

Relating and Contrasting Plain and Prefix Kolmogorov Complexity

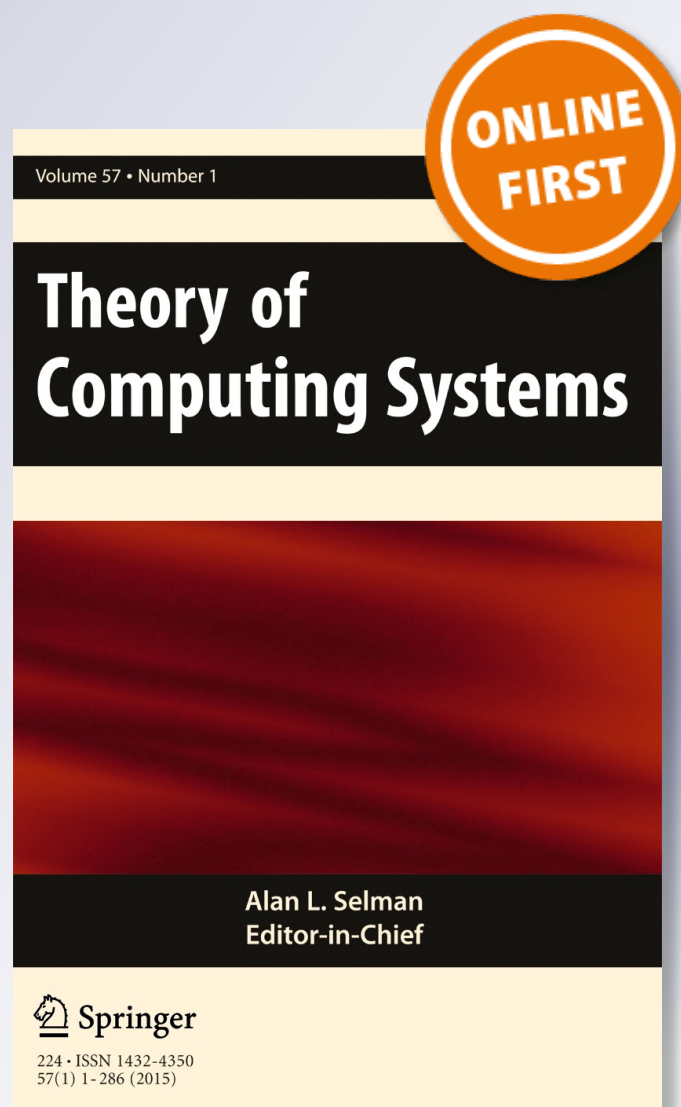
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Relating and Contrasting Plain and Prefix Kolmogorov Complexity

Bruno Bauwens¹

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Abstract In (Bauwens and Shen, J. Symb. Log. **79**(2), 620–632, 2013) a short proof is given that some strings have maximal plain Kolmogorov complexity but not maximal prefix-free complexity. We argue that the proof technique is useful to simplify existing proofs and to solve open questions. We present a short proof of a result due to Robert Solovay that relates plain and prefix complexity:

$$K(x) = C(x) + CC(x) + O(CCC(x))$$

$$C(x) = K(x) - KK(x) + O(KKK(x)),$$

(here $CC(x)$ denotes $C(C(x))$, etc.). We show that there exist ω such that $\liminf C(\omega_1 \dots \omega_n) - C(n)$ is infinite and $\liminf K(\omega_1 \dots \omega_n) - K(n)$ is finite, i.e. the infinitely often C -trivial reals are not the same as the infinitely often K -trivial reals, answering Question 1 in Barmpalias (Bull. Symb. Log. **19**(3), 2013). We answer a question from Bienvenu (Laurent Bienvenu, personal communication 2011): some 2-random sequence has a family of initial segments with bounded plain deficiency (i.e. $|x| - C(x)$ is bounded) and unbounded prefix deficiency (i.e. $|x| + K(|x|) - K(x)$ is unbounded). Finally, we show that there exists no monotone relation between probability and expectation bounded randomness deficiency, answering Question 1 in Bienvenu et al. (Proceedings of the Steklov Institute of Mathematics, **274**(1), 34–89, 2011).

Keywords Kolmogorov complexity · Martin-Löf randomness · 2-randomness · Randomness deficiency · Infinitely often C and K -trivial sequences

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1 Introduction

Plain Kolmogorov complexity $C(x)$ of a bitstring x was independently defined by Ray Solomonoff [19] and later by Andrei Kolmogorov [10] as the minimal length of a program that produces x on a Turing machine. Programs are strings of zeros and ones written on a work tape; the beginning and end of the program is marked by blanc symbols. During the execution, the Turing machine (which we call *plain* machine) can scan the beginning and end of the program and use its length as additional information during the computation. Kolmogorov complexity on such a machine is called *plain* complexity (see [9, 13] for details).

A closely related notion of complexity was introduced by Leonid Levin [11, 12] and Gregory Chaitin [6] and has many applications in the study of algorithmic randomness. Imagine a Turing machine on which programs are presented on a separate 2-symbol input tape (without blanc symbols). During the execution more input is scanned until the machine reaches a halting state, after which an output x is defined. Programs on such a machine are also called *self-delimiting*. During the computation, the length of p is no longer available. Note that the set of programs on which U halts is prefix-free. Kolmogorov complexity on such a machine is called *prefix* complexity $K(x)$.

The difference $|K(x) - C(x)|$ of both complexity measures is $O(\log |x|)$, where $|x|$ denotes the length of x . For many applications this difference is not important. However, for applications in the theory of algorithmic randomness, often $O(1)$ -precise relations are used, and often one raises the question what happens when plain and prefix complexity are exchanged in a result or a definition. The goal of the paper is two-fold. First, we present a simple proof on a result that relates plain and prefix complexity. Secondly, we refine a proof-technique (from [3]) to build strings where plain and prefix complexity behave differently, and apply it to solve three open questions.

Several results are related to one of the oldest questions in algorithmic randomness, raised by Robert Solovay [20] (see [7, page 263]). The maximal plain complexity of a string of length n is $n + O(1)$ and we say that a string has c -maximal complexity if $C(x) \geq |x| - c$. Martin-Löf observed that for no c and no infinite sequence all initial segments x have c -maximal complexity. On the other hand, the class of sequences for which some c and infinitely many initial segments x exist with $C(x) \geq |x| - c$ has measure one. Similar observations hold for prefix complexity, (where the maximal complexity is $n + K(n) + O(1)$). Solovay's question is whether the classes of sequences with infinitely often maximal plain and prefix complexity are the same; in other words, is $\liminf_{x \sqsubseteq \omega} |x| - C(x)$ finite iff $\liminf_{x \sqsubseteq \omega} K(|x|) + |x| - K(x)$ is finite?

