ISSN 1990-4789, Journal of Applied and Industrial Mathematics, 2013, Vol. 7, No. 2, pp. 221–228. © Pleiades Publishing, Ltd., 2013.
Original Russian Text © D.S. Malyshev, 2012, published in Diskretnyi Analiz i Issledovanie Operatsii, 2012, Vol. 19, No. 6, pp. 37–48.

A Study of the Boundary Graph Classes for Colorability Problems

D. S. Malyshev[1,2*]

[1]*National Research University Higher School of Economics, B. Pecherskaya ul. 25/12, Nizhni Novgorod, 603155 Russia*

[2]*Nizhni Novgorod State University, ul. Gagarina 23, Nizhni Novgorod, 603950 Russia*

Received December 29, 2011; in final form, March 24, 2012

Abstract—The notion of a boundary class of graphs is a helpful tool for the computational complexity analysis of graph theory problems in the family of hereditary classes. Some general and specific features for families of boundary classes of graphs for the vertex k-colorability problem and its "limit" variant, the chromatic index problem, were studied by the author earlier. In the present article, these problems are considered in application to the edge version of the colorability problem. It turns out that each boundary class for the edge 3-colorability problem is a boundary class for the chromatic index problem; however, for each $k > 3$, there exist uncountably many boundary classes for the edge k-colorability problem each of which is not a boundary class for the chromatic index problem. We formulate some necessary condition for the existence of boundary classes of graphs for the vertex 3-colorability problem which are not boundary for the chromatic index problem. To the author's mind, this condition is never satisfied and so there are no such boundary classes for the vertex 3-colorability problem.

DOI: 10.1134/S1990478913020117

Keywords: *computational complexity, boundary class of graphs, colorability problem*

INTRODUCTION

Let us continue studying the "borderline" between "easy" and "hard" graph classes for colorability problems for hereditary classes of graphs; i.e., the classes of graphs closed under isomorphism and vertex deletion. It is well known that every hereditary class of graphs $\mathcal{X}$ can be defined by the set of its forbidden induced subgraphs $\mathcal{S}$; this is written as $\mathcal{X} = Free(\mathcal{S})$. If $\mathcal{S}$ is finite then $\mathcal{X}$ is called *finitely defined*.

Let Π be a problem on graphs. A hereditary class of graphs is called Π-*easy* if Problem Π is polynomially solvable in this class; and Π-*hard*, otherwise. In what follows, we assume that P $\neq$ NP, and this condition is not included explicitly in the statements of the assertions. For example, if Problem Π is NP-complete for some hereditary class of graphs then the class is Π-hard.

A hereditary class of graphs $\mathcal{X}$ is called Π-*limit* whenever there is an infinite sequence $\mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \dots$ of Π-hard classes of graphs such that

$$\mathcal{X} = \bigcap_{i=1}^{\infty} \mathcal{X}_i.$$

An inclusion minimal Π-limit class is called a Π-*boundary* class. In [4], V. E. Alekseev introduced the notion of a boundary class of graphs which reveals the meaning of this notion:

Theorem 1. *A finitely defined class of graphs $\mathcal{X}$ is Π-hard if and only if $\mathcal{X}$ contains a Π-boundary class.*

Thus, once the set of Π-boundary classes of graphs is known, we can fully describe the whole set of finitely defined Π-easy classes of graphs. But at present such a description is obtained for no problem Π. At the same time, the difficulties arising in getting such descriptions prompted the author to suggest

*E-mail: dsmalyshev@rambler.ru

that, for some problems on graphs, the set of boundary classes can be rather intricate; therefore, attempts at giving its description are apparently doomed to failure. This idea was confirmed by the fact that, for both of the 3-colorability problems (the vertex and edge versions), in [2], we found some continual sets of the boundary classes. Thus, we constructively proved the assumption of [5] on the existence of problems on graphs with infinitely many boundary classes.

