Vortical Freak Waves in Water Under External Pressure Action

Anatoly Abrashkin[1,2,*] and Alexander Soloviev[1,†]

[1]Institute of Applied Physics, Russian Academy of Sciences, 46 ulitsa Ulyanova, Nizhny Novgorod 603950, Russia
[2]National Research University Higher School of Economics, 25/12 Bolshaja Pecherskaja ulitsa, Nizhny Novgorod 603155, Russia

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A vortical model for freak wave formation in water is presented. The wind action is simulated by nonuniform pressure on the free surface. The motion of the fluid is described by an exact solution of 2D hydrodynamic equations for ideal inviscid fluid in Lagrangian variables. Fluid particles rotate in circles of different radius. The model describes the appearance of a freak wave in the field of the Gerstner wave. The physical parameters of the wave and feasibility of the proposed scenario are discussed.

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Rogue waves are characterized by the amplitude criterion: the height of a rogue wave is two or more times the significant waveheight. Being considered initially for ocean waves, [1–4], nowadays the concept is shifted to other fields of physics such as nonlinear optics [5–7], physics of plasma [8], superfluid helium [9], and Bose-condensate systems [10].

The mechanisms of freak wave formation are not clear enough. Pelinovsky and co-workers [2] proposed a linear model of dispersive focusing of waves. Some other models are weakly nonlinear. They represent a rogue wave as an envelope solution of the nonlinear Schrödinger equation [11–14] or Dysthe equation [15].

The formation of a freak wave is an essentially nonlinear phenomenon [16]. Zakharov and Dyachenko suggest that focusing of ocean waves creates only preconditions for the formation of freak waves, which is a strongly nonlinear effect [17]. Using numerical calculations they demonstrated the formation of a rogue wave from a Stokes wave [17]. Ruban studied numerically two different kinds of rogue waves using a completely nonlinear model for long-crested water waves [18].

Most rogue waves in the ocean occur under storm conditions where wind action must be taken into account [16]. Kharif *et al.* consider the wind action as a linear equation connecting the pressure and the steepness of the wave profile [19]; i.e., the wind action is simulated by a nonuniform pressure distribution.

We present an exact solution of 2D hydrodynamical equations that describes the appearance of a freak wave in the field of a Gerstner wave. The solution is written in Lagrangian variables and belongs to the class of Ptolemaic flows [20,21]. Fluid particles rotate in circles of different radius and drift current is absent. The pressure on the free surface is nonuniform and opposite in phase with the wave profile. The dynamics of free surface and pressure for freak waves are studied. Unlike other models the analyzed freak wave is vortical. The vorticity is located mostly in the neighborhood of its peak. This is the first example of the importance of vorticity effects in freak wave formation.

Exact solution.—The equations of 2D hydrodynamics for waves on the surface of incompressible inviscid fluid in Lagrangian coordinates have the following form [21,22]:

$$\frac{D(X,Y)}{D(a,b)} = \frac{D(X_0,Y_0)}{D(a,b)}, \tag{1}$$

$$X_{tt}X_a + Y_{tt}Y_a = -\frac{1}{\rho}p_a - gY_a, \tag{2}$$

$$X_{tt}X_b + Y_{tt}Y_b = -\frac{1}{\rho}p_b - gY_b, \tag{3}$$

where X, Y are Cartesian coordinates and a, b are the Lagrangian coordinates of the fluid particles; t is time, ρ is fluid density, p is pressure, g is acceleration of gravity, the subscripts mean differentiation by the corresponding variable, and the index "zero" means the value at time $t = 0$.

Equation (1) is the volume conservation equation; Eqs. (2) and (3), are flow equations. Using the cross differentiation it is possible to exclude the pressure and to get the condition of the vorticity conservation along a trajectory:

$$(X_{ta}X_b + Y_{ta}Y_b - X_{tb}X_a - Y_{tb}Y_a)_t = 0. \tag{4}$$

Abrashkin and Yakubovich proposed to introduce complex Cartesian coordinates [20,21] $W = X + iY$ ($\bar{W} = X - iY$) and complex Lagrangian coordinates $\chi = a + ib$ ($\bar{\chi} = a - ib$). Then the Eqs. (1) and (4), are equivalent to the conditions of two Jacobians' conservation [20,21]:

$$\begin{aligned}\frac{D(W,\bar{W})}{D(\chi,\bar{\chi})} &= \frac{D(W_0,\bar{W}_0)}{D(\chi,\bar{\chi})} = D_0(\chi,\bar{\chi}),\\ \frac{D(W_t,\bar{W})}{D(\chi,\bar{\chi})} &= \frac{D(W_{t0},\bar{W}_0)}{D(\chi,\bar{\chi})} = \frac{i}{2}D_0\Omega(\chi,\bar{\chi}).\end{aligned} \tag{5}$$

Here, Ω is the vorticity. The function D_0 defines the dependence of the initial positions of fluid particles W_0 on the Lagrangian variables. This function must not vanish in the flow region.