To answer this question, Solovay investigated whether there was a monotone relation between $C(\cdot)$ and $K(\cdot)$. He found that this was approximately the case by showing

$$\begin{aligned} K(x) &= C(x) + CC(x) + O(CCC(x)) \\ C(x) &= K(x) - KK(x) + O(KKK(x)), \end{aligned}$$

where complexity of a number n is the complexity of the n -bit string $00 \dots 0$ and where $CC(x)$, $KK(x)$, etc, be short for $C(C(x))$, $K(K(x))$, etc. The proof in [20] is

cumbersome and Joseph Miller [15] made some simplifications using symmetry of information for prefix complexity. Here we use this technique to give an even simpler proof. (Readers only interested in this result can directly go to Sections 2 and 3.)

Solovay showed that the continuation of the first equation with terms up to $O(CCCC(x))$ does not hold. He also showed that maximal prefix complexity implies maximal plain complexity, but the reverse is not true: there exist infinitely many n and x of length n such that $n - C(x) \leq O(1)$ and

$$K(n) + n - K(x) \geq \log^{(2)} n - O(\log^{(3)} n). \tag{1}$$

In [3] a simple proof (and generalizations) are presented. Here we further develop the proof technique to solve several open questions.

Despite this negative result, Miller [14, 16] gave a positive answer to Solovay's question: the sequences that have infinitely many initial segments with maximal plain and prefix complexity are the same. The proof is indirect: it shows that both classes coincide with the class of 2-random sequences, i.e. Martin-Löf random sequences relative to the halting problem (the equivalence of the first class with 2-randomness was also shown in [17]). Miller raised the question whether an (elegant) direct proof exists. In [2] simple proofs of these equivalences with 2-randomness are given, but still no direct proof. It is also shown that

$$\liminf_{x \sqsubset \omega} [|x| - C(x)] = \liminf_{x \sqsubset \omega} [K(|x|) + |x| - K(x)] + O(1),$$

by showing both sides equal 2-randomness deficiency (see further). Laurent Bienvenu asked whether for a 2-random sequence, the initial segments for which plain and prefix-free complexity are maximal are the same; more precisely, for 2-random ω , does there exist c and d such that for all n : $n - C(\omega_1 \dots \omega_n) \leq c$ implies $K(n) + n - K(\omega_1 \dots \omega_n) \leq d$? (For some c and d the reverse implication is always true.) We show that this is not the case: for every 3-random sequence (a subset of the 2-random sequences) there are infinitely many initial segments x with $|x| - C(x) \leq O(1)$ for which (1) holds. This makes the existence of a simple direct proof unlikely. We refer to Section 6 for the proof of this result.

In algorithmic information theory, many relations are known between highly random sequences and highly compressible sequences [1, Section 3.5]. The second application of our technique considers one such class called *the infinitely often K -trivial sequences*: the sequences ω for which there exist c and infinitely many n such that $K(\omega_1 \dots \omega_n) \leq K(n) + c$, i.e.

$$\liminf_n [K(\omega_1 \dots \omega_n) - K(n)] \leq O(1)$$

This class contains the computably enumerable sequences and the (weakly) 1-generic sequences. Similar observations hold for the infinitely often C -trivial sequences, i.e. the sequences for which

$$\liminf_n [C(\omega_1 \dots \omega_n) - C(n)] \leq O(1).$$

Question 1 in [1] asks whether both classes coincide. We show that this is not the case.

A last application of the proof technique concerns randomness deficiency for infinite sequences. Suppose one million zeros are prepended to a random string. The new string is still random, but one might argue that it is somehow “less random”. Randomness deficiency quantifies the amount of structure in a random sequence (see [13, Section 3.6.2] and [5]). Let μ denote the uniform measure. Two closely related notions of deficiency exist in the literature.

- A lower semicomputable¹ function $f : \{0, 1\}^\infty \rightarrow \overline{\mathbb{R}}^+$ (i.e. \mathbb{R}^+ extended with $+\infty$) is a *probability bounded randomness test* if for each k

$$\mu \{ \omega : f(\omega) \geq k \} \leq k, .$$

- A measurable function $f : \{0, 1\}^\infty \rightarrow \overline{\mathbb{R}}^+$ is an *expectation bounded randomness test* if

$$\int_{\{0,1\}^\infty} f(\omega)d\omega \leq 1.$$

The first notion is inspired by the notion of confidence in statistical hypothesis testing, while the second is closely related, but mathematically more convenient to handle. There exists a lower semicomputable expectation bounded test f_E that exceeds any other such test g within a constant factor, i.e. for all g there exist c such that $g \leq cf_E$. The logarithm of such a universal test is called *expectation bounded randomness deficiency* d_E . The deficiency depends on the choice of the universal test, but this choice affects the deficiency by at most an additive constant. Similar for probability bounded tests and *probability bounded deficiency* d_P . Both deficiencies are related: $d_E = d_P + O(\log d_P)$, and both deficiencies are finite iff the sequence is Martin-Löf random. We argue that the relationship between plain and prefix complexity is very similar to the relationship between d_P and d_E .

Question 1 in [5] asks whether there exists a monotone relation between probability bounded deficiency and expectation bounded deficiency that holds within additive $O(1)$ terms. If this is not the case then there exist two families of sequences ω_i and ω'_i such that

$$d_P(\omega_i) - d_P(\omega'_i) \rightarrow +\infty$$

for increasing i , and

$$d_E(\omega_i) - d_E(\omega'_i) \rightarrow -\infty, .$$

In Section 7, we translate the main proof technique to deficiencies and construct such sequences. Hence, no monotone relation exists between the deficiencies.

The paper is organized as follows: first we discuss two old results which will be used throughout the paper: Levin’s formula relating plain and prefix complexity and Levin’s formula for symmetry of information. In the next section we present a simple proof for Solovay’s formulas relating C and K . All further results in the paper demonstrate that C and K behave differently and the proofs have a common structure. In Section 4, we repeat the simplest such proof by showing that some strings

¹A non-negative rational function f on $\{0, 1\}^\infty$ is *basic* if $f(\omega)$ is determined by a finite prefix of ω . A function f into $\overline{\mathbb{R}}^+$, is *lower-semicomputable* if there exists a uniformly computable series of (non-negative) basic functions f_i such that $f = \sum_i f_i$.

have maximal plain but non-maximal prefix complexity. Afterwards, in Section 5, we show that the class of infinitely often C and K trivial sequences are different. In Section 6, we show that each 3-random sequence has infinitely many initial segments with maximal plain complexity but non-maximal prefix complexity. Finally, in Section 7, we show that no monotone relationship exists between plain and prefix randomness deficiency. Section 3, Sections 4, 5, 6, and Section 7 can be read independently.

2 Prerequisites

Two results are central in most of our proofs. The first is Levin's symmetry of information [8]: for all x, y

$$K(x) + K(y|x, K(x)) = K(x, y).$$

The conditional variant is given by

$$K(x|z) + K(y|x, K(x|z), z) = K(x, y|z).$$

The second result relates plain and prefix complexity for random strings. For all n -bit x : $C(x) = n + O(1)$ iff $K(x|n) = n + O(1)$. We will use a more general variant.

