

# The Differential Calculus on Quantum Linear Groups.

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## Abstract

The non-commutative differential calculus on the quantum groups  $SL_q(N)$  is constructed. The quantum external algebra proposed contains the same number of generators as in the classical case. The exterior derivative defined in the constructive way obeys the modified version of the Leibnitz rules.

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# 1 Introduction

Recent interest in constructing the differential calculi on the quantum groups stems from the Woronowicz's pioneering work [33]. There he has formulated the general algebraic framework for dealing with the problem. In the subsequent investigations the emphasis was made on two main directions. First, the experience of dealing with such algebras was accumulated while considering the simplest lower dimensional examples (see e.g. [32, 23, 27]). It was soon recognized that the true quantum group differential calculus should be bicovariant, and that this condition is very restrictive. Indeed, the only use of this condition allows one to obtain the unique external algebra construction for the  $SL_q(2)$  Cartan 1-forms [13]. Next, a very close connection was established with the theory of quadratic quantum algebras (quantum spaces) [19, 10, 31]. It was then realized that the condition of unique ordering of higher order monomials (the so-called diamond condition) is very important [21, 18], and that really it is only to be checked for the cubic monomials [20].

Another direction of investigations was finding out the adequate technique for dealing with quantum differential algebras. Here the close connections between the quantum differential calculi and the  $R$ -matrix formulation for quantum groups and algebras [10] were soon established [16, 11, 34] (for further considerations see [6]). It turns out that the  $R$ -matrix technique is highly appropriate in treating the problems arised.

The next stage of investigations was to combine both the lines of research to obtain the concrete differential algebra constructions for known series of quantum groups. Here the substantial progress was achieved for the  $GL_q(N)$  case. Namely, in the series of papers [18, 29, 26, 28, 30] the pair of the nice-looking differential algebras on  $GL_q(N)$  was constructed. But the situation with the  $q$ -deformed series of the simple Lie groups appears to be much more complex. The natural way of obtaining the  $SL_q(N)$ ,  $SO_q(N)$  and  $SP_q(N)$  differential calculi by performing reduction from the  $GL_q(N)$  calculi failed in the quantum case, because one cannot consistently reduce the number of the generating elements in the  $GL_q(N)$  differential algebras constructed (see discussion in [8, 35]). In principle one can treat these nonreduced (or partially reduced) differential calculi as a quantizations of the nonstandard classical calculi on the special groups ( see [22]), but the problem of finding out the deformations of the ordinary calculi still remained open. It is rather natural in this situation to revise once again the basic postulates involved into constructive scheme. The only postulate that seems too restrictive is the classical Leibnitz rule for the exterior derivative [11]

$$d(f \cdot g) = df \cdot g + (-1)^{|f|} f \cdot dg .$$

Indeed, let us remind that the basic vector fields after quantization correspond to the finite shifts rather than to infinitesimal differentiations. The natural Leibnitz rule for them is multiplicative instead of being additive. Correspondingly the Leibnitz rule for differential must take into account this shift property of vector fields.

In this paper we propose the construction of the differential algebra with properly modified Leibnitz rule. We consider the case most close to  $GL_q(N)$  – the  $SL_q(N)$

differential algebra. Here only one Cartan 1-form and one basic vector field should be reduced. The reduction scheme for vector fields was already developed in [28]. We propose the reduction scheme for Cartan 1-forms. We do not discuss here the involution, leading to the unitary reduction of our system. As was shown in [2] this can be done for  $q$  on the circle ( $|q| = 1$ ) for the algebra of vector fields and functions on the quantum group. We believe, that the involution, found in [2] can be continued on the differential forms as well.

The paper is organized as follows. In Section 1 we fix the notations of the  $R$ -matrix technique and formulate the basic postulates of our construction. We believe that it was the consistent use of the  $R$ -matrix technique which allowed us to make the construction through. So it not only led to the simplifying of calculations, but played an important heuristic role. In Section 2 we present the external algebra on  $SL_q(N)$ . This algebra is also supplied with the action of the basic vector fields (or Lie derivatives). We refer to this extended algebra as the differential algebra on  $SL_q(N)$ . Section 3 is devoted to construction of the exterior derivative operator  $d$ . Note that the scheme proposed can be equally applied to  $GL_q(N)$ . In this way one can recover a wide variety of the differential algebras on  $GL_q(N)$ . It seems to us that such a nonuniqueness is due to the nonsemisimplicity of  $GL_q(N)$ .

## 2 The basic principles and notation

The starting point for our consideration is the Hopf algebras  $Fun(GL_q(N))$  and  $Fun(SL_q(N))$  [10]. We present here some facts and definitions for these algebras.

