FOLIATIONS 2012
ed. by Paweł WALCZAK *et al.*
World Scientific, Singapore, 2013
pp. 215–233

Local and global stability of leaves of conformal foliations*

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1. Basic concepts and results

One of the main goals of this work is to apply the previous results of the author [29, 30], and to prove new theorems on local and global leaf stability of conformal foliations of codimension $q > 2$. We also remind our results about local and global stability of compact leaves of foliations with quasi analytical holonomy pseudogroup admitting an Ehresmann connection and corresponding results of other authors.

Local stability of leaves and foliations

The notion of stability of leaves of foliations was introduced by Ehresmann and Reeb, the founders of the theory of foliations.

Remind that a subset of foliated manifold is called *saturated* if it may be represented as a union of some leaves of the foliation.

Definition 1. A leaf L of a foliation (M, F) is said to be *proper* if it is an

*This work was supported by the Federal Target Program 'Scientific and Scientific-Pedagogical Personnel', Project No 14.B37.21.036, and the Russian Federation Ministry of Education and Science, Project N 1.1907.2011.

embedded submanifold of the foliated manifold M. A foliation is proper, if every its leaf is proper. A leaf L is called closed if L is a closed subset of M.

Definition 2. A leaf L of a foliation (M, F) of codimension q is said to be locally stable in sense of Ehresmann and Reeb, if there exists a family of its saturated neighbourhoods W_β, $\beta \in \mathcal{B}$, with the following properties:

(1) there exists a locally trivial fibration $f_\beta : W_\beta \to L$, $\beta \in \mathcal{B}$, with a q-dimensional disk D^q as the typical fiber, whose fibers are transversal to the leaves of the foliation $(W_\beta, \mathcal{F}_{W_\beta})$;

(2) for some $\delta \in \mathcal{B}$ the traces of these neighbourhoods form a base of the topology of a fiber of the fibration $f_\delta : W_\delta \to L$ over $x \in L$ at the point x.

A foliation is refer to be *locally stable* if each its leaf is locally stable.

According to the well-known theorem of Reeb [23, 24], any compact leaf of a foliation with finite holonomy group is locally stable.

The leaf stability of Riemannian foliations

Blumenthal and Hebda [4] introduced a notion of Ehresmann connection for a smooth foliation (M, F) as a smooth q-dimensional distribution $\mathfrak{M}$ on M transverse to (M, F) with the vertical-horizontal property (the precise definition see in Section 2). We showed ([31], Proposition 2), that a complete Cartan foliation admits an Ehresmann connection. It is known examples of Riemannian foliations with an Ehresmann connection whose are not (transversally) complete (Example 1). Thus, the existence of an Ehresmann connection for a Cartan foliation (with fix transverse Cartan geometry) does not imply the completeness of this foliation in general.

The Proposition 1 describes the structure of a saturated neighbourhood of a proper leaf L, and $\Gamma(L, x)$, $x \in L$, is the germ holonomy group usually used in the foliation theory [24].

Proposition 1. *Let L be a proper leaf of a Riemannian foliation (M, F) with an Ehresmann connection $\mathfrak{M}$. Then there exist a bundle like metric g on M relatively which $\mathfrak{M}$ is orthogonal to (M, F) and a family of saturated tubular neighbourhood W_β of the radius $\beta \in (0, r]$, $r > 0$, with the orthogonal projection $f_\beta =: W_\beta \to L$, where $f_\beta = f_r|_{W_\beta}$, satisfying the following conditions:*

(1) the neighbourhood W_β is a smooth fibre space with the projection $f_\beta : W_\beta \to L$, and its structure group is the germ holonomy group $\Gamma(L, a)$, $a \in L$, of L. The the typical fibre is $D_\beta(a) = exp_a(D(0, \beta))$, where $D(0, \beta) \subset \mathfrak{M}_a$, the q-dimensional disk of radius β, on which $\Gamma(L, a)$ naturally acts by isomerties;

(2) the distribution $\mathfrak{M}_\beta = \mathfrak{M}|_{W_\beta}$ is an integrable Ehresmann connection for the submersion $f_\beta : W_\beta \to L$;

(3) the germ holonomy group of an arbitrary leaf $L(z) \subset W_\beta$, $z \in f_\beta^{-1}(a)$, is isomorphic to the stationary subgroup Γ_z of the group $\Gamma(L, a)$ at point z;

(4) the restriction $f_\beta|_{L(z)} : L(z) \to L$ is the covering map, and its set of sheets is bijective to the orbit $\Gamma(L, a) \cdot z$ of the point z under the action of $\Gamma(L, a)$.

The following assertion was proved with the use of Proposition 1.

Theorem 1. *Let (M, F) be a Riemannian foliation of an arbitrary codimension $q \geq 1$ admitting an Ehresmann connection. Then the following three conditions for a leaf L are equivalent:*

(i) L is locally stable leaf;
(ii) L is a proper leaf;
(iii) L is a closed leaf.

For transversally complete Riemannian foliations Theorem 1 and assertions equivalent to Proposition 1 were proved by the author in [32]. Under an additional assumption about the existence of a complementary topological foliation, the local stability of a proper leaf of Riemannian foliation has been proved by Ehresmann [10]. For parallel foliations on a complete Riemannin manifold the equivalence of conditions *(i)-(iii)* of Theorem 1 was proved in [15], where the proof is considerably simpler, due to specificity of the case. In [1] it was proved that a proper leaf of a Riemannin foliation on manifold with a complete bundle like metric is covered by all near leaves.

Theorem 2. *Let (M, F) be a Riemannian foliation of codimension $q \geq 1$ admitting an Ehresmann connection. If there exists a closed leaf L of (M, F) with a finite (germ) holonomy group $\Gamma(L, x)$, $x \in L$, then:*

(1) any its leaf L_α is closed subset of M with a finite holonomy group $\Gamma(L_\alpha, x_\alpha)$, $x_\alpha \in L_\alpha$, and L_α is a locally stable leaf;
(2) the leaf space M/F is a smooth q-dimensional orbifold.

