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Relatively Unstable Attractors

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To the memory of Evgenii Frolovich Mishchenko

Abstract—There are different non-equivalent definitions of attractors in the theory of dynamical systems. The most common are two definitions: the maximal attractor and the Milnor attractor. The maximal attractor is by definition Lyapunov stable, but it is often in some ways excessive. The definition of Milnor attractor is more realistic from the physical point of view. The Milnor attractor can be Lyapunov unstable though. One of the central problems in the theory of dynamical systems is the question of how typical such a phenomenon is. This article is motivated by this question and contains new examples of so-called relatively unstable Milnor attractors. Recently I. Shilin has proved that these attractors are Lyapunov stable in the case of one-dimensional fiber under some additional assumptions. However, the question of their stability in the case of multidimensional fiber is still an open problem.

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1. UNSTABLE MILNOR ATTRACTORS

Let us recall some definitions. In what follows, we consider homeomorphisms of a metric measure space into itself. Let $F\colon X \to X$ be such a homeomorphism and $U \subset X$ be an attracting domain, which means that $\operatorname{Cl} F(U) \subset U$. A *maximal attractor* of F in U is the intersection

$$A_{\max}(F,U) = \bigcap_{n>0} F^n(U).$$

It easily follows from the definition that the maximal attractor is *Lyapunov stable*: for each of its neighborhoods V there is k such that $F^k(U) \subset V$.

A *Milnor attractor* is a minimal closed set that contains the ω-limit sets of almost all points of the phase space.

A simplest example of a Lyapunov unstable Milnor attractor is provided by a diffeomorphism of a circle with a single semistable fixed point, for instance,

$$x \mapsto x + 0.1(\cos x - 1).$$

The point 0 is the Milnor attractor here, but it is not Lyapunov stable.

This example can easily be generalized to diffeomorphisms of manifolds of arbitrary dimension: one can multiply this map in a Cartesian way by a north–south diffeomorphism of a sphere. It follows that in phase spaces of any dimension Lyapunov unstable Milnor attractors unremovably appear in one-parameter families, thus being of codimension at most one. But can they be more typical?

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Problem 1. Is there an open set in the space of diffeomorphisms such that every map in this set has a Lyapunov unstable Milnor attractor?

This problem is one of the central problems in dynamical systems.

2. RELATIVELY UNSTABLE MILNOR ATTRACTOR

Let us say that a homeomorphism of a metric measure space has a *relatively unstable Milnor attractor* if some iterate of this map has an invariant subset such that the restriction of this iterate to that subset has an unstable Milnor attractor. A detailed definition is given at the end of this section.

We say that a set in a function space is *locally dense* if it is dense in some open subset of this space.

In this paper we construct a locally dense subset of the space of skew products over a Bernoulli shift with fibers of any dimension such that all the maps in this subset have relatively unstable Milnor attractors.

Let M be an arbitrary closed manifold of dimension k. Consider a skew product over Σ^{k+2} with fiber M:

$$F\colon X \to X, \quad (\omega, x) \mapsto (\sigma\omega, f_{\omega_0}(x)). \tag{1}$$

Here Σ^l is the space of doubly infinite sequences over an l-element alphabet, $\sigma\colon \Sigma^l \to \Sigma^l$ is the Bernoulli shift, $X = \Sigma^{k+2} \times M$, and f_j, $j = 0, \dots, k+1$, are diffeomorphisms $M \to M$, called *fiber maps*.

We denote by C_k^m the space of ordered sets of maps $f = (f_0, \dots, f_{k+1})$ endowed with the C^m topology.

Let M be a Riemannian manifold, and let d_M and mes be the distance and the volume corresponding to the Riemannian metric. Let d_Σ be the standard metric and P_k be the $\left(\frac{1}{k}, \dots, \frac{1}{k}\right)$ Bernoulli measure on Σ^k. We define d and μ, a metric and a measure on X, as follows:

$$d\big((\omega, x), (\omega', x')\big) = d_\Sigma(\omega, \omega') + d_M(x, x'), \qquad \mu = P_{k+2} \times \mathrm{mes}.$$

Then for homeomorphisms $F\colon X \to X$ the Milnor attractor $A_{\mathrm{M}}(F)$ is well defined.

