



# Cohomology of 3-dimensional color Lie algebras

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Received 26 October 2005

Available online 29 December 2006

Communicated by Gerhard Hiss

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## Abstract

We develop the cohomology theory of color Lie algebras due to Scheunert–Zhang in a framework of non-homogeneous quadratic Koszul algebras. In this approach, the Chevalley–Eilenberg complex of a color Lie algebra becomes a standard Koszul complex for its universal enveloping algebra, providing a constructive method for computation of cohomology. As an application, we compute cohomologies with trivial coefficients of  $\mathbb{Z}_2^n$ -graded 3-dimensional color Lie algebras.

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*Keywords:* Color Lie algebras; Koszul algebras; Quadratic algebras; Cohomology

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## Introduction

The main objects studied in this paper are the unital associative algebras which are universal enveloping algebras of color Lie algebras also called generalized Lie algebras, or, more precisely,  $\Gamma$ -graded  $\varepsilon$ -Lie algebras, or generalized Lie algebras. Since the 1970s, generalized (color) Lie algebras have been an object of constant interest in both mathematics and physics [1,2,7,8,11,13,15–32,37–50]. The main goal of the paper is to describe the cohomology with scalar coefficients of three-dimensional (three generators) color Lie algebras. Our attention is concentrated on those three-dimensional color Lie algebras which are in an essential way graded as Lie color algebras by groups larger than  $\mathbb{Z}_2$ . The classification of such color Lie algebras has been given in [48,50] and presented in terms of commutation relations between generators. We use this classification

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in this paper for computing the cohomology. We believe our results to be useful for better understanding the structure of the variety of color Lie algebras and their representation theory, and also for shedding further light on a general constructive approach to cohomology computation. This work expands the previous works on three-dimensional color Lie algebras and their representations [22,29,30,46,49], where quadratic central elements and involutions (real forms) for these algebras have been computed, and representations for the graded analogues of the Lie algebra  $\mathfrak{sl}(2; \mathbb{C})$ , of the Lie algebra of the group of plane motions, and of the Heisenberg Lie algebras, the three of the non-trivial algebras from the classification, have been considered. It is expected that the representation theory for the three-dimensional color Lie algebras, when developed for all of them, will show deep connections with the cohomology structure for universal enveloping algebras of these algebras described in the present work.

The (co)homology theory for color Lie algebras has been introduced in [44] (see also [43] and [31]). The theory becomes analogous to the cohomology theory of the usual Lie algebras [12] and Lie superalgebras, i.e. color Lie algebras with the grading group  $\mathbb{Z}_2$ ; namely, there are cochain complexes, long exact sequences, homological description of deformations and universal central extensions, etc. In this paper we envelope this cohomology theory in a framework of Koszul algebras. The universal enveloping algebras of color Lie algebras are Koszul, and the Chevalley–Eilenberg complex becomes a standard Koszul complex for these associative algebras. This gives a kind of Koszul duality for different types of finitely generated color Lie algebras (a type is uniquely defined by the grading and a commutator factor), and a direct way to calculate cohomology. We use these techniques to compute the cohomology (with trivial coefficients) for 3-dimensional color Lie algebras classified in [50].

The paper is organized as follows. In Section 1, we give the definition of color Lie algebras and introduce basic notations. In the next section, we first discuss their (co)homologies. Notice that we use another definition of cohomology than the one given in [44], that is, we calculate it via the Ext's of the universal enveloping algebras. The equivalence of these approaches is considered in detail in [8, Theorem 4] (see also [43, Remark 3.1]). In the next Section 2.2, we briefly recall the theory of Koszul algebras and show that the universal enveloping algebras of the color Lie algebras are (non-homogeneous) Koszul. In Section 3, we study the case of abelian color Lie algebras. Their universal enveloping algebras are homogeneous Koszul, and the classical Koszul duality gives the duality on the set of their isomorphism classes. This gives us an easy way to compute the cohomology of abelian color Lie algebras with scalar coefficients and to establish some properties of their homologies with coefficients in cyclic modules. In Section 3.2, we use this to describe the method to compute the algebra of cohomology of arbitrary finite-dimensional color Lie algebra as a homology of a differential graded algebra, that is, the Koszul dual to the universal enveloping algebra of its associated abelian color Lie algebra with the differential dual to the multiplication. In Section 5, we apply this method to compute the cohomologies with scalar coefficient of three-dimensional color Lie algebras.

## 1. Basic notions

Given a commutative group  $\Gamma$  which will be in what follows referred to as the grading group, a commutation factor on  $\Gamma$  with values in the multiplicative group  $k \setminus \{0\}$  of a field  $k$  of characteristic 0 is a map

$$\varepsilon : \Gamma \times \Gamma \mapsto k \setminus \{0\}$$

satisfying three properties:

$$\varepsilon(\alpha + \beta, \gamma) = \varepsilon(\alpha, \gamma)\varepsilon(\beta, \gamma),$$

$$\varepsilon(\alpha, \beta + \gamma) = \varepsilon(\alpha, \beta)\varepsilon(\alpha, \gamma),$$

$$\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = 1.$$

The first two properties simply say that the map is a bihomomorphism or a bilinear map on  $\Gamma$ .

A  $\Gamma$ -graded  $\varepsilon$ -Lie algebra (or a color Lie algebra) is a  $\Gamma$ -graded linear space

$$X = \bigoplus_{\gamma \in \Gamma} X_\gamma$$

with a bilinear multiplication (bracket)  $\langle \cdot, \cdot \rangle : X \times X \rightarrow X$  obeying

*Grading axiom:*  $\langle X_\alpha, X_\beta \rangle \subseteq X_{\alpha+\beta}$ ,

*Graded skew-symmetry:*  $\langle a, b \rangle = -\varepsilon(\alpha, \beta)\langle b, a \rangle$ ,

*Generalized Jacobi identity:*

$$\varepsilon(\gamma, \alpha)\langle a, \langle b, c \rangle \rangle + \varepsilon(\beta, \gamma)\langle c, \langle a, b \rangle \rangle + \varepsilon(\alpha, \beta)\langle b, \langle c, a \rangle \rangle = 0$$

for all  $\alpha, \beta, \gamma \in \Gamma$  and  $a \in X_\alpha, b \in X_\beta, c \in X_\gamma$ . The elements of  $\bigcup_{\gamma \in \Gamma} X_\gamma$  are called homogeneous.