Equations (5) have an exact solution [20,21],

$$W = G(\chi)\exp(i\lambda t) + F(\bar{\chi})\exp(i\mu t), \tag{6}$$

where F, G are analytic functions, and λ, μ are real constants. The trajectories of the fluid particles are epicycloids (hypocycloids) as planet orbits in the Ptolemaic system of the world, so the flows (6) were named Ptolemaic [20,21].

Let us consider the gravitational waves on the surface of an infinitely deep fluid. Suppose that the motion of the fluid is described by the expression (6). In Lagrangian coordinates the flow region corresponds to the domain $b = \mathrm{Im}\chi < 0$.

We study a particular case $\lambda = 0$, $\mu = -\omega$, when fluid particles move in circles. The fluid is motionless at the bottom, so $|F| \to 0$ as $b \to -\infty$. The function G should be bijective, so $G' \neq 0$ in the flow region. One more requirement on the choice of the functions F, G is positiveness of value D_0:

$$D_0 = |G'|^2 - |F'|^2 > 0. \tag{7}$$

The pressure p on the free surface is calculated from Eqs. (2) and (3), and has the following expression:

$$\frac{p - p_0}{\rho} = -g\mathrm{Im}(G + Fe^{-i\omega t}) + \frac{1}{2}\omega^2|F|^2 + \mathrm{Re}\left(e^{i\omega t}\int \omega^2 G'\bar{F}d\chi\right). \tag{8}$$

Here, p_0 is a constant value. The pressure p oscillates periodically. The first term in the expression (8) represents the well-known effect of the "inverted barometer" [23].

Model of a freak wave.—Consider the following solution:

$$W = \chi + \frac{\beta_1}{(\chi - \alpha i)^n} + \left[iAe^{ik\bar{\chi}+i\varphi_0} + \frac{\beta_2}{(\bar{\chi} + \alpha i)^n}\right]e^{-i\omega t}. \tag{9}$$

It belongs to the class of Ptolemaic flows (6). Here A, k, ω, α are positive parameters, $n \geq 2$. When $\beta_1 = \beta_2 = 0$, the expression (9) describes a Gerstner wave. For Ptolemaic flows the superposition principle holds true. If the function F is a sum of functions, the resulting profile qualitatively corresponds to the superposition of profiles defined by these functions. The terms in F, G have one pole of order n, which corresponds to $b = \alpha > 0$, so it is out of the fluid region. The term with the pole in the function F describes a periodically appearing peak. The term with the pole in the function G compensates the peak of the wave profile at the initial moment of time. So the expression (9) corresponds to the peak standing out in the field of a Gerstner wave.

In the solution (9) the value of A is amplitude, ω is frequency, and k is the wave number of a Gerstner wave, $kA \leq 1$. The equality corresponds to the wave with sharp crests on the profile. The parameter φ_0 characterizes the phase shift between the crests of the Gerstner wave and vortex breather. If $\varphi_0 = \pi$ their crests coincide and amplitudes are summarized. If $\varphi_0 = 0$ the breather crest coincides with the trough of the Gerstner wave and their amplitudes are subtracted. This behavior of the solution can be interpreted as wave interference. The parameter α characterizes the peak width. The values β_1, β_2 are not necessarily real numbers. To compensate for the peak at $t = 0$ and reinforce it at another moment, let $\beta_1 = (-1)^m\beta i$, $\beta_2 = \bar{\beta}_1$, $\beta > 0$, $m = n/2$. The value of n should be an even integer number. The dimension of β is L^{n+1}, its value characterizes the peak height.

We consider a particular case $n = 2$, thus $\beta_1 = -\beta i$, $\beta_2 = \beta i$. According to (7) there is a restriction on the value of β. A sufficient condition is formulated as $\beta \leq (1 - kA)\alpha^3/4$. The exact bounding for the β parameter can be calculated numerically.