Let Ξ be a closed subset of Σ^k. We define a *cylinder* in Ξ as an intersection of a cylinder in Σ^k with Ξ. For an arbitrary subset A in Σ^k we denote by A^ε its ε-neighborhood in Σ^k. Then for any cylinder $C \subset \Xi$ the *relative measure* is defined in the following way:

$$\mu_\Xi(C) = \lim_{\varepsilon \to 0} \frac{\mu(C^\varepsilon)}{\mu(\Xi^\varepsilon)}.$$

We prove that in examples below this limit exists. Consider the σ-algebra of subsets of Ξ generated by cylinders. We extend our relative measure, defined on cylinders, to this algebra. The extended measure is called the *relative measure on* Ξ and is denoted by μ_Ξ.

Suppose that Ξ is invariant under an lth power of the Bernoulli shift. Let $Y = \Xi \times M$. On Y we get the measure $\mu_Y = \mu_\Xi \times \mathrm{mes}$ and the metric induced from X. Let F be a skew product (1). Then Y is F^l-invariant. Therefore, the map $F^l|_Y$ has a Milnor attractor defined by the measure and metric described above.

Definition 1. We say that the map (1) has a *relatively unstable Milnor attractor* if the restriction of the lth power of this map to the subset Y described above has an unstable Milnor attractor.

Theorem 1. *For every closed Riemannian manifold M of dimension k and every $m \geq 1$, there exists an open domain in C_k^m and a dense subset $\mathcal{N}$ in this domain such that for each $f \in \mathcal{N}$ the skew product* (1) *has a relatively unstable Milnor attractor.*

The proof for $k = 1$ is given in the next two sections, and for arbitrary k, in Section 5 (except for Theorem 3, which is proved in Section 6).

3. THE STANDARD UNSTABLE MILNOR ATTRACTOR

Consider a skew product (1) over Σ^2 with $M = S^1$. Let the fiber maps f_0 and f_1 have the following structure:

1. The rotation number of both maps is 0. All their fixed points are hyperbolic.
2. The map f_0 is a north–south map with an attractor a_0 and a repeller r_0.
3. The map f_1 has attractors a_1, a_2 and repellers r_1, r_2. Moreover,

$$r_1 = a_0. \tag{2}$$

On the arc $[r_0, r_2]$ (the smallest of the two arcs with the endpoints r_0 and r_2) there are no attractors of the maps f_0 and f_1.

4. The multiplicators of the fiber maps at the point $a_0 = r_1$ satisfy the inequality

$$f_0'(a_0) f_1'(a_0) < 1. \tag{3}$$

Theorem 2. *The skew product*

$$F_0\colon\ X = \Sigma^2 \times S^1 \to X, \quad (\omega, x) \mapsto (\sigma\omega, f_{\omega_0} x) \tag{4}$$

with the fiber maps f_0 and f_1 described above has a Lyapunov unstable Milnor attractor.

A more general Theorem 3 is proved below.

Definition 2. The Milnor attractor described in Theorem 2 is called a *standard unstable attractor.*

Example 1. Consider the skew product (1) with $k = 1$ and $M = S^1$. Suppose that in the tuple $f = (f_0, f_1, f_2)$ the maps f_0 and f_1 are the same as in Theorem 2 and f_2 is close to f_1, but the fixed points of f_2 differ from a_0. The base of this skew product, the space Σ^3, consists of all bi-infinite sequences of symbols from the alphabet $\{0, 1, 2\}$. Let $\Xi \subset \Sigma^3$ be the set of the bi-infinite sequences of 0 and 1 with the $\left(\frac{1}{2}, \frac{1}{2}\right)$ Bernoulli measure. We will prove below that this measure is the relative measure in the sense of Section 2. Let $Y = \Xi \times M$. Then the skew product (1), restricted to Y, has an unstable Milnor attractor; therefore, in the whole phase space this map has a relatively unstable Milnor attractor.

In the proof of Theorem 1 we construct attractors in the same way as in the example above.

4. RELATIVELY UNSTABLE ATTRACTORS OF SKEW PRODUCTS WITH CIRCLE FIBER

In this section the maps f_j are orientation-preserving diffeomorphisms of a circle.

4.1. Construction of the domain Ω. The space of all skew products (1) with $k = 1$ is the space of tuples of fiber maps; each tuple has the form $f = (f_0, f_1, f_2)$ and each map is a diffeomorphism of a circle. In other words, this space is $(\mathrm{Diff}\, S^1)^3$.