Theorem 2 may be proved by analogy with the author's proof of similar Theorem 2 in [32]. Here we give a new proof of this statement.

Corollary 1. *Let (M, F) be a Riemannian foliation of codimension $q \geq 1$ with an Ehresmann connection. If any its leaf is a closed subset of M, then (M, F) is locally stable and the leaf space M/F is a smooth q-dimensional orbifold.*

In the case, when (M, F) is a Riemannian foliation on a Riemannian manifold with complete bundle like metric the statement of Corollary 1 was proved by Reinhart [21].

Theorem 3. *Let (M, F) be a Riemannian foliation of codimension $q \geq 1$ admitting an Ehresmann connection. If there exists a closed leaf L of (M, F) with a finite fundamental group $\pi_1(L, x)$, then any its leaf L_α is closed with a finite fundamental group $\pi_1(L_\alpha, x_\alpha)$, $x_\alpha \in L_\alpha$, and (M, F) is a locally stable Riemannian foliation.*

Based on statements of this section, Theorem 1 in [30] and the paper [28] we ask the following.

Question: *For a Riemannian foliation (M, F) with an Ehresmann connection $\mathfrak{M}$ there is a bundle like metric g such that $\mathfrak{M}$ is a orthogonal distribution to (M, F). Does there exist a transversally complete bundle like metric $\widetilde{g}$, which is $\mathfrak{M}$-conformal to g?*

Criterions of the local stability of leaves of conformal foliations

Using Theorems 1 and 2 and results of our previous paper [30] we prove the following two criterions of the local leaf stability for conformal foliations.

Theorem 4. *Let (M, F) be a conformal foliation of codimension $q > 2$ admitting an Ehresmann connection. Then a leaf L of (M, F) is locally stable if and only if L is a proper leaf with inessential holonomy group (in sense of Section 2).*

Theorem 5. *Let (M, F) be a proper non-Riemannian conformal foliation of codimension $q > 2$ admitting an Ehresmann connection. Then the following three conditions for a leaf L of (M, F) are equivalent:*

(i) L is locally stable;
(ii) L is an unclosed leaf;
(iii) L has a finite holonomy group $\Gamma(L, x)$.

The problem of local stability of compact foliations

A foliation is called compact, if every its leaf is compact. Epstein [11] proved that any leaf of a compact foliation (M, F) has a finite holonomy group iff the leaf space M/F is Hausdorff. Reeb showed that a codimension one compact foliation has a Hausdorff leaf space. Millett [18] put out the conjecture that all holonomy groups of a compact foliation on a compact manifold are finite. As it was said, according to the famous Reeb's theorem a compact leaf with a finite holonomy group is locally stable. Therefore the Millett's conjecture is called *a problem of local stability.* Now it is known that for $q = 2$ the Millett's conjecture is valid unlike the case $q = 3$. If the foliated manifold M is not compact, the analog of the Millett's conjecture is not true for compact foliations (M, F) of codimension 2. Different criterions of local stability of a compact foliations were proved [9, 18, 33, 34]. Among them there is Rummer's characterization of a compact locally stable foliation by the existence of a Riemannian metric with respect to which every leaf is a minimal submanifold. Epstein stated that (M, F) is a compact locally stable foliation iff there exists a Riemannian bundle like metric g on M such that the volume function of leaves is locally bounded.

The leaf local stability takes an important place in works on partially hyperbolic diffeomorphisms with compact central foliations [5, 13].

Definition 3. Pseudogroup of local diffeomorphisms $\mathcal{H}$ of a manifold N is quasi analytical, if for any open subset U in N and an element $h \in \mathcal{H}$, the condition $h|_U = id_U$ implies $h = id_{D(h)}$, where $D(h)$ is the connected domain of definition of h containing U.

The results of our works ([33], Theorem 5 and [34], Theorem 8.1) imply the following criterion of the local leaf stability of compact foliations (M, F) without assumption of compactness of M.

Theorem 6. *All holonomy groups of a compact foliation (M, F) are finite if and only if it satisfies the following two conditions:*

(1) there exists an Ehresmann connection for (M, F);
(2) the holonomy pseudogroup $\mathcal{H}(M, F)$ of this foliation is quasi analytical.

The effectivity of this criterion is confirmed by the following corollary.

Corollary 2. *Compact complete Cartan foliations and compact complete G-foliations of a finite type are locally stable. Leaf spaces of those foliations are smooth orbifolds.*

In the case when M is compact, Corollary 2 implies Theorem 1 of Wolak [26] for complete compact G-foliations of a finite type.

Lawson put out the following problem ([17], Problem 14):

Characterize the foliations of compact manifolds in which every leaf is compact.

Theorem 7 decides this problem for conformal foliations (M, F) of codimension $q > 2$ without assumption of compactness of the foliated manifold M.

Theorem 7. *Any compact conformal foliation of codimension $q > 2$ is a locally stable Riemannian foliation, the leaf space of which is a smooth q-dimensional orbifold.*

The analogous theorem for a compact transversely holomorphic foliation of codimention 2 was proved by Walczak [25].

Remark 1. If all leaves of a conformal foliation (M, F) of codimension $q > 2$ are closed subsets of M, then (M, F) is a Riemannian foliation, which is not local stable in general (see Example 2).

Global stability of a compact leaf of foliations with quasi analytical pseudogroup

For a leaf L of a foliation (M, F) with an Ehresmann connection $\mathfrak{M}$, Blumenthal and Hebda introduced a holonomy group $H_{\mathfrak{M}}(L, x)$ (its definition is given in Section 2).

Our results from [33, 34] implies the following statement.

Proposition 2. *Let (M, F) be a foliation with an Ehresmann connection $\mathfrak{M}$ and L be any its leaf. The natural group epimorphism $\chi : H_{\mathfrak{M}}(L, x) \to \Gamma(L, x)$ of holonomy groups is isomorphism if and only if the holonimy pseudogroup $\mathcal{H} = \mathcal{H}(M, F)$ of this foliation is quasi analytical.*

Application of Theorem 1 of Blumenthal – Hebda [4] and Proposition 2 allowed us to obtain the following assertion about the global stability of some compact leaves.