The investigation of the structure of boundary classes of graphs for the colorability problems was continued in [3, 7], where we considered the vertex k-colorability problems and the corresponding "limit" problem (the chromatic index problem). Recall that the chromatic index of a graph G is the least number of the colors with which it is possible to color the vertices of G so that every two adjacent vertices are painted with different colors. In the vertex k-colorability problem (the *k-VC problem*), it is required to determine whether the chromatic index of the graph G is greater than a fixed number k. In the chromatic index problem (*CI-problem*), we must answer the same question provided that k is given together with the graph G.

In [3, 7], the focus is on the common and different features of the boundary classes of graphs for the k-colorability and chromatic index problem. In [7], for each $k > 3$, there is found a continual collection of k-VC-boundary classes of graphs none of which is CI-boundary. On the other hand, in [3], we found a CI-boundary class that is k-VC-boundary for no k.

The present article is mainly devoted to the comparative analysis of boundary classes of graphs for the edge k-colorability problem and the chromatic index problem. Recall that a *proper edge k-coloring* (*edge k-coloring*) of a graph G is a mapping

$$f : E(G) \to \{1, 2, \dots, k\}$$

such that $f(e_1) \neq f(e_2)$ for every adjacent edges e_1 and e_2 in G. In the vertex case, the definitions of the chromatic index of a graph G, of the edge k-colorability problem, and the chromatic index problem are formulated similarly.

In this article, we prove that every 3-EC-boundary class is also CI-boundary. This result sharply contrasts the circumstance that, for each $k > 3$, there exists a continuum of k-EC-boundary classes each of which is not CI-boundary. We discuss the difficulties that hinder discovering a CI-boundary class that is k-EC-boundary for no k. Finally, we outline a way for answering the question about the existence of 3-VC-boundary classes that are not CI-boundary (recall that an answer to this question was not obtained in [3, 7]). Namely, we prove that if such a class exists then it contains a 4-VC-boundary class. This inclusion is unlikely (in our opinion); therefore, most probably, there is no such a class $\mathcal{X}$.

We adopt the following notations:

$[\mathcal{X}]$ is the *hereditary closure* of $\mathcal{X}$; i.e., the set of all graphs isomorphic to induced subgraphs of graphs in $\mathcal{X}$;

$\mathcal{T}$ is the set of the graphs whose each connected component is a tree with at most three leaves;

$\mathcal{D}$ is the set of line graphs to graphs in $\mathcal{T}$;

A_k ($k \geq 3$) is the graph with vertex set $\{x, y, x_1, x_2, \dots, x_{k-1}, y_1, y_2, \dots, y_{k-1}\}$ in which the sets $\{x, x_1, x_2, \dots, x_{k-1}\}$ and $\{y, y_1, y_2, \dots, y_{k-1}\}$ form the two cliques and there are edges

$$(x_1, y_1),\ \ (x_2, y_2),\ \ \dots,\ \ (x_{k-1}, y_{k-1});$$

B_k ($k \geq 3$) is the graph with vertex set $\{x, y, x_1, x_2, \dots, x_{k-2}, y_1, y_2, \dots, y_k\}$ in which the sets $\{x, x_1, x_2, \dots, x_{k-2}\}$, $\{y, x_1, x_2, \dots, x_{k-2}\}$, and $\{y_1, y_2, \dots, y_k\}$ form the three cliques and there are the edges

$$(x_1, y_1), (x_2, y_2), \dots, (x_{k-2}, y_{k-2}), (x, y_{k-1}), (y, y_k);$$

P_n is a simple path with n vertices.

1. ON THE CARDINALITY OF THE SET OF BOUNDARY GRAPH CLASSES FOR THE EDGE k-COLORABILITY PROBLEM

We need some graph transformations preserving the edge k-colorability. Let G be a graph in which two vertices are chosen such that there exists an automorphism of G taking these vertices to each other. The operation of the *replacement* of an edge $e = (a, b)$ by the graph G consists in deleting e from some graph with the subsequent identification of the vertex a with one of the chosen vertices in G and the identification of b with the other chosen vertex in G. Clearly, the graph obtained by edge replacement is independent of which of the chosen vertices in G is identified with a.

