (where $b$ would rather go to zero). Instead, in this limit $V_{1/2b} \to V_0 = 1$ and $B_5(x|q) \to B_4(q) \stackrel{\mathrm{AGT}}{=} \exp\left(-\frac{\mathcal{F}(\epsilon_1,\epsilon_2)}{\epsilon_1\epsilon_2}\right)$. Only after one picks up the $\epsilon_2^{-2}$ terms in the equation, they combine into a Schrödinger-like equation with $\epsilon_1$, playing the role of the Planck constant, and the semiclassical expansion in small $\epsilon_1$ can be considered, along the lines of [11].

### 4.3. Examples of different gauge theories

Thus, we have established that the monodromies of the wave function of the Schödinger equation (88) with the Plank constant $\hbar \equiv \epsilon_1$, i.e., $\oint \mathrm{d}S = \oint \mathrm{d}\log\psi(x)/\epsilon_1$ are described (83) by the YY function $F(\epsilon_1)$ (i.e., by the Nekrasov function $\mathcal{F}(\epsilon_1,\epsilon_2)$ at $\epsilon_2 \to 0$). In particular, the quantization condition of this Schrödinger equation implies that the $B$-period, which is nothing but the Bohr/Sommerfeld integral equals $2\pi\hbar(n+1/2)$.

This construction was first discussed for the periodic Toda case (= pure gauge theory) in [11]. In the $SU(2)$ case, one can easily reproduce the corresponding construction from Section 2.3.4, the potential in the Schrödinger equation from (49) is just $V(x) = \Lambda^2 \cosh x$, therefore (84) satisfies

$$\left(-\epsilon_1^2\partial_x^2 + \Lambda^2 \cosh x\right)\psi(x) = E\psi(x). \tag{90}$$

More examples were considered in [15] and in [19] (further details will appear in [47]). In particular, the case of the gauge theory with adjoint matter hypermultiplet with mass $m$, is described by the Calogero model. In the $SU(2)$ case, it is obtained from Eq. (36) and leads to the Schrödinger equation with an elliptic potential

$$\left(-\epsilon_1^2\partial_x^2 + m(m-\epsilon_1)\wp(x)\right)\psi(x) = E\psi(x). \tag{91}$$

Of course, Eq. (83) becomes really very restrictive in the $SL(N)$ case with $N > 2$, when there are many $A_I$- and $B_I$-cycles and many periods $a_I$. However, this case is related to conformal blocks of $W_N$ algebras [2,4]. An analysis of surface operators in these models can also be easily performed, but this is beyond the scope of the present paper (see recent papers [22,23] devoted to this case).

Still, one has to expect that the entire construction of this section is directly generalized. Indeed, it was proposed and partly checked in [11,15], that, in the $SU(N)$ case, the role of the Schrödinger equation is played by the Fourier transform of the Baxter equation for the corresponding integrable system. For instance, the pure gauge $SU(N)$ theory is described by the periodic Toda chain on $N$ sites, and the corresponding Baxter equation is given by

$$P_N(\lambda)Q(\lambda) = Q(\lambda + i\hbar) + Q(\lambda - i\hbar) \tag{92}$$

where $P_N(\lambda)$ is a polynomial of degree $N$ with the coefficients being the conserved quantities, and $Q(\lambda)$ is the Baxter $Q$-operator. Thus, the corresponding Schrödinger equation is of the form

$$\left[P_N\left(i\hbar\frac{\partial}{\partial x}\right) + \cosh x\right]\psi(x) = 0. \tag{93}$$

At $N = 2$, $P_2(\lambda) = (\lambda^2 - E)/\Lambda^2$, and one obtains (90).

The Calogero case is more involved; however, there is also the equation in the separated variables in this case, which can be considered as the substitute of (88), (see, e.g., for $N = 3$ [15, Eq. (55)]).

However, the most intriguing is the case of the theory with fundamental matter hypermultiplets with masses $m_a$. As expected from SW theory, this case is described by the (non-compact) $sl(2)$ (XXX) chain [34]. The Baxter equation in this case is [15]

$$P_N(\lambda)Q(\lambda) = K_+(\lambda)Q(\lambda + i\hbar) + K_-(\lambda)Q(\lambda - i\hbar) \tag{94}$$

where $K_\pm(\lambda) = \prod_a^{N_\pm}(\lambda - m_a)$ and $N_+ + N_- = N_f$ is the number of matter hypermultiplets (the answer does not depend on how one parts these hypermultiplets into two sets $N_+$ and $N_-$). Note that in the case of $N = 2$, one does not come to the Schrödinger equation (86) of the previous subsection. However, the checks of the first terms (in particular, those done in [19]) shows that both the equations lead to the same result! This finding means that the Gaudin magnet, which, corresponding to (86), gives rise to the same results as the XXX chain, at least, in the case of $N = 2$. This point definitely deserves further investigation.