Theorem 8. *Let (M, F) be a smooth foliation with an Ehresmann connection and quasi analytical holonomy pseudogroup. Then the existence a compact leaf L with a finite germ holonomy group $\Gamma(L, x)$ (or finite fundamental group $\pi_1(L, x)$) guarantees compactness of every leaf L_α of this foliation and finiteness of the holonomy group $\Gamma(L_\alpha, x_\alpha)$, $x_\alpha \in L_\alpha$, (or finite fundamental group $\pi_1(L_\alpha, x_\alpha)$) and the local stability of (M, F).*

In particular, Theorem 8 implies the global stability of a compact leaf with finite germ holonomy group of complete Cartan foliations. For complete G-foliations of finite type the analogous result belongs to Wolak [26].

Global leaf stability of conformal foliations

Let (M, F) be a foliation. A *saturated set* is a union of leaves.

Theorem 9. *Let (M, F) be a conformal foliation of codimension $q > 2$ admitting an Ehresmann connection. If there exists a closed leaf L of (M, F) with a finite holonomy group $\Gamma(L, x)$ (or a finite fundamental group $\pi_1(L, x)$)), then any its leaf L_α is closed with a finite holonomy group $\Gamma(L_\alpha, x_\alpha)$, $x_\alpha \in L_\alpha$, (respectively, a finite fundamental group $\pi_1(L_\alpha, x_\alpha)$) and (M, F) is a locally stable Riemannian foliation, the leaf space of which is a smooth q-dimensional orbifold.*

Corollary 3 ([2]). *Let (M, F) be a complete conformal foliation of codimension $q > 2$. If there exists a compact leaf with a finite holonomy group, then any its leaf is compact with a finite holonomy group.*

Corollary 3 belonging to Blumenthal [2] was a unique known result about the leaf stability of a conformal foliation.

Corollary 4. *Let (M, F) be a conformal foliation of codimension $q > 2$ admitting an Ehresmann connection. If any its leaf is a closed subset of M, then (M, F) is a locally stable Riemannian foliation.*

Remark 2. We constructed an example of a complete transversally affine foliation (M, F) of an arbitrary codimension $q \geq 2$ with an Ehresmann connection such that (M, F) satisfies conditions of both Theorem 9 and Corollary 4, but it is not locally stable (Example 3). Thus, statements on stability of noncompact leaves (Theorems 9 and Corollary 4) can not be generalized to all complete Cartan foliations unlike statements on compact leaves (Corollary 2 and Theorem 8)

The following assertion about global stability of a compact leaf with a finite germ holonomy group was proved by us without assumption of completeness or the existence of an Ehresmann connection of the foliation (M, F).

Theorem 10. *If a conformal foliation (M, F) of codimension $q > 2$ on a compact manifold M has a compact leaf L with a finite holonomy group*

$\Gamma(L)$, *then any its leaf is compact with a finite holonomy group and* (M, F) *is a locally stable compact Riemannian foliation.*

The analogous theorem for holomorphic foliations of codimension k on compact complex Kaehler manifolds was proved by Pereira [20].

Remark 3. Theorems 3, 8–10 are some analogous of the well-known Reeb global stability theorem [23], according to which a smooth codimension one foliation (M, F) of a closed manifold M containing a compact leaf with a finite fundamental group has only compact leaves with finite fundamental groups.

2. Cartan foliations. Holonomy and completeness

Ehresmann connection for foliations

Remind the notion of an Ehresmann connection belongs to Blumenthal and Hebda [4]. At that we use a term *a vertical-horizontal homotopy* introduced earlier by Hermann [14].

Let $(M, \mathfrak{F})$ be a foliation of arbitrary codimension $q \geq 1$. A distribution $\mathfrak{M}$ on a manifold M is called *transversal* to a foliation $\mathfrak{F}$ if for any $x \in M$ the equality $T_xM = T_x\mathfrak{F} \oplus \mathfrak{M}_x$ holds, where $\oplus$ stands for a direct sum of vector spaces. Vectors from $\mathfrak{M}_x$, $x \in M$, are called horizontal. A piecewise smooth curve σ is horizontal (or $\mathfrak{M}$-horizontal) if each of its smooth segments is an integral curve of the distribution $\mathfrak{M}$. A distribution TF tangent to leaves of the foliation (M, F) is called vertical. One says that a curve h is vertical if h is contained in the leaf of the foliation (M, F).

A *vertical-horizontal homotopy* (v.h.h. for shot) is a piecewise smooth map $H : I_1 \times I_2 \to M$, where $I_1 = I_2 = [0, 1]$, such that for any $(s, t) \in I_1 \times I_2$ the curve $H|_{I_1 \times \{t\}}$ is horizontal and the curve $H|_{\{s\} \times I_2}$ is vertical. A pair of curves $(H|_{I_1 \times \{0\}}, H|_{\{0\} \times I_2})$ is called *a base of the v.h.h. H*. Two paths (σ, h) with common origin $\sigma(0) = h(0)$, where σ is a horizontal path and h is vertical one, are called an *admissible pair of paths.*

A distribution $\mathfrak{M}$ transversal to a foliation (M, F) is called an *Ehresmann connection for* (M, F) if for any admissible pair of paths (σ, h) there exists a v.h.h. with a base (σ, h).

