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Construction of global-in-time solutions to Kolmogorov-Feller pseudodifferential equations with a small parameter using characteristics

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Using an idea going back to Madelung, we construct global in time solutions to the transport equation corresponding to the asymptotic solution of the Kolmogorov-Feller equation describing a system with diffusion, potential and jump terms. To do that we use the construction of a generalized delta-shock solution of the continuity equation for a discontinuous velocity field. We also discuss corresponding problem of asymptotic solution construction (Maslov tunnel asymptotics).

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1 Introduction

The goal of the present paper is to describe a new approach for constructing global in time solutions of parabolic PDE with a small parameter. It is well-known that the construction of asymptotic solutions of linear equations with a small parameter is based on the WKB method [18].

For parabolic equations, the global-in-time asymptotic solutions were first constructed by Maslov [15], [17]. His construction, called the tunnel canonical operator, is a modification of the canonical operator construction also developed by him [15]. The latter resembles the construction of Fourier integral operators [12] and is based on the use of integral representations, the only distinction is that the tunnel canonical operator is based on the use of the integral transformation with heat kernel in contrast to the Fourier transformation. The asymptotic solutions constructed in the framework of the theory of tunnel canonical operator have been justified by the author.

In the present paper, we propose another construction based on the use of characteristics but not of the integral transformations. Thus, our construction is pointwise in contrast to the tunnel canonical operator.

The idea of this construction goes back to Madelung [14] who noted that the system of equations arising in the WKB method (Hamilton-Jacobi equation + transport equation) can be reduced to the form of a system of gas dynamics equations (in the one-dimensional case). Here the key point is the fact that the transport equation that does not have a divergence form is transformed into the continuity equation.

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A technique for constructing the solution of the continuity equation in the discontinuous velocity field has been developed in the last few years. And this permits constructing global asymptotic solutions for parabolic PDE with a small parameter.

Moreover, the pointwise construction presented in the paper can be used to solve the Cauchy problems for parabolic PDE backward in time. This is possible, because the Hamilton flow is invertible in time for stationary Hamiltonians.

In the present paper, we consider a new approach to the construction of singular (i.e., containing the Dirac δ -function as a summand) solutions to the continuity equation and to show how these solutions can be used to construct the global in time solution of the Cauchy problem for Kolmogorov-Feller-type equations with diffusion, potential and jump terms. The relation between the solutions of the continuity equation and the system consisting of the Hamilton-Jacobi equation plus the transport equation has been well studied before in the case of a smooth action functional.

The velocity field u is determined as the family of velocities of points on the projections of the trajectories of the Hamiltonian system corresponding to the Hamilton-Jacobi equation. As it was mentioned by E. Madelung [14], in this velocity field, the squared solution of the transport equation (denoted by ρ) satisfies the continuity equation

$$\rho_t + (\nabla, u\rho) + a\rho = 0 \quad (1.1)$$

with some additional term $a\rho$, which is defined below (a is equal to 0 if the Hamiltonian is formally self-adjoint). The main obstacle to the extension of this correspondence globally in time is the fact that in general the solution of the Hamilton-Jacobi equation are smooth only locally in time. The loss of smoothness is equivalent to the appearance of singularities of the velocity field mentioned above. Till recent times there was no method for constructing formulas for solutions of the continuity equation for a discontinuous velocity field. It is clear that the continuity equation has the divergent form and this very important property allows precisely to introduce the concept of a global solution in spite of singularities in the velocity field. The divergent form as itself does not play any role in Madelung's approach but it is very important for our construction because we deal with singular solutions.

In the present paper we generalize Madelung's approach to the case in which the singular support of the velocity field is a stratified manifold transversal to the velocity field trajectories. This holds, for example, in the case where the space is one-dimensional under the condition that, for any $t \in [0, T]$, the velocity field singular support is a discrete set without limit points.

In the present paper we briefly present the contents of [4] and the theory of Maslov's tunnel canonical operator. Although these constructions have been known for a rather long time, they still remain insufficiently known and we are compelled to present them for the reader convenience.

It is interesting to note that the generalization of Madelung's idea, first suggested for the Schrödinger equation, turned to be possible for equations of another class.

Namely, let Λ^n be the Lagrangian manifold corresponding to the solution (see [15], [17], [18]) and let $\pi: \Lambda^n \rightarrow \mathbb{R}_x^n$ be the projection mapping.

All the points of $\pi^{-1}(x) \in \Lambda^n$ make contributions to the construction of an asymptotic oscillating solution (of the type of the WKB-solution $\varphi(x, t) \exp(iS(x, t)/h)$).

But if we consider nonoscillating solutions of the form

$$\varphi(x, t) \exp(-S(x, t)/h),$$

just as for equations of heat conduction type with a small parameter h at the second-order derivative, see Section 3 and [16], [17], then only the point from the set $\bar{\gamma} \in \pi^{-1}(x)$ at which

$$S(\bar{\gamma}, t) = \min_{\bar{\gamma} \in \pi^{-1}(x)} S(\gamma, t)$$

makes a contribution to the solution. Hence the points of the Lagrangian manifold making contributions to the construction of a nonoscillating solution form a surface with (shock wave-type) jumps, which results in discontinuities of ∇S and hence in discontinuities of the velocity field for the transport equation.

The last equation does not have a divergent form in contrast to the continuity equation, which arises in the Madelung construction. So it turned out that Madelung's idea was initially adopted to the global constructions involving generalized solutions to the continuity equation.

In Section 2, we describe the definition and construction of these types of solutions. The conclusion is that these solutions can be constructed by means of characteristics. This allows us partly to change the direction of time and to solve some inverse problems. We will discuss this problem in detail in subsequent papers.

2 Generalized solutions of the continuity equation

Here we follow the approach developed in [4], where the solution of the continuity equation is understood in the sense of an integral identity, which, in turn, follows from the fact that relation (1.1) can be understood in the sense of the distributional space $\mathcal{D}'(\mathbb{R}_{x,t}^{n+1})$. The first step in this way has been done in [9], see also [3], [6], [7] where the approach based on smooth approximations of the solutions was used.