Any color Lie algebra  $X$  can be embedded in its universal enveloping algebra  $U(X)$  in such a way that, for homogeneous  $a \in X_\alpha$  and  $b \in X_\beta$ , the bracket  $\langle \cdot, \cdot \rangle$  becomes the  $\varepsilon$ -commutator  $\langle a, b \rangle = ab - \varepsilon(\alpha, \beta)ba$  which in particular becomes commutator  $[a, b] = ab - ba$  when  $\varepsilon(\alpha, \beta) = 1$  and anticommutator  $\{a, b\} = ab + ba$  when  $\varepsilon(\alpha, \beta) = -1$  (see [40]).

Before proceeding with the main results and constructions of this work, let us consider an example of one of the non-commutative algebras appearing in the classification, the color analogue of the Heisenberg Lie algebra, which is a three-dimensional  $\mathbb{Z}_2^3$ -graded generalized Lie algebra. Its universal enveloping algebra is the algebra with three generators  $e_1, e_2$  and  $e_3$  satisfying defining commutation relations

$$\begin{aligned} e_1e_2 + e_2e_1 &= e_3, \\ e_1e_3 + e_3e_1 &= 0, \\ e_2e_3 + e_3e_2 &= 0. \end{aligned} \tag{1}$$

When anticommutators in the left-hand side of (1) are changed into commutators, we indeed have the relations between generators in the universal enveloping algebra of the Heisenberg Lie algebra.

This algebra can be seen as the universal enveloping algebra of a  $\mathbb{Z}_3^n$ -graded  $\varepsilon$ -Lie algebra with the commutation factor given by  $\varepsilon(\alpha, \beta) = (-1)^{\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3}$ , for all  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$  and  $\beta = (\beta_1, \beta_2, \beta_3)$  in  $\mathbb{Z}_3^n$ . Now take  $X$  to be a  $\mathbb{Z}_3^3$ -graded linear space

$$X = X_{(1,1,0)} \oplus X_{(1,0,1)} \oplus X_{(0,1,1)}$$

with the homogeneous basis  $e_1 \in X_{(1,1,0)}$ ,  $e_2 \in X_{(1,0,1)}$ ,  $e_3 \in X_{(0,1,1)}$ . The homogeneous components graded by the elements of  $\mathbb{Z}_2^3$  different from  $(1, 1, 0)$ ,  $(1, 0, 1)$  and  $(0, 1, 1)$  are zero and so are omitted. If the  $\mathbb{Z}_2^3$ -graded bilinear multiplication  $\langle \cdot, \cdot \rangle$  turns  $X$  into a  $\mathbb{Z}_2^3$ -graded generalized Lie algebra, then with the above commutation factor  $\langle e_i, e_i \rangle = 0$ ,  $i = 1, 2, 3$  and

$$\langle e_1, e_2 \rangle = c_{12}e_3, \quad \langle e_2, e_3 \rangle = c_{23}e_1, \quad \langle e_3, e_1 \rangle = c_{31}e_2.$$

When  $a$  and  $b$  are in different homogeneous subspaces, it follows that  $\langle a, b \rangle = \langle b, a \rangle$ , whereas  $\langle a, b \rangle = -\langle b, a \rangle$  if  $a$  and  $b$  belong to the same one. Now put  $c_{12} = 1$ ,  $c_{23} = 0$  and  $c_{31} = 0$ . The  $\mathbb{Z}_2^3$ -graded  $\varepsilon$ -Lie algebra  $X$  so defined is the color analogue of the Heisenberg Lie algebra having the algebra defined by relations (1) as its universal enveloping algebra.

The color analogue of the Heisenberg Lie algebra (1) is one of the algebras in the classification of three-dimensional color Lie algebras with injective commutation factor, the class of algebras we work with in this paper. The condition of injectivity for a commutation factor simply means that for each fixed  $\alpha$  corresponding to a non-zero homogeneous component the maps  $\varepsilon(\alpha, \cdot) : \Gamma \rightarrow k \setminus \{0\}$  are injective. In terms of the grading and commutation relations this means that one cannot reduce the number of non-zero homogeneous components by simply adjoining some of them into one component of higher dimension without changing the algebra structure. For example in case of the color analogue of the Heisenberg Lie algebra we have three one-dimensional homogeneous components. The commutation factor can be conveniently represented by a  $3 \times 3$  matrix

$$\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

indicating the values of the commutation factor, that is the type of the bracket between the three non-zero homogeneous components, with 1 corresponding to a commutator and  $-1$  to an anti-commutator. In terms of the matrix of the commutation factor the injectivity simply means that there are no equal rows (and by symmetry columns) in the matrix.

The following simple lemma implies, that together with the known cases of Lie algebras and superalgebras, the color Lie algebras with injective commutation factor and three homogeneous components considered in this paper cover all three-dimensional color Lie algebras with commutation factors taking values in  $\{1, -1\}$ .

**Lemma 1.** *Let  $g$  be a color Lie algebra having at most two non-zero homogeneous components,  $g = g_i \oplus g_j$ ,  $i, j \in \Gamma$ . Then either  $g$  is abelian or is a usual (i.e.  $\mathbb{Z}_2$ -graded) Lie superalgebra.*

**Proof.** Obviously, if the whole  $g$  is concentrated in the same component  $g_i$ , then it is either abelian or is an ordinary Lie algebra  $g = g_0$ .