Consider a pair of maps $f_0^0 \in \mathrm{Diff}\, S^1$, $f_1^0 \in \mathrm{Diff}\, S^1$ satisfying conditions 1–4 of Section 3. The maps f_0^0 and f_1^0 admit neighborhoods V_0 and V_1 in $\mathrm{Diff}\, S^1$ such that all the pairs of maps $f_0 \in V_0$,

$f_1 \in V_1$ satisfy conditions 1–4 of Section 3, except for condition (2). Each map $f_1 \in V_1$ has a repeller close to a_0. These repellers lie on an arc $W \ni a_0$. We require the neighborhood V_1 to be so small that all the maps inverse to $f_1 \in V_1$ contract on the arc W.

Now we define a domain $\Omega \subset (\operatorname{Diff} S^1)^3$ as a set $\Omega \subset V_0 \times V_1^2$ that consists of the triplets $f = (f_0, f_1, f_2)$ such that the attractor of the map f_0 belongs to the arc W and lies between those repellers of the maps f_1 and f_2 that belong to this arc.

It is clear that the set Ω of tuples with such properties is open.

4.2. Construction of the set $\mathcal{N}$. Let us denote by $\mathcal{N}$ the subset of Ω that consists of all tuples (f_0, f_1, f_2) such that the repeller of some composition of the maps f_1 and f_2 coincides with the attractor of the map f_0. We recall that this attractor is close to a_0 by the construction of Ω.

Lemma 1. *The set $\mathcal{N}$ is dense in Ω.*

Proof. In the one-dimensional case the well-known Hutchinson lemma [1] can be formulated in the following way.

Lemma 2. *Let maps f_1^{-1} and f_2^{-1} be contracting on the arc W, and let r_1 and r_2 be their attractors on that arc. Suppose that $[r_1, r_2] \subset f_1^{-1}([r_1, r_2]) \cup f_2^{-1}([r_1, r_2])$. Then for any interval $U \subset [r_1, r_2]$ there is a composition g_U of these maps such that*

$$g_U[r_1, r_2] \subset U.$$

Note that the map g_U contracts on $[r_1, r_2]$. Therefore, it has a single attracting fixed point r_U. The point r_U is a repeller for the inverse map $f_U = g_U^{-1}$. Notice that the map f_U is a finite composition of the maps f_1 and f_2.

Now consider an arbitrary ordered set $f \in \Omega$, $f = (f_0, f_1, f_2)$. We prove that slightly changing solely f_0 we can get a tuple that belongs to $\mathcal{N}$. First we take the attractor a_0 of the map f_0, a small neighborhood U of this attractor, and a composition f_U of the maps f_1 and f_2 that has a repeller $r_U \in U$. We replace f_0 by the map $\widetilde{f}_0$ that is close to f_0 and has an attractor coinciding with r_U. The tuple $(\widetilde{f}_0, f_1, f_2) \in \mathcal{N}$ is the required one. $\square$

4.3. Relatively unstable attractors. Here we finish the proof of Theorem 1 in the case of one-dimensional fiber.

We prove that for every tuple $f \in \mathcal{N}$ the map (1) has a relatively unstable Milnor attractor on some subset. Let $f = (f_0, f_1, f_2) \in \mathcal{N}$ and f_U be the composition of the maps f_1 and f_2 such that the repeller of f_U coincides with the attractor of f_0. Denote by w the word of 1 and 2 corresponding to f_U. Let l be the length of this word. Consider the immersion $\zeta\colon \Sigma^2 \to \Sigma^3$ defined in the following manner. In each sequence of zeros and ones we replace every 1 by the word w and every 0 by the word of l zeros. At the zero position of the new sequence, we have the first letter of the word that replaces the symbol that stood at the zero position in the old sequence. This construction allows us to examine the relative measure on the set

$$\Xi = \zeta(\Sigma^2). \tag{5}$$

This subset is invariant under the lth iterate of the Bernoulli shift.

Now let $Y = \Xi \times S^1$. Denote the $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ Bernoulli measure on Σ^3 by P_3 and the one-dimensional Lebesgue measure on the circle by m_1. Then $\mu = P_3 \times m_1$ is the measure on X defined above. Consider the induced measure $P_\Xi = \zeta_* P_2$ on $\zeta(\Sigma^2)$. Let $\widetilde{\mu}_Y = P_\Xi \times m_1$.