We choose the corresponding  $R$ -matrix [15]  $R \in Mat_N(\mathbf{C})^{\otimes 2}$  in the form

$$R = q \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ji} \otimes e_{ij} + \lambda \sum_{j < i} e_{jj} \otimes e_{ii}, \quad (2.1)$$

where  $i, j = 1, \dots, N$  and  $\lambda = q - \frac{1}{q}$ . In what follows we will also use the shorthand notation  $R$  for the matrix  $R \otimes I \in Mat_N(\mathbf{C})^{\otimes 3}$ , where  $I \in Mat_N(\mathbf{C})$  is the unit matrix. One can easily distinguish, in the context of each formula, whether  $R$  belongs to  $Mat_N(\mathbf{C})^{\otimes 2}$  or to  $Mat_N(\mathbf{C})^{\otimes 3}$ . The  $R$ -matrix (2.1) satisfies the Yang-Baxter equation and the Hecke condition, respectively,

$$R R' R = R' R R', \quad (2.2)$$

$$R^2 = \mathbf{I} + \lambda R. \quad (2.3)$$

Here  $R' = I \otimes R$ , and  $\mathbf{I} = I \otimes I$ . It is worthwhile to establish the connection with other  $R$ -matrix conventions of frequent use:

$$\begin{aligned} \text{our } R &= \hat{R}_{12} = P_{12} R_{12} = R_{12}^+ P_{12}, \\ \text{our } R^{-1} &= R_{12}^- P_{12}. \end{aligned}$$

Here  $P \in Mat_N(\mathbf{C})^{\otimes 2}$  is the permutation matrix and all the notation  $\hat{R}_{12}$ ,  $R_{12}$ ,  $R_{12}^\pm$  is given in Ref. [10].

The unital associative algebra  $Fun(GL_q(N))$  is generated by  $N^2$  elements  $T = (t_{ij})_{i,j=1}^N$ . The multiplication and comultiplication in it are defined, respectively, by

$$RTT' = TT'R, \quad (2.4)$$

$$\Delta(t_{ij}) = t_{ik} \otimes t_{kj}, \quad (2.5)$$

where  $T$  means  $T \otimes I$  in (2.4) and  $T' = I \otimes T$ .

The  $q$ -deformed Levi-Civita tensor  $\epsilon_q^{i_1 \dots i_N}$  ( $= \epsilon_q^{1 \dots N}$  in brief notation) satisfies the following characteristic relations:

$$\epsilon_q^{1 \dots N} R_i = -\frac{1}{q} \epsilon_q^{1 \dots N}, \quad 1 \leq i \leq N, \quad (2.6)$$

$$\epsilon_q^{i_1 \dots i_N} \Big|_{i_1=1, \dots, i_k=k, \dots, i_N=N} = 1.$$

Here  $R_i = I^{\otimes(i-1)} \otimes R \otimes I^{\otimes(N-i-1)}$  (note:  $R_1 = R$ ,  $R_2 = R'$ ). The quantum determinant of  $T$ ,  $det_q T$ , defined through the relation

$$\epsilon_q^{1 \dots N} T_1 T_2 \dots T_N = T_1 T_2 \dots T_N \epsilon_q^{1 \dots N} = \epsilon_q^{1 \dots N} \cdot det_q T, \quad (2.7)$$

where  $T_k = I^{\otimes(k-1)} \otimes T \otimes I^{\otimes(N-k)}$ , is the central element of the algebra  $Fun(GL_q(N))$ . This can be checked by the following formula

$$\Psi^{N+1} \epsilon_q^{1 \dots N} R_N^{\pm 1} \dots R_1^{\pm 1} = q^{\pm 1} \Psi^1 \epsilon_q^{2 \dots N+1}, \quad (2.8)$$

where  $\Psi = (\psi^i)_{i=1}^N \in \mathbf{C}^N$  is an arbitrary vector. The Hopf algebra  $Fun(SL_q(N))$  is then obtained by adding one more relation

$$det_q T = 1. \quad (2.9)$$

to (2.4). Finally, the antipodal mapping  $S(\cdot)$  on  $Fun(GL_q(N))$  and  $Fun(SL_q(N))$  (for its explicit form see [10]) satisfies the relations

$$S(T)T = TS(T) = I, \quad (2.10)$$

therefore, in what follows we prefer using the notation  $T^{-1}$  rather than  $S(T)$ .