Let $\mathfrak{M}$ be an Ehresmann connection for a foliation (M, F). Then for any admissible pair of paths (σ, h) there exists a unique v.h.h. H with base (σ, h). We say that $\widetilde{\sigma} := H|_{I_1 \times \{1\}}$ is the result of the *transfer of the path* σ *along* h *with respect to the Ehresmann connection* $\mathfrak{M}$. It is denoted by $\sigma \overset{h}{\to}> \widetilde{\sigma}$. Take any point $x \in M$. Denote by Ω_x the set of horizontal curves with the origin at x. An action of the fundamental group $\pi_1(L, x)$ of the leaf $L = L(x)$ on the set Ω_x is defined by the following a way:

$$\Phi_x : \pi_1(L, x) \times \Omega_x \to \Omega_x : ([h], \sigma) \mapsto \widetilde{\sigma},$$

where $[h] \in \pi_1(L, x)$ and $\widetilde{\sigma}$ is the result the transfer of σ along h relatively $\mathfrak{M}$. The quotient group $H_{\mathfrak{M}}(L, x) = \pi_1(L, x)/Ker(\Phi_x)$ of the kernel $Ker(\Phi_x)$ of the action Φ_x in $\pi_1(L, x)$ is a *group of* $\mathfrak{M}$*-holonomy of a leaf* L [4].

Cartan foliations

Notions belonging to Cartan geometry can be found in [16] and [6]. The definition of Cartan geometry $\xi = (P(N, H), \omega)$ of type (G, H) is equivalent to specifying the following objects:

(1) a Lie group G and its closed Lie subgroup H with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively;
(2) a principal H-bundle $\pi : P \to M$;
(3) a $\mathfrak{g}$-values 1-form ω on P called *a Cartan connection* having the following properties:
   (i) $\omega(A^*) = A$ for any $A \in \mathfrak{p}$, where A^* is the fundamental vector field corresponding to A;
   (ii) $R_a^*\omega = Ad_H(a^{-1})\omega$, $\forall a \in H$, where Ad_H is the adjoint representation of the Lie subgroup H in the Lie algebra $\mathfrak{g}$ of G;
   (iii) for any $u \in P$ the map $\omega_u : T_u(P) \to \mathfrak{g}$ is bijection.

Further we assume that Cartan geometry ξ of a type (G, H) is effective, i.e., the left action of the group G on G/H is effective. At that the Blumenthal's definition of a Cartan foliation [3] and our one [31] are equivalent.

Let N be q-dimensional manifold and M be a smooth n-dimensional manifold, $0 < q < n$. Unlike M the connectedness of the topological space N is not assumed. An N-cocycle is the set $\{U_i, f_i, \{k_{ij}\}\}_{i,j \in J}$ such that:

(1) The family $\{U_i, i \in J\}$ forms an open cover of M.
(2) The mappings $f_i : U_i \to N$ are submersions into N with connected fibers.
(3) If $U_i \cap U_j \neq \emptyset$, $i, j \in J$, then a diffeomorphism $k_{ij} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$ is well-defined and satisfies the equality $f_i = k_{ij} \circ f_j$.

Definition 4. Let a foliation (M, F) be given by an N-cocycle $\{U_i, f_i, \{k_{ij}\}\}_{i,j \in J}$. If the manifold N admits an effective Cartan geometry such that every local diffeomorphism k_{ij} is an isomorphism of the

Cartan geometries induced on open subsets $f_i(U_i \cap U_j)$ and $f_j(U_i \cap U_j)$, then we refer to (M, F) as *a Cartan foliation defined by the* (N, ξ)*-cocycle* $\{U_i, f_i, \{k_{ij}\}\}_{i,j \in J}$.

At the beginning we represent in the following statement about different interpretations of the holonomy groups of Cartan foliations, which was established in the previous work of the author ([30], Proposition 5).

Proposition 3. *Let* (M, F) *be an arbitrary Cartan foliation defined by* (N, ξ)*-cocycle* $\{U_i, f_i, \{k_{ij}\}\}_{i,j \in J}$ *and* $\pi : \mathcal{R} \to M$ *be the projection of the foliated H-bundle over* (M, F) *with lifted foliation* $(\mathcal{R}, \mathcal{F})$*. For each leaf* $L = L(x)$ *of* (M, F) *consider the leaf* $\mathcal{L} = \mathcal{L}(u)$*, where* $u \in \mathcal{R}$, $\pi(u) = x \in U_i$*, of the lifted foliation* $(\mathcal{R}, \mathcal{F})$ *and* $v = f_i(x)$*. Then the germ holonomy group* $\Gamma(L, x)$ *of L is isomorphic to the following groups:*

- *the subgroup* $H(\mathcal{L}) := \{a \in H \mid R_a(\mathcal{L}) = \mathcal{L}\}$ *of H;*
- *the group of covering transformations of the regular covering* $\pi|_{\mathcal{L}} : \mathcal{L} \to L$.
- *the group of germs at point v of local isomorphisms from the holonomy isotropy subpseudogroup* $\mathcal{H}_v(N, \xi)$.

If in conditions of the Proposition 3 we consider an other point $u' \in \pi^{-1}(x)$ and the leaf $\mathcal{L}' = \mathcal{L}'(u')$, then the group $H(\mathcal{L}')$ must be conjugated to $H(\mathcal{L})$ in H. Therefore, the following definition makes sense.

Definition 5. Refer to the holonomy group of a leaf L of a Cartan foliation as *relatively compact* or *inessential* if the corresponding subgroup $H(\mathcal{L})$ of the Lie group H is relatively compact. Otherwise the holonomy group of a leaf is called *essential.*

Completeness

Let $\mathfrak{M}$ be a smooth q-dimensional distribution on M transverse to a Cartan foliation (M, F) of codimension q and $\widetilde{\mathfrak{M}}$ be a smooth distribution on $\mathcal{R}$ transverse to the lifted foliation $(\mathcal{R}, \mathcal{F})$ such that $\pi_{*u}(\widetilde{\mathfrak{M}}_u) = \mathfrak{M}_{\pi(u)}$, $u \in \mathcal{R}$. Denote by $\mathfrak{X}(\mathcal{R})$ the set of smooth vector fields on $\mathcal{R}$ and by $\mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R})$ the subset of smooth vector fields tangent to $\widetilde{\mathfrak{M}}$. A Cartan foliation (M, F) is called *complete* (or $\mathfrak{M}$-complete) if any $\omega_{\mathcal{R}}$-constant vector field $X \in \mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R})$ is complete, where $\omega_{\mathcal{R}}$ is $\mathfrak{g}$-valued base 1-form on $\mathcal{R}$ induced by Cartan connection ω [2, 31]. As was proved by us in [31] (Proposition 2), if (M, F) is a $\mathfrak{M}$-complete Cartan foliation, then $\mathfrak{M}$ is an Ehresmann connection for (M, F). It is naturally true for conformal and Riemannian foliations. An $\mathfrak{M}$-complete Riemannian foliation with a bundle like metric, where $\mathfrak{M}$ is the complementary orthogonal distribution to this foliation, is called a transversally complete one.