First, consider the case  $i = 0$ . Then  $\varepsilon(j, j) = \varepsilon(0, j)\varepsilon(j, j)$ . Hence  $\varepsilon(0, j) = 1$ . Because  $\varepsilon(0, 0) = 1$ , we have that  $g$  is either a Lie algebra or a  $\mathbb{Z}_2$ -graded Lie algebra depending on the sign of  $\varepsilon(j, j) = \pm 1$ .

The case  $j = 0$  is completely analogous. So, we may assume that  $i, j \neq 0$ . Since  $i + j \neq i, j$ , we have  $\langle g_i, g_j \rangle \subseteq g_{i+j} = 0$ . Suppose that  $g$  is not abelian, that is, say,  $\langle g_i, g_i \rangle \neq 0$ . Then  $\langle g_i, g_i \rangle \subseteq g_j$ , so,  $j = i + i$ . Thus,  $\varepsilon(i, j) = \varepsilon(i, i)^2 = 1$ , and hence  $\varepsilon(j, j) = \varepsilon(i, j)^2 = 1$ . Then  $g$  again becomes  $\mathbb{Z}_2$ -graded, with odd component either  $g_i$  or zero, depending on the sign of  $\varepsilon(i, i) = \pm 1$ .  $\square$

## 2. Koszul algebras and color Lie algebras

In this section, we describe the general construction and methods of cohomology theory of color Lie algebras. Here we assume that the ground field  $k$  has characteristics different from 2 and 3.

### 2.1. (Co)homology of color Lie algebras

In this subsection, we first consider associative  $k$ -algebras and modules over them. All algebras  $R$  are assumed to be *augmented*, that is, admitting a surjective homomorphism (augmentation)  $R \rightarrow k$ ; its kernel (augmentation ideal) is denoted by  $R_+$ . All modules over associative algebras below are assumed to be right-sided.

Let us introduce some short notations for (co)homologies of modules over augmented associative algebras. For a module  $M$  over a  $k$ -algebra  $R$ , we will denote by  $H_i(M) = H_i(R, M)$  its homology, i.e., the vector space  $\text{Tor}_i^R(M, k)$ . By  $H_i R$  we will denote the vector space  $\text{Tor}_i^R(k, k)$ . In the same way, we define  $H^i(M) = H^i(R, M) := \text{Ext}_R^i(k, M)$  and  $H^i R := H^i(R, k) := \text{Ext}_R^i(k, k)$ .

**Definition 2.** Let  $g$  be a (color) Lie algebra, let  $R = U(g)$  be its universal enveloping algebra, and let  $M$  be an  $R$ -module. By the  *$i$ th homology* of  $g$  with coefficients in  $M$  we will mean the vector space  $H_i(g, M) := H_i(R, M)$ , by  *$i$ th cohomology* we will mean  $H^i(g, M) := H^i(R, M)$ . By  *$i$ th (co)homology of  $g$  with trivial coefficients* we will mean the vector spaces  $H_i g := H_i(g, k) = H_i R$  and  $H^i g := H^i(g, k) = H^i R$ .

Because of [43, Remark 3.1] (see also [8, Theorem 4]), these cohomology groups are the same as the ones defined in [44], at least in the case when the ground field is of zero characteristics.

### 2.2. Koszulity

We will call a linear space over  $k$ , a  $k$ -algebra, or  $k$ -algebra module *positively graded*, if it is  $\mathbb{Z}_+$ -graded and finite-dimensional in every component. A positively graded associative algebra  $R = R_0 \oplus R_1 \oplus \dots$  is called *connected*, if its zero component  $R_0$  is  $k$ ; a connected algebra is called *standard*, if it is generated by  $R_1$  and a unit. All modules over positively graded algebras below are assumed to be  $\mathbb{Z}$ -graded and right-sided. For a graded module  $M$  over a connected algebra, its (co)homology vector space is again graded, that is,  $H_i(M) = \bigoplus_{j \in \mathbb{Z}} H_i(M)_j$ ,  $H^i(M) = \bigoplus_{j \in \mathbb{Z}} H^i(M)_j$  (where the second index  $j$  means the internal  $\mathbb{Z}$ -grading) and for (co)homologies with trivial coefficients we have the following duality isomorphisms of graded vector spaces:

$$H_i(R) = H^i(R)^*.$$

**Definition 3.** A graded module  $M$  over a standard associative algebra is called

- linear (of degree  $d$ ), if it is generated in degree  $d$ , i.e.,  $H_0(M)_j = 0$  for  $j \neq d$ ;
- quadratic, if it is linear of degree  $d$  and all its relations may be chosen in degree  $d + 1$ , i.e.,  $H_1(M)_j = 0$  for  $j \neq d + 1$ ;
- Koszul, if it has linear free resolution, i.e.,  $H_i(M)_j = 0$  for all  $i \geq 0$ ,  $j \neq i + d$ .

**Definition 4.** [33] A standard algebra  $R$  is called (homogeneous) Koszul if the trivial module  $k_R$  is Koszul, i.e., every homology module  $H_i R$  is concentrated in degree  $i$ .

Notice that the notion of homogeneous Koszul algebra has a lot of applications and a wide theory (see [3,4,34,35] and references therein).

Priddy in his original paper [33] had also introduced the notion of Koszulity for some non-graded algebras. Let  $A$  be an augmented associative algebra minimally generated by a finite-dimensional vector space  $V$  (that is, the augmentation ideal is generated by  $V$ ). Then  $A$  is called *almost quadratic* if its relations are non-homogeneous non-commutative polynomials of degree two, that is,  $A$  is isomorphic to a quotient algebra of the free algebra  $F = T(V)$  by an ideal  $I$  generated by a subspace of relations  $R \subset F_1 \oplus F_2 = V \oplus (V \otimes V)$ .