We specially note that the integral identities in [4] can be derived without using the construction of nonconservative products [19], of the nonsmooth and generalized functions (or measure solutions [2], [11], [20]), and the value of the velocity on the discontinuity lines (surfaces) is not given a priori but is calculated. In the case considered in [1], the integral identities exactly coincide in form with the identities derived using the construction of a nonconservative product (measure solutions) in the situation described at the end of above introduction, which we shall now make more precise.

First, we consider an $n - 1$ -dimensional surface γ_t moving in \mathbb{R}_x^n , which is determined by the equation

$$\gamma_t = \{x; t = \psi(x)\},$$

where $\psi \in C^1(\mathbb{R}^n)$, and $\nabla\psi \neq 0$ in the domain in \mathbb{R}_x^n where we work.

This is equivalent to determining a surface by an equation of the form

$$S(x, t) = 0$$

($S \in C^1$ in both variables, $S(x, t) = 0$, $\nabla_{x,t} S|_{S=0} \neq 0$) under the condition that

$$\frac{\partial S}{\partial t} \neq 0.$$

We remain that the situation with $\frac{\partial S}{\partial t} = 0$, can also be covered by making the change of variables $x'_i = x_i - c_i t$ with appropriately chosen c_i , $i = 1, \dots, n$, solving the problem with the moving surface and then returning to the original variables. Possible generalizations are considered later in this section.

Next, we assume that ζ belongs to $C_0^\infty(\mathbb{R}^n \times \mathbb{R}_+^1)$. Then, by definition,

$$\langle \delta(t - \psi(x)), \zeta(x, t) \rangle = \int_{\mathbb{R}^n} \zeta(x, \psi(x)) dx,$$

where δ is the Dirac delta function and $\langle \cdot, \cdot \rangle$ is the distributional pairing (with respect to the variable $t \in \mathbb{R}_+^1$ and $x \in \mathbb{R}^n$).

Let $\delta(t - \psi(x))$ be applied to the test function $\eta \in C_0^\infty(\mathbb{R}^n)$, then

$$\langle \delta(t - \psi(x)), \eta(x) \rangle = \int_{\gamma_t} \eta d\omega_\psi,$$

where $d\omega$ is the Leray form [10] on the surface $\{t = \psi(x)\}$ such that $-d\psi d\omega_\psi = dx_1 \dots dx_n$.

One can show that (see [5], [10])

$$\langle \delta(t - \psi(x)), \eta(x) \rangle = \int_{\gamma_t} \frac{\eta(x)}{|\nabla\psi|} d\sigma.$$

First, we assume that the solution ρ to Equation (1.1) has the form

$$\rho(x, t) = R(x, t) + e(x)\delta(t - \psi(x)), \tag{2.1}$$

where $R(x, t)$ is a piecewise smooth function with possible discontinuity at $\{t = \psi(x)\}$:

$$R = R_0(x, t) + H(t - \psi(x))R_1(x, t),$$

$e \in C(\mathbb{R}^n)$ and has a compact support, $\psi \in C^2$ and $\nabla\psi \neq 0$ for $x \in \text{supp } e$, and $H(z)$ is the Heaviside function.

It is clear that the term

$$e(x)\delta'(t - \psi(x))$$

appears in (1.1) if we differentiate the distribution $\delta(t - \psi(x))$ with respect to t . Hence it is necessary to have in (1.1)

$$(\nabla, \rho u) = -e(x)\delta'(t - \psi) + \text{smoother summands},$$

since $\nabla\delta(t - \psi) = -\nabla\psi\delta'(t - \psi)$. Then we must have

$$\rho u = \frac{e\nabla\psi}{|\nabla\psi|^2}\delta(t - \psi) + \text{smoother summands}.$$

Now we formulate an integral identity, defining a generalized solution to the continuity equation.

We set $\Gamma = \{(x, t); t = \psi(x)\}$; this is an n -dimensional surface in $\mathbb{R}^n \times \mathbb{R}_+^1$. Let

$$u(x, t) = u_0(x, t) + H(t - \psi)u_1(x, t),$$

where ψ is the same function as before, and $u_0, u_1 \in C(\mathbb{R}^n \times \mathbb{R}_+^1)$.

Let us consider Equation (1.1) in the sense of distributions. For all $\zeta(x, t) \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}_+^1)$, $\zeta(x, 0) = 0$, we have

$$\left\langle \frac{\partial \rho}{\partial t} + (\nabla, \rho u), \zeta \right\rangle = -\langle \rho, \zeta_t \rangle - \langle \rho u, \nabla \zeta \rangle.$$

Substituting the singular terms for ρ and ρu calculated above, we come to the following definition.

Definition 2.1 A function $\rho(x, t)$ determined by relation (2.1) is called a *generalized δ -shock wave type solution* to (1.1) on the surface $\{t = \psi(x)\}$ if the integral identity

$$\int_0^\infty \int_{\mathbb{R}^n} (R\zeta_t + (uR, \nabla\zeta) + aR\zeta) dx dt + \int_\Gamma \frac{e}{|\nabla\psi|} \frac{d}{dn_\perp} \zeta(x, t) dx = 0 \tag{2.2}$$

holds for all test functions $\zeta(x, t) \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}_+^1)$, $\zeta(x, 0) = 0$, $\frac{d}{dn_\perp} = \left(\frac{\nabla\psi}{|\nabla\psi|}, \nabla\right) + |\nabla\psi| \frac{\partial}{\partial t}$.

We have also the relation

$$\int_{\mathbb{R}^n} \frac{e}{|\nabla\psi|} \frac{d}{dn} \zeta(x, \psi) dx = \int_\Gamma \frac{e}{|\nabla\psi|} \frac{d}{dn_\perp} \zeta(x, t) dx.$$

We note that the vector n_\perp is orthogonal to the vector $(\nabla\psi, -1)$, which is the normal to the surface Γ , i.e., $\frac{d}{dn_\perp}$ lies in the plane tangent to Γ .