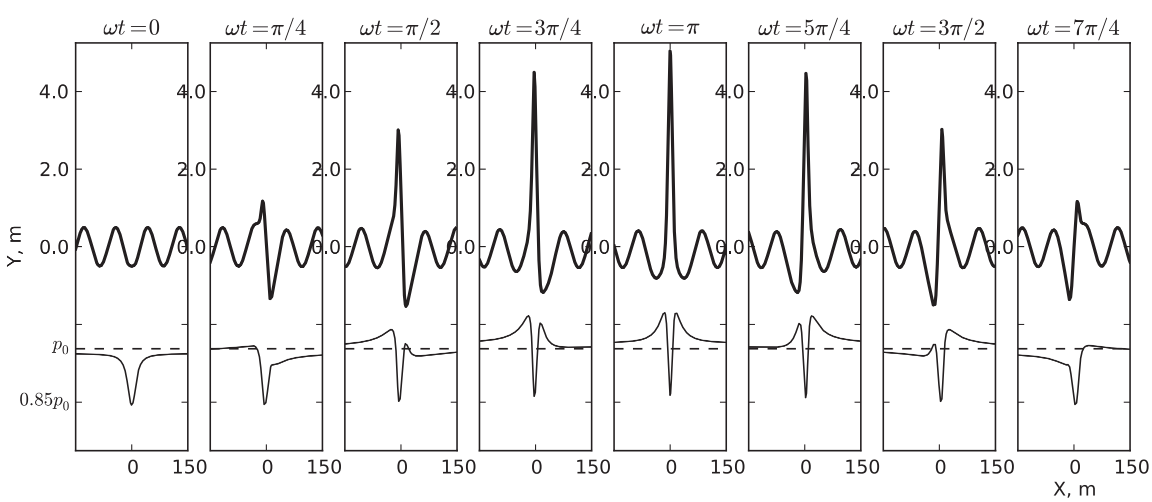

FIG. 1. Formation of a freak wave.

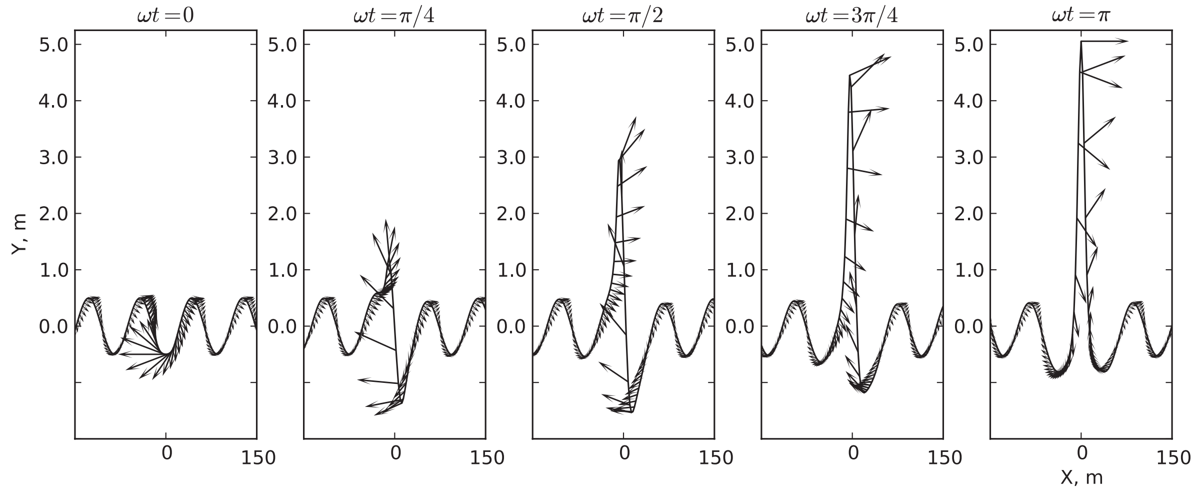

FIG. 2. The field of velocities for a freak wave on the free surface.

Figure 1 represents the dynamics of a freak wave (9) for the case $A = 0.5$ m, $k = 0.074\ \mathrm{m}^{-1}$, $\alpha = 12$ m, $\beta = 328\ \mathrm{m}^3$, $\omega = \sqrt{gk} = 0.85\ \mathrm{s}^{-1}$; the wave length is $\lambda = 84.9$ m. The phase shift is $\varphi_0 = \pi$. The peak height is $h = 2\beta/\alpha^2 + A \approx 5.1$ m. At the moment $t = 0$ there is no peak and the wave profile corresponds to the Gerstner wave exactly. Next time the peak raises up to a maximum value at the moment $t = \pi/\omega$, then it decreases and in a period disappears. This motion is periodic. The peak height is greater than the amplitude of the Gerstner wave in eight times. So such a peak can be considered as a freak wave. The pressure has nonstationary character. In Fig. 1 the lower curves represent the deviation of the pressure on the free surface from atmospheric pressure p_0.