There are two graded algebras associated to  $A$ . The first one, denoted here by  $A^0$ , is a quotient algebra of  $A$  by an ideal generated by the projection of  $R$  on  $F_2 = V \otimes V$ . It is homogeneous quadratic. Also, the algebra  $A$  is obviously filtered by total degree, and the associated graded algebra  $\mathbf{gr} A$  is again a standard algebra generated by  $V$ .

In these terms, the definition of Priddy looks as follows.

**Definition 5.** [33] An almost quadratic algebra  $A$  is called Koszul if both graded algebras  $A^0$  and  $\mathbf{gr} A$  are isomorphic to each other and Koszul.

A discussion on this concept could be found in [6,34,35].

Priddy had also introduced the following special kind of Koszul algebras. We extend his definition to our non-homogeneous case. In modern terms, the definition looks as follows.

**Definition 6.** An almost quadratic algebra  $A = F/I$  is Poincaré–Birkhoff–Witt (PBW for short), if, for some choice of bases  $v_1, \dots, v_n$  of  $V$  and  $r_1, \dots, r_m$  of  $R$ , the set  $r = \{r_1, \dots, r_m\}$  is a Groebner basis of the ideal  $I$  w.r.t. the degree-lexicographical order with  $v_1 > \dots > v_n$ .

**Theorem 7.** Every almost quadratic PBW algebra is Koszul.

**Proof.** By theorem of Priddy [33], all graded (= homogeneous) PBW algebras are Koszul.

Let  $r_i = f_i + l_i$  for  $i = 1, \dots, m$ , where  $l_i \in V$  is the linear part and  $f_i \in V \otimes V$  is the homogeneous quadratic part, and let  $t_i$  be the leading monomial of  $r_i$ . By the Diamond Lemma (see [51]), the condition that  $r$  is a Groebner basis means that, if  $t_i v_p = v_q t_j$  for some  $i, j, p, q$ , then the  $s$ -polynomial  $s = r_i v_p - v_q r_j$  may be reduced to zero w.r.t.  $r$ . In particular, its high-degree part (of degree 3)  $s' = f_i v_p - v_q f_j$  may be reduced to an element of degree less than 3 w.r.t.  $r$ , that is, it is reducible to zero w.r.t. the set  $f = \{f_1, \dots, f_m\}$  of relations of  $A^0$ . This means that  $f$  is Groebner basis of relations of  $A^0$ ; in particular, the previously mentioned result of Priddy implies that  $A^0$  is Koszul. Also, by the definition of Groebner bases, the algebras  $A$  and  $A^0$  have the same linear basis consisting of all the monomials on generators without submonomials equal to  $t_i$ . Because the degree-lexicographical order is an extension of the partial order “ $\ll$ ” given by degrees of elements of  $F$ , the associated graded algebra  $\mathbf{gr} A$  w.r.t. the filtration induced by “ $\ll$ ” has the same linear basis. Thus, it is isomorphic to  $A^0$ .  $\square$

The terminology is due to the fact that the universal enveloping algebras of finite-dimensional Lie (super)algebras are PBW. In this connection, the following re-formulation of Poincaré–Birkhoff–Witt Theorem [40,41] for color Lie algebras is not surprising.

**Theorem 8.** *Let  $R = U(g)$  be the universal enveloping algebra of a finite-dimensional color Lie algebra  $g$ . Then the almost quadratic algebra  $R$  is PBW (in particular, Koszul). The associated graded quadratic algebra is the universal enveloping algebra of some abelian color Lie algebra.*

**Proof.** Let  $v_1, \dots, v_n$  be a  $\Gamma$ -homogeneous basis of  $V$ , let  $r$  be the set  $f_{ij} = v_i v_j - \varepsilon(v_i, v_j)v_j v_i - \sum_k c_{ij}^k$ , and let  $G$  be the set of the leading monomials of these elements. By the PBW-Theorem [40,41], the set  $B$  of monomials on the variables  $v_1, \dots, v_n$  whose all degree two submonomials do not belong to  $G$ , is linear independent modulo  $I$ . By definition, this means that the set  $r$  is a Groebner basis of  $I$ . It follows that the associated graded algebra has the relations  $h_{ij} = v_i v_j - \varepsilon(v_i, v_j)v_j v_i$ . By definition, it is the universal enveloping algebra of the abelian color Lie algebra with the same  $\Gamma$ -homogeneous basis and the same commutation factor as  $g$ .  $\square$

**Corollary 9.** *Let  $g$  be an abelian color Lie algebra. Then its universal enveloping algebra  $R = U(g)$  is homogeneous Koszul.*

### 3. The case of abelian color Lie algebras

#### 3.1. Koszul duality

For any standard quadratic algebra  $A$  with the space of generators  $V = A_1$  and the space of relations  $R \subset V \otimes V$ , its quadratic dual algebra  $A^!$  is, by definition, a standard quadratic algebra generated by the space  $V^*$  dual to  $V$  with the set of relations  $R^\perp \subset V^* \otimes V^*$  (the annihilator of  $R$  in the dual space to  $V \otimes V$ ). The importance of this algebra is due to the following fact [33]: if the algebra  $A$  is Koszul, then  $H_i A \simeq H^i A \simeq A_i^!$ .