Now let us turn to the differential algebra of extensions of  $Fun(GL_q(N))$  and  $Fun(SL_q(N))$ . First, we should fix the basic principles of our construction:

**A. The bicovariance condition.** Following [33] we require that a differential algebra should possess the bicomodule structure with respect to the underlying quantum group. By this condition we guarantee that the left and right translations in a quantum group do not affect the structure of its differential calculus. From this point it looks most natural to use, say, right-invariant and left-adjoint vector fields  $L = (l_{ij})_{i,j=1}^N$  and

Cartan 1 - forms  $\Omega = (\omega_{ij})_{i,j=1}^N$  in addition to  $T$ s as the generating elements for differential algebra<sup>1</sup>. The left and right  $Fun(GL_q(N))$ -coactions in this case read:

$$\delta_L(x_{ij}) = t_{ik}t_{lj}^{-1} \otimes x_{kl} \quad , \quad \delta_R(x_{ij}) = x_{ij} \otimes 1 \quad , \quad (2.11)$$

where by  $X = (x_{ij})_{i,j=1}^N$  we understand either  $L$  or  $\Omega$ .

In case of the  $SL_q(N)$  - differential algebra the number of independent Cartan 1-forms should be reduced by 1. This can only be achieved in a bicovariant manner with the use of the  $q$ -deformed trace [10, 24] (see also [34, 28, 12]). Here we define this operation and present several useful formulae

$$Tr_q(X) = Tr(\mathcal{D}X) \quad , \quad \mathcal{D} = diag\{q^{-N+1}, q^{-N+3}, \dots, q^{N-1}\} \quad . \quad (2.12)$$

The  $Tr_q$ -operation possesses the invariance property

$$Tr_q \delta_L(X) = 1 \otimes Tr_q X \quad , \quad (2.13)$$

and also

$$\begin{aligned} Tr_{q(2)}(RXR^{-1}) &= Tr_{q(2)}(R^{-1}XR) = I \cdot Tr_q X \quad , \\ Tr_{q(1,2)}(Rf(X,R)R^{-1}) &= Tr_{q(1,2)} f(X,R) \quad , \\ Tr_{q(2)}R^{\pm 1} &= q^{\pm N} I \quad , \quad Tr_q I = [N]_q \quad . \end{aligned} \quad (2.14)$$

Here the index in parantheses denotes the number of the matrix space in which the operation  $Tr_q$  acts, and  $[N]_q = \frac{q^N - q^{-N}}{\lambda}$ .

**B. The ordering condition.** We suppose that multiplication in the differential algebra is defined by relations quadratic in  $T$ ,  $\Omega$  and  $L$  and these relations allow us to order lexicographically any quadratic monomial of the generators. Moreover, they should allow the *unique* ordering for any higher order monomial of  $T$ ,  $\Omega$  and  $L$ . The latter is the so-called *diamond (or confluence) condition* (see e.g. [7]). It guarantees us that the Poincaré series of the classical differential algebra do not change under quantization. The direct check of this condition consists in the use of the Diamond Lemma [7]. Such calculations appear to be very cumbersome already in the  $N = 2$  case (see discussion in subsection 3.8 of the Ref. [21]). and it seems hard to generalize them for any  $N$ . The alternative way we shall advocate here is in noticing that the quadratical relations for  $T$ ,  $\Omega$ ,  $L$  express in fact the action of some representation of the braid group on the differential algebra. The diamond condition is then the consequence of the braid group defining relations and, hence, it should follow from the general properties (2.2), (2.3) of the  $R$ -matrix. Examples of such formal  $R$ -matrix manipulations are presented in [14] and in Section 2 of the present paper.

**C.** The last but not the least condition is that the differential algebra is to be supplied with the differential complex structure. In other words, we should define the  $\mathbf{C}$ -linear differential mapping  $d$  on it. Taking into account the discussion above we choose the following set of its characteristic properties:

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<sup>1</sup> For the left-invariant and right-adjoint generators all the constructions proceed analogously.

- $d$  is of degree 1 with respect to the natural  $\mathbf{Z}$ -grading on the algebra of the differential forms;
- $d$  satisfies the *nilpotence condition*:  $d^2 = 0$ .

Now let us proceed to the construction of such differential algebra.

### 3 The differential algebra

We arrange the main result of this section in

**Theorem 1:** *For general values of the deformation parameter  $q$  ( $[2]_q \neq 0$ ,  $[N]_q \neq 0$ ,  $[N]_q \neq -\lambda q^N$ ,  $[N \pm 1]_q \neq \pm q^{N \mp 4}$ ) the  $GL_q(N)$ -differential algebra defined as*

$$RTT' = TT'R, \quad (3.1)$$

$$R\Omega R\Omega + \Omega R\Omega R^{-1} = \kappa_q(\Omega^2 + R\Omega^2 R), \quad (3.2)$$