Let G be an arbitrary graph and let $e = (a, b)$ be its edge, $k \geq 3$. Further, let G'_e (G''_e) stand for the result of removing e from G, the addition of vertices a' and b' and the edges (a, a'), (a', b'), and (b', b), and the replacement of the edge (a', b') by the graph A_k (B_k) in which the chosen vertices are two vertices of degree $k - 1$ in A_k (B_k).

Lemma 1. *A graph G is edge k-colorable if and only if so are the graphs G'_e and G''_e for its arbitrary edge $e = (a, b)$.*

Proof. It is easy to construct a k-coloring of the edges of each of the graphs G'_e and G''_e from a k-coloring of the edges of G. We give its construction for the more difficult case; i.e., for G''_e and leave that for G'_e to the reader.

Consider two copies of a clique with k vertices in each of which the vertices are numbered from 1 to k. Color the edges of each of the copies properly so that the edges of both cliques incident to pairs of vertices with the same numbers have identical colors. For each number i, we have a color col_i that is not involved in the coloring of the edges incident to the vertices with number i. Therefore, there exists some k-coloring of the edges of the graph H obtained by adding all edges incident to the vertices and having the same numbers.

Choose some edge (x, y) in H belonging to one of the k-cliques and remove it from H. Denote the color of (x, y) in the k-coloring of the edges of H by col. Consider a k-coloring of the edges of G in which e has color col (such a coloring obviously exists). It generates a coloring of the edges in $E(G''_e) \cap E(G)$. Color the edges (a, a') and (b', b) of G''_e with the color col. Note that the vertices in

$$V(G''_e) \setminus (V(G) \cup \{a, b\})$$

generate a subgraph isomorphic to H without the edge (x, y). Therefore, we may extend the current partial coloring of the edges of G''_e to some its edge k-coloring.

Prove now that, in every k-coloring of the edges of G'_e and G''_e, the edges (a, a') and (b', b) are colored with identical colors. This will imply the lemma. Suppose, on the contrary, that there exists a proper edge coloring for at least one of the graphs G'_e and G''_e with k colors so that (a, a') and (b', b) are colored with different colors col_1 and col_2.

Consider the set of the edges of the subgraph A_k of G'_e (respectively, B_k of G''_e) colored with col_1. Each vertex in $V(A_k) \setminus \{a'\}$ (respectively, $V(B_k) \setminus \{a'\}$)) must be incident to some edge colored with col_1 (this follows form the fact that all vertices of A_k have degree k in G'_e (respectively, of B_k in G''_e), and the edge (b, b') is assigned to a color $\mathrm{col}_2 \neq \mathrm{col}_1$). Hence, the set $V(A_k) \setminus \{a'\}$ (respectively, $V(B_k) \setminus \{a'\}$) must contain an even number of elements; a contradiction.

Lemma 1 is proved. □

Given a binary sequence π of length l, refer as the $\pi(k)$-*sheaf* ($k \geq 3$) to the graph obtained from the simple path P_{4l+2} by replacement of its edges. For each $i \in \{1, 2, \ldots, l\}$, the edges $2i$ and $(4l + 2 - 2i)$ of this path are replaced by A_k if $\pi_i = 0$ or by B_k if $\pi_i = 1$. The $\pi(k)$-*transformation* of a graph G consists in the replacement of each of its edges by the $\pi(k)$-sheaf. Let $G_{\pi(k)}$ denote the graph obtained by the $\pi(k)$-transformation of G.

Lemma 1 implies

Lemma 2. *For an arbitrary finite binary sequence π, the graph $G_{\pi(k)}$ is edge k-colorable if and only if G is edge k-colorable.*

Let $T_{\pi(s)}$ for $s \geq 3$ (respectively, $T'_{\pi(s)}$ for $s \geq 3$) denote the graph obtained by applying the $\pi(s)$-transformation to $K_{1,s}$ (respectively, $K_{1,3}$). Let π be an infinite binary sequence and let $\pi^{(l)}$ stand for a subsequence π consisting of its first l terms. Denote by $\mathcal{T}_{\pi(s)}$ (respectively, $\mathcal{T}'_{\pi(s)}$) the set of the graphs whose each component is a induced subgraph in $\bigcup_{l=1}^{\infty}\{T_{\pi^{(l)}(k)}\}$ (respectively, $\bigcup_{l=1}^{\infty}\{T'_{\pi^{(l)}(k)}\}$). Obviously, for any infinite binary sequence π, we have the inclusion $\mathcal{T}'_{\pi(s)} \subseteq \mathcal{T}_{\pi(s)}$ with equality attained only for $s = 3$.