Let  $V$  be a  $n$ -dimensional vector space spanned by  $v_1, \dots, v_n$ . Let  $[n]$  be the  $n$ -element set  $\{1, \dots, n\}$  and  $P$  be a set of unordered pairs of integers  $\{(i, j) \mid i, j \in [n], i < j\}$ . For any two subsets  $J \subset [n], Q \subset P$  let us define a standard associative algebra  $A_{J,Q} = T(V)/I_{J,Q}$ , where the ideal  $I_{J,Q}$  is generated by the following set of quadratic elements:

$$\begin{aligned} v_i v_j - v_j v_i, & \quad (i, j) \in Q; \\ v_i v_j + v_j v_i, & \quad (i, j) \in P - Q; \\ v_i^2, & \quad i \in J. \end{aligned}$$

Such relations can be put into a one-to-one correspondence with unordered graphs. Let  $\Gamma = (V, E)$  be a graph with a set of vertices  $V = \{v_i\}_{i \in [n]}$  and a set of edges  $E$ , where  $v_i$  and  $v_j$  are connected by an edge if and only if they anticommute, that is if  $(i, j) \in P - Q$  or  $i = j \in J$  (if the coefficient field is not of characteristic 2, then  $i = j \in J$  is equivalent to anticommutativity of a generator with itself).

The following statement is obtained easily from definitions.

**Proposition 10.** *The following relation holds:*

$$A_{J,Q}^! \cong A_{[n]-J, P-Q}.$$

The associative algebra  $A_{J,Q} = T(V)/I_{J,Q}$  can be also viewed as a universal enveloping algebra of an abelian  $\mathbb{Z}_2^n$ -graded (color) generalized Lie algebra. The following statement follows from the constructions in [38].

**Proposition 11.** *Let  $g$  be an  $n$ -dimensional abelian color Lie algebra with commutation factor taking the values in the set  $\{-1, 1\}$ . Then its universal enveloping algebra  $A = U(g)$  is isomorphic to the algebra  $A_{J,Q}$  for some  $J, Q$ .*

*Conversely, every algebra  $A_{J,Q}$  with  $n$  generators is isomorphic to the universal enveloping algebra of some  $n$ -dimensional abelian color Lie algebra.*

3.2. (Co)homology with trivial coefficients

In view of the previous subsection, the dimensions of (co)homologies of any abelian color Lie algebra  $g$  are equal to the dimensions of homogeneous components of a suitable PBW-algebra  $A_{J,Q} = (U(g))^!$ . To calculate these dimensions, it remains to show a formula for the generating function for these dimensions, that is, for the Hilbert series

$$A_{J,Q}(z) = \sum_{i=0}^{\infty} z^i \dim(A_{J,Q})_i.$$

**Proposition 12.** *Let  $q$  be the cardinality of elements in the set  $Q$ . Then*

$$A_{J,Q}(z) = \frac{(1+z)^q}{(1-z)^{n-q}}.$$

**Proof.** Since the algebra  $A_{J,Q}$  is PBW, as a graded vector space it is isomorphic to the span of the monomials which have no submonomials of the form  $\{v_i v_j \mid i < j\} \cup \{v_i^2 \mid i \in Q\}$ . It follows that its Hilbert series is equal to the Hilbert series of any algebra spanned by the same set of monomials, in particular, to the one of the algebra  $\Lambda(k^q) \otimes S(k^{n-q})$ , where  $\Lambda(k^q)$  and  $S(k^{n-q})$  are, respectively, the exterior algebra and the symmetric algebra. Hence

$$A_{J,Q}(z) = \Lambda(k^q)(z)S(k^{n-q})(z) = \frac{(1+z)^q}{(1-z)^{n-q}}. \quad \square$$

**Corollary 13.** *Let  $g$  be an  $n$ -dimensional abelian color Lie algebra such that in its  $\Gamma$ -homogeneous relations there are exactly  $q$  squares of the generators (that is, there are exactly  $q$  relations of the form  $\langle v_i, v_i \rangle = 0$  with  $\varepsilon(i, i) = -1$ ). Then its universal enveloping algebra  $A = U(g)$  has Poincaré series of the following form:*

$$P_A(z) := \sum_{i=0}^{\infty} z^i \dim H_i A = \frac{(1+z)^{n-q}}{(1-z)^q}.$$

*In particular, the homology algebra is finite-dimensional iff there are no squares among the  $\Gamma$ -homogeneous relations.*



### 3.3. Further Koszul properties

In this subsection, we describe some additional properties of the algebras  $A_{J,Q}$ . The subsequent section does not depend on them.

A homogeneous Koszul algebra may also have other homological properties depending on the structure of its ideals and cyclic Koszul modules. Such a theory is connected to the concept of the so-called *Koszul filtration*. For commutative Koszul algebras, this concept has been studied in several papers [5,9,10]. The non-commutative version of the theory has been developed in [36].

**Definition 14.** [36] Let  $R$  be a standard algebra. A set  $\mathbf{F}$  of degree-one generated right-sided ideals in  $R$  is called a Koszul filtration if  $0 \in \mathbf{F}$ ,  $R_+ \in \mathbf{F}$ , and for every  $0 \neq I \in \mathbf{F}$  there are  $I \neq J \in \mathbf{F}$  and  $x \in R_1$  such that  $I = J + xR$  and the ideal  $(x : J) := \{a \in R \mid xa \in J\}$  lies in  $\mathbf{F}$ .

As in the commutative case, every algebra admitting Koszul filtration is Koszul, as well as every ideal  $I \in \mathbf{F}$  and its cyclic module  $R/I$ .

The smallest possible Koszul filtration consisting of the complete flag of degree-one generated right-sided ideals  $0 = I_0 \subset \dots \subset I_n = R_+$  is called the Groebner flag: it corresponds to the sequence  $x_1, \dots, x_n$  of generators of  $V = A_1$ . An algebra having a Groebner flag is called initially Koszul.

Applying a criterion for initially Koszul algebras given in [36], we obtain the following results for color Lie algebras.

**Proposition 15.** Let  $g$  be a  $n$ -dimensional abelian color Lie algebra. Then its universal enveloping algebra  $A = U(g)$  is initially Koszul.

Moreover, it is initially Koszul w.r.t. any sequence  $x_1, \dots, x_n$  of homogeneous generators of  $g$ , that is,  $A$  is universally initially Koszul [5] and in particular it is strongly Koszul [14].