3. Proofs of statements for Riemannian foliations

Proof of Proposition 1. Let (M, F) be a Riemannian foliation with an Ehresmann connection $\mathfrak{M}$ defined by N-cocycle $\{U_i, f_i, \{k_{ij}\}\}_{i,j \in J}$ and $\mathfrak{M}$ be a transverse distribution to (M, F). Then, as known (see, for instance [30], Proposition 1) there exists such Riemannian metric g on M and g_N on N, that:

(i) the distribution $\mathfrak{M}$ is orthogonal to the foliation (M, F), and every submersion $f_i : U_i \to V_i = f_i(U_i)$ is Riemannian;
(ii) any geodesic γ on the Riemannian manifold (M, g), which is tangent to $\mathfrak{M}$ at one point, is tangent to $\mathfrak{M}$ at every point;
(iii) for every admissible pair of paths of the form (σ, h), where σ is $\mathfrak{M}$-horizontal geodesic, the result $\widetilde{\sigma}$ of the transfer $\sigma \overset{h}{\to}> \widetilde{\sigma}$ is also $\mathfrak{M}$-horizontal geodesic of the same length as σ, i.e. $l(\widetilde{\sigma}) = l(\sigma)$.

The metric g is a bundle like one in terminology of Reinhart [22]. Denote by d the distance function of the Riemannian manifold (M, g).

Suppose now that a leaf L is proper. Let S be a connected open relatively compact subset in the leaf L. Then ([19], p. 73) there are such $\varepsilon > 0$ and an open contractible neighbourhood V_ε satisfying the following properties:

(i) For any $y \in V_\varepsilon$ there exists a unique $x =: f(y) \in S$ and a unique vector $X \in \mathfrak{M}_x$ such that $y = exp_x(X)$ and $||X||_x = d(y, S)$.
(ii) The orthogonal projection $f : V_\varepsilon \to S$ thus defined is trivial fibration whose typical fiber is the open disc $D(0, \varepsilon)$ in $\mathfrak{M}_a$, where a is a fixed point in S.

One say that V_ε is *a tubular neighbourhood of* S with radius ε. As a leaf L is proper, according to ([24], Theorem 4.11) without loss generality we assume that $L \cap V_\varepsilon = S$.

There exists a normal convex neighbourhood $B(a, 2r) \subset V_\varepsilon$ at a with radius $2r$. The set of convex neighbourhoods $B(a, \beta)$, $\beta \in (0, r]$ forms a base of the topology of M at point a. Put $W_\beta := \cup L_\alpha$, where $L_\alpha \cap B(a, \beta) \neq \emptyset$. Then every W_β is an open saturated neighbourhood of L ([24], Theorem 4.10).

We shall use notation $D_\beta(a) = exp_a(D(0,\beta))$, where $D(0,\beta) \subset \mathfrak{M}_a$. Remark that $D_\beta(a)$ is the diffeomorphism image of $D(0,\beta)$. Show that the family of neighbourhoods $\{W_\beta \,|\, \beta \in (0,r]\}$ satisfies to Definition 2.

Define a map $f_\beta : W_\beta \to L$ by the following a way. Take any point $z \in W_\beta$. According to the definition of W_β there is a point $y \in L(z) \cap D_\beta(a)$. There is a unique horizontal geodesic γ jointing $a = \gamma(0)$ with $y = \gamma(1)$, and $\gamma(s) \in D_\beta(a)$, $\forall s \in [0,1]$. Connect y with z by a piecewise smooth path h in the leaf $L(z)$. Then (γ^{-1}, h) is an admissible pair of paths and there exists the transfer $\gamma^{-1} \overset{h}{\to\!>} \widetilde{\gamma}^{-1}$. Put $f_\beta(z) = \widetilde{\gamma}(0) \in L$. Show that this definition takes meaning.

Let h' be an other piecewise smooth path in the leaf $L(z)$ connecting $h'(0) = y$ with $h'(1) = z$ and $\gamma^{-1} \overset{h'}{\to\!>} \gamma'^{-1}$. Then $f_\beta(z) = \gamma'(0) \in L$. Consider the transfer $\gamma'^{-1} \overset{h^{-1}}{\to\!>} \widehat{\gamma}^{-1}$, then $\widehat{\gamma}^{-1}(1) = y$. According the above property (iii) $l(\widehat{\gamma}) = l(\gamma)$. Therefore $d(\gamma(0), \widehat{\gamma}(0)) \leq d(a,y) + d(y, \widehat{\gamma}(0)) = 2l(\gamma) < 2\tau$ and $\widehat{\gamma}(0) \in V_\varepsilon \cap L(z) = S$. So it is necessary that $\gamma(0) = \widehat{\gamma}(0) = y$ and $\widehat{\gamma} = \gamma$. It implies the equality $\widetilde{\gamma} = \gamma'$. Thus f_β does not depend of the choice of the path connecting y with z.

Consider the case, when there is an other point $y' \in L(z) \cap D_\beta(a)$. Let k be a path connecting y' with z in L and σ be a horizontal geodesic in $D_\beta(a)$ joints a with y'. By analogy with the above arguments we see that the result of the transfer of γ^{-1} along $h \cdot k^{-1}$ is equal to σ. Therefore the result of the transfers of γ^{-1} along h and σ^{-1} along k are coincided, i.e. $\widetilde{\gamma} = \widetilde{\sigma}$ and $f_\beta(z) = \widetilde{\gamma}(0) = \widetilde{\sigma}(0)$. Thus, the map $f_\beta : W_\beta \to L$ is really defined. It is not difficult to check that $f_\beta : W_\beta \to L$ is a surjective submersion. Note that the foliation $F_\beta = F|_{W_\beta}$ is an integrable Ehresmann connection for the submersion f_β. Therefore $f_\beta : W_\beta \to L$ is the projection of the locally trivial fibration whose typical fiber is the open q-dimensional disc $D_\varepsilon(a)$ on which the holonomy group $\Gamma(L,a)$ naturally acts by isometries. Thus, the statements (1) and (2) are valid.