Lemma 1 (Folklore) *For all j and x*

$$|j - C(x)| = \Theta(|j - K(x|j)|)$$

Proof The Lemma implies Levin's formula

$$C(x) = K(x|C(x)) + O(1),$$

and in fact, it is equivalent to it: for any j it implies $K(x|j) = C(x)$ up to terms $O(\log |j - C(x)|)$, and by the triangle inequality:

$$|j - K(x|j)| = |j - C(x)| + O(\log |j - C(x)|).$$

□

3 Relating Plain and Prefix Complexity

Recall that $KK(x)$, $CC(x)$, etc, are short for $K(K(x))$, $C(C(x))$, etc.

Theorem 1

$$\begin{aligned} K(x) &= C(x) + CC(x) + O(CCC(x)) \\ C(x) &= K(x) - KK(x) + O(KKK(x)). \end{aligned} \tag{2}$$

Proof Using symmetry of information we have

$$K(x) = K(x, K(x)) = KK(x) + K(x|K(x), KK(x)) + O(1).$$

The last term equals $K(x|K(x) - KK(x)) + O(KKK(x))$. Setting $j = K(x) - KK(x)$ the equality can be rewritten as

$$j = K(x|j) + O(KKK(x)).$$

Thus $C(x) = j + O(KKK(x))$ by Lemma 1, i.e. we have shown (2).

We obtain the first equation of the theorem from the second by showing that

$$CC(x) = KK(x) + O(KKK(x)) \tag{3}$$

$$KKK(x) \leq O(CCC(x)). \tag{4}$$

For (3), note that $a = b - c + O(d)$ implies $C(a) = C(b) + O(K(c) + d)$. Applying this to (2) we obtain

$$C(C(x)) = C(K(x)) + O(K(KK(x)) + KKK(x)).$$

Substituting $x \leftarrow K(x)$ in (2) gives

$$C(K(x)) = K(K(x)) + KK(K(x)) + O(KKK(x)).$$

Combining both equations implies (3).

It remains to show that (3) implies (4). Using $K(a) \leq K(b) + K(b - a) + O(1)$:

$$K(KK(x)) \leq K(CC(x)) + K(KK(x) - CC(x)) + O(1)$$

The first term on the right is bounded by $2C(CC(x)) + O(1)$. For the second, note that $K(d) \leq O(\log d)$ for any number d , hence

$$KKK(x) \leq 2CCC(x) + O(\log KKK(x)), \tag{5}$$

i.e. (4). □

Remark 1 The proof implies that $K(x) = C(x) + O(CC(x))$ and $KK(x) = CC(x) + O(CCC(x))$. Alexander Shen raised the question whether $KKK(x) = CCC(x) + O(CCCC(x))$? This does not hold. The proof is cumbersome and uses a topological argument from [18], see Appendix A.²

4 Contrasting Maximal Plain and Prefix Complexity

To get used to the main proof technique for the remainder of this paper, we start by showing the subsequent variant of Solovay's theorem.

² For later use in the appendix, note that the proof above also implies

$$CC(x), CK(x), KC(x), KK(x),$$

are all equal within error $O(CCC(x))$ and error $O(KKK(x))$. (Indeed, to relate $KK(x)$ to $KC(x)$, apply $K(\cdot)$ to (2).) Moreover, for all $U, V, W, X, Y, Z \in \{C, K\}$ we have that $UVW(x) \leq O(XYZ(x))$. Indeed, by applying $C(a) = C(b) + O(\log(a - b))$ on the equalities above, we obtain that $CYZ(x) = CCC(x) + O(\log CCC(x))$. In the same way one shows that $KYZ(x) = KKK(x) + O(\log KKK(x))$. The result follows now from (5).

Theorem 2 (Solovay [20], Bauwens and Shen [3]) *There exist infinitely many x such that $|x| - C(x) \leq O(1)$ and $K(|x|) + |x| - K(x) \geq \log^{(2)} |x| - O(1)$.*

The main technique is to combine the two results from Section 2 with a third result: Peter Gács' quantification of incomputability of Kolmogorov complexity [8]. He showed that for all lengths, there are x such that $K(K(x)|x)$ is close to $\log |x|$ (and similar for plain complexity); if complexity were computable, then this would be bounded by $O(1)$. The following tight variant from [3] will be used:

Theorem 3 *For some c and all l there exists an n such that $\log n = 2^l$, $K(n) \geq (\log n)/2$ and $K(K(n)|n) \geq l - c$.*

Lemma 2 *If n satisfies the conditions of Theorem 3, then*

$$\log^{(2)} n = \log K(n) + O(1) = K(K(n)|n) + O(1).$$

Proof Indeed, dropping additive $O(1)$ terms, the left equality follows from

$$\log^{(2)} n \leq \log((\log n)/2) \leq \log K(n) \leq \log(2 \log n) \leq \log^{(2)} n.$$

It remains to show that $K(K(n)|n) \leq \log^{(2)} n$. Indeed, $K(K(n)|n) \leq K(K(n)|\log^{(2)} n)$, and using $\log^{(2)} n = \log K(n)$ this follows from $K(i|\log i) \leq \log i$.³ □

We informally explain why some strings have maximal plain complexity but non-maximal prefix complexity. There exist plain machines U for which a string w exist such that $U(wx) = x$ for all x . If x has $O(1)$ -maximal plain complexity, then wx is an $O(1)$ -shortest program for x . In a similar way, there exists a prefix machine V such that for some w we have $V(wx|x) = x$ for all x ; indeed, V just copies the input from the program tape and uses the condition $|x|$ to know when to stop this operation.

If the length of x is not available in the condition, there might not exist such a trivial program. To decide when to halt the copying procedure, the length of x must somehow be represented in the program in self-delimited form. If the length of the program is minimal (within an $O(1)$ constant), this encryption of the length should also be minimal. Mathematically, this corresponds to the following observations for x of length n : $K(x) = K(n, x)$, (here and below we omit $O(1)$ terms); and by symmetry of information

$$K(n, x) = K(n) + K(x|n, K(n)).$$

Thus, any shortest program for x can be reorganized into a concatenation of two self-delimiting programs: the first computes n and the second uses n and the length of the first program to compute x . The prefix deficiency is $K(n) + n - K(x) = n - K(x|n, K(n))$ and this is different from the plain deficiency which is close to

³ For the proof in the appendix note that this argument implies $K(K(n)|\log^{(2)} n) = \log^{(2)} n$. By Lemma 1 this implies $C(K(n)) = \log^{(2)} n$.

$n - K(x|n)$ by Lemma 1. This explains why small prefix deficiency implies small plain deficiency, but not vice versa. In particular the deficiencies can only be different if $K(K(n)|n)$ is non-negligible, and this might indeed happen because of Theorem 3.