Lemma 3. *The class $\mathcal{T}_{\pi(k)}$ is k-EC-limit for every infinite binary sequence π with $k \geq 3$.*

Proof. It is known that all graphs with vertex degrees at most k form a k-EC-hard class [10] for each $k \geq 3$. Let $\mathcal{X}_{\pi^{(l)}(k)}$ denote the set of the graphs obtained by applying the $\pi^{(l)}(k)$-transformation to these graphs. This and Lemma 2 imply that Problem k-PP is NP-complete in the class $\mathcal{X}_{\pi^{(l)}(k)}$ for each l. Thus, for each l, the class

$$\mathcal{Y}^{(l)}_{\pi(k)} = \left[\bigcup_{i=l}^{\infty}\mathcal{X}_{\pi^{(i)}(k)}\right]$$

is k-EC-hard. Obviously,

$$\mathcal{Y}^{(1)}_{\pi(k)} \supseteq \mathcal{Y}^{(2)}_{\pi(k)} \supseteq \dots, \qquad \mathcal{T}_{\pi(k)} = \bigcap_{l=1}^{\infty}\mathcal{Y}^{(l)}_{\pi(k)}.$$

Consequently, the class $\mathcal{T}_{\pi(k)}$ is k-EC-limit for each $k \geq 3$. Lemma 3 is proved. □

Call a vertex x of some graph G with degrees of all vertices at most k *k-eliminable* if one of the following holds:

(a) x belongs to a connected component of G isomorphic to a induced subgraph (which is possibly improper) of either A_k or B_k or G contains an isthmus after removing which x belongs to the connected component with the same property;

(b) there is at most one neighbor of degree k for x.

The meaning of the notion of a k-eliminable vertex is that G and the result of deleting the vertex x from G are simultaneously either edge k-colorable graphs or not. In case (a), this is obvious (it follows from the fact that A_k and B_k have edge k-coloring and, after removing the isthmus from the graph with all vertex degrees at most k, its edge k-colorability is equivalent to the k-colorability of each of the corresponding components); in case (b), the assertion follows from Lemma 4.

Lemma 4 [11]. *Suppose that a vertex v in some graph G and all neighboring vertices have degrees at most k, and also at most one vertex in a neighborhood of v has exactly degree k. If the result of removing v from G is an edge k-colorable graph then G is an edge k-colorable graph.*

It is well known that many problems NP-complete in the general case become polynomially solvable for trees. The same is true for families of graphs close to trees in some qualitative or quantitative measure. In other words, if the measure grows slowly for graphs in some class (for example, it is bounded above by some absolute constant for every graph) then we can expect that the problem is efficiently solvable for these classes.

One of such measures is the treewidth of graphs defined as follows: A *k-tree* is a graph that can be obtained from a $(k+1)$-clique (which is regarded as a simplest k-tree) by the following recursive rule: "to some k-tree G, add a new vertex and k edges incident to the new vertex and the vertices of a k-clique of G." The *treewidth* of a graph is the minimal k such that this graph is a subgraph (which is not necessary induced) in a k-tree.

Many problems on graphs are efficiently solvable for the graphs for which the degrees of all vertices and the treewidth are bounded above by an a priori constant. The same holds for the k-EC-problem.