Remark that $(W_\beta, F|_{W_\beta})$ is a suspended foliation with $f_\beta : W_\beta \to L$ as transverse fibration. This foliation is defined by suspension of the natural group homomorphism $\pi_1(L,a) \to \Gamma(L,a)$, where $\Gamma(L,a)$ is considered as a subgroup of the diffeomorphism group of $D_\beta(a)$. Therefore, thanks to the quasi analyticity of $\Gamma(L,a)$ the properties (3) and (4) also take place. □

Proof of Theorem 1. It is follows from Definition 2 that a locally stable leaf L is proper. In conformity with Proposition 1 the family of saturated neighbourhoods W_β, $\beta \in (0,\varepsilon]$, of a proper leaf L satisfies Definition 2, i.e.,

(i) is equivalent to (ii).

Let L be a proper leaf of a Riemannian foliation (M,F) admitting an Eresmann connection. We proved ([30], Theorem 1) that the holonomy pseudogroup of Riemannian foliation admitting an Eresmann connection is complete and the closure of any its leaf is a minimal set. As a nontrivial minimal set contains only improper leaves it is necessary that L be a closed leaf. As it is well known (see for instance [19], p. 22), every closed leaf is proper. Thus, (ii) is equivalent to (iii). □

Proof of Theorem 2. Let $\mathfrak{M}$ be an Ehresmann connection for a Riemannian foliation (M,F). Consider the foliated H-bundle $\pi : \mathcal{R} \to M$, where H is $O(q)$ (or $H = SO(q)$, if (M,F) is a transversally orientable Riemannian foliation), with the lifted foliation $(\mathcal{R},\mathcal{F})$ over the given conformal foliation (M,F). Observe that the induced distribution $\widetilde{\mathfrak{M}} = \{\widetilde{\mathfrak{M}}_u \,|\, u \in \mathcal{R}\}$, where $\widetilde{\mathfrak{M}}_u = \{X \in T_u\mathcal{R} \,|\, \pi_{*u}(X) \in T_xM \,, x = \pi(u)\}$ is an Ehresmann connection for $(\mathcal{R},\mathcal{F})$.

Let L be a closed leaf with a finite holonomy group $\Gamma(L,x)$. Take an arbitrary point $u \in \pi^{-1}(x)$. Denote by $\mathcal{L}$ the leaf of $(\mathcal{R},\mathcal{F})$ passing through u. In accordance with Proposition 2, the finiteness of $\Gamma(L,x)$ implies that the restriction $\pi|_{\mathcal{L}} : \mathcal{L} \to L$ is a finite sheet covering. As the closed leaf L is proper, $\mathcal{L}$ is a proper leaf of the lifted foliation. Thus the e-foliation $(\mathcal{R},\mathcal{F})$ with an Ehresmann connection has a proper leaf. Thanks this, by analogy with the proof of Proposition 4.4 from the Conlon's work [8] it is not difficult to show that leaves of $(\mathcal{R},\mathcal{F})$ are coincided with fibres of a locally trivial bundle $\pi_b : \mathcal{R} \to W$. Therefore every leaf of the lifted foliation is proper and closed.

Take any point $y \in M$. Let $y \in L_\alpha$ and $\mathcal{L}_\alpha$ is a leaf of $(\mathcal{R},\mathcal{F})$ over L_α, i.e. $\mathcal{L}_\alpha \subset \pi^{-1}(L_\alpha)$. Therefore the intersection $\pi^{-1}(y) \cap \mathcal{L}_\alpha$ is discrete and closed subset of $\pi^{-1}(y)$. The fiber $\pi^{-1}(y)$ is compact, because it is diffeomorphic to the compact Lie group H. It implies the finitness of the set $\pi^{-1}(y) \cap \mathcal{L}_\alpha$. As the group of the covering transformations of the regular covering $\pi_{\mathcal{L}_\alpha} : \mathcal{L}_\alpha \to L_\alpha$ is bijective to the set $\pi^{-1}(y) \cap \mathcal{L}_\alpha$, it is finite. In accordance with Proposition 3 the holonomy group $\Gamma(L_\alpha, y)$ of L_α is also finite. The formula

$$R^W : W \times H \to W : (w,a) \mapsto \pi_b(R_a(u)), \forall (w,a) \in W \times H, \forall u \in \pi_b^{-1}(w),$$

defines a smooth action of the group H on the base manifold W (see for example, [31], Proposition 4). Observation that the stationary group H_v, $v \in W$, of the action R^W is isomorphic to the group of covering transformations of the covering $\pi|_{\mathcal{L}_\alpha} : \mathcal{L}_\alpha \to L_\alpha$, where $\mathcal{L}_\alpha = \pi_b^{-1}(v)$, implies the finiteness of all stationary groups. Therefore the orbit space W/H of this action is a smooth q-dimensional orbifold.

Denote by $f : M \to M/F : L \mapsto [L]$ and $f^W : W \to W/H$ the corresponding projections. The map

$$\kappa : M/F \to W/H : [L] \mapsto \pi_b(\pi^{-1}(f^{-1}[L])), \ \forall [L] \in M/F,$$

is well defined and is a bijection. Remind that both f and f^W are open mappings. The submersions π and π_b are also open maps. Thank this, $\kappa : M/F \to W/H$ is a homeomorphism of the topological spaces. Thus we have a natural identification of M/F with N/H through κ, hence M/F is a q-dimensional orbifold. □

Proof of Corollary 1. Let (M, F) be a Riemannian foliation all leaves of which are closed subsets of M. By Theorem of Epstein-Millett-Tishler [12] there exists a saturated dense G_δ subset of M formed by leaves without holonomy. Therefore there exists a closed leaf L of (M, F) with the trivial holonomy group. The application of Theorem 2 finishes the proof. □

Proof of Theorem 3. Suppose that there exists a closed leaf L with a finite fundamental group $\pi_1(L, x)$. As the holonomy group $\Gamma(L, x)$ is a group homomorphism image of $\pi_1(L, x)$, the group $\Gamma(L, x)$ is finite. According to Theorem 2 every leaf L_α of (M, F) is closed and has a finite holonomy group $\Gamma(L_\alpha, x_\alpha)$, where $x_\alpha \in L_\alpha$, and (M, F) is a locally stable foliation.