We can give a geometric definition of the field $\frac{d}{dn_\perp}$. The trajectories of this vector field are curves lying on the surface Γ , and they are orthogonal to all sections of this surface produced by the planes $t = \text{const}$. Furthermore, it is clear that the expression $\frac{1}{|\nabla\psi|}$ is an absolute value of the normal velocity of a point on γ_t , i.e., on the cross-section of Γ by the plane $t = \text{const}$, and the expression $\frac{1}{|\nabla\psi|} \cdot \frac{\nabla\psi}{|\nabla\psi|} \stackrel{\text{def}}{=} \vec{V}_n$ is the vector of normal velocity of a point on γ_t . Thus, we have another representation:

$$\int_\Gamma \frac{e}{|\nabla\psi|} \frac{d}{dn_\perp} \zeta(x, t) dx = \int_\Gamma e \left((\vec{V}_n, \nabla) + \frac{\partial}{\partial t} \right) \zeta(x, t) dx,$$

where $V_n = \pi^*(v_n)$, v_n is the normal velocity of a point on γ_t , and π^* is induced by the projection mapping $\pi : \Gamma \rightarrow \mathbb{R}_x^n$.

It follows from the latter definition that the following relations must hold:

$$\begin{aligned} R_t + (\nabla, Ru) + aR\zeta &= 0, \quad \text{for all points } (x, t) \notin \Gamma, \\ ([R] - |\nabla\psi|[Ru_n]) + \left(\frac{d}{dn}\right)^* \frac{e}{|\nabla\psi|} &= 0, \quad \text{for all points } (x, t) \in \Gamma. \end{aligned} \tag{2.3}$$

The last relation can be rewritten in the form

$$\mathcal{K}E + \frac{d}{dn}E = [Ru_n]|\nabla\psi| - [R], \tag{2.4}$$

where $E = e/|\nabla\psi|$, the factor $\mathcal{K} = \left(\nabla, \frac{\nabla\psi}{|\nabla\psi|}\right) = \text{div } \nu$ (ν is the normal on the surface $\{t = \psi(x)\}$) and, as is known, is the mean curvature of the cross-section of the surface Γ by the plane $t = \text{const}$, $\frac{d}{dn} = \left(\frac{\nabla\psi}{|\nabla\psi|}, \nabla\right)$.

Now we assume that there are two surfaces

$$\Gamma_i = \{(x, t); t = \psi_i(x)\}$$

in $\mathbb{R}^n \times \mathbb{R}_+^1$, $i = 1, 2$, whose intersection is a smooth surface

$$\hat{\gamma} = \{(x, t); (t = \psi_1) \cap (t = \psi_2)\}$$

belonging to the third surface $\Gamma_{(3)} = \{(x, t); t = \psi_3(x)\}$. Further, we assume that the surface $\Gamma_{(3)}$ is a continuation of the surfaces $\Gamma^{(i)}$ in the following sense. We let $n_\perp^{(i)}$ denote curves on the surfaces Γ_i and we assume that each point (\hat{x}, \hat{t}) on the surface $\hat{\gamma}$ is associated with a graph consisting of the trajectories $n_\perp^{(1)}$ and $n_\perp^{(2)}$ entering (\hat{x}, \hat{t}) and the trajectory $n_\perp^{(3)}$ leaving this point (i.e., the trajectories $n_\perp^{(i)}$ fiber the surface $\Gamma^{(i)}$). We also assume that the surface (stratified manifold) $\Gamma_\cup = \Gamma_{(1)} \cup \Gamma_{(2)} \cup \Gamma_{(3)}$ consists of points belonging to these graphs. Next, we assume that $u(x, t)$ is a piecewise smooth vector field whose trajectories enter Γ_\cup .

Definition 2.2 Let

$$u(x, t) = u_0(x, t) + \sum_{i=1}^3 H(t - \psi_i)u_{1i}(x, t),$$

where ψ is the same function as before, and $u_0, u_{1i} \in C(\mathbb{R}^n \times \mathbb{R}_+^1)$. The function $\rho(x, t)$ determined by the relation

$$\rho(x, t) = R(x, t) + \sum_{i=1}^3 e_i(x)\delta(t - \psi_i(x)),$$

where $R(x, t) \in C^1(\mathbb{R}^n \times \mathbb{R}_+^1) \setminus \{\cup \Gamma_t^{(i)}\}$, is called a generalized δ -shock wave type solution to (2.2) corresponding to the stratified manifold Γ_\cup if the integral identity

$$\int_0^\infty \int_{\mathbb{R}^n} (R\zeta_t + (uR, \nabla\zeta) + aR\zeta) dx dt + \sum_{i=1}^3 \int_{\Gamma^{(i)}} \frac{e_i}{|\nabla\psi_i|} \frac{d}{dn_\perp^{(i)}} \zeta(x, t) dx = 0 \tag{2.5}$$

holds for all test functions $\zeta(x, t) \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}_+^1)$, $\zeta(x, 0) = 0$, $\frac{d}{dn_\perp^{(i)}} = \left(\frac{\nabla\psi_i}{|\nabla\psi_i|}, \nabla\right) + |\nabla\psi_i| \frac{\partial}{\partial t}$.

As above this relation implies the first equation from (2.3) outside Γ_\cup , equations of the type of the second equation in (2.3) on strata $\Gamma^{(i)}$ and the Kirchhoff type relation on $\hat{\gamma}$:

$$(e_1 + e_2)|_{\hat{\gamma}} = e_3|_{\hat{\gamma}}. \tag{2.6}$$

Now we consider the case with $\text{codim } \Gamma > 1$. First, we note that the second integral in (2.2) can be written as

$$\int_{\Gamma} \frac{e}{|\nabla\psi|} \frac{d}{dn_{\perp}} \zeta(x, t) dx = \int_{\Gamma} e \left(\left(\frac{\nabla\psi}{|\nabla\psi|^2}, \nabla \right) + \frac{\partial}{\partial t} \right) \zeta(x, t) dx.$$

We note that if the surface Γ is determined by the equation $S(x, t) = 0$ rather than by the simpler equation $\{t = \psi(x)\}$ presented at the beginning of this section, then

$$\vec{V}_n = -\frac{S_t}{|\nabla S|} \cdot \frac{\nabla S}{|\nabla S|} = -\frac{S_t}{|\nabla S|^2} \nabla S$$

and, of course, the new vector field $\frac{d}{dn_{\perp}} = (\vec{V}_n, \nabla) + \frac{\partial}{\partial t}$ remains tangent to Γ .