Lemma 5 [6]. *For every fixed d and t, the k-EC-problem is polynomially solvable for the totality of graphs for which the degrees of all vertices do not exceed k and whose treewidth is at most t.*

A sufficient condition for the boundedness of the treewidth is given by

Lemma 6 [8]. *For every graphs $G_1 \in \mathcal{T}$ and $G_2 \in \mathcal{D}$ and each natural d, there exists a natural $C(d, G_1, G_2)$ such that the treewidth of each graph in $Free(\{G_1, G_2\})$ with degrees of all vertices at most d does not exceed $C(d, G_1, G_2)$.*

Lemma 7. *Let $\mathcal{X}$ be a k-EC-boundary class. Then $\mathcal{X}$ has the following two properties*:

(i) *either $\mathcal{T} \subseteq \mathcal{X}$ or $\mathcal{D} \subseteq \mathcal{X}$*;

(ii) *if a graph $G \in \mathcal{X}$ contains a k-eliminable vertex x then there exists a graph $G' \in \mathcal{X}$ for which G is an induced subgraph and x is not k-eliminable.*

Proof. On the contrary, suppose that (i) fails. Assume that $\mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \dots$ is a sequence of k-EC-hard classes of graphs converging to a k-EC-boundary class $\mathcal{X}$. Since none of the graphs having a vertex of degree at least $k+1$ is edge k-colorable, we may assume that, for each j, the degree of each vertex of graphs in $\mathcal{X}_j$ is at most k. Since the sequence $\{\mathcal{X}_j\}$ converges to $\mathcal{X}$, there exist j' and graphs $G_1 \in \mathcal{T}$ and $G_2 \in \mathcal{D}$ such that $\mathcal{X}_{j'} \subseteq Free(\{G_1, G_2\})$. But it follows from Lemmas 5 and 6 that $\mathcal{X}_{j'}$ is k-EC-easy. Therefore, every sequence converging to $\mathcal{X}$ contains a k-EC-easy class; a contradiction to the fact that the class is k-EC-boundary. Hence, the assumption fails.

Consider the set of the graphs in $\mathcal{X}_i$ not having k-eliminable vertices. The hereditary closure of the set of these graphs (denoted by $\mathcal{X}_i'$) is a k-EC-hard class. Nevertheless,

$$\bigcap_{i=1}^{\infty} \mathcal{X}_i' = \mathcal{X}$$

because

$$\bigcap_{i=1}^{\infty} \mathcal{X}_i' \subseteq \bigcap_{i=1}^{\infty} \mathcal{X}_i$$

and $\mathcal{X}$ is a k-EC-boundary class. Hence, $\mathcal{X}$ is the hereditary closure of graphs without k-EC-eliminable vertices. Thus, this class satisfies (ii).

Lemma 7 is proved. □

Lemma 8. *For every infinite binary sequence π, there exists a k-EC-boundary subclass in $\mathcal{T}_{\pi(k)}$; moreover, every such class includes $\mathcal{T}'_{\pi(k)}$.*

Proof. That $\mathcal{T}_{\pi(k)}$ is a limit class was proved in Lemma 3. Hence, by the definition of a boundary class of graphs, there exists a k-EC-boundary subclass in $\mathcal{T}_{\pi(k)}$. Consider one of these subclasses and denote it by $\mathcal{X}$. Since $\mathcal{D} \nsubseteq \mathcal{T}_{\pi(k)}$ (and hence $\mathcal{D} \nsubseteq \mathcal{X}$), by Lemma 7, we have $\mathcal{T} \subseteq \mathcal{X}$. Therefore, the graph $iK_{1,3}$ belongs to $\mathcal{X}$ for each i.

Now, prove that the following holds for each i:

$$\left[\bigcup_{j=1}^{\infty} \{iT'_{\pi^{(j)}(k)}\}\right] \subseteq \mathcal{X}.$$

This will imply $\mathcal{T}'_{\pi(k)} \subseteq \mathcal{X}$. Suppose the contrary. Then, for some j, the graph $iT'_{\pi^{(j)}(k)}$ does not belong to $\mathcal{X}$. Among the induced subgraphs of $iT'_{\pi^{(j)}(k)}$ containing $iK_{1,3}$ as an induced subgraph and lying in $\mathcal{X}$, choose an inclusion maximal subgraph. Obviously, such a graph exists. Denote it by G. Clearly,

$$G \neq iT'_{\pi^{(j)}(k)}.$$

There exists a vertex in G whose degree in G is distinguished from zero and less than its degree in $iT'_{\pi^{(j)}(k)}$. Then this vertex is nonhanging in G and belongs to some its induced subgraph A_k or to some its induced subgraph B_k. It is easy that then this subgraph must contain a k-EC-eliminable vertex x.

