MASS TRANSPORT GENERATED BY A FLOW OF GAUSS MAPS

VLADIMIR I. BOGACHEV, ALEXANDER V. KOLESNIKOV

Abstract. Let $A \subset \mathbb{R}^d$, $d \geq 2$, be a compact convex set and let $\mu = \varrho_0\, dx$ be a probability measure on A equivalent to the restriction of Lebesgue measure. Let $\nu = \varrho_1\, dx$ be a probability measure on $B_r := \{x\colon |x| \leq r\}$ equivalent to the restriction of Lebesgue measure. We prove that there exists a mapping T such that $\nu = \mu \circ T^{-1}$ and $T = \varphi \cdot \mathrm{n}$, where $\varphi\colon A \to [0, r]$ is a continuous potential with convex sub-level sets and n is the Gauss map of the corresponding level sets of φ. Moreover, T is invertible and essentially unique. Our proof employs the optimal transportation techniques. We show that in the case of smooth φ the level sets of φ are governed by the Gauss curvature flow $\dot{x}(s) = -s^{d-1} \frac{\varrho_1(s\mathrm{n})}{\varrho_0(x)} K(x) \cdot \mathrm{n}(x)$, where K is the Gauss curvature. As a by-product one can reprove the existence of weak solutions to the classical Gauss curvature flow starting from a convex hypersurface.

Keywords: optimal transportation, Monge–Kantorovich problem, Monge–Ampère equation, Gauss curvature flow, Gauss map.

AMS Subject Classification: 53C44, 49Q15, 49Q20, 28D05

1. Introduction

The goal of this paper is to introduce a new class of transformations of measures on $\mathbb{R}^d$ which (heuristically) have the form $T = \varphi \cdot \nabla\varphi/|\nabla\varphi|$ with some function φ. Our work is motivated by two intensively developing areas: optimal transportation and curvature flows, and establishes an interesting link between these areas. Optimal transportation can be described as a problem of optimization of a certain functional associated with a pair of measures. The quadratic transportation cost $W_2^2(\mu, \nu)$ between two probability measures μ, ν on $\mathbb{R}^d$ is defined as the minimum of the following functional (the Kantorovich functional):

$$m \mapsto \int_{\mathbb{R}^d \times \mathbb{R}^d} |x_1 - x_2|^2\, dm(x_1, x_2), \quad m \in \mathcal{P}(\mu, \nu), \tag{1.1}$$

where $\mathcal{P}(\mu, \nu)$ is the set of all probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with the marginals μ and ν; here $|v|$ denotes the Euclidean norm of $v \in \mathbb{R}^d$. The problem of minimizing (1.1) is called the mass transportation problem. This formulation is due to Kantorovich [14]. A detailed discussion of the mass transportation problem in this setting can be found in [19]. In many cases there exists a mapping $T\colon \mathbb{R}^d \to \mathbb{R}^d$, called the optimal transport between μ and ν (or a solution to the Monge problem), such that $\nu = \mu \circ T^{-1}$ and

$$W_2^2(\mu, \nu) = \int_{\mathbb{R}^d} |x - T(x)|^2\, \mu(dx).$$

The minimization of the latter integral in the class of measurable mappings T such that $\mu \circ T^{-1} = \nu$ is called the Monge problem. If T is a solution to the Monge problem, then the image of μ under the mapping $x \mapsto (x, T(x))$ to $\mathbb{R}^d \times \mathbb{R}^d$ minimizes the Kantorovich functional. However, it may happen that the Monge problem has no solution, while the Kantorovich problem is always solvable. It is worth mentioning that the first rigorous results related to existence of optimal mappings were obtained in the classical work of A.D. Alexandroff [1] on convex surfaces with prescribed curvature! If μ and ν are absolutely continuous, then, as show by Brenier [6] and McCann [16], there exists an optimal transportation T which takes μ to ν. Moreover, this mapping is μ-unique and has the form $T = \nabla W$, where W is convex. Under broad assumptions, W solves the following nonlinear PDE (the Monge–Ampère equation):

$$\varrho_\nu(\nabla W) \det D_a^2 W = \varrho_\mu,$$

where ϱ_μ and ϱ_ν are densities of μ and ν and $D_a^2 W$ is the absolutely continuous part of the distributional derivative of $D^2 W$. At present the optimal transportation theory attracts attention of researchers from the most diverse fields, including probability, partial differential equations, geometry, and infinite-dimensional analysis (see Villani's book [24] and papers [2] and [21]).