Therefore, in this more general case, using this new vector \vec{V}_n , we can again rewrite the integral identity from Definition 2.1 as

$$\int_0^{\infty} \int_{\mathbb{R}^n} (R\zeta_t + (uR, \nabla\zeta) + aR\zeta) dx dt + \int_{\Gamma} e \left((\vec{V}_n, \nabla) + \frac{\partial}{\partial t} \right) \zeta(x, t) dx = 0. \tag{2.7}$$

This form of integral identity can easily be generalized to the case in which Γ is a smooth surface in \mathbb{R}^{n+1} of codimension > 1 .

In this case, instead of \vec{V}_n , we can use a vector \vec{v} that is transversal to Γ and such that the field $(\vec{v}, \nabla) + \frac{\partial}{\partial t}$ is tangent to Γ . We note that the vector \vec{v} is uniquely determined by this condition, which can be treated as “the calculation of the velocity value on the discontinuity” from the viewpoint of [11] and [20].

Moreover, in this case, the expression for ρ does not contain the Heaviside function, and it is assumed that the trajectories of the field u are smooth, nonsingular outside Γ , and transversal to Γ at each point of Γ . In this case, the function ρ has the form

$$\rho(x, t) = R(x, t) + e(x)\delta(\Gamma),$$

where $R \in C^1(\mathbb{R}^{n+1} \setminus \Gamma)$, $e \in C^1(\Gamma)$, and the function $\delta(\Gamma)$ is determined by

$$\langle \delta(\Gamma), \zeta(x, t) \rangle = \int_{\Gamma} \zeta \omega,$$

where ω is the Leray form on Γ . If $\Gamma = \{S_1(x, t) = 0 \cap \dots \cap S_k(x, t) = 0\}$, $k \in [1, n]$, then ω is determined by the relation, see [10], p. 274,

$$dt dx_1 \dots dx_n = dS_1 \dots dS_k \omega.$$

In this case, we assume that the functions S_k are sufficiently smooth (for example, $C^2(\mathbb{R}^n \times \mathbb{R}_+^1)$) and their differentials on Γ are linearly independent.

Moreover, we can assume that the inequality

$$J = \frac{\mathcal{D}(S_1, \dots, S_n)}{\mathcal{D}(t, x_1, \dots, x_{n-1})} \neq 0$$

holds. This inequality is an analog of $S_t \neq 0$ at the beginning of this section and allows us to write ω in the form

$$\omega = J^{-1} dx_k \dots dx_n.$$

The integral identity, an analog of (2.7), has the form

$$\int_0^{\infty} \int_{\mathbb{R}^n} (R\zeta_t + (uR, \nabla\zeta) + aR\zeta) dx dt + \int_{\Gamma} e \left((v, \nabla) + \frac{\partial}{\partial t} \right) \zeta(x, t) \omega = 0.$$

Integrating the latter relation by parts, we obtain equations for determining the functions e and R similarly to (2.4).

Now we assume that the singular support of the velocity field is the stratified manifold $\bigcup \Gamma_i$ with smooth strata Γ_t of codimensions $n_i \geq 1$.

We also assume that the velocity field trajectories are transversal to $\bigcup \Gamma$ and are entering trajectories.

Then the general solution of Equation (1.1) has the form

$$\rho(x, t) = R(x, t) + \sum e_i \delta(\Gamma_i),$$

where $R(x, t)$ is a function smooth outside $\bigcup \Gamma_i$, e_i are functions defined on the strata Γ_i , and the sum is taken over all strata Γ_i .

The integral identities determining such a generalized solution have the form

$$\int_0^\infty \int_{\mathbb{R}^n} (R\zeta_t + (uR, \nabla\zeta) + aR\zeta) dx dt + \sum_i \int_{\Gamma_i} e_i \left[\left((v_i, \nabla) + \frac{\partial}{\partial t} \right) \zeta(x, t) \right] \omega_i = 0. \quad (2.8)$$

This implies that, outside $\bigcup \Gamma_i$, the function R satisfies the continuity equation

$$R_t + (\nabla, uR) + aR = 0,$$

and, on the strata Γ_j for $n_j = 1$, equations of the form (2.4) hold, which contain the values of R brought to Γ_t along the trajectories. For $n_l = n - k$, $k > 1$, on the strata Γ_l , we have the equations

$$\frac{\partial}{\partial t} e_l \mu_l + (\nabla, v_l e_l \mu_l) = F_l \mu_l, \quad (2.9)$$

where μ_l is the density of the measure ω_l with respect to the measure on Γ_l which is left-invariant with respect to the field $\frac{\partial}{\partial t} + \langle v_l, \nabla \rangle$, and F_l is defined by the following construction. Denote a ε -neighborhood of Γ_l by Γ_l^ε and denote its boundary by $\partial\Gamma_l^\varepsilon$. Let us consider the integral appearing after integration by parts:

$$\int_{\partial\Gamma_l^\varepsilon} \zeta \rho u_{nl} \omega_l^\varepsilon,$$

where u_{nl} is the normal component of the velocity u on $\partial\Gamma_l^\varepsilon$, ω_l^ε is the Leray measure on $\partial\Gamma_l^\varepsilon$, ζ is a test function. Passing to the limit as $\varepsilon \rightarrow 0$ we obtain

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial\Gamma_l^\varepsilon} \zeta \rho u_{nl} \omega_l^\varepsilon = \int_{\Gamma_l} \zeta F_l \omega_l.$$

It is well-known that outside $\bigcup \Gamma_i$ the function $R(x, t)$ can be calculated using the famous Cauchy formula

$$R(x, t) = \rho_0(x, t) \left| \frac{Dx}{Dx_0} \right|^{-1} \exp \left(- \int_0^t a dt' \right) \quad (2.10)$$

where ρ_0 is a constant along the trajectories of the field u outside $\bigcup \Gamma_i$, $\left| \frac{Dx}{Dx_0} \right|$ is the jacobian of the mapping corresponding to the shift along the trajectories of u and the integral under exponent is calculating along the trajectories of the field u .