$$R\Omega R^{-1}T = T\Omega', \quad (3.3)$$

$$RLRL = LRLR, \quad (3.4)$$

$$RLRT = q^{\frac{2}{N}}TL', \quad (3.5)$$

$$R^{-1}\Omega RL = LR\Omega R^{-1}, \quad (3.6)$$

where

$$\kappa_q = \frac{\lambda q^N}{[N]_q + \lambda q^N}, \quad (3.7)$$

admits the consistent reduction to  $SL_q(N)$ . This reduction is achieved by adding three more relations

$$\det_q T = 1, \quad Tr_q \Omega = 0, \quad Det L = 1, \quad (3.8)$$

to (3.1)-(3.6). Here

$$Det L = q^{1-N}(R_1 R_2 \dots R_{N-1} L_1)^N \epsilon_q^{1\dots N} = q^{1-N}(L_1 R_1 R_2 \dots R_{N-1})^N \epsilon_q^{1\dots N}. \quad (3.9)$$

**Proof:** It is not difficult to check the bicovariance condition for (3.1-3.8) by using the commutation properties of  $T$ s (2.4) and the definitions for left and right transitions on the quantum group (2.5),(2.11). Here we only mention the transformation properties of  $Det L$ :

$$\delta_L(Det L) = 1 \otimes Det L, \quad \delta_R(Det L) = Det L \otimes 1.$$

The validity of the ordering condition for the quadratic monomials of  $T$ ,  $\Omega$ ,  $L$  is to be verified by rewriting relations (3.1-3.6) in matrix components. We can rather convince ourselves in this by noticing that relations (3.1), (3.2), (3.4) contain the right number of the commutation relations for  $T$ s,  $\Omega$ s and  $L$ s due to their symmetry property

$$\begin{aligned} P_q^\pm(RTT' - TT'R)P_q^\pm &\equiv P_q^\pm(RLRL - LRLR)P_q^\pm \equiv 0, \\ P_q^\pm(R\Omega R\Omega + \Omega R\Omega R^{-1} - \kappa_q(\Omega^2 + R\Omega^2 R))P_q^\mp &\equiv 0. \end{aligned}$$

Here  $P_q^\pm = (\pm R + q^{\mp 1})/[2]_q$  are the quantum symmetrizer and antisymmetrizer, respectively (see [15, 10]).

Now, let us concentrate on checking the diamond condition for monomials cubic in  $T$ ,  $\Omega$  and  $L$ . First, we choose the suitable full set of such monomials:

$$\begin{aligned} & (R' R \Omega)^3, \quad T(R' \Omega)^2, \quad R T T' \Omega'', \quad R' R^{-1} \Omega R'^{-1} R \Omega R' R L, \quad T T' T'' \\ & (R' R L)^3, \quad T(R' L)^2, \quad R T T' L'', \quad R'^{-1} R^{-1} \Omega (R' R L)^2, \quad T \Omega' R' L R'. \end{aligned} \quad (3.10)$$

Here  $T'' = T_3 = I^{\otimes 2} \otimes T$  and the same is for  $\Omega''$  and  $L''$ . The combinations (3.10) are constructed so that one can apply the 'commutation rules' (3.1-3.6) for any adjacent pair of the generators entering into them. We interpret this operation as the ( $q$ -)permutation of a pair of generators. Applying the  $q$ -permutations three times to the monomials (3.10) we arrange their entries in an inverse order. Obviously, this reordering can be performed in two different ways, depending on whether we first permute the left pair of generators or the right one. The diamond condition states that in both cases the result must be the same. We demonstrate how the calculations proceed in the most complex case of the  $(R' R \Omega)^3$ -reordering. This example was already considered in [14] and here we present a simpler derivation.

The calculations proceed as follows:

$$\begin{aligned} (R' R \Omega)^3 &= R R' \underline{R \Omega R \Omega} R' R \Omega \\ &\quad \downarrow^{1 \leftrightarrow 2 \text{ perm.}} \\ & -R \Omega R R' \underline{R \Omega R \Omega} R'^{-1} \quad + \kappa_q R R' (\Omega^2 + R \Omega^2 R) R' R \Omega \\ &\quad \downarrow^{2 \leftrightarrow 3 \text{ perm.}} \\ & \underline{R \Omega R \Omega} R' R \Omega R^{-1} R'^{-1} \quad - \kappa_q R \Omega R R' (\Omega^2 + R \Omega^2 R) R'^{-1} \\ &\quad \downarrow^{1 \leftrightarrow 2 \text{ perm.}} \\ & -\Omega R R' \Omega R R'^{-1} \Omega R^{-1} R'^{-1} \quad + \kappa_q (\Omega^2 + R \Omega^2 R) R' R \Omega R^{-1} R'^{-1}, \end{aligned} \quad (3.11)$$