The study of curvature flows is a very popular subject in geometry. The theory of Ricci flows attracted particular interest after the famous works of G. Perelman on the Poincaré conjecture. The theory of geometrical flows began, however, with flows of embedded manifolds. Let $F_0\colon\ M^{d-1} \to \mathbb{R}^d$ be a smooth embedding of a smooth compact Riemannian manifold M^{d-1} (without boundary). Denote by A the enclosed body: $\partial A = F_0(M)$. We say that an evolution

$$F(\,\cdot\,,\,\cdot\,)\colon\ M \times [0, T) \to \mathbb{R}^d$$

is a geometrical flow if $F_0 = F(\cdot, 0)$ and

$$\frac{\partial}{\partial t} F(x,t) = -g(F(x,t)) \cdot \mathrm{n}(F(x,t)), \tag{1.2}$$

where $g\colon\ M \to \mathbb{R}$ is some curvature function and n is the outer unit normal vector. If $g = H$ is the mean curvature, then F is called the mean curvature flow. If $g = K$ is the Gauss curvature, then F is called the Gauss curvature flow. The simplest case of the curvature flow is given by the following planar flow:

$$\frac{\partial x(t,s)}{\partial t} = -k(x) \cdot \mathrm{n}(x),\ x \in \mathbb{R}^2,\ M = S^1.$$

Consider the flow of closed curves $t \mapsto x(t, \cdot)$. Under this flow the enclosed volume decreases with the constant speed -2π. In addition, any non-convex curve becomes convex in finite time and then remains convex. Finally, any curve shrinks to a point in finite time; the shape of any curve becomes more and more rotund (see [10] and [12]). Any multi-dimensional mean curvature flow or Gauss curvature flow starting from a convex surface preserves convexity and shrinks the surface to a point (see [13] and [23]). In [9] and [7], equation (1.2) in the case of the mean curvature was investigated from the PDE's point of view. It turns out that the surfaces driven by (1.2) can be obtained as level sets of a function $u(t,x)$ which satisfies a nonlinear degenerate second order parabolic equation of the Monge–Ampère type. A solution of this equation is in general understood in some weak sense (viscosity solutions). For the PDE approach and viscosity solutions, see the recent book [11]. Concerning Gauss flows, see [3].

In this paper, we establish the existence and uniqueness of a special measure transportation mapping between two probability measures μ and ν. It has the following heuristic expression:

$$T = \varphi \frac{\nabla \varphi}{|\nabla \varphi|}.$$

The potential φ has convex sub-level sets. Note that this transportation mapping may not be a gradient. Nevertheless, T can be obtained as a degenerate limit of some transportation mappings which are constructed by means of the optimal transportation techniques. The limiting potential satisfies a degenerate Monge–Ampère equation (see the proof of the main theorem). In addition, we show that the resulting limit is naturally connected with the Gauss flow. The level sets of the potential φ can be associated to a special Gauss flow associated to the measures μ and ν according to

$$\dot{x}(s) = -s^{d-1} \ \frac{\varrho_1(s\mathrm{n})}{\varrho_0(x)} K(x) \cdot \mathrm{n}(x).$$

In the case $\varrho_1(x) = \frac{C_{d,r}}{|x|^{d-1}}$ and $\varrho_0(x) = \frac{1}{\mathcal{H}^d(A)}$ we obtain a weak solution to

$$\dot{x}(s) = -c \ K(x) \cdot \mathrm{n}(x),$$

which is the classical Gauss flow starting from some initial convex hypersurface.

Finally, we note that in [15], [17], [22] the reader can find other interesting links between mass transportation and geometrical flows (in particular, the Ricci flows). Some analogs of the presented results in the case of a manifold will be considered in our forthcoming joint paper with F.-Y. Wang.