This formula implies that the limit as $\varepsilon \rightarrow 0$ of the above integral exists.

We note that it follows from the above that the function R is determined independently of the values of v_i on the strata under the condition that the field trajectories enter $\bigcup \Gamma_i$.

In conclusion, we consider the case where the coefficient a has a singular support on $\bigcup \Gamma_i$, i.e.,

$$a = f(u).$$

In this case, we set

$$a\rho = \check{a}\rho + \sum f(v_i) e_i \delta(\Gamma_i),$$

where $\check{a} = f(u)$ outside $\bigcup \Gamma_i$. We note that such a choice of the definition of the term $a\rho$ is not unique in this case. But, first, it is consistent with the common concept of measure solutions (see [2], [11]) and, second, it is of no importance for the construction of the solution outside $\bigcup \Gamma_i$ for the case in which the trajectories u enter $\bigcup \Gamma_i$.

In this case, identity (2.8) takes the form

$$\int_0^\infty \int_{\mathbb{R}^n} (R\zeta_t + (uR, \nabla\zeta) + f(u)R\zeta) dx dt + \sum_i \int_{\Gamma_i} e_i \left[\left((v_i, \nabla) + \frac{\partial}{\partial t} + f(v_i) \right) \zeta(x, t) \right] \omega_i = 0, \tag{2.11}$$

and Equation (1.7) can be rewritten in the form

$$\frac{\partial}{\partial t} (e_l \mu_l) + (\nabla, v_l e_l \mu_l) + f(v_l) = F_l \mu_l. \tag{2.12}$$

All the afore said gives the following statement.

Theorem 2.3 *Let that the following conditions be satisfied for $t \in [0, T], T > 0$:*

- (1) $\bigcup \Gamma_i$ is a stratified manifold with smooth strata Γ_i ;
- (2) the trajectories of the field u are smooth outside $\bigcup \Gamma_i$, enter $\bigcup \Gamma_i$ and do not intersect outside $\bigcup \Gamma_i$;
- (3) Equations (2.12) are solvable on the strata Γ_i ;
- (4) the Kirchhoff laws are satisfied on the intersections of strata Γ_i .

Then there exists a general solution to the continuity Equation (1.1) with $a = f(u)$ in the sense of the integral identity (2.11).

3 The Maslov tunnel asymptotics

We recall that the asymptotic solutions of a general Cauchy problem for an equation with pure imaginary characteristics was first constructed by Maslov [15]. In the present paper, we consider only the following Cauchy problem

$$-h \frac{\partial u}{\partial t} + P \left(\frac{\partial}{\partial x}, -h \frac{\partial}{\partial x} \right) u = 0, \quad u(x, t, h)|_{t=0} = e^{-S_0(x)/h} \varphi^0(x), \tag{3.1}$$

where $P(x, \xi)$ is the (smooth) symbol of the Kolmogorov–Feller operator [5], $S_0 \geq 0$ is a smooth function, $\varphi^0 \in C_0^\infty$, $h \rightarrow +0$ is a small parameter characterizing the frequency and the amplitude of jumps of the Markov stochastic process having transition probability given by $P(x, \xi)$. To be more precise, we can have in the mind the following form of $P(x, \xi)$:

$$P(x, \xi) = (A(x)\xi, \xi) + V(x) + \int_{\mathbb{R}^n} (e^{i(\xi, \nu)} - 1) \mu(x, d\nu),$$

where $A(x)$ is positive definite smooth matrix, $V(x)$ is a smooth function and its second derivatives are assumed to be uniformly bounded, and $\mu(x, d\nu)$ is a family of bounded measures smooth with respect to x in the sense that the functions $x \mapsto \mu(x, B)$ are smooth for all measurable sets B . The symbol $P(x, \xi)$ can also depend on t , we will be more precise later on.

Locally in t , an asymptotic solution of problem (3.1) can be constructed according to the scheme of the WKB method, see [15]: the solution is constructed in the form

$$u = e^{-S(x,t)/h} \sum_{i=0}^\infty (\varphi_i(x, t) h^i$$

in the sense of asymptotic series. In this case, for the functions $S(x, t)$ and $\varphi_0(x, t)$ we obtain the following problems:

$$\frac{\partial S}{\partial t} + P \left(x, \frac{\partial S}{\partial x} \right) = 0, \quad S(x, t)|_{t=0} = S_0(x), \tag{3.2}$$

$$\frac{\partial \varphi_0}{\partial t} + \left(\nabla_{\xi} P \left(x, \frac{\partial S}{\partial x} \right), \nabla \varphi_0 \right) + \frac{1}{2} \sum_{ij} \frac{\partial^2 P}{\partial \xi_i \partial \xi_j} \frac{\partial^2 S}{\partial x_i \partial x_j} \varphi_0 = 0, \quad (3.3)$$

$$\varphi_0(x, t)|_{t=0} = \varphi^0(x).$$

As is known, the solution of problem (3.2) is constructed using the solutions of the Hamiltonian system assumed to exist and to be smooth

$$\begin{aligned} \dot{x} &= \nabla_{\xi} P(x, p), & x|_{t=0} &= x_0, \\ \dot{p} &= -\nabla_x P(x, p), & p|_{t=0} &= \nabla S_0(x_0). \end{aligned} \quad (3.4)$$

This solution is smooth on the support of $\varphi_0(x, t)$ for all t such that the Jacobian $|Dx/Dx_0| \neq 0$ for $x_0 \in \text{supp } \varphi^0(x)$. We let g_H^t denote the translation mapping along the trajectories of the Hamiltonian system (3.4).