It follows from Lemma 7 that, in $\mathcal{X}$, there is a graph G' in which G is a proper induced subgraph and x is not k-EC-eliminable. From G' remove all vertices not lying in $iT'_{\pi^{(j)}(k)}$. The so-obtained graph G'' belongs to $\mathcal{X}$ (since the class is hereditary) and x is not k-EC-eliminable in G''. To see this, observe that the k-EC-eliminability of x in G' is influenced only by the vertices belonging to

$$V(iT'_{\pi^{(j)}(k)}) \cap V(G').$$

Then G is a proper induced subgraph in G''; therefore, the graph G is not inclusion maximal; a contradiction.

Consequently, the assumption is false that $iT'_{\pi^{(j)}(k)}$ exists. Lemma 8 is proved. □

Thus, by Lemma 8, every 3-EC-boundary subclass in $\mathcal{T}_{\pi(3)}$ includes the class $\mathcal{T}'_{\pi(3)} = \mathcal{T}_{\pi(3)}$. Therefore, for every infinite binary sequence π, the class $\mathcal{T}_{\pi(3)}$ is 3-EC-boundary. Apparently, $\mathcal{T}_{\pi(s)}$ is an s-EC-boundary class for each $s \geq 3$. Though we cannot prove that (the proof of Lemma 8 implies that it suffices to verify that every s-EC-boundary subclass in $\mathcal{T}_{\pi(s)}$ includes the set $\bigcup_{i=1}^{\infty}\{iK_{1,s}\}$), Lemma 8 allows us to prove

Theorem 2. *Given $k \geq 3$, the set of boundary classes for the k-EC-problem is continual.*

Proof. Note that, for any two different infinite binary sequences π_1 and π_2, there is no k-EC-boundary class that is simultaneously a subset and $\mathcal{T}_{\pi_1(k)}$ and $\mathcal{T}_{\pi_2(k)}$. Indeed, by Lemma 8, the contrary means that

$$\mathcal{T}'_{\pi_1(k)} \cup \mathcal{T}'_{\pi_2(k)} \subseteq \mathcal{T}_{\pi_1(k)} \cap \mathcal{T}_{\pi_2(k)},$$

which is impossible. Hence, since the set of infinite binary sequences is continual, the set of k-EC-boundary cubes is also continual for each $k \geq 3$. This completes the proof. □

2. COMPARATIVE ANALYSIS OF FAMILIES OF BOUNDARY CLASSES FOR COLORABILITY PROBLEMS

In Theorem 2, we established that the set of k-EC-boundary classes is continual. This helps us to completely answer the question posed in the introduction about the existence of k-EC-boundary classes that are not CI-stable. Indeed, the class $\mathcal{T}_{\pi(3)}$ is 3-EC-boundary for every infinite binary sequence π. But, for every such a sequence π, we have the chain of inclusions

$$\mathcal{T}_{\pi(3)} = \mathcal{T}'_{\pi(3)} \subset \mathcal{T}'_{\pi(4)} \subset \mathcal{T}'_{\pi(5)} \subset \dots,$$

and every k-EC-boundary subclass in $\mathcal{T}_{\pi(k)}$ includes $\mathcal{T}'_{\pi(k)}$. Below we prove that every 3-EC-boundary class is necessarily CI-boundary. Hence, $T_{\pi(3)}$ is CI-boundary for every infinite binary sequence π. By the above inclusion, this implies that, for each $k \geq 4$, there exists a continual family of k-EC-boundary classes of graphs that are not boundary for the chromatic index problem.

Theorem 3. *Every* 3-EC-*boundary class is* CI-*boundary.*

Proof. It is obvious that every 3-EC-hard class in CI-hard. Hence, every 3-EC-boundary class is necessarily CI-limit. Show that, in fact, each such a class $\mathcal{X}$ is CI-boundary.