2. Main result

Throughout we assume that $d \geq 2$ and denote by $\mathcal{H}^n$ the n-dimensional Hausdorff measure. For Lebesgue measure we often use another common notation dx. Let $\operatorname{Int} A$ denote the interior of a set A.

Recall that, given a compact smooth orientable $(d-1)$-dimensional surface M in $\mathbb{R}^d$, one has the Gauss map $\mathrm{n}\colon\ M \to S^{d-1}$, where $\mathrm{n}(x)$ is the global unit outer normal vector field. Let $D\mathrm{n}\colon\ TM_x \to TS^{d-1}_{\mathrm{n}(x)}$ be the differential of n. Choose an orthonormal basis $\{e_2, \ldots, e_d\} \subset TM_x$. Then the matrix $D\mathrm{n}$ can be written as $\langle \partial_{e_i}\mathrm{n}, e_j \rangle$, where $\partial_{e_i}\mathrm{n}$ are the usual partial derivatives of n. The determinant of $D\mathrm{n}(x)$ is called the Gauss curvature and is denoted throughout by $K(x)$.

Below we deal with the case where M is a surface (possibly, non-smooth) of the form $M = \partial V$, where V is a convex compact set. In this case the normal $\mathrm{n}(x)$ is well-defined almost everywhere on M. More precisely, for an arbitrary point $x \in M$, let us set

$$N_{M,x} := \{\eta \in S^{d-1};\ \forall z \in V,\ \langle \eta, z - x \rangle \leq 0\}.$$

If $N_{M,x}$ contains a single element $\mathrm{n}(x)$, then $\mathrm{n}(x)$ is the unit normal in the usual sense. We shall use the fact that one has $\mathcal{H}^{d-1}(S) = 0$, where

$$S = \{x\colon\ N_{M,x} \text{ contains more than one element}\}.$$

Hence the Gauss map $\mathrm{n}(x)$ is well-defined $\mathcal{H}^{d-1}$-almost everywhere on M. Moreover, one can show that $K(x)$ is well-defined $\mathcal{H}^{d-1}$-almost everywhere on M (but this fact is not used below).

We shall consider the following Hausdorff distance between nonempty compact sets:

$$\operatorname{dist}(B_1, B_2) = \max\bigl(\sup_{x \in B_1} \operatorname{dist}(x, B_2),\ \sup_{x \in B_2} \operatorname{dist}(x, B_1)\bigr).$$

Theorem 2.1. *Let $A \subset \mathbb{R}^d$ be a compact convex set and let $\mu = \varrho_0\,dx$ be a probability measure on A equivalent to the restriction of Lebesgue measure. Let $\nu = \varrho_1\,dx$ be a probability measure on $B_r = \{x\colon |x| \le r\}$ equivalent to the restriction of Lebesgue measure. Then, there exist a Borel mapping $T\colon A \to B_r$ and a continuous function $\varphi\colon A \to [0,r]$ with convex sub-level sets $A_s = \{\varphi \le s\}$ such that $\nu = \mu \circ T^{-1}$ and*

$$T = \varphi \cdot \mathrm{n} \quad \mathcal{H}^d\text{-almost everywhere,}$$

where $\mathrm{n} = \mathrm{n}(x)$ is a unit outer normal vector to the level set $\{y\colon \varphi(y) = \varphi(x)\}$ at the point x.

If φ is smooth, the level sets of φ are moving according to the following Gauss curvature flow equation:

$$\dot{x}(s) = -s^{d-1}\,\frac{\varrho_1(s\mathrm{n})}{\varrho_0(x)} K(x) \cdot \mathrm{n}(x) \tag{2.1}$$

where $x(s) \in \partial A_{r-s}$, $0 \le s \le r$, $x(0) \in \partial A$ is any initial point.