and in another way

$$\begin{aligned} (R' R \Omega)^3 &= R' R \Omega R R' \underline{R \Omega R \Omega} \\ &\quad \downarrow^{2 \leftrightarrow 3 \text{ perm.}} \\ & -R' \underline{R \Omega R \Omega} R' R \Omega R^{-1} \quad + \kappa_q R' R \Omega R R' (\Omega^2 + R \Omega^2 R) \\ &\quad \downarrow^{1 \leftrightarrow 2 \text{ perm.}} \\ & \Omega R R' \underline{R \Omega R \Omega} R'^{-1} R^{-1} \quad - \kappa_q R' (\Omega^2 + R \Omega^2 R) R' R \Omega R^{-1} \\ &\quad \downarrow^{2 \leftrightarrow 3 \text{ perm.}} \\ & -\Omega R R' \Omega R R'^{-1} \Omega R^{-1} R'^{-1} \quad + \kappa_q \Omega R R' (\Omega^2 + R \Omega^2 R) R'^{-1} R^{-1}. \end{aligned} \quad (3.12)$$

Here we employ subsequently Eqs. (2.2) and (3.2) through all calculations. It remains to compare the  $\kappa_q$ -terms arising under transformations (3.11) and (3.12). Here we need one more formula [14]

$$R \Omega^2 R \Omega - \Omega R \Omega^2 R = 0. \quad (3.13)$$

It is derived as follows: denoting the l.h.s. of (3.13) as  $U$  and using (3.2) twice we get

$$U + \kappa_q R U R = 0. \quad (3.14)$$

Now, dividing  $U$  into a sum of  $q$ -symmetric and  $q$ -antisymmetric parts  $U_{\pm}$ :

$$U_{\pm} = U \pm RUR^{\pm 1}, \quad P_q^{\pm}U_+P_q^{\mp} = P_q^{\pm}U_-P_q^{\pm} = 0, \quad U = \frac{(1 + R^{-2})}{[2]_q^2} (U_+ + U_-),$$

we transform (3.14) into a couple of relations

$$(I + \kappa_q R^2)U_+ = 0, \quad (1 - \kappa_q)U_- = 0.$$

Then, under restrictions  $(1 + \kappa_q R^2) \not\sim P_q^{\pm}$ ,  $\kappa_q \neq 1$ , or, equivalently,  $[N \pm 1]_q \neq \pm q^{N \mp 4}$ ,  $[N]_q \neq 0$  we get the desired relation (3.13).

Now, one can compare the  $\kappa_q$ -terms in (3.11) and (3.12), turning all the  $\Omega^2$  entries to the left. The result is the same in both cases and, thus, the diamond condition on  $(R' R \Omega)^3$  is satisfied. The same calculations, although simpler, can be carried out for all other monomials of (3.10), and we leave them as an exercise.

It remains to check the consistency of the  $SL_q(N)$ -reduction. The centrality of  $\det_q T$  is easily proved by the use of relation (2.8). Next, the application of  $Tr_{q(2)}$  to Eq. (3.2) and subsequent use of the Hecke relation (2.3) give

$$[Tr_q \Omega, \Omega]_+ + \lambda q^N \Omega^2 = \kappa_q ([N]_q + \lambda q^N) \Omega^2 + \kappa_q Tr_q \Omega^2,$$

whereof we conclude that  $Tr_q \Omega$  anticommutes with  $\Omega$  under conditions that

- the parameter  $\kappa_q$  is chosen as in (3.7);
- the quadratic scalar combination  $Tr_q \Omega^2$  identically vanishes.

The last is the direct consequence of (3.2). It is derived like follows. Applying  $Tr_{q(1,2)}(\dots)$  and  $Tr_{q(1,2)}(\dots R^{-1})$  operations to (3.2) and using (2.14),(2.3) we get the system of linear relations on the quadratic scalars  $(Tr_q \Omega)^2$  and  $Tr_q \Omega^2$ :

$$\begin{aligned} 2(Tr_q \Omega)^2 - [N]_q \kappa_q Tr_q \Omega^2 &= 0, \\ -\lambda (Tr_q \Omega)^2 + (2q^N - \kappa_q (q^N + q^{-N})) Tr_q \Omega^2 &= 0. \end{aligned}$$

The determinant of this system:  $\frac{q^N [2]_q^2 [N]_q}{[N]_q + \lambda q^N}$  - does not vanish under conditions of the Theorem and, hence, we conclude

$$(Tr_q \Omega)^2 = Tr_q \Omega^2 = 0.$$

Then, applying the  $Tr_{q(2)}$  operation to (3.3), (3.6) we find that  $Tr_q \Omega$  is the (graded) central element in the algebra (3.1)-(3.7).