We recall that the plot

$$\Lambda_0^n = \{x = x_0, p = \nabla S_0(x_0)\}$$

is the initial Lagrangian manifold corresponding to Equation (3.2), and $\Lambda_t^n = g_H^t \Lambda_0^n$ is the Lagrangian manifold corresponding to Equation (3.2) at time t . Let $\pi : \Lambda_t^n \rightarrow \mathbb{R}_x^n$ be the projection of Λ_t^n on \mathbb{R}_x^n , which is assumed to be proper. For this property to hold, it is sufficient to assume that the trajectories of the system (3.4) do not go to infinity during a finite time.

The point $\alpha \in \Lambda_t^n$ is said to be essential if

$$\hat{S}(\alpha, t) = \min_{\beta \in \pi^{-1}(\alpha)} \hat{S}(\beta, t)$$

and nonessential otherwise. Here \hat{S} is the action on Λ_t^n determined by the formula

$$\hat{S}(\beta, t) = \int_0^t p dx - H dt,$$

where the integral is calculated along the trajectories of the system (3.4) the projection of its origin being $x_0 = \beta$. As is known that

$$S(x, t) = \hat{S}(\pi^{-1}x, t)$$

at regular points where the projection π is bijective.

The global in time asymptotic solution of problem (3.1) is given by the Maslov tunnel canonical operator.

To define this operator, following [15], [16] we introduce the set of essential points $\bigcup \gamma_{it} \subset \Lambda_t^n$. This set is closed because the projection π is proper, i.e., that for all x the set of p such that $(x, p) \in \Lambda_t^n, \pi(x, p) = x$ is finite.

Suppose that the open domains $U_j \subset \Lambda_t^n$ form a locally finite covering of the set $\bigcup \gamma_{it}$. If the set U_j consists of regular points, then we set

$$u_j = e^{-S_j(x,t)/h} \varphi_{0j}(x, t) \quad (3.5)$$

where

$$\varphi_{0j}(x, t) = \psi_{0j}(x, t) \left| \frac{Dx_0}{Dx} \right|^{1/2},$$

$\psi_{0j}(x, t)$ being the solution of the equation

$$\frac{\partial \psi_{0j}}{\partial t} + (P_{\xi}(x, \nabla S_j), \nabla \psi_{0j}) - \frac{1}{2} \text{tr} \frac{\partial^2 P}{\partial x \partial \xi}(x, \nabla S_j) \psi_{0j} = 0, \quad (3.6)$$

which exists and is smooth whenever $|Dx/Dx_0| \neq 0$. The solution u_j in the domain containing essential (non-regular) points (at which $d\pi$ is degenerate) is given in the following way: the canonical change of variables is

performed so that the nonregular points become regular, then we determine a fragment of the solution in new coordinates by formula (3.5) and return to the old variables, applying the “quantum” inverse canonical transformation to the solution obtained in the new coordinates.

The Hamiltonian determining this canonical transformation has the form

$$H_\sigma = \frac{1}{2} \sum_{k=1}^n \sigma_k p_k^2,$$

where $\sigma_1, \dots, \sigma_n = \text{const} > 0$.

The canonical transformation to the new variables is given by the translation by the time -1 along the trajectories of the Hamiltonian H_σ . One can prove (see [15], [16]) that the family of sets σ for which the change of variables takes a nonregular point into a regular is not empty.

Next, the solution near the essential point is determined by the relation

$$u_j = e^{\frac{1}{h} \hat{H}_\sigma} \tilde{u}_j, \quad (3.7)$$

where \tilde{u}_j is given by formula (3.5) in the new variables and

$$\hat{H}_\sigma = \frac{1}{2} \sum_{k=1}^n \sigma_k \left(-h \frac{\partial}{\partial x_k} \right)^2.$$

On the intersections of singular charts (containing singular points) and nonsingular charts (without singular points), we must match S_j and ψ_{0j} . This can be done by applying the Laplace method to the integral whose kernel is a fundamental solution for the operator $-h \frac{\partial}{\partial t} + \hat{H}_\sigma$. This integral appears if we write the right-hand side of (3.7) in detail. In this case, since the solution is real, the Maslov index which is well-known [15] to appear in hyperbolic problems does not appear. The complete representation of the solution of problem (3.1) is obtained by summing functions of the type (3.5) and (3.7) over all the domains U_j , for more detail, see [15], [16]. Here we only say the corresponding sum is (locally) finite because we assumed that the projection π is proper.

The asymptotics thus constructed is justified, i.e., the proximity between the exact and asymptotic solutions of the Cauchy problem (3.1) is proved [5], [15]. More precisely it is proved that at the points of the set $\pi(\bigcup \gamma_{it})$ where the projection π is bijective the following estimate holds:

$$u(x, t, h) - u_j = O(h).$$

In the preceding section we noted that the values of the solution of the continuity equation at nonregular points are independent of the values of the solution on the singularity support (of course, the inverse influence takes place) by the condition that the velocity field trajectories enters the singular support.

In the case of the canonical operator whose construction has briefly been described above, the relation between the solutions at essential and nonessential point is also unilateral, namely, the essential points are “bypassed” using (3.7), but the values of the functions u_j on the singularity support do not determine the values at the regular points.

It is clear that the values of the functions S_j and ψ_{0j} at regular points are defined by characteristics via the initial data. The trajectories of the Hamiltonian system also come to points on Λ_t^n such that the projection mapping at these points is singular. At these points, we cannot define ψ_{0j} by characteristics directly and we must use an auxiliary construction (see (2.7)) which allows us to determine the values of an asymptotic solution at the projections of singular points onto x -space. In this auxiliary construction, the values of S_j and ψ_{0j} in a neighborhood of the singular points on Λ_t^n are used to determine the values of u_j at the projections of singular points. Thus, there is a similarity between this and the preceding sections: we define the values of an asymptotic solution by characteristics outside its singular support and then define the values of the asymptotic solution at the singular points using the already defined values at regular points in our auxiliary construction.