Proposition 1. *The relative measure on the subset $\Xi \subset \Sigma^3$ coincides with the induced measure $\zeta_* P_2$.*

Proof. *Step* 1. We consider a natural embedding $\Sigma^2 \subset \Sigma^3$ and prove that the relative measure on Σ^2 coincides with the Bernoulli measure P_2. For this purpose we consider ε-neighborhoods of the set Σ^2. Note that the ε-neighborhoods of every point, and consequently of any set in the space Σ^3, coincide for $3^{-n} \le \varepsilon < 3^{-n+1}$. Therefore, it is enough to consider the sequence $\varepsilon_n = 3^{-n}$. For every n, the ε_n-neighborhood of the subset Σ^2 consists of 2^{2n+1} pairwise disjoint balls of volume $3^{-(2n+1)}$ each. This implies $P_3((\Sigma^2)^\varepsilon) = \left(\frac{2}{3}\right)^{2n+1}$.

Now consider a cylinder in the subset $\Sigma^2 \subset \Sigma^3$ and suppose that this cylinder has m fixed positions. Suppose n is so large that all these positions lie inside the interval $[-n, n]$. Then the ε_n-neighborhood of the cylinder in question consists of 2^{2n+1-m} disjoint balls of radius ε_n, and the measure of this neighborhood equals $\left(\frac{2}{3}\right)^{2n+1} \cdot 2^{-m}$. Therefore, the relative measure of this cylinder equals 2^{-m} and coincides with its P_2-measure.

Step 2. Suppose that w is a word of length l and ζ is the corresponding immersion $\Sigma^2 \to \Sigma^3$ defined at the beginning of this subsection. Each point $\omega \in \zeta(\Sigma^2)$ is a sequence consisting of clusters of length l, one of which begins from the zero position. Every cluster either consists solely of zeros or coincides with the word w. When $3^{-nl} \le \varepsilon < 3^{-(n-1)l}$, the ε-neighborhoods of the points $\omega \in \zeta(\Sigma^2)$ coincide. Continuing as above, we conclude that the relative measure of the cylinder with m fixed clusters equals 2^{-m}; so this measure coincides with the measure $\zeta_* P_2$ for that cylinder. This proves that the relative measure on Ξ coincides with $\zeta_* P_2$. $\square$

Proposition 2. *Suppose that the map* (1) *of the space* $\Sigma^3 \times S^1$ *onto itself belongs to the set* $\mathcal{N}$. *Then it has a relatively unstable Milnor attractor.*

Proof. Suppose that w is a word corresponding to a composition f_w of the fiber maps f_1 and f_2, and that the repeller of f_w coincides with the attractor a_0 of the map f_0. Denote by l the length of the word w. Then the map

$$F_0\colon\ \Sigma^2 \times S^1 \to \Sigma^2 \times S^1, \quad (\omega, x) \mapsto (\sigma\omega, g_{\omega_0}(x)), \qquad g_0 = f_0^l, \quad g_1 = f_w,$$

satisfies all conditions of Theorem 2. Therefore, its Milnor attractor is unstable.

Consider the restriction $F^l|_Y$. It leaves the set $Y = \Xi \times S^1$, $\Xi = \zeta(\Sigma^2)$, invariant. The homeomorphism $\zeta \times \mathrm{id}$ conjugates the maps F_0 and $F^l|_Y$. It converts the measure μ on $\Sigma^2 \times S^1$ into the measure μ_Y on Y as well. The map F_0 has an unstable Milnor attractor by Theorem 2. It follows that the Milnor attractor of the map $F^l|_Y$ is unstable too. Thus the map F has a relatively unstable Milnor attractor.

This proves the proposition and, therefore, Theorem 1 for $k = 1$. $\square$

5. MULTIDIMENSIONAL CASE

In this section we prove Theorem 1 for an arbitrary closed Riemannian manifold M of dimension k.