Finally, to construct the central element from  $Ls$ , we use the following trick suggested in [2, 28, 9] (see also [35]). Consider the matrix  $Z = LT$ . It behaves like  $T$  under the left and right transitions in  $GL_q(N)$ . Moreover, it possesses the similar algebraic properties:

$$\begin{aligned} R Z Z' &= Z Z' R, \\ R^{-1} \Omega R Z &= Z \Omega', \\ R L R Z &= q^{\frac{2}{N}} Z L'. \end{aligned}$$



Hence,  $DetL = det_q Z \cdot (det_q T)^{-1}$  is central in the algebra (3.1)-(3.6). Now, let us show that  $DetL$  indeed depends only on  $L$ :

$$\begin{aligned} \epsilon_q^{1\dots N} \cdot DetL &= (L_1 T_1)(L_2 T_2) \dots (L_N T_N) \epsilon_q^{1\dots N} \cdot (det_q T)^{-1} \\ &= q^{N-1} L_1 (R_1 L_1 R_1) \dots (R_N \dots R_1 L_1 R_1 \dots R_N) \epsilon_q^{1\dots N} . \end{aligned}$$

The expression (3.9) for  $DetL$  is then extracted by using (3.4), (2.2) and performing induction in  $N$ . **Q.E.D.**

**Comment:** Among relations (3.1)-(3.6) only (3.3) is a completely new relation. Formula (3.2) were proposed for  $N = 2$  case in [13] and for general  $N$  in [14] as commutation relations for Cartan 1-forms on  $SL_q(N)$ . Formulae (3.4), (3.5) appeared in [1] as the algebra of functions on the cotangent bundle of  $GL_q(N)$ . The algebra of vector fields (3.4)-(3.6) is suggested in [28, 35] for the differential calculus on  $GL_q(N)$  and  $SL_q(N)$ . The definition of quantum determinant  $DetL$  can also be found in these works and in [9]. Note also the recent work [4] where the external algebra (3.1)-(3.3) has been given in components for  $N = 2$  case. A really new point in our approach is that all these formulae are consistently combined into a single algebra.

**Remark 1:** Besides the algebra (3.1)-(3.8) there exist three more differential algebras on  $SL_q(N)$ . They can be obtained from (3.1)-(3.6) by the substitutions of two types:

$$\mathbf{S1} : \quad R \leftrightarrow R^{-1}, \quad \kappa_q \leftrightarrow \kappa_{\frac{1}{q}} \quad \text{in (3.2)} ; \quad (3.15)$$

$$\mathbf{S2} : \quad R \leftrightarrow R^{-1}, \quad q \leftrightarrow q^{-1} \quad \text{in (3.3) - (3.6), (3.9)} . \quad (3.16)$$

For  $N = 2$  the substitution (3.15) is trivialised. Indeed, the relations (3.2) and  $\mathbf{S1} \cdot (3.2)$  in case  $N = 2$  differ by the term proportional to  $P_q^-(\Omega^2 + R\Omega^2 R) \sim P_q^-\Omega^2 P_q^- \sim Tr_q \Omega^2 \cdot P_q^-$  and since the scalar relation  $Tr_q \Omega^2$  is contained both in (3.2) and  $\mathbf{S1} \cdot (3.2)$ , it follows that relations (3.2) and  $\mathbf{S1} \cdot (3.2)$  for  $N = 2$  are identical. This result agrees with the statement of [4] that there exist only two different external algebra structures on  $SL_q(2)$ . We should stress here that this mechanism does not work for  $N > 2$ , where we have 4 noncoinciding differential algebras.

**Remark 2:** The very limited number of the  $q$ -deformations for the differential calculus on  $SL(N)$  seems to be an effect of the simplicity property of  $SL(N)$ . By contrast, one could derive a lot of the quantized versions for  $GL(N)$  case. For instance, if we take away the condition of the existence of  $SL_q(N)$ -reduction, then it is no need of fixing parameters  $\kappa_q$  and  $q^{\frac{2}{N}}$  in Eqs. (3.2), (3.5). Another possibility is to use for  $T$  and  $\Omega$  the commutation rules different from (3.3):

$$R\Omega RT = T\Omega' . \quad (3.17)$$

Algebras of that type were considered in [21, 18, 29, 26, 22, 28, 30, 35, 14].

**Remark 3:** A few words on the interpretation of the basic vector fields  $L$  are in order. It is very natural to suppose that the algebra of the classical vector fields  $V$  behaves under the quantization like  $U_q g$  and, hence, is not quadratic. On the other

hand, simple quadratic relations are achieved for another-type generators  $L^+$ ,  $L^-$  [10] and  $L$  [25, 3, 2, 17]. These generators constitute finite shifts on the quantum group and can be viewed as some ‘exponentiated’ form of infinitesimal vector fields  $L = I + \lambda V + O(\lambda^2)$ . That is why the  $SL_q(N)$  reduction for  $L$  is performed not by the  $Tr_q$ -like condition but by its exponentiated  $Det$ -like form. It is also natural from this point of view that the quantities  $Z = LT$  obtained from  $T$ s by finite  $L$ -shifts behave algebraically like  $T$ s.