For appropriate n the discussion explains how we construct x ; it should contain $K(n)$ and then be filled up further with bits independent from n and $K(n)$ until the plain complexity is n . This is the approach in [3], here we take advantage of the fact that the program with largest computation time of length at most n can also compute $K(n)$ from n . The proof below is even shorter than that of [3, Corolary 6].

Proof As discussed above, we choose n , the length of x , such that

$$K(K(n)|n) = \log^{(2)} n + O(1). \tag{6}$$

By Theorem 3 and Lemma 2, there exist infinitely many such n . Let $x = B(n)$ be the program of length at most n with maximal running time on a plain machine. We drop $O(1)$ terms. Note that $C(B(n)) = n = |B(n)|$. It remains to show $K(B(n)) \leq n + K(n) - \log^{(2)} n$ and this follows from

$$K(B(n)|n, K(n)) \leq n - \log^{(2)} n,$$

(because of symmetry of information: $K(B(n)) = K(n, B(n)) = K(n) + K(B(n)|n, K(n))$). From n and $B(n)$ we can compute $K(n)$, thus $n = C(B(n)) = K(B(n)|n)$ also equals

$$K(K(n), B(n)|n) = K(K(n)|n) + K(B(n)|K(n), K(K(n)|n), n).$$

Applying (6) twice implies $n = \log^{(2)} n + K(B(n)|K(n), n)$. □

Remark 2 As a corollary it follows that $K(x) = C(x) + CC(x) + CCC(x) + O(CCCC(x))$ is false. To show it contradicts Theorem 2 note that $CCCC(x) \leq O(\log^{(3)}(n))$. Let x satisfy the conditions of the theorem and choose y of length n with maximal plain and prefix complexity. Now $K(x) - K(y) \geq \log^{(2)} n - O(\log^{(3)} n)$.

For similar reasons the following inequality is not an equality

$$K(x) \leq K(C(x)) + C(x),$$

see also Remark 5 below.

Remark 3 Miller generalized Solovay's theorem [15]. The proof above also implies this generalization.

Theorem 4 *If a co-enumerable set (i.e. the complement can be algorithmically enumerated) of strings contains a string of each length, then it also contains infinitely many strings x such that $K(|x|) + |x| - K(x) \geq \log^{(2)} |x| - O(1)$.*

This theorem also implies that the set of strings with maximal prefix complexity is not co-enumerable.

Proof Suppose n satisfies the conditions of Theorem 3. Let x be the lexicographically first string of length n in the set. We show that x can be computed from $B(n+c)$

for some constant c , and this suffices because we know from the proof above that $K(B(n + c)) \leq n + K(n) - \log^{(2)} n + O(c)$.

Consider a list of all strings of length n and remove the strings outside the set using an enumeration of its complement. The moment the last string was removed can be computed with a program of length $n + O(1)$ on a plain machine (by the total number of removed strings prepended with zeros to have an n -bit number). Thus, this moment must be before $B(n + c)$ for large c . \square

Remark 4 The proof above can be used to contrast *computational depth* with plain and prefix complexity. In [4, Tentative⁴ definition 1] the computational depth of a string x with precision c is given by the minimal computation time of a plain program for x of length at most $C(x) + c$:

$$depth_{C,c}(x) = \min \{t : |p| \leq C(x) + c \text{ and } U(p) = x \text{ in } t \text{ steps} \} .$$

In a similar way, computational depth $depth_{K,c}(x)$ with prefix machines can be defined.⁵ With this assumption it follows easily that there exists a computable f such that $depth_{K,c+2\log|x|}(x) \leq f(depth_{C,c}(x))$ and that $depth_{C,c+2\log|x|}(x) \leq f(depth_{K,c}(x))$ for x of large length. The subsequent proposition shows that with higher precision, the equivalence is not possible. Let $BB(n)$ be the maximal computation time of a program of length at most n on a plain machine (i.e. the computation time of $B(n)$).

Proposition 1 *There exist a c and infinitely many x such that $depth_{C,c}(x)$ is bounded by a computable function of x (and in fact bounded by a constant for an appropriate universal machine) and $depth_{K,\log^{(2)}|x|-c}(x)$ exceeds $BB(|x| - c)$.*

Proof Consider the proof of Theorem 2. Rather than choosing x to be $B(n)$, we fix some appropriate c (see further), and choose x to be the lexicographically first n -bit string such that $C(x) \geq n - 2$ and no self-delimiting program of length $n + K(n) - c$ outputs x in at most $BB(n)$ steps. x exist because for large d there are at most $O(2^{n-d})$ strings of length n with complexity $n + K(n) - d$ (see [7, Theorem 3.7.6 p. 129], this also follows from the coding theorem). By construction $C(x) \geq n - O(1)$ thus a trivial program of x on a plain machine is shortest within $O(1)$. Hence, the depth of x is small on a plain machine. Because x can be computed from $B(n)$, the proof above guarantees that for infinitely many n we have $K(x) \leq K(B(n)) + O(1) \leq n + K(n) - \log^{(2)} n + O(1)$. Fix such an n . To have $depth_{K,\log^{(2)}n-e}(x) < BB(n)$, we need a program for x that computes x in time less than $BB(n)$ of length $n + K(n) - \log^{(2)} n + O(1) + (\log^{(2)} n - e) = n + K(n) + O(1) - e$. For large e this contradicts the choice of x , and hence the depth is at least $BB(n - O(1))$. \square

⁴ Although it was called “tentative” definition, this version is simpler than the others and is more often used in literature.

⁵ We assume in all these definitions that the machine U is universal in the sense that for each other machine V there exist w such that $U(wp) = V(p)$ each time $V(p)$ is defined and that simulating V by U in this way increases the computation time by a computable function.

Remark 5 There exist infinitely many x such that $K(K(x)|x, C(x)) \geq \log^{(2)} n - O(1)$. Indeed, let n be as in Theorem 3. Let x be a string of length n having maximal prefix (and hence plain) complexity such that $K(K(n)|x, n) \geq K(K(n)|n) - O(1)$. This implies

$$K(K(x)|x, C(x)) = K(n + K(n)|x, n) = K(K(n)|x, n) \geq K(K(n)|n) \geq \log^{(2)} n$$

up to $O(1)$ terms.

On the other hand $K(C(x)|x, K(x))$ must be very small and it is an *open question* whether it is bounded by a constant. In particular this would imply that the inequality

$$K(x) \leq K(C(x)) + K(x|C(x), K(C(x)))$$

is an equality, which is also an *open question*.

5 Infinitely Often C and K Trivial Sequences

In the previous section we explained why a minimal self-delimiting program for a string can contain more information than a minimal plain program. This suggest that the classes of infinitely often C and K trivial sequences might be different. The following theorem illustrates this.