Suppose on the contrary that there exists a CI-boundary class $\mathcal{Y} \subset \mathcal{X}$. Let

$$\mathcal{Y}_1 \supseteq \mathcal{Y}_2 \supseteq \mathcal{Y}_3 \dots$$

be a sequence of CI-hard classes converging to $\mathcal{Y}$. In proving Lemma 7, we proved that every k-EC-boundary class consists of the graphs the degree of each vertex in which is at most k. Since $\mathcal{Y} \subset \mathcal{X}$, the class $\mathcal{Y}$ for $k = 3$ has the same property. Note that the set of all graphs with degrees of vertices at most 3 is a finitely defined class. This follows from the fact that the corresponding forbidden subgraphs result by adding to all graphs with four vertices x_1, x_2, x_3, and x_4 one more vertex y and edges

$$(x_1, y), \qquad (x_2, y), \qquad (x_3, y), \qquad (x_4, y)$$

(it can be verified that there are exactly eleven such prohibitions). Therefore, there exists i such that the class $\mathcal{Y}_i$ contains none of these 11 graphs; i.e., it consists of the graphs with degrees of vertices at most 3.

A well-known theorem by V. G. Vizing [1] implies that the chromatic index of each graph in $\mathcal{Y}_i$ is at most 4. Therefore, the class $\mathcal{Y}_j$ is k-EC-easy for all $k > 3$ and $j \geq i$. Consequently, the class $\mathcal{Y}_j$ is 3-EC-hard for all $j \geq i$. Hence, the class $\mathcal{Y}$ is 3-EC-limit; therefore, the class $\mathcal{X}$ ($\mathcal{Y} \subset \mathcal{X}$) cannot be 3-EC-boundary; a contradiction.

The proof of Theorem 3 is complete. □

Unfortunately, we were unable to find examples of CI-boundary classes that are k-EC-boundary for no k (recall that, for the vertex version of the colorability problem, such an example was found in [3]). Apparently, the obtention of such examples (or, at least, a nonconstructive proof of the existence) is inevitably preceded by the establishment of the NP-completeness of the CI-problem for the hereditary class $\mathcal{X}$ such that, for each k, the graphs in $\mathcal{X}$ with degrees of vertices at most k form a CI-easy class. At the moment, only a few cases are known of intractability of the chromatic index problem (see the survey [9, 10]), and they all fail to satisfy the above condition.

Recall that, in [3, 7], the question remained open of the existence of 3-VC-boundary classes that are not CI-boundary. Unfortunately, we have not answered this question in this article either. Nevertheless, we formulate some necessary condition for the existence of such a class and, possibly, it will simplify answering the question:

Theorem 4. *For a* 3-VC-*boundary class* $\mathcal{X}$ *to include a proper* CI-*boundary subclass* $\mathcal{Y}$*, it is necessary that* $\mathcal{X}$ *contain a* 4-VC-*boundary subclass.*

Proof. Let $\mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \mathcal{X}_3 \ldots$ and $\mathcal{Y}_1 \supseteq \mathcal{Y}_2 \supseteq \mathcal{Y}_3 \ldots$ be some sequences of 3-VC- and CI-hard classes respectively converging to the classes $\mathcal{X}$ and $\mathcal{Y}$. Demonstrate that all $\mathcal{X}_i$ are necessarily 4-VC-hard. This implies that $\mathcal{X}$ is a 4-VC-limit class. Hence, $\mathcal{X}$ contains a 4-VC-boundary subclass.

If there exists a 4-VC-boundary class $\mathcal{X}_{i^*}$ then the infinite sequence $\mathcal{X}_1, \mathcal{X}_2, \ldots$ contains only finitely many 4-VC-hard terms. Therefore, for each $j \geq i^*$ given a graph $G \in \mathcal{X}_j$, it is possible to determine in a polynomial time of its vertices whether it is true that the chromatic index of G is at least 5.