Now we note that the function $S(x, t)$ is such that

$$S(x, t)|_{U_j} = S_j(\pi^{-1}(\alpha), t)$$

is globally determined and continuous at points of the domain $\pi(\bigcup \gamma_{it}) \subset \mathbb{R}_x^n$. We denote this set by $\bigcup \Gamma_i$ and assume that this is a stratified manifold with smooth strata Γ_{it} of different codimensions. We note that, for example, if the inequality $\nabla(S_i(x, t) - S_j(x, t)) \neq 0$ holds while we pass from one branch $\Lambda_t^n \cap \bigcup \gamma_{it}$ to another, then the set $\pi\{(\tilde{S}_i - \tilde{S}_j) = 0\}$ generates a smooth stratum of codimension 1. In the one-dimensional case, all strata are points or curves on the (x, t) -plane (under the above assumptions about the singularities being discrete).

Now we consider the equation for ψ_{0j}^2 . We denote this function by ρ and then obtain

$$\frac{\partial \rho}{\partial t} + (\nabla, u\rho) + a\rho = 0, \quad (3.8)$$

where $u(x, t) = \nabla_\xi P(x, \nabla S)$ and $a = -\text{tr} \frac{\partial^2 P}{\partial x \partial \xi}(x, \nabla S)$.

If the condition

$$\text{Hess}_\xi P(x, \xi) > 0$$

is satisfied, then it follows from the implicit function theorem that $\nabla S(x, t) = F(x, u(x, t))$, where $F(x, u)$ is a smooth function and

$$a = f(x, u),$$

where $f(x, z)$ is again a smooth function.

Thus, we have proved the following theorem.

Theorem 3.1 *Suppose that the following conditions are satisfied for $t \in [0, T]$, $T > 0$:*

- (1) *There exists a smooth solution of the Hamiltonian system (3.4).*
- (2) *The singularities of the velocity field $u = \nabla_\xi P(x, \nabla S)$ form a stratified manifold with smooth strata and $\text{Hess}_\xi P(x, \xi) > 0$.*
- (3) *There exists a generalized solution ρ of the Cauchy problem for Equation (3.8) in the sense of the integral identity (2.9).*

Then at the points of $\pi(\bigcup \gamma_{it})$ where the projection π is bijective, the asymptotic solution of the Cauchy problem (3.1) has the form

$$u = \exp(-S(x, t)/h)(\sqrt{\rho} + O(h)).$$

This theorem is a global in time analog of the corresponding Madelung observation [14] about local solutions of Schroedinger type equations.

4 Particular cases

The theorem stated in the previous section requires that some assumptions are satisfied. The most restrictive is the item 3 in Theorem 3, that is the existence of the global generalized solution to the continuity equation. Under the above made assumptions it is possible to construct this solution using characteristics, but only in the case where the structure of the singular support of u is not changing in time—all sections of the stratified manifold introduced above by planes $t = \text{const}$ smoothly depend on t . A more complicate situation arises when the singularities of the velocity field change their structure. In this case the problem of the construction of a global in time generalized solution to the continuity equation has not been solved yet. The obstacle is that in this case usually one has no global in time expression for the velocity field u . In turn this does not allow to apply formula (2.10) to construct global solution to the continuity equation. In the multi-dimensional case as far as we know there is only one result concerning to shock wave generation [3] which allows to construct global in time approximations of the shock wave formation process. But this is slight different from the construction that we needs here. In the one dimensional case the situation is better and we have all needed formulas.

We begin with the spatially homogeneous case. Here the problem is equivalent to the one of constructing a formula for a global solution to conservation law equation

$$\frac{\partial v}{\partial t} + \frac{\partial P(v, t)}{\partial x} = 0. \quad (4.1)$$

Here $P(-h\frac{\partial}{\partial x}, t)$ is the same operator as in (3.1) but assumed to be independent of x with the symbol $P(\xi, t)$ and $v = \partial S / \partial x$. The velocity field u in this case is $P_\xi(v, t)$. In [6] a construction of the global solution to the continuity equation where the velocity field is given by the solution of Equation (4.1) was given. Because the set of singular points is discrete by our assumptions, without loss of generality one can consider the case where only one point of singularity appears. Denote the corresponding (smooth) initial condition by u_0 , the instant where the singularity appears by t^* and the point of singularity by x^* .

The first step of construction suggested in [6], [7] is that we change u_0 in a small neighborhood of origin x_0^* of the trajectory coming to x^* when $t = t^*$. We denote this new part of initial data $u_1(x_0)$ for $x_0 \in (x_0^* - \beta, x_0^* + \beta)$, $\beta \rightarrow 0$ and assume

$$\varepsilon\beta^{-1} \longrightarrow 0, \quad \varepsilon \longrightarrow 0. \quad (4.2)$$

We define the function $u_1 = u_1(x_0, t)$ as a solution of implicit equation

$$P'_\xi(u_1, t) = -K(t)x_0 + b(t). \quad (4.3)$$

The latter equation is solvable under the condition $\text{Hess}_\xi P(x, \xi) > 0$, formulated above.

The functions $K(t)$ and $b(t)$ are defined from the condition of continuity of the characteristics flow, i.e.,

$$u_1(x_0^* - \beta, t) = u_0(x_0^* - \beta, t), \quad u_1(x_0^* + \beta, t) = u_0(x_0^* + \beta, t).$$

It is easy to check that this choice of u_1 provides that the Jacobian $|Dx/Dx_0|$ is identically equal to 0 for $t = t^*$ and $x_0 \in (x_0^* - \beta, x_0^* + \beta)$. Here we remove from usual topological concept of general position considering the situation of identical equality that can be destroyed by small perturbation. But this construction follows from the algebraic concept and allows to present the solution of (4.1) in the form of linear combination of Heaviside functions (see [6]).