Theorem 5 *There exists a sequence ω for which $K(\omega_1 \dots \omega_N) - K(N) \leq O(1)$ for infinitely many N , and for which $C(\omega_1 \dots \omega_N) - C(N)$ tends to infinity.*

Proof Recall that $B(n)$ is a program of length at most n with maximal running time on a plain machine. ω consists of zeros, except at small neighborhoods before indexes 2^n for all large n , and in these neighborhoods strings $w_n = B(n + \log^{(2)} n)$ are placed, see Fig. 1; more precisely $\omega_{2^n - |w_n|} \dots \omega_{2^n} = 1w_n$ (the prepended one in $1w_n$ allows us to identify the beginning of w_n).

We show that $C(\omega_1 \dots \omega_N) - C(N) \geq \log^{(3)} N - O(1)$ for all N , which obviously tends to infinity. Fix any N and let n be such that $2^n \leq N < 2^{n+1}$. The initial segment $\omega_1 \dots \omega_N$ computes w_n , thus $C(\omega_1 \dots \omega_N) \geq C(w_n) \geq n + \log^{(2)} n$ (here and below we omit terms $O(1)$). On the other hand we have $C(N) \leq \log N = n$, hence

$$C(\omega_1 \dots \omega_N) - C(N) \geq (n + \log^{(2)} n) - n = \log^{(2)} n = \log^{(3)} N.$$

It remains to construct c and infinitely many N such that $K(\omega_1 \dots \omega_N) \leq K(N) + c$. The idea is to choose for infinitely many n some N such that $2^n \leq N < 2^{n+1} - |w_{n+1}|$ and such that some shortest program for N can compute w_n with $O(1)$ of information; thus it can also compute w_1, w_2, \dots, w_{n-1} and $\omega_1 \dots \omega_N$ with $O(1)$ bits of information.

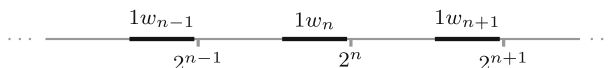


Fig. 1 Construction of ω in the proof of Theorem 5

As one might guess, we choose n such that $K(K(n)|n) = \log^{(2)} n$. Let us compute $K(w_n|n, K(n))$ in a similar way as before. We drop $O(1)$ terms:

$$\begin{aligned} n + \log^{(2)} n &= C(w_n) = K(w_n|n) = K(K(n), w_n|n) \\ &= K(K(n)|n) + K(w_n|K(n), K(K(n)|n), n) \\ &= \log^{(2)} n + K(w_n|K(n), n). \end{aligned}$$

Thus $K(w_n|n, K(n)) = n$.

Let N be the integer whose binary expansion equals the first $n-2$ bits of a program witnessing this equation (i.e. a program of length at most $n + O(1)$ computing w_n from n and $K(n)$) prepended with the string “10”. Prepending “10” guarantees that $2^n \leq N < 2^{n+1} - |w_{n+1}|$ for large n . By construction, if n and $K(n)$ are given, N can compute w_n with $O(1)$ bits of information. Thus it also computes w_1, \dots, w_{n-1} and $\omega_1 \dots \omega_N$. On the other hand, every shortest program for N can also compute n and $K(n)$ with $O(1)$ bits of information. Indeed,

$$K(N) = K(N, n) = K(n) + K(N|n, K(n));$$

thus on a universal prefix machine, there exists a $O(1)$ -shortest program for N that is the concatenation of two self-delimiting programs and the length of the first is $K(n)$. Together:

$$K(N) = K(n, K(n), N) = K(w_1, \dots, w_n, n, K(n), N) \geq K(\omega_1 \dots \omega_N).$$

□

6 Contrasting Plain and Prefix Complexity in 3-random Sequences

Theorem 6 *For every 3-random sequence ω there are a c and infinitely many j such that $j - C(\omega_1 \dots \omega_j) \leq c$ and $K(j) + j - K(\omega_1 \dots \omega_j) \geq \log^{(2)} j - c$.*

We conjecture that the result holds for all 2-random sequences. It is possible to present the proof in a game structure, but both the game and the strategy are quite complicated. We give a proof that has the same core structure as the other proofs above. In the proof we use two lemmas. The first roughly states that randomness deficiency of a string is bounded by the deficiency of an initial segment.

Lemma 3 *Let $j = |x|$ and $n = |xy|$*

$$j - K(x|j) \leq n - K(xy|j, n) + O(1)$$

Proof We omit $O(1)$ terms. Observe that $K(xy|j, n) = K(x, y|j, n)$, and this is

$$\leq K(x|j, n) + K(y|j, n) \leq K(x|j) + n - j,$$

because $K(y|y) \leq |y|$ for all strings y and $|y| = n - j$ is computable from the condition. The inequality of the lemma follows after rearranging. □

Let a and b be two strings of the same length. Let $XOR(a, b)$ denote the bitwise XOR operator on these strings. The following lemma states that if a is incompressible, and b is incompressible given a , then also b is incompressible relative to $XOR(a, b)$. In fact, we will use a generalization which states that if an extension bw is incompressible given a , then this extension is incompressible given $XOR(a, b)$.

Lemma 4 *Let a and b be strings of equal length ℓ , let w be any string, let $n = |bw|$, and let i be any number. If*

$$K(a|\ell, n, i) \geq \ell - c \quad \text{and} \quad K(bw|a, n, i) \geq n - c,$$

then

$$K(bw|XOR(a, b), n, i) \geq n - O(c).$$

Proof In the lemma all complexities are conditional to i . The proof of the conditional form follows the unconditional one, presented here. We first consider the case where w is the empty string, the proof for non-empty w follows the same structure and will be presented afterwards. We need to show that for all c, ℓ, a, b such that $|a| = |b| = \ell$, $K(a|\ell) \geq \ell - c$ and $K(b|a) \geq \ell - c$ we have

$$K(b|XOR(a, b)) \geq \ell + O(c). \tag{7}$$

Indeed,

$$K(a, b|\ell) = K(a|\ell) + K(b|a, \ell, K(a|\ell)) + O(1).$$

By assumption $K(a|\ell) \geq \ell - c$, thus $K(a|\ell) = \ell + O(c)$ and the last term simplifies to $K(b|a, \ell) + O(c)$ and this equals $\ell + O(c)$. Hence $K(a, b|\ell) = 2\ell + O(c)$. Let $xor = XOR(a, b)$. Because $a = XOR(b, xor)$ we have up to additive terms $O(c)$:

$$2\ell = K(a, b|\ell) \leq K(xor, b|\ell) \leq K(xor|\ell) + K(b|xor, \ell) \leq \ell + K(b|xor),$$

and this implies (7).

We modify the equations above for the case where w is not empty. Let $n = |bw|$ and recall that $|a| = \ell$. We start with

$$K(a, b, w|\ell, n) = K(a|\ell, n) + K(b, w|a, K(a|\ell, n), n) \geq \ell + n - O(c).$$

Note that because $\ell = |b|$ we have $K(bw, \dots|\ell, \dots) = K(b, w, \dots|\ell, \dots)$. The left-hand side also equals

$$K(xor, b, w|\ell, n) \leq K(xor|\ell, n) + K(b, w|xor, \ell, n) \leq \ell + K(b, w|xor, n),$$

hence $K(b, w|xor, n) \geq n - O(c)$. □

In the proof, we also use the characterization of 2-random sequences with plain complexity. This theorem was proven independently by J. Miller in [14] and by A. Nies, F. Stephan, and S. Terwijn in [17].