5.1. Standard unstable attractor in the general case. Consider a skew product over Σ^2 with fiber M and fiber maps f_0^0 and f_1^0 that are defined in the following way. Both maps are of Morse–Smale class, and their non-wandering sets consist of fixed points only. The map f_0^0 has a repeller r_0 with a repelling neighborhood W, and f_1^0 has a repeller $r_1 \in W$ for which W is also a repelling domain. Let f_0^0 have an attractor (a stable equilibrium position) a_0. We assume that it attracts all points of the manifold M, except for the points of a stratified manifold S of dimension at most $k-1$. Let f_1^0 have a repeller $r_2 = a_0$. Suppose that

$$\|df_0^0(a_0)\| \cdot \|df_1^0(a_0)\| < 1. \tag{6}$$

We also require that for every point $q \in S$ there is a finite word w such that the corresponding composition of the fiber maps $f_w = f^0_{w_{|w|-1}} \circ \ldots \circ f^0_{w_0}$ takes the point q to some point outside S.

Theorem 3. *Consider a skew product*

$$F_0\colon \Sigma^2 \times M \to \Sigma^2 \times M, \quad (\omega, x) \mapsto \big(\sigma\omega, f^0_{\omega_0}(x)\big), \tag{7}$$

where the fiber maps f_0^0 and f_1^0 are the same as above. Then the Milnor attractor of the map F_0 coincides with the set

$$A_{\mathrm{M}} = \Sigma^2 \times \{a_0\}$$

and is Lyapunov unstable.

The proof is given in Section 6.

5.2. Construction of the domain Ω in the general case. Let the maps f_0^0 and f_1^0 be the same as in Section 5.1. We consider the neighborhoods V_0 and V_1 of f_0^0 and f_1^0 in the space $\mathrm{Diff}\, M$ and define the set $\Omega \subset (\mathrm{Diff}\, M)^{k+2}$ in the following manner.

We take the set of tuples $f = (f_0, \dots, f_{k+1}) \in V_0 \times V_1^{k+1}$ with the following properties:

1. The maps f_j have a common repelling region W, the same as for f_0^0 and f_1^0.

2. There is a domain $\Omega_0 \in M$ such that $f_j^{-1}(\Omega_0) \subset \Omega_0$ and each f_j^{-1} contracts on Ω_0, $j = 1, \dots, k+1$.

3. There is a domain $\Omega_1 \subset \Omega_0$ such that

$$\Omega_1 \subset \bigcup_1^{k+1} f_j^{-1}(\Omega_1).$$

4. The map f_0 has an attractor a_0 that attracts all points of the manifold M, except for the points of a stratified manifold S of dimension at most $k-1$.

5. The attractor a_0 belongs to Ω_1.

The definition of the domain Ω is completed. Obviously, the set of tuples $f = (f_0, \dots, f_{k+1}) \in V_0 \times V_1^{k+1}$ that satisfy conditions 1–5 is nonempty and open.

5.3. The set $\mathcal{N}$ and its density. By definition, the set $\mathcal{N}$ consists of tuples $f \in \Omega$ such that the attractor a_0 of f_0 coincides with the repeller of some composition of the maps $f_1, \dots, f_{k+1}$.

Proposition 3. *The set $\mathcal{N}$ is dense in Ω_1.*

Proof. We use the following Hutchinson lemma.

Lemma 3 (the Hutchinson lemma [1])**.** *Consider a k-dimensional manifold M, a domain $\Omega_0 \subset M$, a positive number $q < 1$, and maps $g_1, \dots, g_{k+1}$ with the following properties*:

$$g_l(\Omega_0) \subset \Omega_0, \tag{8}$$

$$\mathrm{Lip}\, g_l|_{\Omega_0} \le q. \tag{9}$$

Suppose there is also a subdomain Ω_1, included with its closure in Ω_0, such that

$$\Omega_1 \subset \bigcup_1^{k+1} g_l(\Omega_1). \tag{10}$$

Then for any domain U that has a nonempty intersection with Ω_1, there exists a map g in the semigroup $G^+(g_1, \dots, g_{k+1})$ such that

$$g(\Omega_0) \subset U. \tag{11}$$

By the Hutchinson lemma, conditions 2 and 3 for the tuple of maps inverse to $f_1, \dots, f_{k+1}$ imply the following: for any domain U that has a nonempty intersection with the domain Ω_1, there is a composition g_U of the maps $f_1^{-1}, \dots, f_{k+1}^{-1}$ such that $g_U(\Omega_0) \subset U$. Note that the map g_U contracts. Consequently, it has a single attracting fixed point $r_U \in U$. Then the map $f_U = g_U^{-1}$ has a repeller at r_U.