We shall use notations introduced above. Let $\pi : \mathcal{R} \to M$ be the projection of the foliated H-bundle over M, where H is equal to $O(q)$ or $SO(q)$. Using Proposition 3 we see that $\pi|_{\mathcal{L}} : \mathcal{L} \to L$ is a finite sheet covering map onto L, so the fundamental group $\pi_1(\mathcal{L}, u)$, $\pi(u) = x$, is also finite. Therefore the universal covering map $f : \mathcal{L}^0 \to \mathcal{L}$ is a finite sheet covering.

Let L_α be any leaf of (M, F), $x_\alpha \in L_\alpha$ and $u_\alpha \in \pi^{-1}(x_\alpha)$. Denote by $\mathcal{L}_\alpha$ the leaf of lifted foliation $(\mathcal{R}, \mathcal{F})$ passing through u_α. In accordance with Proposition 3 the group of covering transformations of the regular covering map $\pi|_{\mathcal{L}_\alpha} : \mathcal{L}_\alpha \to L_\alpha$ is isomorphic to the group $\Gamma(L_\alpha, x_\alpha)$, hence it is a finite sheet covering map.

In the proof of Theorem 2 we have showed that the existence of an Ehresmann connection for (M, F) implies the existence of an Ehresmann connection for the lifted foliation $(\mathcal{R}, \mathcal{F})$. Therefore the leaves $\mathcal{L}$ and $\mathcal{L}_\alpha$ are diffeomorphic. So the universal covering map $f^0_\alpha : {\mathcal{L}_\alpha}^0 \to \mathcal{L}_\alpha$ is also a finite sheet covering. Therefore $\pi|_{\mathcal{L}_\alpha} \circ f^0_\alpha : {\mathcal{L}_\alpha}^0 \to L_\alpha$ is the finite sheet universal covering map. It implies the finiteness of the fundamental group $\pi_1(L_\alpha, x_\alpha)$. □

4. Proofs of other theorems and their corollaries

Proof of Theorem 4. If (M, F) is a Riemannian foliation, the assertion of Theorem 3 follows from Theorem 1.

Further by an attractor of a foliation (M, F) we understand a nonempty closed saturated subset $\mathcal{M}$ of M admitting an open saturated neighbourhood $\mathcal{U}$ such that the closure in M of any leaf from $\mathcal{U} \setminus \mathcal{M}$ contains $\mathcal{M}$. The neighbourhood $\mathcal{U}$ is called a basin of $\mathcal{M}$ and denoted by $\mathcal{A}ttr(\mathcal{M})$. If, moreover, $\mathcal{A}ttr(\mathcal{M}) = M$, then the attractor $\mathcal{M}$ is called *global.*

Assume now that (M, F) is non-Riemannian conformal foliation of codimension $q > 2$ with an Ehresmann connection $\mathfrak{M}$. By our Theorem 4 from [30], (M, F) is complete. Therefore, in accordance with Theorem 5 proved by us in [29] (M, F) has the following properties:

- there exists a global attractor $\mathcal{M}$ of this foliation, that is either one closed leaf or the union of two closed leaves, or the nontrivial minimal set;
- the induced foliation (M_0, F_{M_0}), where $M_0 = M \setminus \mathcal{M}$, is Riemannian.

Note that the restriction $\mathfrak{M}_0$ of the distribution $\mathfrak{M}$ onto the open saturated subset M_0 is an Ehresmann connection for the Riemannian foliation (M_0, F_{M_0}).

Assume that L is a local stable leaf of (M, F). Agreeably to Definition 2 any local stable leaf is proper and is not an attractor. If $\mathcal{M}$ is nontrivial minimal set, then any leaf from $\mathcal{M}$ is improper. Therefore, the leaf L belongs to the Riemannian foliation (M_0, F_{M_0}) admitting an Ehresmann connection $\mathfrak{M}_0$. So in accordance with our Theorem 3 from [30] the holonomy group of the leaf L is inessential.

Converse, suppose that L is a proper leaf with inessential holonomy group of non-Riemannian conformal foliation (M, F) of codimension $q > 2$. Then by Theorem 5 from [30] it is necessary $L \subset M_0$. Thus, L is a proper leaf of the Riemannian foliation (M_0, F_{M_0}) with an Ehresmann connection. According to Theorem 1 L is a local stable leaf. □

Proof of Theorem 5. Consider a proper non-Riemannian conformal foliation (M, F) of codimension $q > 2$. Then there exists a global attractor $\mathcal{M}$ which is either a closed leaf with an essential holonomy group or the union of two closed leaves with essential holonomy groups ([30], Theorem 6).

Put $M_0 = M \setminus \mathcal{M}$. As it was observed in the proof of Theorem 3, in this case (M_0, F_{M_0}) is a Riemannian foliation with an Ehresmann connection on the open saturated subset M_0 of M. Therefore (M_0, F_{M_0}) is proper foliation.

Application of Theorem 1 and Corollary 1 allowed us to state that every leaf L of (M_0, F_{M_0}) is locally stable and closed in M_0 with a finite holonomy group. Hence the closure $Cl(L)$ of L in M is equal to $Cl(L) = L \cup \mathcal{M}$ and L is locally stable unclosed leaf of (M, F) with a finite holonomy group. Thus, $L \subset M_0$ iff L is an unclosed leaf or, equivalent, L has a finite holonomy group.