**Corollary 16.** Let  $W$  be a subset of the set of homogeneous generators of an  $n$ -dimensional abelian color Lie algebra  $g$ , and let  $M = U(g)/WU(g)$ . Then the graded vector space  $H_i(g, M)$  is concentrated in degree  $i$  for all  $i \geq 0$ .

## 4. The case of arbitrary finite-dimensional color Lie algebras

### 4.1. Priddy's method to calculate the cohomology

Let  $g$  be an  $n$ -dimensional color Lie algebra, and let  $g_{Ab}$  be the associated abelian color Lie algebra. Denote  $A = U(g)$ ,  $\bar{A} = U(g_{Ab})$ , and  $A^1 = \bar{A}^1$ . Let  $V = \bar{A}_1$  be the vector space of generators of  $\bar{A}$ ; we can identify  $V$  with  $g \subset A$ . The multiplication  $\mu : V \otimes V \rightarrow V$  in  $A$  induces the dual map  $d_1 : V^* \rightarrow V^* \otimes V^*$ , whose image may be identified with  $A_2^1$ , that is, we have  $d_1 : A_1^1 \rightarrow A_2^1$ . Extending  $d_1$  to the differential  $d$  of the whole algebra  $A^1$  by the Leibniz rule

$$d(ab) = d(a)b + (-1)^{|a|}ad(b),$$

where  $|\cdot|$  denotes the  $\mathbb{Z}_+$ -grading degree in  $A^1$ , and setting  $d1 = 0$ , we may consider the algebra  $A^1$  as a differential graded algebra  $(A^1, d)$ .

The following theorem was proved by Priddy for arbitrary Koszul algebra  $A$ .

**Theorem 17.** *In the notation above,  $H^\bullet g = H^\bullet(A^1, d)$ .*

The first corollary is well known for classical Lie algebras.

**Corollary 18.** [44] *Let  $g$  be a finite-dimensional color Lie algebra. Then*

$$H^0 g = k, \quad H^1 g = (g/[g, g])^*.$$

**5. Main result**

**Theorem 19.** *The cohomologies of all non-abelian three-dimensional color Lie algebras over  $\mathbb{C}$  with injective commutation factor taking values in  $\{-1, 1\}$  are described in Table 1.*

**Comments on Table 1**

The number in brackets in the first column denotes the number in the classification table in [30]. Among the 27 cases in [30], 12 color Lie algebras are abelian. Their homology is described in Corollary 13. The rest 15 cases are listed in our Table 1. The graphs in the second column denote the associated graded abelian algebras as described in Section 3.1. We do not repeat the graphs in the subsequent lines if the algebras have the same associated abelian algebras. In the tables  $\{a, b\} = ab + ba$  and  $\{a, b\} = ab - ba$ . The Betti numbers  $h_n = \dim H^n(g)$  are given in the last column in the form of their generating function (Poincaré series)  $P_g(z) = \sum_{n \geq 0} h_n z^n$ . For infinite-dimensional algebras  $H^\bullet(g)$ , we also give the explicit values for  $h_n, n \geq 1$ .

The *proof* of Theorem 19 is given in Table 2. We calculate the cohomology using Theorem 17. The numbering of cases is the same as in the previous Table 1. The linear bases of the differential graded algebras (that is, the universal enveloping algebras of the dual abelian Lie algebras) for the cohomology calculation are given in the second column. If an entry in the second column is empty, then it means that it is the same as the preceding first non-empty entry. The values of the differentials on the generators  $f_i = 2e_i^*$  (where  $\{e_i^*\}$  is the basis of  $g^*$  dual to a fixed basis  $\{e_i\}$  of  $g$ ) are given in the third column. In the last three columns, we give the bases of the vector spaces of cocycles  $Z^n$ , coboundaries  $B^n$ , and cohomology  $H^n$ . Sometimes we omit the case  $n = 1$ , because the groups  $H^1 g$  are described in Corollary 18. Notice that the multiplication structure of the algebra  $H^\bullet(g)$  is induced by the multiplication in the initial differential graded algebra  $A^1$ .

Table 1  
Cohomologies with trivial coefficients





No.	Graphs	Relations	Betti numbers $h_n := \dim H^n(g)$ , Poincaré series $P_g(z)$
1 (16)		$\{e_1, e_2\} = \mu e_2, \mu \in \mathbb{C} \setminus 0$ $\{e_1, e_3\} = e_3$ $\{e_2, e_3\} = 0$	$P_g(z) = \begin{cases} 1 + z, & \mu \neq -1, \\ 1 + z + z^2, & \mu = -1 \end{cases}$
2 (17)		$\{e_1, e_2\} = e_2$ $\{e_1, e_3\} = 0$ $\{e_2, e_3\} = 0$	$P_g(z) = 1 + 2z + z^2$
3 (1)		$\{e_1, e_2\} = e_3$ $\{e_1, e_3\} = e_2$ $\{e_2, e_3\} = e_1$	$P_g(z) = 1 + z^3$

Table 1 (continued)