Theorem 7 *A sequence ω is 2-random if and only if there exists a c and infinitely many n such that $C(\omega_1 \dots \omega_n) \geq n - c$.*

The proof of the theorem relativizes to the halting problem θ' , i.e., a sequence is 3-random if and only if there are a c and infinitely many n such that $CH(\omega_1 \dots \omega_n) \geq n - c$.

Proof of Theorem 6 Let ω be 3-random. By Lemma 1, it suffices to construct infinitely many j such that

$$K(\omega_1 \dots \omega_j | j) \geq j - O(1) \tag{8}$$

and $K(\omega_1 \dots \omega_j | j, K(j)) \leq j - \log^{(2)} j + O(1)$. (Indeed, the last inequality implies $K(\dots) \leq j + K(j) - \log^{(2)} j + O(1)$ for the same reasons as in the proof of Theorem 2.) The second inequality follows from

$$K(\omega_1 \dots \omega_{\log^{(2)} j} | j, K(j)) \leq O(1). \tag{9}$$

Overview of the proof. The idea is to encrypt $\omega_1 \dots \omega_{\log^{(2)} j}$ in j , using an encryption key that can be computed from $K(j)$. To construct j , we first fix n , and then define i (which will be close to j information-wise) such that $K(i)$ contains maximal information relative to i . Next, let the “encryption” q be $XOR(\omega_1 \dots \omega_{\log K(i)}, \langle K(i) \rangle)$; i.e., the binary representation of $K(i)$ is used as the encryption key. j will be defined as an encoding of both i and q . Finally, we show that the construction implies that $K(i)$ and $K(j)$ can be computed from each other with $O(1)$ bits of information. This implies that from j and $K(j)$ we can compute $K(i)$ and the encryption q with $O(1)$ bits of information, and hence also $\omega_1 \dots \omega_{\log K(i)}$. At the same time, the construction must imply that n, i, q, j do not contain any information about ω .

Requirements for n, i and q . We choose infinitely many triples (n, i, q) and start with formulating five requirements from which (8) and (9) follow. Let $\langle \cdot, \cdot \rangle$ be a computable bijective pairing function from numbers and strings to numbers. For later use we assume that $\log \langle k, x \rangle = \log k + O(|x|)$ for all k and x .

Equation 8 with $j = \langle i, q \rangle$, follows from Lemma 3 and

- (a) $K(\omega_1 \dots \omega_n | i, q, n) \geq n - O(1)$,
- (b) $\langle i, q \rangle \leq n$ for large n .

Equation 9 follows from:

- (A) $K(\omega_1 \dots \omega_{\log K(i)} | K(i), q) \leq O(1)$,
- (B) $\log K(i) = \log^{(2)} \langle i, q \rangle + O(1)$,
- (C) $K(i, q) = K(i) + \log K(i) + O(1)$.

Indeed, for all z , (C) implies $K(z | i, q, K(i)) = K(z | j, K(j)) + O(1)$.

Construction of n and i . Fix an n such that $CH(\omega_1 \dots \omega_n) \geq n - c$, using the relativized version of Theorem 7. By Lemma 1:

$$KH(\omega_1 \dots \omega_n | n) \geq n - O(1). \tag{10}$$

From now on we only use complexities that are conditional to n . For notational simplicity we drop n from the condition, thus $K(a) \equiv K(a | n)$, $K(a | b) \equiv K(a | b, n)$, etc.

Let i be the largest number such that

- (i) $K(K(i)|i) \geq \log^{(2)} i - c$ and $K(i) \geq (\log i)/2$, where c is the constant from Theorem 3.
- (ii) $\langle i, x \rangle \leq n$ for all x of length at most $1 + \log^{(2)} i$.

Such i exists because also the conditional version of Theorem 3 holds. In fact, for increasing choices of n , we find infinitely many such i . By Lemma 2, the first condition implies

$$\log K(i) = \log^{(2)} i + O(1). \tag{11}$$

Note that i and $K(i)$ can be computed from $\mathbf{0}'$ and n , hence (10) implies

$$K(\omega_1 \dots \omega_n | i, K(i)) \geq n - O(1). \tag{12}$$

Construction of q . q is given by the bitwise XOR-function of $K(i)$ in binary, and the initial segment of ω with the same length:

$$q = XOR(\omega_1 \dots \omega_{\log K(i)}, \langle K(i) \rangle).$$

Because $XOR(a, XOR(a, b)) = b$ this implies (A).

Recall that all complexities implicitly have n in the condition and that $K(K(i)|i) \geq \log^{(2)} i + O(1)$. Together with (12), this can be applied to Lemma 4 (with $l = \log K(i) = \log^{(2)} i + O(1)$, $bw = \omega_1 \dots \omega_n$ and $a = \langle K(i) \rangle$) and we conclude that $K(\omega_1 \dots \omega_n | i, q, n) \geq n - O(1)$, i.e. condition (a).

For large n , we have large i , and hence $|q| = \log K(i) \leq \log(2 \log i) = 1 + \log^{(2)} i$. By choice of i (the second condition) this implies (b). We assumed that the pairing function satisfies $\log \langle i, q \rangle = \log i + O(|q|) = \log i + O(\log^{(2)} i)$. Thus $\log^{(2)} \langle i, q \rangle = \log^{(2)} i + O(1)$. By (11) this implies (B).

It remains to show (C). Note that

$$K(i, q) = K(i) + K(q|i, K(i)).$$

The last term equals $K(\omega_1 \dots \omega_{\log K(i)} | i, K(i))$. By (12) and Lemma 3 this is at least $\log K(i) + O(1)$, and in fact it is equal to this, because $K(z|z) \leq |z|$ for all z . \square

7 Contrasting Expectation and Probabilistically Bounded Deficiency

Recall from the introduction that there exist two different notions of randomness deficiency for a sequence ω . We start by showing that the two notions are related.

Proposition 2

$$d_P(\omega) = \sup\{k : d_E(\omega|k) \geq k\} + O(1) \tag{6}$$

⁶ Conditional probability bounded deficiency is defined in the natural way: it is the logarithm of a multiplicatively maximal function $f(\cdot|k)$ that is lower semicomputable uniformly in k , such that for each k the function is a probability bounded test.

This characterization is closely related to a characterization of plain complexity in terms of prefix complexity (see [13, Lemma 3.1.1 p. 203]):

$$C(x) = \min \{k : K(x|k) \leq k\} + O(1).$$

Many results relating and contrasting prefix and plain complexity on one side, can be translated to results about expectation and probability bounded deficiency. (In these results $d_E(\cdot)$ corresponds to $K(\cdot)$ and $d_P(\cdot)$ to $C(\cdot)$).