Let w be the word corresponding to the composition f_U and l be its length.

For an arbitrary point $f = (f_0, f_1 \dots, f_{k+1}) \in \Omega$ we take its neighborhood $\mathcal{U}_f$ and prove that there is a point of the set $\mathcal{N}$ in this neighborhood. Suppose that $a = a_0(f_0) \in \Omega_1$ is the attractor of the map f_0. Let $U_a \subset \Omega_1$ be a neighborhood of a such that for any $b \in U_a$ there is a map g_0 for which $g = (g_0, f_1, \dots, f_{k+1}) \in \mathcal{U}_f$ and b is the attractor of g_0. By the Hutchinson lemma, there exists a composition F_{U_a} of the maps $f_1, \dots, f_{k+1}$ that has a repeller $r \in U_a$. Now we choose $g = (g_0, f_1, \dots, f_{k+1})$ so that $g \in \mathcal{U}_f$ and the attractor of g_0 is r. The tuple g belongs to $\mathcal{N}$ by definition and to the domain $\mathcal{U}_f$ by construction. $\square$

5.4. Relative instability. Suppose that $f = (f_0, f_1 \dots, f_{k+1}) \in \mathcal{N}$ and a composition f_w of the last $k+1$ maps of the tuple has a repeller that coincides with the attractor of f_0. The word w that generates this composition exists by the definition of the set $\mathcal{N}$. Let $l = |w|$. Consider a skew product (7) with $f_0^0 = f_0$ and $f_0^1 = f_w$. We define the immersion $\zeta\colon \Sigma^2 \to \Sigma^{k+2}$ in the following way. In each sequence of zeros and ones we replace every 1 by the word w and every 0 by the word of l zeros. At the zero position of the new sequence, we have the first letter of the word that replaces the symbol that stood at the zero position of the old sequence. A statement analogous to Proposition 1 is proved identically, except that 3 is replaced by $k+2$.

The proof of Theorem 1 in the multidimensional case is concluded in the same way as for dimension one; we merely replace the reference to Theorem 2 by the reference to Theorem 3.

6. PROOF OF THE THEOREM ON THE STANDARD UNSTABLE ATTRACTOR

In this section we prove Theorem 3. We must show that the Milnor attractor $A_{\mathrm{M}}(F_0)$ coincides with the set $A = \Sigma^2 \times \{a_0\}$ and is not Lyapunov stable.

6.1. The attractor is included in A. In order to show that the set $A = \Sigma^2 \times \{a_0\}$ includes the Milnor attractor, it is enough to prove that the basin of attraction of A has measure no less than $1 - \delta$, where δ may be an arbitrary small positive number.

Consider a sequence $\omega = \dots \omega_{-n} \dots \omega_0 \dots \omega_n \dots \in \Sigma^2$. The subsequence $\omega_0 \omega_1 \dots$ is called the *right* (or *future*) *part* of ω, and the subsequence $\dots \omega_{-2}\omega_{-1}$ is called the *left* (or *past*) *part.*

Lemma 4. *For any $\delta \in (0,1)$ and any neighborhood $U_0 \subset M$ of the point a_0, there exists a set $B \subset \Sigma^2$ of measure $1 - \delta$ and a neighborhood $U \subset U_0$ of a_0 such that every point of the set $B \times U$ is attracted to A under the iterates of F_0. Moreover, one may require that the set B contain, with each sequence $\omega \in B$, all the sequences with the same right part.*

This statement was first formulated and proved by S. Minkov (unpublished). Below we present the proof that differs from his original argument in detail but follows the same ideas. Now we will use Lemma 4 to prove the theorem.

Proposition 4. *The Milnor attractor of the map F_0 is included in A.*

Proof. Proving this proposition, we follow the method suggested by Yu. Kudryashov [2].

Let U be a neighborhood of the point a_0, as in the lemma, small enough to be an attracting domain for f_0^0. For any point $q \in M$ there is a finite word $w(q)$ of zeros and ones such that the image of q under the corresponding composition $f_{w(q)}$ of fiber maps does not belong to S. If $q \notin S$, then $w(q)$ may be the empty word or the word consisting solely of zeros.

By the hypothesis of the theorem, some iterate of the map f_0^0 takes the point $t = f_{w(q)}(q) \notin S$ into the domain U. Therefore, any point of the fiber can be taken to a point of the set U by a composition of the fiber maps. When the point q comes to U, so does a small neighborhood of q. Consider the union of these neighborhoods for all points of the fiber: this union covers M, and there is a finite covering as long as M is compact.