### 4.4. Perturbative limit of gauge theories

This construction, obtained indirectly from the beta-ensemble representation of the conformal block, can be also tested immediately for the first terms in $\hbar$ and $\Lambda$. It has been done for various cases in [11,19]. Note, however, that, for the perturbative contribution, i.e., in the leading order in $\Lambda$ it can be checked exactly in $\hbar$. Indeed, let us first look at Eq. (90): its perturbative limit is described by the Liouville equation [34] (one has to first shift $x \to x - 2\log\Lambda$ and then consider small $\Lambda$ in Eq. (90), then only one of the exponents remains):

$$\left(-\epsilon_1^2\partial_x^2 + \Lambda^2 \exp(-x)\right)\psi(x) = E\psi(x). \tag{95}$$

In this limit, the $A$- and $B$-cycles are degenerate. Note that the cycles in (83) and the corresponding curve are determined completely by the semiclassical limit of the Schrödinger equation, i.e., by the corresponding Seiberg–Witten curve, which becomes rational in this limit. In particular, the $A$-cycle degenerates into a pair of marked points on the curve, while the $B$-cycle extends from the turning point $x_c$, $E = \Lambda^2\exp(-x_c)$ to infinity (encircling them), and the corresponding monodromy of the wave function is determined by logarithm of the ratio of asymptotics at infinity:

$$\begin{gathered}\psi(x) \underset{x\to\infty}{\longrightarrow} -\frac{\pi}{\sin\frac{2\pi\lambda}{\epsilon_1}}\left[\frac{1}{\Gamma\left(1+\frac{2\lambda}{\epsilon_1}\right)}\left(\frac{\Lambda}{\epsilon_1}\right)^{2\lambda/\epsilon_1} e^{x\lambda/\epsilon_1} - \frac{1}{\Gamma\left(1-\frac{2\lambda}{\epsilon_1}\right)}\left(\frac{\Lambda}{\epsilon_1}\right)^{-2\lambda/\epsilon_1} e^{-x\lambda/\epsilon_1}\right] \\ \frac{1}{\epsilon_1}\oint_B dS = \oint_B \frac{\partial\log\psi}{\partial x} = \log\frac{c_+(\lambda)}{c_-(\lambda)}.\end{gathered} \tag{96}$$

Here, $\lambda \equiv \sqrt{-E} = a$ is purely imaginary and

$$c_{\pm}(\lambda) = \pm \frac{1}{\Gamma\left(1 \pm \frac{2\lambda}{\epsilon_1}\right)} \left(\frac{\Lambda}{\epsilon_1}\right)^{\pm 2\lambda/\epsilon_1}. \tag{97}$$

Formula (96) coincides with the perturbative expression for the derivative of the YY function w.r.t. $a$, and the quantization condition imposed on the Bohr–Sommerfeld integral

$$\log \frac{c_+(\lambda)}{c_-(\lambda)} = 2\pi i n, \quad n \in \mathbb{Z} \tag{98}$$

coincides with [10, Eq. (6.6)].

In fact, the functions $c_{\pm}(\lambda)$ are proportional to the Harish–Chandra functions, which determine the Plancherel measure on the set of irreducible unitary representations contributing to the Whittaker model. Moreover, the $S$-matrix in the integrable system is determined by the Harish–Chandra functions, (see further details and references in [48]).

Thus, it is clear, that our consideration can be easily generalized to a generic situation, and the perturbative result is still determined by the logarithm of the ratio of two asymptotics, i.e., by the ratio of two Harish–Chandra functions. Indeed, for instance, the perturbative limit of the $SL(N)$-Toda case is described by the conformal (non-periodic) Toda system [34], and the corresponding Harish–Chandra functions are [48]

$$c_w(\vec{\lambda}) \sim \prod_{\vec{\alpha} \in \Delta_+} \frac{1}{\Gamma\left(1 - \frac{w(\vec{\lambda})\cdot\vec{\alpha}}{\epsilon_1}\right)} \tag{99}$$

with $w$ being an element of the Weyl group, and $\Delta_+$ here is the set of all positive roots. Choosing the basis (for the $sl(N)$ algebra) $\vec{\lambda} \cdot \vec{\alpha} = a_i - a_j$ for all $i, j = 1, \ldots, N,\ i < j$, one easily gets the proper ratios of the Harish–Chandra functions:

$$\frac{c_{i,+}(\vec{\lambda})}{c_{i,-}(\vec{\lambda})} = -\left(\frac{\Lambda}{\epsilon_1}\right)^{\frac{2Na_i}{\epsilon_1}} \prod_{j \neq i} \frac{\Gamma\left(1 - \frac{a_i - a_j}{\epsilon_1}\right)}{\Gamma\left(1 + \frac{a_i - a_j}{\epsilon_1}\right)} \tag{100}$$

for all $i = 1, \ldots, N$. This result immediately leads to the quantization conditions, coinciding with those of [10, Eq. (6.6)].