Let $\mathcal{X}'_j$ denote all kinds of graphs in $\mathcal{X}_{i^*+j}$ with chromatic index at most 4. It is easy to see that, for each j, the 3-VC-problem for $\mathcal{X}_{i^*+j}$ is polynomially reducible to the same problem for $\mathcal{X}'_j$; moreover, the class $\mathcal{X}'_j$ is 4-VC-easy. Therefore, the class $\mathcal{X}'_j$ is 3-VC-hard for each j. Obviously,

$$\bigcap_{j=1}^{\infty} \mathcal{X}'_j \subseteq \mathcal{X},$$

and, since $\mathcal{X}$ is a 3-VC-boundary class, this intersection is exactly $\mathcal{X}$. Since $\mathcal{Y} \subset \mathcal{X}$, there is j^* such that $\mathcal{Y}_{j^*} \subseteq \mathcal{X}'_1$. Therefore, in the sequence $\mathcal{Y}_1, \mathcal{Y}_2, \ldots$, each term starting from the j^*th one consists of graphs with chromatic index at most 4. This means that the CI- and 3-VC-problems are polynomially equivalent for each such term. Thus, $\mathcal{Y}$ is a 3-VC-limit class; a contradiction to the fact that the class $\mathcal{X}$ is not 3-VC-boundary. Hence, the initial assumption fails.

Theorem 4 is proved. □

We conjecture that no 3-VC-boundary class can include a 4-VC-boundary class; therefore, each such a class is necessarily CI-boundary by Theorem 4.

ACKNOWLEDGMENTS

The author is grateful to the anonymous referee for suggesting that the notion of a k-EC-eliminable vertex be specified which facilitates the understanding of some parts of the article.

The author was supported by the Russian Foundation for Basic Research (projects nos. 11–01–00107–a and 12–01–00749–a), the Federal Target Program "Scientific and Scientific-Pedagogical Personnel for Innovative Russia" for 2009–2012 (State Contract no. 14.V37.21.0393), the Laboratory of Algorithms and Technologies for Networks Analysis at the National Research University "Higher School of Economics", and the Government of the Russian Federation (project no. 11.G34.31.0057).

REFERENCES

1. V. G. Vizing, "An Estimate of the Chromatic Class of a p-Graph," Diskret. Anal. **3**, 25–30 (1964).
2. D. S. Malyshev, "Continual Sets of Boundary Classes of Graphs for Colorability Problems," Diskret. Anal. Issled. Oper. **16** (5), 41–51 (2009).
3. D. S. Malyshev, "On Intersection and Symmetric Difference of Families of Boundary Classes in the Problems on Coloring and on the Chromatic Index," Diskret. Mat. **24** (2), 75–78 (2011) [Discrete Math. Appl. **21** (5–6), 645–649 (2011)].
4. V. E. Alekseev, "On Easy and Hard Hereditary Classes of Graphs with Respect to the Independent Set Problem," Discrete Appl. Math. **132**, 17–26 (2004).
5. V. E. Alekseev, R. Boliac, D. V. Korobitsyn, and V. V. Lozin, "NP-Hard Graph Problems and Boundary Classes of Graphs," Theor. Comput. Sci. **389**, 219–236 (2007).
6. H. L. Bodlaender, "Dynamic Programming on Graphs with Bounded Treewidth," in *Lecture Notes in Computer Science*, Vol. 317: *Automata, Languages, and Programming (Tampere, 1988). Proceedings A* (Berlin, Springer, 1988), pp. 105–118.
7. N. Korpeilainen, V. V. Lozin, D. S. Malyshev, and A. Tiskin, "Boundary Properties of Graphs for Algorithmic Graph Problems," Theor. Comput. Sci. **412**, 3545–3554 (2011).
8. V. V. Lozin and D. Rautenbach, "On the Band-, Tree- and Clique-Width of Graphs with Bounded Vertex Degree," Discrete Math. **18**, 195–206 (2004).
9. R. Machado and C. M. H. de Figueiredo, "Complexity Separating Classes for Edge-Coloring and Total-Coloring," J. Braz. Comput. Soc. **17**, 281–285 (2011).
10. R. Machado, C. M. H. de Figueiredo, and K. Vuskovic, "Chromatic Index of Graphs with no Cycle with a Unique Chord," Theor. Comput. Sci. (2010) **411**, 1221–1234.
11. A. Schrijver, *Combinatorial Optimization—Polyhedra and Efficiency* (Springer, Berlin, 2003).