The remark that L is a local stable leaf of (M, F) iff $L \subset M_0$ leads to the finish of the proof. □

Proof of Theorem 7. As (M, F) is a compact conformal foliation (M, F) of codimension $q > 2$, it has not an attractor. According to Theorem 2 proved by us in [29] in this case (M, F) is a Riemannian foliation. It is well known (see for instance, [19], Proposition 3.7) a codimension q compact Riemannian foliation is locally stable, and its leaf space M/F admits a structure of smooth q-dimensional orbifold. □

Proof of Theorem 9. According to our Theorem 4 proved in [30] for non-Riemannian conformal foliation (M, F) of codimension $q > 2$ the existence of an Ehresmann connection is equivalent to its completeness. Therefore Theorem 5 of [30] implies that if a conformal foliation (M, F) admitting an Ehresmann connection has a closed leaf with a finite holonomy group $\Gamma(L, x)$ or a finite fundamental group $\pi_1(L, x)$, then (M, F) is a Riemannian foliation. So the statements of Theorem 9 follow from Theorem 2 and Theorem 3 respectively. □

Proof of Corollary 4. If all leaves of conformal foliation (M, F) of codimension $q > 2$ are closed, then (M, F) has not attractors. Thus (M, F) is a Riemannian foliation satisfying Corollary 1. □

Proof of Theorem 10. Let (M, F) be a conformal foliation of codimension $q > 2$ on a compact manifold M. Emphasize that a finite holonomy group is inessential. Therefore in accordance with Theorem 4 proved by the author in [29] the existence of a compact leaf with a finite holonomy group implies that (M, F) is a Riemannian foliation. As M is compact, a bundle like metric g on M relatively the foliation (M, F) is complete. Then the q-dimensional distribution orthogonal to TF is an Ehresmann connection for this foliation. Therefore the assertion of Theorem 10 follows from Theorem 9 (and also from Theorem 8). □

5. Examples

Example 1 ([27]). Let $E^3 = E^1 \times E^2$ be an 3-dimensional Euclidian space. Put $M = E^3 \setminus (E^1 \times \{(0,0)\})$, where $(0,0) \in E^2$. Then the foliation (M, F), where $F = \{E^1 \times \{(x,y)\} \mid (x,y) \in E^2 \setminus (0,0)\}$, is not transversally complete Riemannian foliation admitting an Ehresmann connection.

Example 2. Consider a product of circles $S^1 \times S^1$. Let $p : S^1 \times S^1 \to S^1 : (x,y) \mapsto x$ be the canonical projection onto the first multiplier. Put $M = (S^1 \times S^1) \setminus \{(a,b)\}$, where $(a,b) \in S^1 \times S^1$ and $p_M = p|_M$. Let (M, F) be the foliation formed by fibres of the submersion $p_M : M \to S^1$.

The foliation (M, F) is Riemannian, and every its leaf is a closed subset of M with the trivial holonomy group. This foliation has a compact leaf diffeomorphic to the circle S^1 with finite holonomy group. As there exists a noncompact leaf $L_0 = p_M^{-1}(a)$, the foliation (M, F) is not locally stable. Remark that (M, F) does not admit an Ehresmann connection. This example shows that the existence of an Ehresmann connection is the essential condition in Theorems 1–3, 6 and Corollary 1.

Example 3. Denote by f_A the linear transformation of the plane R^2 having the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in the canonical basis. Define an action of the group the integral numbers $\mathbb{Z}$ on the product $R^1 \times R^2 \times R^m \cong R^{3+m}$, where m is an arbitrary nonnegative integer number, by the following formula

$$\Phi : \mathbb{Z} \times R^1 \times R^2 \times R^m \to R^1 \times R^2 \times R^m : (n, t, z, w) \mapsto (t - n, (f_A)^n(z), w)$$

for all $n \in \mathbb{Z}$ and $(t, z, w) \in R^1 \times R^2 \times R^m$. As the action Φ of $\mathbb{Z}$ is proper and free, the $(m+3)$-manifold of orbits $M = R^1 \times_{\mathbb{Z}} (R^2 \times R^m)$ is defined. Let $p : R^1 \times R^2 \times R^m \to M$ be the projection on the orbit space. We get a foliation (M, F) of codimension m+2 covered via p by the trivial foliation $F_{tr} = \{R^1 \times \{(z,w)\} \mid (z,w) \in R^2 \times R^m\}$. Note that the other trivial foliation $F'_{tr} = \{\{t\} \times R^2 \times R^m \mid t \in R^1\}$ of the product $R^1 \times R^2 \times R^m$ is projected by p onto a simple foliation (M, F') transversal to (M, F). Therefore the distribution $\mathfrak{M} = TF'$ tangent to (M, F') is an integrable Ehresmann connection for (M, F). Moreover, it is not difficult to see that (M, F) is $\mathfrak{M}$-complete transversally affine foliation. Observe that all leaves of (M, F) are closed subsets in M.

Let $r : R^1 \times R^2 \times R^m \to R^2 \times R^m$ be the projection onto the multiplier. For any point $v \in M$ there exists a point $(z, w) \in r(p^{-1}(v)) \in R^2 \times R^m$. Emphasize that a leaf $L = L(v)$ is compact and diffeomorphic to the circle S^1 iff $z = (x, 0) \in R^2$ in the the coordinates defined by the canonical basis of R^2. Note that any neighbourhood of a point $(z, w) = ((x, 0), w)$ in $R^2 \times R^m$ contains a subset of the form $\{((x + ny, x), w) \mid n \in \mathbb{Z}\}$ for some $y > 0$. So does not exist a neighbourhood of a point $((x, 0), w)$ invariant relatively the action Φ of the group $\mathbb{Z}$ and belonging to an ε-neighbourhood of $((x, 0), w)$ in the usual topology in $R^2 \times R^m$. It means that any compact leaf $L = L(v)$ is not stable unlike noncompact leaves diffeomorphic to R^1.

Thus, the constructed foliation (M, F) is not locally stable.

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Received January 16, 2013.