Finally, consider the case of gauge theory with adjoint matter, described by the Calogero model (and restricted here only for the $N = 2$ case, i.e., Eq. (91)). The perturbative limit is given by the trigonometric Calogero–Moser–Sutherland model [34], i.e., by equation

$$\left(-\epsilon_1^2 \partial_x^2 + \frac{m(m-\epsilon_1)}{\sinh^2 x}\right) \psi(x) = E\psi(x). \tag{101}$$

The solution to this equation has asymptotics ($\lambda^2 = -E$, i.e., $\lambda$ is again pure imaginary)

$$\psi(x) \overset{x\to\infty}{\sim} \frac{\pi}{\sin \pi \left(\frac{\lambda}{\epsilon_1} + \frac{m}{\epsilon_1}\right)} \frac{e^{-x\lambda/\epsilon_1}}{\Gamma\left(-\frac{\lambda}{\epsilon_1}\right)\Gamma\left(\frac{m}{\epsilon_1} + \frac{\lambda}{\epsilon_1}\right)} + \frac{\pi}{\sin \pi \frac{m}{\epsilon_1}} \frac{e^{x\lambda/\epsilon_1}}{\Gamma\left(\frac{\lambda}{\epsilon_1}\right)\Gamma\left(\frac{m}{\epsilon_1} - \frac{\lambda}{\epsilon_1}\right)} \tag{102}$$

i.e., the Harish–Chandra functions are

$$c_{\pm}(\lambda) \sim \frac{1}{\Gamma\left(\pm\frac{\lambda}{\epsilon_1}\right)\Gamma\left(\frac{m}{\epsilon_1} \mp \frac{\lambda}{\epsilon_1}\right)}. \tag{103}$$

The logarithm of their ratio again equals to the derivative of the perturbative part of the YY function, and the corresponding quantization condition coincides with [10, Eq. (6.9)].

## 5. Conclusion

In this paper, we collected some knowledge about the degenerate conformal blocks and their possible application to the study of AGT relations. The main application so far is that the insertion of the degenerate primary and appropriate restriction of the additional intermediate dimension converts the conformal block into a "wave function", which, in the limit $\epsilon_2 \to 0$, provides the Seiberg–Witten representation for the one-parameter deformed prepotential $F(\epsilon_1) = \mathcal{F}|_{\epsilon_2=0}$, also playing the role of the YY function. The differential equation for the degenerate conformal block turns into a Schrödinger-like equation, which can be also related to the Baxter quantization of the spectral curve, arising in the SW representation of the original prepotential $F_{SW} = \mathcal{F}|_{\epsilon_{1,2}=0}$.

Thus, despite this the "wave function" itself (i.e., degenerate conformal block) is perfectly well defined for the double-epsilon deformation, when both $\epsilon_1$ and $\epsilon_2$ are non-vanishing, its interpretation is found only in the limit of $\epsilon_2 \to 0$.

Let us remind here that the Nekrasov function $\mathcal{F} = \mathcal{F}(\epsilon_1, \epsilon_2)$ is a double deformation of the original Seiberg–Witten prepotential $F_{SW}$ in two directions,: introducing non-vanishing string coupling $g_s = \sqrt{-\epsilon_1\epsilon_2} \neq 0$ [49] and the non-vanishing screening charge $b = \sqrt{-\epsilon_1/\epsilon_2} \neq 1$. In the matrix-model approach to AGT relations [50]: the string coupling governs the topological recursion [40], while the second deformation turns the matrix model into the beta-ensemble with $\beta = b^2$. The understanding is, unfortunately, much worse if the double deformation is taken symmetrically for both non-vanishing $\epsilon_{1,2} \neq 0$, which is more natural in the original definition of Nekrasov functions [30].

These two deformations are also different from the point of view of an integrable systems [33,51,34]: the first, corresponding to $\epsilon_1 \neq 0$, seems to be equivalent to a standard quantum-mechanical quantization of a classical integrable system, while the second, associated with $\epsilon_2 \neq 0$ is rather like switching on the flows of the quasiclassical hierarchy [52] (in an already quantized problem!). In particular, turning on the non-vanishing $\epsilon_2$ adds the non-stationary term $\partial/\partial \log \Lambda$ to the stationary Schrödinger equation (49), and this result is exactly the deformation of the original integrable system in the direction of the time-variable $\log \Lambda$ [51,53]. It would be extremely important to extend the role of degenerate conformal blocks (or surface operator insertions on the other side of the AGT relation) to better understand the second deformation for $\epsilon_2 \neq 0$.