Each element of this subcovering can be taken into U by a finite composition of fiber maps. Hence, for every point $q \in M$ there exists a word $\widetilde{w}(q)$ such that $f_{\widetilde{w}(q)}$ takes q to a point of the set U and the length of the word $\widetilde{w}(q)$ is at most N. We assume that the length of any word $\widetilde{w}(q)$ equals N: since U is an attracting region for f_0^0, one can supplement the word $\widetilde{w}(q)$ by zeros on the right if the length of this word is less than N.

Now we fix some point $p \in M$ and estimate the measure of the set

$$G(p) = \bigl\{\omega \in \Sigma^2 \bigm| \operatorname{dist}[F_0^n(\omega, p), A] \to 0 \ (n \to \infty)\bigr\}.$$

The set $G(p)$ consists of all sequences ω such that the points (ω, p) are attracted to A under the iterates of F_0.

Consider an arbitrary sequence $\eta = \dots \eta_{-m} \dots \eta_m \dots$ from Σ^2. We divide the right part of this sequence into clusters of length N: the first cluster begins from the zero position. The mth cluster of the sequence η is called *good* with respect to the point $p \in M$ whenever the image of the point p under the composition corresponding to the word $\eta_0 \dots \eta_{Nm-1}$ belongs to U.

Proposition 5. *Let a set $H(p)$ consist of the sequences that have at least one good cluster with respect to the point p. Then this set has the Bernoulli measure* 1.

Proof. We will estimate the measure of $\Sigma^2 \setminus H(p)$. Each sequence in $\Sigma^2 \setminus H(p)$ has a "bad" first cluster; therefore, there are no more than $2^N - 1$ options for this cluster. Whichever word we chose for the first cluster, there would be no more than $2^N - 1$ variants for the second one. These variants depend on the image of p under the composition corresponding to the first cluster. Continuing in the same way, we see that for each cluster we can find at least one word of length N that makes this cluster good provided that we know all the previous clusters. Since $\lim_{m\to\infty}\bigl(\frac{2^N-1}{2^N}\bigr)^m = 0$, the measure of $\Sigma^2 \setminus H(p)$ equals zero. Note that it depends on the previous clusters whether the cluster is good or not, but it does not depend on the symbols that stand to the right of this cluster. The proof of Proposition 5 is complete. $\square$

The set $H(p)$ can be represented in the form of a disjoint union:

$$H(p) = \bigsqcup_{i\in\mathbb{N}} H_i.$$

Here the sets H_i consist of sequences such that their first good cluster has number i. We denote by H_i' the subset of H_i defined by the following property: in each sequence of H_i', to the right from the ith cluster one has the right part of some sequence from the set B. In other words, $H_i' = H_i \cap \sigma^{-iN}(B)$.

Suppose that ω belongs to H_i. This means that there is some restriction on the symbols to the left of the iNth position. If a sequence belongs to $\sigma^{-iN}(B)$, the right "tail" that begins from the iNth position is actually the right part of some sequence in B, so there is a restriction on the symbols to the right of the iNth position. Therefore, $P_2(H_i') = P_2(H_i)P_2(B) = (1-\delta)P_2(H_i)$. Note that the union $H'(p) = \bigsqcup_{i\in\mathbb{N}} H_i'$ is disjoint and $P_2(H(p)) = 1$. This implies that $P_2(H'(p)) = 1 - \delta$. Since $H'(p) \subset G(p)$, we obtain an estimate

$$P_2(G(p)) \ge 1 - \delta.$$

Now we integrate in p using Fubini's theorem and get the likewise estimate for the measure of the attraction basin of A. Since δ in the lemma can be chosen arbitrary small, this basin has the full measure. It follows that $A_{\mathrm{M}} \subset A$.

The proof of Proposition 4 is complete. $\square$

6.2. The attractor coincides with A and is Lyapunov unstable. The Milnor attractor cannot be a proper subset of A. Note that for the Bernoulli shift the ω-limit set of almost every sequence coincides with Σ^2. We have already shown that for almost every point of the phase space the distance between its images under the iterates of F_0 and the projections of these images onto A tends to zero. These two statements imply that A is the ω-limit set for almost all points; therefore, $A_{\mathrm{M}} = A$.