## 4 Exterior derivative

We shall define the differential mapping  $d$  on external algebra (3.1)-(3.3) in a constructive way.

1. Define the action of  $d$  on the generators  $T$  and  $\Omega$  as

$$dT = \Omega T, \quad d\Omega = \Omega^2. \quad (4.1)$$

2. For the Cartan 1-forms we postulate the ordinary Leibnitz rule to be satisfied

$$d \cdot \Omega = \Omega^2 - \Omega \cdot d \quad (4.2)$$

Using (3.13) it is straightforward to check that this prescription agrees with the commutation relations for  $\Omega$ s (3.2). Besides, due to (4.2) the action of exterior derivative on  $T$  and on any function  $F$  of  $\Omega$  is nilpotent:  $d^2T = d^2F(\Omega) = 0$ . Leaving apart the mathematical reasonings, we would like to stress that it is rather natural to keep the classical Leibnitz picture for infinitesimal objects like  $\Omega$ .

3. Using **1.** and **2.** we are able to calculate the exterior derivative action on any monomial of  $T$  and  $\Omega$  which is of first order in  $T$ . Namely, we should at first turn all  $\Omega$ s to the left by using commutation relations (3.3), and then apply (4.1) and (4.2) to get:  $d(F(\Omega)T) = dF(\Omega)T + F(-\Omega)\Omega T$ . In this way we get automatically the consistency of the differential mapping with the algebraic relations (3.3) and the nilpotence of  $d$  on any monomial of that type.
4. The next step is to construct the differential mapping for the general quadratic monomial of  $T$ :  $TT'$ . We stress here that since under quantization we obtain the finite shifts  $L$  acting on  $T$  rather than differentiation, it is very reasonable to get the modified Leibnitz rules for  $T$ . The action of  $d$  should take into response the algebraic relations (3.1):

$$R d(TT') = d(TT') R.$$

Note also that the expression for  $d(TT')$  must be of 1-st order in  $\Omega$ . The general ansatz satisfying both these conditions reads

$$d(TT') = f(R)(\Omega + R\Omega R)TT'. \quad (4.3)$$

Here  $f(R)$  is a function of  $R$  and the combination  $\Omega + R\Omega R$  commutes with the  $R$ -matrix due to Hecke conditions (2.3). The exact form of the function  $f(R)$  is dictated by the nilpotence condition:

$$0 = d^2(TT') = f \left\{ (\Omega^2 + R\Omega^2 R) - f(\Omega + R\Omega R)^2 \right\} TT' .$$

Using (3.2) it is straightforward to obtain

$$(\Omega + R\Omega R)^2 = (I + \kappa_q R^2)(\Omega^2 + R\Omega^2 R) ,$$

and, hence,  $d$  is nilpotent on  $TT'$  if we put<sup>2</sup>

$$f(R) = (I + \kappa_q R^2)^{-1} . \quad (4.4)$$

Using (3.3), (4.1), (4.2), (4.3) we can obtain now how  $d$  acts on any monomial of  $T$  and  $\Omega$  that is quadratic in  $T$ , and again the nilpotence of  $d$  is guaranteed by (4.2).

Thus, we gave the detailed consideration of the the first few steps in constructing the differential mapping  $d$ . Generalizing this procedure to monomials of any order in  $T$  we get

**Theorem 2:** *For the external algebra (3.1)-(3.3) presented in Theorem 1 there exists the left acting differential mapping  $d$ , defined by (4.1), (4.2) and*

$$d(T_1 T_2 \dots T_k) = \{I + \kappa_q (S_k(I) - I)\}^{-1} S_k(\Omega) T_1 T_2 \dots T_k , \quad (4.5)$$

where

$$S_k(X) = X + \sum_{i=1}^{k-1} R_i \dots R_2 R_1 X R_1 R_2 \dots R_i . \quad (4.6)$$

In particular,

$$d(\det_q T) = \frac{1}{q^{N-1}(1 - \kappa_q) + [N]_q \kappa_q} \text{Tr}_q \Omega \det_q T , \quad (4.7)$$

$$d(\text{Tr}_q \Omega) = \text{Tr}_q \Omega^2 = 0 , \quad (4.8)$$

which guarantees the compatibility of  $d$  with the reduction conditions (3.8). This differential mapping is commutative with the action of the basic vector fields  $L$ :

$$[d, L] = 0 . \quad (4.9)$$

**Proof:** As in the case  $k = 2$  we start with the following general ansatz

$$d(T_1 T_2 \dots T_k) = f_k S_k(\Omega) T_1 T_2 \dots T_k . \quad (4.10)$$

---

<sup>2</sup> Note that under restrictions of Theorem 1 the matrix  $(I + \kappa_q R^2)$  is invertible.