Also interesting, though much more straightforward, is the extension of above discussion from the four-point Virasoro conformal blocks on sphere (and one-point conformal block on torus) to generic situation with an arbitrary number of punctures, arbitrary genus and, moreover, for arbitrary chiral algebras. In all these cases, the surface operator insertions provide an exhaustive description of the $\epsilon_1 \neq 0$ deformation. All of them are well defined also for $\epsilon_2 \neq 0$, but their role in the description of the doubly deformed prepotentials $\mathcal{F}(\epsilon_1, \epsilon_2)$ still remains to be revealed. In all these cases, the discrete $\epsilon_1 \leftrightarrow \epsilon_2$ symmetry is explicitly broken by the choice of the particular degenerate primary, used for the insertions.

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## Appendix

For illustrative purposes, we list here the first few multi-resolvents in the Gaussian $\beta$-ensemble [54]:

$$
\begin{aligned}
\rho_1(x) &= N\left(\frac{1}{x} + g_s^2\frac{1+(N-1)\beta}{x^3} + g_s^4\frac{3-5\beta+3\beta^2+5\beta N-5\beta^2N+2N^2\beta^2}{x^5} + \cdots\right)\\
&\xrightarrow{\epsilon_2\to\infty} N\left(\frac{1}{x} + \frac{(N-1)}{x^3}\epsilon_1^2 + \frac{2N^2-5N+3}{x^5}\epsilon_1^4 + \frac{5N^3-22N^2+32N-15}{x^7}\epsilon_1^6 + \cdots\right)\\
\rho_2(x,y) &= Ng_s^2\left(\frac{1}{x^2y^2} + g_s^2\frac{3-3\beta+3N\beta}{x^4y^2} + g_s^2\frac{3-3\beta+3N\beta}{x^2y^4} + 2g_s^2\frac{1-\beta+\beta N}{x^3y^3} + \cdots\right)\\
&\xrightarrow{\epsilon_2\to\infty} Ng_s^2\left(\frac{1}{x^2y^2} + \epsilon_1^2(N-1)\left\{\frac{3}{x^2y^4} + \frac{2}{x^3y^3} + \frac{3}{x^4y^2}\right\} + \epsilon_1^4\left\{(15-25N+10N^2)\left(\frac{1}{x^6y^2} + \frac{1}{x^2y^6}\right)\right.\right.\\
&\quad \left.\left. + \frac{15-27N+12N^2}{x^4y^4} + \frac{12-20N+8N^2}{x^5y^3} + \frac{12-20N+8N^2}{x^3y^5}\right\}\cdots\right)\\
\rho_3(x,y,z) &= \frac{2Ng_s^4}{x^2y^2z^2}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 2g_s^2(1-\beta+\beta N)\left\{\frac{3}{x^3} + \frac{3}{y^3} + \frac{3}{z^3} + \frac{2}{xyz}\right.\right.\\
&\quad \left.\left. + \frac{3}{x^2y} + \frac{3}{xy^2} + \frac{3}{x^2z} + \frac{3}{xz^2} + \frac{3}{y^2z} + \frac{3}{yz^2}\right\} + \cdots\right)\\
\rho_4(x,y,z,w) &= \frac{2Ng_s^6}{x^2y^2z^2w^2}\left(\frac{3}{x^2} + \frac{3}{y^2} + \frac{3}{z^2} + \frac{3}{w^2} + \frac{4}{xy} + \frac{4}{xz} + \frac{4}{xw} + \frac{4}{yz} + \frac{4}{yw} + \frac{4}{zw} + \cdots\right).
\end{aligned}
\tag{104}
$$

In the paper, we use the resolvents of the DF $\beta$-ensemble and do not require the Gaussian model resolvents themselves. However, the Gaussian resolvents, which are much simpler, nevertheless show an important property that is true for any potential: the resolvents $\rho_k$ behaves as $g_s^{2k-2}$ as $\epsilon_2 \to 0$ (still being functions of $\epsilon_1$). This fact implies that the naturally normalized quantities would be $\beta^k\rho_k$ rather than $\rho_k$. Exactly these quantities enter the topological recursion (loop equations). Therefore, these $\beta^k\rho_k$ are the quantities that remain finite in the limit of $\beta \to \infty$, while $\rho_k/\rho_1 \to \beta^{1-k} \to 0$ for $k \geq 2$, which was used in (76).

Note also that, after $\epsilon_2$ is set equal to zero, there still exists a genus counting; however, the t'Hooft limit now corresponds to the double scaling limit of $N \to \infty$ and $N\epsilon_1^2 =$ fixed.

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