Proof For the \geq -direction we need to show that the supremum of the supremum defines a lower-semicomputable probability bounded test. d_E is lower semicomputable, thus also the supremum is lower semicomputable, and it remains to show that the measure where it exceeds ℓ is bounded by $O(2^{-\ell})$. By definition we have $\int 2^{d_E(\omega|k)} d\omega \leq 1$ for all k , thus the measure of ω such that $d_E(\omega|k) \geq k$ is at most 2^{-k} . If the supremum exceeds ℓ for some ω , then $d_E(\omega|k) \geq k$ for some $k \geq \ell$. The total measure for which this can happen is at most $2^{-\ell} + 2^{-\ell-1} + \dots \leq O(2^{-\ell})$.

For the \leq -direction note that every probability bounded test f defines a family of expectation bounded tests $g(\cdot|k)$ such that $g(\omega|k) = 2^k$ iff $f(\omega) \geq 2^k$. Indeed the condition implies $\int f(\omega|k) d\omega \leq 2^k \cdot 2^{-k} = 1$. Obviously, if f is lower semicomputable, the tests $g(\cdot|k)$ are lower semicomputable uniformly in k . If f is the universal test corresponding to d_P , then $d_P(\omega) \geq k$ implies $f(\omega) \geq 2^k$, which implies $g(\omega|k) \geq 2^k$ thus $d_E(\omega|k) \geq k - O(1)$. \square

The question was raised in [5, Question 1] whether the two deficiencies are related by a monotone function, or *does there exist two families of sequences ω^ℓ and ω'^ℓ such that*

$$d_A(\omega^\ell) - d_A(\omega'^\ell) \rightarrow \infty$$

for $\ell \rightarrow \infty$ and

$$d_P(\omega^\ell) - d_P(\omega'^\ell) \rightarrow -\infty.$$

We show this is indeed the case.

Theorem 8 *There exist families of sequences ω^ℓ and ω'^ℓ such that*

$$\begin{aligned} |d_P(\omega^\ell) - d_P(\omega'^\ell)| &\leq O(1) \\ d_E(\omega^\ell) - d_E(\omega'^\ell) &\geq \ell - O(1). \end{aligned}$$

The positive answer to the question above follows by prepending $\ell/2$ zeros to ω'^ℓ for all ℓ . This decreases the complexities in the definition of $d_P(\omega'^\ell)$ and $d_E(\omega'^\ell)$ by $\ell/2 + O(\log \ell)$ and hence increases these deficiencies by the same amount; and this is enough for the question.

Before presenting the proof, we show two lemmas that play the same role as symmetry of information and Levin's result relating plain and prefix complexity (i.e. Lemma 1).

Lemma 5 (Symmetry of deficiency) *For all ω and all x that belong to a prefix-free computably enumerable set of strings, we have*

$$d_E(x\omega) = |x| - K(x) + d_E(\omega|x, K(x)) + O(1),$$

here $x\omega$ denotes concatenation of x and ω . The $O(1)$ -term depends on the choice of the computably enumerable set.

The proof uses a characterization of expectation bounded deficiency in terms of prefix Kolmogorov complexity (see for example [5, Proposition 2.22]):

Theorem 9 $d_E(\omega|z) = \sup_n \{n - K(\omega_1 \dots \omega_n|z)\} + O(1)$

Proof of Lemma 5 Let x be a member of the prefix-free computably enumerable set. From xy we can compute x by enumerating the prefix-free set until an initial segment of xy appears and this segment must be x . Symmetry of information implies

$$K(xy) = K(x, y) + O(1) = K(x) + K(y|x, K(x)) + O(1),$$

i.e.

$$|xy| - K(xy) = |x| - K(x) + |y| - K(y|x, K(x)).$$

If we take on both sides the supremum of y over all prefixes of ω , we *almost* obtain the equation of the lemma; the problem is that in the definition of $d_E(x\omega)$ we also need to consider prefixes z of x . It remains to verify that

$$|z| - K(z) \leq |x| - K(x) + O(1)$$

for all prefixes z of x . In general this is false, but for x in a prefix-free enumerable set it holds. For any z and x , let $P(x|z) = 2^{-|x|+|z|}$ if x is an extension of z that belongs to the prefix-free set, otherwise let $P(x|z) = 0$. Note that $\sum_x P(x|z) \leq 1$ and $P(x|z)$ is lower-semicomputable, hence the coding theorem implies $K(x|z) \leq -\log P(x|z) + O(1) \leq |x| - |z| + O(1)$. Symmetry of information implies

$$K(x) \leq K(x, z) \leq K(z) + K(x|z) + O(1) \leq K(z) + |x| - |z| + O(1),$$

and this implies the equation above. □

The analogue of Lemma 1 for deficiencies of sequences is

Lemma 6 *For all j and ω*

$$|j - d_E(\omega|j)| = \Theta |j - d_P(\omega)|.$$

Proof For fixed random ω , the map $t \rightarrow d_E(\omega|t)$ maps points at distance d to points at distance $O(\log d)$. Hence, the map has a unique fixed point t within precision $O(1)$, i.e. $d_E(\omega|t) = t + O(1)$ for some t . This implies that t is $O(1)$ -close to the minimal s such that $d_E(\omega|s) \geq s$, i.e. $d_P(\omega)$. Our observation implies that $d_E(\omega|t + d) = t + O(\log d)$, thus for $j = t + d$ we have $j - d_E(\omega|j) = j - d_P(\omega) + O(\log(j - d_P(\omega)))$, and this implies the lemma. □

Proof of Theorem 8 For each ℓ we choose a k such that $\log^{(2)} k \leq \ell$ and $K(K(k)|k) \geq \log k - c$ where c is the constant from Theorem 3. By Lemma 2

$$\ell = \log^{(2)} k + O(1) = \log K(k) + O(1). \tag{13}$$

We choose ω such that

$$d_P(\omega|k, K(k)) \leq 1.$$

Let $0^k 1 \omega$ be the sequence that starts with k zeros, followed by a one and followed by ω . Let $0^k 1 \langle K(k) \rangle \omega$ be $0^k 1$ followed by $K(k)$ in binary, followed by ω . The theorem follows from the values of the expectation and probability bounded deficiencies of these strings, given in the table below (recall that $\ell = \log^{(2)} k + O(1)$):

α	$d_E(\alpha)$	$d_P(\alpha)$
$0^k 1 \omega$	$k - K(k) + O(1)$	$k + O(1)$
$0^k 1 \langle K(k) \rangle^\ell \omega$	$k - K(k) + \ell + O(1)$	$k + O(1)$

It remains to prove that the values in the table are correct.