No.	Graphs	Relations	Betti numbers $h_n := \dim H^n(g)$ , Poincaré series $P_g(z)$
4 (2)		$\{e_1, e_2\} = e_3$ $\{e_1, e_3\} = e_2$ $\{e_2, e_3\} = 0$	$P_g(z) = 1 + z + z^2 + z^3$
5 (3)		$\{e_1, e_2\} = e_3$ $\{e_1, e_3\} = 0$ $\{e_2, e_3\} = 0$	$P_g(z) = 1 + 2z + 2z^2 + z^3$
6 (19)		$[e_1, e_2] = \mu e_2, \mu \neq 0$ $[e_1, e_3] = e_3$ $\{e_2, e_3\} = 0$ $e_3^2 = 0$	$P_g(z) = \begin{cases} 1 + z + z^{k+1} + z^{k+2}, & \\ \mu = -1/k, & k \geq 1; \\ 1 + z, & \text{otherwise} \end{cases}$
7 (20)		$[e_1, e_2] = 0$ $[e_1, e_3] = 0$ $\{e_2, e_3\} = 0$ $2e_3^2 = e_1$	$P_g(z) = 1 + 2z + z^2$
8 (21)		$[e_1, e_2] = e_2$ $[e_1, e_3] = 0$ $\{e_2, e_3\} = 0$ $e_3^2 = 0$	$h_n = 2, n \geq 1;$ $P_g(z) = \frac{1+z}{1-z}$
9 (22)		$[e_1, e_2] = 0$ $[e_1, e_3] = e_3$ $\{e_2, e_3\} = 0$ $e_3^2 = 0$	$P_g(z) = 1 + 2z + 2z^2 + z^3$
10 (24)		$[e_1, e_2] = \mu e_2$ $[e_1, e_3] = e_3$ $[e_2, e_3] = 0$ $e_2^2 = e_3^2 = 0, \mu \in \mathbb{C} \setminus 0$	<p>For <math>\mu \notin \mathbb{Q}, P_g(z) = 1 + z.</math>                      If <math>\mu = p/q, p, q \in \mathbb{Z} \setminus 0, q &gt; 0, \gcd(p, q) = 1,</math>                      then <math>h_n = 1</math> for <math>n &gt; 1,</math> if either  <math>p = -2s, n = \lambda r, \lambda r + 1,</math> or  <math>p = 2s + 1, n = (2\lambda + 1)r, (2\lambda + 1)r + 1,</math> or  <math>p = -2s - 1, n = 2\lambda r, 2\lambda r + 1,</math>                      where <math>n \geq 2, \lambda \in \mathbb{Z}_+, r =  p  + q, s &gt; 0;</math>                      otherwise, <math>h_n = 0,</math></p> $P_g(z) = \begin{cases} 1 + z, & p = 2s, \\ \frac{1+z}{1-z^{2r}}, & p = -2s, \\ 1 + z + z^r \frac{1+z}{1-z^{2r}}, & p = 2s + 1, \\ \frac{1+z}{1-z^{2r}}, & p = -2s - 1, \end{cases}$ <p>where <math>r =  p  + q, s &gt; 0.</math></p>
11 (25)		$[e_1, e_2] = 0$ $[e_1, e_3] = 0$ $[e_2, e_3] = 0$ $e_2^2 = 0, 2e_3^2 = e_1$	$h_n = 2, n \geq 1;$ $P_g(z) = \frac{1+z}{1-z}$

(continued on next page)

Table 1 (continued)


No.	Graphs	Relations	Betti numbers $h_n := \dim H^n(g)$ , Poincaré series $P_g(z)$
12 (26)		$[e_1, e_2] = e_2$ $[e_1, e_3] = 0$ $[e_2, e_3] = 0$ $e_2^2 = e_3^2 = 0$	$h_n = 2, n \geq 1;$ $P_g(z) = \frac{1+z}{1-z}$
13 (5)		$\{e_1, e_2\} = e_3$ $\{e_1, e_3\} = e_2$ $[e_2, e_3] = 0$ $e_2^2 = e_3^2 = 0$	$h_n = 1, n \geq 0;$ $P_g(z) = \frac{1}{1-z}$
14 (6)		$\{e_1, e_2\} = 0$ $\{e_1, e_3\} = 0$ $[e_2, e_3] = e_1$ $e_2^2 = e_3^2 = 0$	$h_n = 2, n \geq 1;$ $P_g(z) = \frac{1+z}{1-z}$
15 (7)		$\{e_1, e_2\} = e_3$ $\{e_1, e_3\} = 0$ $[e_2, e_3] = 0$ $e_2^2 = e_3^2 = 0$	$h_n = 2, n \geq 1;$ $P_g(z) = \frac{1+z}{1-z}$

Table 2  
Cohomology computation

No.	Basis of $A_n^1$	$d \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$	Basis of $Z^n$	Basis of $B^n$	Cocycles representing $H^n$
1 (16)	$f_i, n = 1;$ $f_i f_j,$ $i < j, n = 2;$ $f_1 f_2 f_3, n = 3$	0 $\mu^{-1} f_1 f_2$ $f_1 f_3$	$f_1, n = 1;$ $f_1 f_2, f_1 f_3,$ $n = 2, \mu \neq -1;$ $f_1 f_2, f_1 f_3, f_2 f_3,$ $n = 2, \mu = -1;$ $f_1 f_2 f_3, n = 3$	$f_1 f_2, f_1 f_3,$ $n = 2;$ $f_1 f_2 f_3,$ $n = 3$	$f_1, n = 1;$ $f_2 f_3,$ $n = 2, \mu = -1$
2 (17)		0 $\mu^{-1} f_1 f_2$ 0	$f_1, f_3, n = 1;$ $f_1 f_2, f_1 f_3,$ $n = 2;$ $f_1 f_2 f_3, n = 3$	$f_1 f_2,$ $n = 2;$ $f_1 f_2 f_3,$ $n = 3$	$f_1, f_3, n = 1;$ $f_1 f_3,$ $n = 2, \mu = -1$
3 (1)		$f_2 f_3$ $f_1 f_3$ $f_1 f_2$	$f_1 f_2, f_1 f_3,$ $f_2 f_3, n = 2;$ $f_1 f_2 f_3, n = 3$	$f_1 f_2, f_1 f_3,$ $f_2 f_3, n = 2$	$f_1 f_2 f_3, n = 3$
4 (2)		0 $f_1 f_3$ $f_1 f_2$	$f_1, n = 1;$ $f_1 f_2, f_1 f_3,$ $f_2 f_3, n = 2;$ $f_1 f_2 f_3, n = 3$	$f_1 f_2, f_1 f_3,$ $n = 2$	$f_1, n = 1;$ $f_2 f_3, n = 2;$ $f_1 f_2 f_3, n = 3$
5 (3)		0 0 $f_1 f_2$	$f_1, f_2, n = 1;$ $f_1 f_2, f_1 f_3,$ $f_2 f_3, n = 2;$ $f_1 f_2 f_3, n = 3$	$f_1 f_2,$ $n = 2$	$f_1, f_2, n = 1;$ $f_1 f_3, f_2 f_3,$ $n = 2;$ $f_1 f_2 f_3, n = 3$