Denote by (1) the sequence that consists only of ones. Then in the invariant fiber $\{(1)\} \times M$ the points are repelled from $\{(1)\} \times \{a_0\}$ under the iterates of F_0 since a_0 is a repeller of f_1^0. Therefore, these points repel from $A = A_{\mathrm{M}}(F_0)$ and the Milnor attractor is unstable. This completes the proof of Theorem 3. $\square$

6.3. Proof of Lemma 4. Let exp be the exponential mapping of the neighborhood T of zero in the tangent space T_{a_0} to the fiber M. This mapping takes each vector ξ to the endpoint of a geodesic arc of length $|\xi|$ that begins at a_0 and lies on M. We assume that the neighborhood T is so small that the exponential map is a diffeomorphism. Let $D_j \colon T_{a_0}M \to T_{a_0}M$ be the differential of the map f_j^0 at the point a_0, $j = 0, 1$. We denote by $d(x)$ the distance from x to a_0 on M. In the domain $\exp T$ this function is well defined. Let the neighborhood T' be so small that

$$d\big(\exp(1+\varepsilon)D_j\xi\big) \geq d\big(f_j^0(\exp\xi)\big). \tag{12}$$

We can choose such a neighborhood for every $\varepsilon > 0$ since the maps f_j^0 are C^1-smooth. Suppose that $D'_j = (1+\varepsilon)D_j$ and ε is chosen so small that

$$\|D'_0\| \cdot \|D'_1\| < 1. \tag{13}$$

Such an ε exists by assumption (6).

We can rewrite inequality (12) in the following way:

$$d(\exp D'_j\xi) \geq d\big(f_j^0(\exp\xi)\big) \qquad \forall \xi \in T'. \tag{14}$$

Note that

$$d(\exp D'_j\xi) \leq \|D'_j\| d(\exp\xi). \tag{15}$$

Denote by $g(\omega)$ the function

$$g(\omega) = \ln\|D'_{\omega_0}\|.$$

By (13), it follows that the space average of g is negative. Since the Bernoulli shift is ergodic, for almost all ω the time averages of g are negative too. Let

$$K_n(\omega) = \frac{1}{n}\sum_{k=0}^{n-1} g(\sigma^k\omega).$$

For almost all ω, we have $\lim K_n(\omega) < 0$. Consequently, for almost all ω,

$$\Sigma_0^{n-1} g(\sigma^k\omega) \to -\infty.$$

We denote by π the natural projection onto the fiber along the base. By induction on n with the help of (14) and (15) we get

$$d\big(\pi F_0^n(\omega,x)\big) = d\big(f_{\omega_{n-1}} \circ \ldots \circ f_{\omega_0}(x)\big) \leq e^{\Sigma_0^{n-1} g(\sigma^k\omega)} d(x) = e^{nK_n(\omega)} d(x) \to 0 \tag{16}$$

under the condition that $x \in \exp T'$ and

$$f_{\omega_{m-1}} \circ \ldots \circ f_0(x) \in \exp T'$$

for $m = 1, \ldots, n-1$.

For almost all ω, the quantity $\exp[nK_n(\omega)]$ tends to zero. According to Egorov's theorem, for any small positive δ there exists a set $B \subset \Sigma^2$ of measure $1-\delta$ on which this convergence is uniform. It follows that there is a number N such that for every $\omega \in B$ and every $n \geq N$ the estimate $\exp[nK_n(\omega)] < 1$ holds.

Now let the neighborhood U of the point a_0 be a small ball such that the points of $\Sigma^2 \times U$ do not leave the set $\Sigma^2 \times \exp T'$ under the first N iterates of the map F_0. In this case the images of all points of the set $B \times U$ under the Nth iterate of F_0 belong to the set $\Sigma^2 \times U$ by the definition of N and by (16). Under further iterations the images of these points do not leave the set $\Sigma^2 \times U$ either. Therefore, estimate (16) holds.

Since $\exp[nK_n(\omega)]$ tends to zero for $\omega \in B$, the points of the set $B \times U$ are attracted to the set $\Sigma^2 \times \{a_0\}$.

Notice that the set B can be expanded by including all sequences with the same right part as the right part of some sequence in B and with arbitrary left parts.

This concludes the proof of Lemma 4. □

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