The second step of our construction of an approximation is a modification of the definition of characteristics. We set

$$\dot{x} = (1 - B)P'_\xi(u_1(x_0, t), t) + Bc, \quad x_0 \in (x_0^* - \beta, x_0^* + \beta), \quad (4.4)$$

and

$$\dot{x} = P'_\xi(u_0, t),$$

where x_0 does not belong to $(x_0^* - \beta, x_0^* + \beta)$,

$$c = \frac{P(v(x_0^* + \beta, t), t) - P(v(x_0^* - \beta, t), t)}{v(x_0^* + \beta, t) - v(x_0^* - \beta, t)}.$$

The initial data for (4.4) are as follows:

$$x|_{t=0} = x_0 + A\varepsilon, \quad \varepsilon > 0.$$

The function B in (4.4) has the form $B = B((t - t^*)/\varepsilon)$ and $B(z)$ is smooth, monotone and increasing from 0 to 1 for $z \in (-\infty, \infty)$. Similarly to [6], [7] one can prove that there exist an $A = \text{const}$ such that the Jacobian $|Dx/Dx_0|$ calculated using the above introduced characteristics is not equal to zero, but it is of order $O(\varepsilon)$ when $t \geq t^* + O(\beta)$ when $x_0 \in (x_0^* - \beta, x_0^* + \beta)$. Using the velocity field generated by \dot{x} we can construct global in time (smooth) solution of the continuity equation in the form (2.10). After that, passing to the limit as $\varepsilon \rightarrow 0$ we will obtain the generalized solution of the continuity equation in the sense of definition from Section 2 just like it was done in [7].

Spatially inhomogeneous one dimensional case. We will follow the scheme introduced above. The case under consideration can be treated in the same way as the previous one with modifications. Firstly, we will assume that the symbol $P = P(x, \xi)$ does not depend on t . In this case this assumption (which means that the mapping g_P^t is invertible w.r.t. time) will be used to construct the insertion to initial data. In the previous case we did it using the implicit function theorem, see (4.3).

Let Λ_0^1 be a smooth nonsingular (w.r.t. the projection π) curve in the (x, p) space, which is a Lagrangian manifold corresponding to initial data for our problem. We consider the Lagrangian manifold $\Lambda_{t^*}^1 = g_P^{t^*} \Lambda_0^1$ and assume that there is only one point singular with respect to the projection onto x -axis and its projection is x^* . Let β be the same as above. Let us set $t_1^* = t^* + \beta$. Because of the assumption that $P''_{\xi\xi}$ is positive, we have that for $t = t_1^*$ the Lagrangian manifold $\Lambda_{t_1^*}^1$ has two parts which contain essential points and these parts form a shock wave type curve with the jump at the point x_1^* where $S_{\text{left}}(x_1^*, t_1^*) = S_{\text{right}}(x_1^*, t_1^*)$. We connect these parts by a vertical line and thus obtain a new Lagrangian manifold, which is a piecewise smooth continuous curve with two angle points (ends of the vertical part, the distance between them of order β). We denote this manifold by $\hat{\Lambda}_{t_1^*}^1$ and apply the mapping $g_P^{-t_1}$ for sufficiently small t_1 to this manifold. This mapping obviously exists and is a diffeomorphism because our Hamiltonian P does not depend on t . We consider the obtained manifold $g_P^{-t_1} \hat{\Lambda}_{t_1^*}^1$ as the new Lagrangian manifold corresponding to our problem for $t = t_1^* - t_1$ changing the manifold $\Lambda_{t_1^*}^1$ by $g_P^{-t_1} \hat{\Lambda}_{t_1^*}^1$. As it was said above the latter manifold is piecewise smooth curve with two angle points and all points of the curve outside of the part between these angle points are regular. Moreover there exist a sufficiently small t_1 such that the part of the curve between these angle points contains only regular points-these statements are a consequence of the positivity of P'' , its stationarity and the possibility to choose t_1 small enough (and independent on ε).

Denote the projections of the mentioned above angle points on the manifold $g_P^{-t_1} \hat{\Lambda}_{t_1^*}^1$ to the x -axis by $a_1 < a_2$. We note that $|a_1 - a_2|$ is of order β .

Like in the previous example we introduce the new characteristics system

$$\begin{aligned} \dot{x} &= (1 - B)P'_\xi(x(x_0, t), p(x_0, t)) + Bc, \\ \dot{p} &= -(1 - B)P'_x(x(x_0, t), p(x_0, t)), \quad x_0 \in (a_1, a_2), \end{aligned} \tag{4.5}$$

and

$$\dot{x} = P'_\xi(x, p), \quad \dot{p} = -P'_x(x, p) \tag{4.6}$$

when x_0 does not belong to (a_1, a_2) . We have set

$$c = \frac{P(v(x(a_2, t), p(a_2, t))) - P(x(a_1, t), p(a_1, t))}{p(x(a_2, t), t) - p(x(a_1, t), t)}. \tag{4.7}$$

The initial data for (4.5), (4.6) are as follows:

$$\begin{aligned} x|_{t=0} &= x_0 + A\varepsilon, \\ p|_{t=0} &= p_0(x_0), \end{aligned}$$

where $(x_0, p_0(x_0)) = g_P^{-t_1} \hat{\Lambda}_{t_1^*}^1$. The expression on the right-hand side of (4.7) is the direct analog of the well-known Rankine-Hugoniot expression for the velocity of the shock propagation. In the case under consideration it is the velocity of the point \tilde{x} on x -axis, where $S_{\text{left}}(\tilde{x}, t) = S_{\text{right}}(\tilde{x}, t)$.

By the assumption we have only one singular point if we are considering the family of manifolds $\Lambda_t^1, t \in [0, t^*]$. We also have by construction that the Jacobian $J = Dx/Dx_0$ calculated using the solutions of the system (4.5) is not equal to zero. More precisely we have

$$\lim_{\varepsilon \rightarrow 0} J = H(t^* - t)J_0,$$

where J_0 is the Jacobian calculated using the solutions of (4.5) for $B = 0$ ($J_0 = 0$ when $t = t^*$ by construction) and

$$J \geq H(t^* - t)J_0 + C\varepsilon,$$

where $C = \text{const} > 0$. This statement directly follows from (4.5) if we take the properties of the function B into account. This means that the velocity field, generated by the projections of the solution of the system (4.5), (4.6) onto the x -axis has nonintersecting trajectories for $\varepsilon > 0$. Thus we can use it to construct solutions of the continuity equation. It remains to note that just like in [7] it is easy to check that the limits of these solutions will satisfy to the integral identities introduced in Section 2 as the definition of generalized solutions to continuity equation.

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