Table 2 (continued)

No.	Basis of $A_n^1$	$d \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$	Basis of $Z^n$	Basis of $B^n$	Cocycles representing $H^n$
6 (19)	$f_1 f_2 f_3^{n-2};$ $f_1 f_3^{n-1};$ $f_2 f_3^{n-1};$ $f_3^n$	0 $\mu^{-1} f_1 f_2$ $f_1 f_3$	$f_2 f_3^{n-1},$ $\mu^{-1} = 1 - n;$ $f_1 f_3^{n-1};$ $f_1 f_2 f_3^{n-2};$	$f_1 f_2 f_3^{n-2},$ $\mu^{-1} \neq 2 - n;$ $f_1 f_3^{n-1}$	$f_2 f_3^{n-1},$ $\mu^{-1} = 1 - n;$ $f_1 f_2 f_3^{n-2},$ $\mu^{-1} = 2 - n$
7 (20)		$f_3^2$ 0 0	$f_2 f_3^{n-1};$ $f_3^n$	$f_2 f_3^{n-1},$ $f_3^n, n \geq 3;$ $f_3^2, n = 2$	$f_1, f_3, n = 1;$ $f_2 f_3, n = 2$
8 (21)		0 $f_1 f_2$ 0	$f_1 f_2 f_3^{n-2};$ $f_1 f_3^{n-1};$ $f_3^n$	$f_1 f_2 f_3^{n-2}$	$f_1 f_3^{n-1};$ $f_3^n$
9 (22)		0 0 $f_1 f_3$	$f_1 f_2 f_3^{n-2};$ $f_1 f_3^{n-1}$	$f_1 f_3^{n-1}, n \geq 3;$ $f_1 f_2 f_3^{n-2},$ $n \geq 4$	$f_1 f_2 f_3^{n-2},$ $n \leq 3;$ $f_1 f_3^{n-1},$ $n \leq 2$
10 (24)	$f_2^k f_3^l,$ $k + l = n;$ $f_1 f_2^k f_3^l,$ $k + l = n - 1$	0 $\mu^{-1} f_1 f_2$ $f_1 f_3$	$f_1 f_2^k f_3^l,$ $k + l = n - 1;$ $f_2^k f_3^l,$ $k + l = n,$ $(-1)^{k+1} k = l\mu$	$f_1 f_2^k f_3^l,$ $k + l = n - 1,$ $(-1)^{k+1} k \neq l\mu$	$f_2^k f_3^l,$ $k + l = n,$ $k = (-1)^k l\mu;$ $f_1 f_2^k f_3^l,$ $k + l = n - 1,$ $(-1)^{k+1} k = l\mu$
11 (25)		$f_3^2$ 0 0	$f_2^k f_3^l,$ $k + l = n$	$f_2^k f_3^{l+2},$ $k + l = n - 2$	$f_2^n;$ $f_2^{n-1} f_3$
12 (26)		0 $f_1 f_2$ 0	$f_3^n;$ $f_1 f_2^k f_3^l,$ $k + l = n - 1$	$f_1 f_2^k f_3^l,$ $k + l = n - 1,$ $k \geq 1$	$f_3^n;$ $f_1 f_3^{n-1}$
13 (5)	$f_2^k f_3^l,$ $k + l = n;$ $f_1 f_2^k f_3^l,$ $k + l = n - 1$	0 $f_1 f_3$ $f_1 f_2$	$f_1 f_2^k f_3^l,$ $k + l = n - 1;$ $(f_2^2 + f_3^2)^t,$ $n = 2t$	$f_1 f_2^k f_3^l,$ $k + l = n - 1,$ $n = 2t;$ $l f_1 f_2^{k+1} f_3^{l-1}$ $- k f_1 f_2^{k-1} f_3^{l+1},$ $k + l = n,$ $k, l > 0,$ $n = 2t + 1;$ $f_1 f_2^{n-2} f_3,$ $n = 2t + 1$	$(f_2^2 + f_3^2)^t,$ $n = 2t;$ $f_1 f_2 f_3^{n-2},$ $n = 2t + 1$
14 (6)		$f_2 f_3$ 0 0	$f_2^k f_3^l,$ $k + l = n$	$f_2^{k+1} f_3^{l+1},$ $k + l = n - 2$	$f_2^n, f_3^n$
15 (7)		0 0 $f_1 f_2$	$f_1 f_2^k f_3^l,$ $k + l = n - 1,$ $f_2^n$	$f_1 f_2^k f_3^l,$ $k + l = n - 1,$ $k \geq 1$	$f_2^n,$ $f_1 f_3^{n-1}$

## Acknowledgments

The authors thank Daniel Larsson for many useful comments. The first author was partially supported by Russian Foundation of Basis Research project 05-01-01034. The research was also partially supported by the Crafoord Foundation, The Royal Physiographic Society in Lund, The Swedish Foundation for International Cooperation in Research and High Education (STINT) and The Royal Swedish Academy of Sciences. The authors gratefully acknowledge hospitality of Mittag-Leffler Institut during the Non-commutative Geometry Program 2003–2004, where the main part of this research was performed. The first author is grateful to the Department of Mathematics, Lund Institute of Technology for hospitality during his visit there.

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