Figure 2 shows velocities of fluid particles at different moments of time. The values of the parameters are the same as for Fig. 1. During formation of freak waves the velocities on the front and back slopes of the peak have opposite directions (see moments $t = \pi/4/\omega$, $t = \pi/2/\omega$). So the wave profile collapses and rises. When the velocity in the highest point becomes horizontal the freak wave begins to decrease.

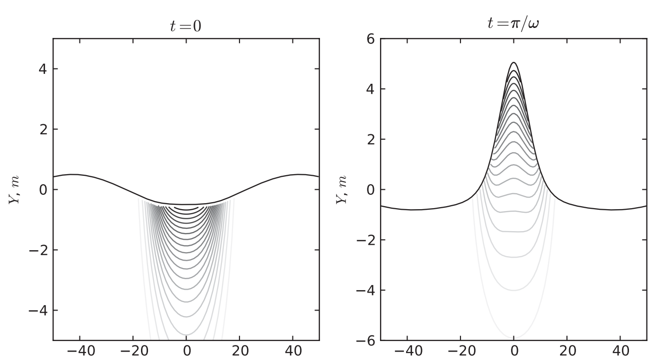

FIG. 3. Isolines of vorticity.

The vorticity of the flow (9) equals to

$$\Omega = \frac{2\omega |F'|^2}{|G'|^2 - |F'|^2}.$$

Far away from the peak location the value of Ω is near to the vorticity of the Gerstner wave. For a Gerstner wave of weak steepness ($kA < < 1$) the flow outside of the freak wave is practically potential. The vorticity localizes in the small neighbourhood of the peak location. So the freak wave is strongly vortical.

Figure 3 represents isolines of vorticity in the neighborhood of the peak location at two moments of time. The light gray isolines correspond to the smaller vorticity, the dark gray ones—to greater vorticity. With the lapse of time the isolines of vorticity become convex near the peak axis. So the formation of the freak wave in our model is connected with bending of vorticity isolines.

As seen from Fig. 1, the minimum of the pressure on the free surface corresponds to the particle $a = b = 0$.

Because of the nonstationarity of the pressure we estimate the pressure at two qualitatively different moments of time. At the moment $t = 0$, when there is no peak, the

TABLE I. Parameters of maximal freak wave.

A, m	kA	λ, m	h, m
0.25	0.07	21.2	4.8
0.5	0.13	23.5	5.4
2.0	0.33	38.0	8.8
4.0	0.45	56.1	12.3
1.25	0.69	11.4	1.8
1.25	0.75	10.5	1.4
1.25	0.88	9.0	0.7

TABLE II. Wave parameters for given amplitude A.

A, m	kA	α, m	h, m	$\|p(0)\|/\alpha$, mm Hg/m
4.0	0.44	15.0	7.6	6.6
4.0	0.40	20.0	6.0	5.0
4.0	0.33	30.0	4.7	3.3
4.0	0.27	40.0	4.2	2.5

pressure is $p(0)$ and at the moment $t = \pi/\omega$, when the peak height is maximal, the pressure is $p(\pi)$. The case $p(0) = p(\pi) = -100$ mm of mercury (mm Hg) is studied.

Table I represents examples of the freak wave parameters obtained by numerical calculations. Here, $\lambda = 2\pi/k$ is the length of the Gerstner wave.

If the steepness is near to 1, then the value of h tends to 0. The peak cannot form on a steep wave. The ratio between the height of the peak and Gerstner wave amplitude is $(1/A - k)\alpha/2$. It can be very large. For small steepness the maximum waves are located in the range of lengths $\lambda \sim 20, \ldots, 60$ m.

Our estimations of the wave parameters are provided for the pressure deviation of order of $|p(0)| = 100$ mm Hg. The value of the pressure gradient can vary in dependence on the breather width that is of the order of α. Table II represents a relation between the average pressure gradient $|p(0)|/\alpha$ and the value of α for given amplitude of Gerstner wave $A = 4$ m. These conditions do not correspond to the maximal freak wave. When the pressure drop takes place on larger scales (for bigger values of α) the height of the vortex breather decreases.

The considered values of pressure, pressure gradient, and the breather width correspond to the tornadoes.

Conclusions.— Our model can explain the formation of freak waves under external pressure action. For deviation of pressure of order of 100 mm Hg, which corresponds to the tornado conditions, the peak height can achieve 4–12 m on a uniform wave of weak steepness. If pressure deviation acts during a period of the wave then the freak wave may appear. Such waves satisfy the amplitude criterion for freak waves. The constructed solution is periodic and can be considered as a vortical breather.

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*abrash@hydro.appl.sci-nnov.ru
†alexandre.solovyov@mail.ru

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