Here  $f_k$  is a function of  $R_1, \dots, R_k$  to be specified below, and

$$R_i f_k = f_k R_i, \quad R_i S_k(\Omega) = S_k(\Omega) R_i, \quad i = 1, \dots, k-1. \quad (4.11)$$

The first of relations (4.11) is the restriction on the possible form of  $f_k$ , while the last is the direct consequence of the Yang-Baxter equation (2.2) and the Hecke condition (2.3). In virtue of Eqs. (4.11) we have

$$R_i d(T_1 \dots T_k) = d(T_1 \dots T_k) R_i, \quad i = 1, \dots, k-1,$$

and, thus, ansatz (4.10) is compatible with relations (3.1) of the external algebra. The nilpotence condition  $d^2(T_1 \dots T_k) = 0$  leads to the relation

$$S_k(\Omega^2) - f_k(S_k(\Omega))^2 = 0.$$

It remains to compute the quantity  $(S_k(\Omega))^2$ . This calculation is based on the essential use of Eqs. (3.2), (2.2) and (2.3) is rather lengthy. We present here the result

$$(S_k(\Omega))^2 = \{I + \kappa_q(S_k(I) - I)\} S_k(\Omega^2).$$

Hence, the function  $f_k$  is to be chosen as in (4.5). Note that with this choice  $f_k$  satisfies the conditions (4.11).

In obtaining the formula (4.7) one should employ the properties (2.6) of the  $q$ -deformed Levi-Civita tensor, and also

$$\epsilon_q^{1\dots N} S_N(X) = q^{1-N} Tr_q X \epsilon_q^{1\dots N}.$$

The check of the compatibility of condition (4.9) with the algebra (3.4)-(3.6) is straightforward. **Q.E.D.**

**Remark 1:** Using (4.1), (4.2), (4.5), and (3.3), (3.2) one can derive the explicit form of the modified Leibnitz rules. These rules appear in the modified form for  $T$ ,  $\Omega$ -polynomials, for which the left acting exterior derivative should cross  $T$  under evaluation. For the quadratic polynomials we have

$$\begin{aligned} d(T T') &= (I + \kappa_q R^2)^{-1} \{R^2 dT T' + T dT'\} \\ d(T \Omega') &= (1 - \kappa_q) T d\Omega' + dT \Omega' + \{(1 - \kappa_q) R^2 - I\} \Omega^2 T. \end{aligned}$$

Here the term  $\Omega^2 T$  may be treated either as  $d\Omega T$  or as  $\Omega dT$ . Note that the operator  $R^2$ , being the generating element of the braid group  $B_2$ , plays a particular role in these formulae. This observation is further approved if we evaluate the action of  $d$  on the monomials of any order in  $T$ :

$$d(T_1 \dots T_k) = \left\{ I + \kappa_q \sum_{i=1}^{k-1} B_{k,i} \right\}^{-1} \sum_{i=1}^k B_{k,i} T_1 \dots dT_i \dots T_k, \quad (4.12)$$

$$B_{k,i} = (R_i R_{i+1} \dots R_{k-1})(R_{k-1} R_{k-2} \dots R_i), \quad B_{k,k} = I^{\otimes k}, \quad i = 1, \dots, k-1.$$

Here  $\{B_{k,i}\}_{i=1}^k$  is the set of generating elements for the braid group  $B_k$ .

**Remark 2:** Note that in constructing the differential mapping  $d$ , essential are the self-commutation relations for  $T$  (3.1) and  $\Omega$  (3.2). The explicit form of the cross-commutation relations for  $T$  and  $\Omega$  (3.3) is not relevant. We only should be aware of that these relations allow us to turn all  $\Omega$ s to the left in any monomial of  $T$  and  $\Omega$ . Thus, the algorithm described can be applied equally to the external algebras considered in [21, 18, 29, 26, 22, 28, 30, 35, 14] and satisfying the cross-multiplication relations of the type (3.17). In this way one can search for all the external algebraic structures on  $GL_q(N)$  compatible with the ordinary Leibnitz prescriptions. It turns out that only two external algebras obtained in the references above satisfy this condition. The first of these algebras is defined by relations (3.1), (3.17) and (3.2) where one must put  $\kappa_q = 0$ . The second algebra is obtained from the first if one makes substitution  $R \leftrightarrow R^{-1}$  in all formulae. This result agrees with the quasiclassical considerations of Ref. [5].

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