To prove this theorem we develop an approach based on the optimal transportation techniques. For every $t \ge 0$, we consider a mapping T_t that takes μ to ν and maximizes the functional

$$F \mapsto \int \langle x, F(x)\rangle |F(x)|^t\,\mu(dx) \tag{2.2}$$

in the class of mappings F with $\mu\circ F^{-1} = \nu$. Equivalently, it minimizes the functional

$$F \mapsto \int \bigl|x - F(x)|F(x)|^t\bigr|^2\,\mu(dx)$$

in the class of mappings F with $\mu \circ F^{-1} = \nu$. For $t = 0$ (2.2) becomes the classical Monge–Kantorovich problem. For $t \ne 0$ standard arguments from the Monge–Kantorovich theory show that the set

$$\Bigl\{\bigl(x, T_t(x)|T_t(x)|^t\bigr),\ x \in A\Bigr\}$$

is cyclically monotone, hence belongs to the graph of the gradient of some convex function W_t (see [24, Chapter 2]). This can be shown, for instance, by a cyclical permutation of small balls (see [24]). More formally, this can be obtained by variation of the corresponding Lagrange functional (see [8]).

If the reader does not want to be concerned with the cyclical monotonicity or calculus of variations, we note that ∇W_t is just the optimal transportation of μ to $\nu \circ S_t^{-1}$, where $S_t(x) = x|x|^t$. This can be taken for a definition of W_t.

One has the following relations:

$$T_t = \frac{\nabla W_t}{|\nabla W_t|^{\frac{t}{1+t}}},\ \ \nabla W_t(x) = T_t(x)|T_t(x)|^t.$$

Clearly, $|T_t(x)| \le r$ since T_t transforms μ into ν.

Throughout the paper we choose W_t in such a way that $\min_{x\in A} W_t(x) = 0$. Define a new potential function φ_t by

$$W_t = \frac{1}{t+2}\varphi_t^{t+2}.$$

One has

$$T_t = \varphi_t \frac{\nabla \varphi_t}{|\nabla \varphi_t|^{\frac{t}{t+1}}}.$$

We show below that the limits

$$\lim_{t\to\infty} \varphi_t = \varphi, \ \lim_{t\to\infty} T_t = T$$

exist almost everywhere (for a suitable sequence $t_n \to \infty$) and then we prove that T is the desired mapping.

Lemma 2.2. *One has*

$$\varphi_t \le (2+t)^{\frac{1}{2+t}} \big(\mathrm{diam}(A)\big)^{\frac{1}{2+t}} r^{\frac{1+t}{2+t}},$$

$$\int_A |\nabla \varphi_t(x)|\, dx \le \int_{\partial A} \varphi_t \, d\mathcal{H}^{d-1} \le (2+t)^{\frac{1}{2+t}} \big(\mathrm{diam}(A)\big)^{\frac{1}{2+t}} r^{\frac{1+t}{2+t}} \ \mathcal{H}^{d-1}(\partial A).$$

Proof. By the convexity of W_t we have

$$W_t(x) - W_t(y) \le \langle x - y, \nabla W_t(x) \rangle.$$

Choosing y_0 in such a way that $W_t(y_0) = 0$, we find $W_t(x) \le \mathrm{diam}(A)|\nabla W_t(x)|$ for every $x \in A$. Since $W_t = \frac{1}{2+t}\varphi_t^{t+2}$, we obtain

$$\varphi_t \le (t+2)\mathrm{diam}(A)|\nabla \varphi_t|.$$

Let $\alpha = \frac{1+t}{2+t}$. Note that $\frac{1-\alpha}{\alpha} = \frac{1}{1+t}$, hence $\varphi_t |\nabla \varphi_t|^{\frac{1-\alpha}{\alpha}} = |T_t| \le r$. Therefore, one has

$$\begin{aligned}
\varphi_t = \varphi_t^{1-\alpha}\varphi_t^{\alpha} &\le \big((2+t)\mathrm{diam}(A)\big)^{1-\alpha} |\nabla \varphi_t|^{1-\alpha} \varphi_t^{\alpha} \\
&\le \big((2+t)\mathrm{diam}(A)\big)^{1-\alpha} \Big[\varphi_t |\nabla \varphi_t|^{\frac{1-\alpha}{\alpha}}\Big]^{\alpha} \\
&\le \big((2+t)\mathrm{diam}(A)\big)^{1-\alpha} r^{\alpha} = \big((2+t)\mathrm{diam}(A)\big)^{\frac{1}{2+t}} r^{\frac{1+t}{2+t}}.
\end{aligned}$$