The values of $d_E(\cdot)$ in the first column are obtained from Lemma 5. In the first case, the prefix-free set is the set of strings $0^m 1$ for all m , thus

$$d_E(0^k 1 \omega) = k - K(k) + d_E(\omega|k, K(k)) + O(1).$$

In the second case, the prefix-free set is the set of all strings $0^m 1 z$ for all m and all z of length $\log^{(2)} m$. Recall that $K(k, K(k)) = K(k) + O(1)$, thus

$$d_E(0^k 1 \langle K(k) \rangle \omega) = k + \log^{(2)} k - K(k) + d_E(\omega|k, K(k)) + O(1).$$

Recall that $l = \log^{(2)} k + O(1)$. This finishes the proof for the values of $d_E(\cdot)$ in the first column.

To evaluate $d_P(\cdot)$ we use Lemma 6. Hence, let us compute $d_E(0^k 1 \omega|k)$. Again we use Lemma 5:

$$d_E(0^k 1 \omega|k) = k - K(0^k 1|k) + d_E(\omega|K(0^k 1|k), k) + O(1) = k + d_E(\omega|k) + O(1).$$

This implies $d_P(0^k 1 \omega) = k + O(1)$. For the second case, note that $K(0^k 1 \langle K(k) \rangle|k) = K(K(k)|k) + O(1) = \log k + O(1)$ by choice of k . With similar reasoning we determine $d_P(0^k 1 \langle K(k) \rangle \omega)$:

$$d_E(0^k 1 \langle K(k) \rangle \omega|k) = (k + \log^{(2)} k) - K(K(k)|k) + d_E(\omega|K(k), k) + O(1).$$

This equals $k + O(1)$ by (13). □

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Appendix A: $KKK(x) = CCC(x) + O(CCCC(x))$ does not hold

Proposition 3 *There exist infinitely many x such that $CCCC(x) \leq O(\log^{(5)} |x|)$ and*

$$|CCC(x) - KKK(x)| \geq \Omega(\log^{(4)} |x|).$$

We use our main technique to contrast $C(\cdot)$ and $K(\cdot)$ to disprove that $KKK(x) = CCC(x) + O(CCCC(x))$. However, we will use the variant presented in the proof of [3, Corollary 6]. It is combined with a topological argument which was inspired by [18]. We start with a definition and a lemma.

Definition 1 A set S of numbers is c -dense in a superset A if for each $a \in A$ there is an $s \in S$ such that $|a - s| \leq c$.

Lemma 7 *If S is c -dense in an interval of size k , then the set*

$$\{K(k) : k \in S\}$$

is $O(\log c)$ -dense in some interval of size $\Omega(\log k - \log c)$.

Proof of Proposition 3 Let T be the set defined by the lemma. Note that the function $K(\cdot)$ maps points at distance d to points at distance $O(\log d)$, hence T is $O(\log c)$ -dense in $[\min T, \max T]$. It remains to show that the maximum of this set differs from its minimum by at least $\log k - O(\log c + \log^2 k)$. Let r be the minimal number in the interval of size k (in which S is dense) that ends with $\log k - 2$ zeros. By assumption r is at c distance of an element in S . On the other side, if the $\log k - 2$ last zeros of r are changed, the corresponding number remains always in the interval of size k , and for one such change the complexity of r must increase by at least $\log k - O(\log^2 k)$. (Otherwise, to many short descriptions exist of such modified r and we could use this to obtain a shorter description for r .) This element is c -close to an element in S , thus the difference of the minimum and the maximum of $K(k)$ over S is at least $\log k - O(\log c + \log^2 k)$. □

Proof of Proposition 3 For infinitely many n we construct strings x_i of length n such that

1. The values $K(x_i)$ are dense in an interval of size $\Omega(\log^{(2)} n)$, while all values $C(x)$ are contained in an interval of size $O(\log^3 n)$.
2. The values $KKK(x_i)$ are dense in an interval of size at least $\Omega(\log^{(4)} n)$, while all values $CCC(x_i)$ are contained in an interval of size $O(\log^5 n)$.
2. $CCCC(x_i) \leq O(\log^{(5)} n)$.

2 and 3 imply Proposition 3. By Lemma 7 we can already observe that 1 implies 2, thus it remains to show 1 and 3.

We start the construction by identifying two strings y and z of length n such that $K(y) - K(z) \geq \log^{(2)} n$ and $C(y) = C(z) = n$ (here and below we omit $O(1)$ terms). More specifically our construction implies $K(y) = K(n) + n - \log^{(2)} n$ and $K(z) = K(n) + n$. We use the construction of [3, Corollary 6] (which slightly differs

from the proof of Theorem 2). Let us repeat this construction. As usual, let n be such that $K(K(n)|n) \geq \log^{(2)} n$. For the proof of item 3, note that n exists for all values of $\log^{(2)} n$, and we choose such values that satisfy

$$C(\log^{(2)} n) \leq O(\log^{(5)} n) \tag{14}$$

(there exist infinitely many such n). Let z of length n be such that $C(z|K(n), n) \geq n$. Let y be the concatenation of $K(n)$ in binary and the last $n - \log^{(2)} n$ bits of z . By Lemma 2, the length of $K(n)$ in binary is $\log^{(2)} n$, thus $|y| = n$. The string y is the same constructed string as in the proof of [3, Corollary 6], and there it is shown using symmetry of information that $C(y) = n$ and $K(y) = K(n) + n - \log^{(2)} n$.

What happens if for some $i \leq \log^{(2)} n$ in this construction only the last i bits from $K(n)$ and the first $n - i$ bits from z are chosen? Let x_i be the string obtained in this way. Note that x_{i+1} is obtained from x_i by removing the last bit and prepending the $i + 1$ -th last bit of $K(n)$. This implies that $K(x_i) = K(x_{i+1}) + O(1)$. For $i = 0$ we have $x_i = y$ and $K(x_i) = K(n) + n - \log^{(2)} n$, and for $i = \log^{(2)} n$ we have $x_i = z$ and thus $K(x_i) = K(n) + n$. This implies that the values of $K(x_i)$ are $O(1)$ -dense in an interval of size $\log^{(2)} n$. Using symmetry of information in a similar way as before, one can show that $C(x_i) = n + O(\log i)$ (we use that any i -bit segment of $K(n)$ is $O(\log i)$ incompressible given i). Recall that $i \leq \log^{(2)} n$, thus this implies that all values $C(x_i)$ are contained in an interval of size $O(\log^{(3)} n)$, and this finishes the proof of item 1.

We show item 3. Recall that $C(x_i) = n + O(\log^{(3)} n)$, so we need to show that $CCC(n) \leq O(\log^{(5)} n)$. We know that $C(\log^{(2)} n) \leq O(\log^{(5)} n)$ but unfortunately, $CC(n)$ can contain much more information than $\log^{(2)} n$. We take another approach by showing that $CC(K(n)) \leq O(\log^{(5)} n)$ and $CCC(n) \leq O(CCK(n))$. The last inequality follows from footnote 2. For the first, note from footnote 3 that $C(K(n)) = \log^{(2)} n$, and by (14) this implies $CC(K(n)) \leq O(\log^{(5)} n)$. \square

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