By using the convexity of W_t again, we get

$$0 \le \mathrm{div}\Big(\frac{\nabla W_t}{|\nabla W_t|}\Big) = \mathrm{div}\Big(\frac{\nabla \varphi_t}{|\nabla \varphi_t|}\Big),$$

where under $\mathrm{div}\Big(\frac{\nabla W_t}{|\nabla W_t|}\Big)$ we understand the distributional derivative of the vector field $\frac{\nabla W_t}{|\nabla W_t|}$. Integrating with respect to $\varphi_t dx$ we obtain

$$0 \le \int_A \mathrm{div}\Big(\frac{\nabla \varphi_t}{|\nabla \varphi_t|}\Big)\varphi_t \, dx = -\int_A |\nabla \varphi_t| \, dx + \int_{\partial A} \varphi_t \Big\langle n_A, \frac{\nabla \varphi_t}{|\nabla \varphi_t|}\Big\rangle \, d\mathcal{H}^{d-1}.$$

Hence

$$\int_A |\nabla \varphi_t| \, dx \le \int_{\partial A} \varphi_t \, d\mathcal{H}^{d-1}.$$

Applying the above uniform estimate for φ_t we complete the proof. In fact, one could do these calculations in the case of smooth densities, where φ_t has a better regularity, and then approximate our densities by smooth ones (the corresponding optimal transports converge to ∇W_t). □

Corollary 2.3. *There exists a sequence $\{t_n\} \to \infty$ such that $\{\varphi_{t_n}\}$ converges almost everywhere to a finite function φ.*

Proof. By Lemma 2.2, every sequence $\{\varphi_{t_n}\}$ is bounded in $W^{1,1}(A)$. By the compactness of the embedding $W^{1,1}(B) \subset L^1(B)$ for any ball $B \subset A$ and the diagonal argument (in fact, since in the present situation A is convex, the embedding of the whole space $W^{1,1}(A)$ is compact), we obtain the claim. □

Lemma 2.4. *There exists a sequence $\{t_n\} \to \infty$ such that*

$$\lim_{t_n\to\infty} |\nabla\varphi_n|^{\frac{1}{1+t_n}} = 1$$

almost everywhere, where $\varphi_n := \varphi_{t_n}$.

Proof. By Lemma 2.2 one has $\varphi_t \le (t+2)\mathrm{diam(A)}|\nabla\varphi_t|$. Hence

$$C_t|T_t|^{\frac{1}{2+t}} \le |\nabla\varphi_t|^{\frac{1}{1+t}},$$

where $C_t^{-1} = (2+t)^{\frac{1}{2+t}}\big(\mathrm{diam}(A)\big)^{\frac{1}{2+t}}$. Changing variables one gets the following estimate for any $\delta > 0$:

$$\mu\Big(C_{t_n}^{-1}|\nabla\varphi_n|^{\frac{1}{1+t_n}} \le 1-\delta\Big) \le \mu\Big(|T_{t_n}|^{\frac{1}{2+t_n}} \le 1-\delta\Big) = \nu\Big(|x|^{\frac{1}{2+t_n}} \le 1-\delta\Big).$$

Hence

$$\mu\Big(1 - C_{t_n}^{-1}|\nabla\varphi_n|^{\frac{1}{1+t_n}} \ge \delta\Big) \to 0.$$

This implies that $\big(1-C_{t_n}^{-1}|\nabla\varphi_n|^{\frac{1}{1+t_n}}\big)^+$ tends to zero in μ-measure as $t_n \to \infty$. Passing to an almost everywhere convergent subsequence one can assume additionally that

$$\underline{\lim}_{t_n\to\infty}|\nabla\varphi_n|^{\frac{1}{1+t_n}} \ge 1 \tag{2.3}$$

almost everywhere. Since $\sup_t \|\nabla\varphi_t\|_{L^1(dx)} < \infty$ by Lemma 2.2, we see that the sequence $\{|\nabla\varphi_n|^{\frac{1}{1+t_n}}\}$ is bounded in $L^p(A)$ for any $p < \infty$. Moreover, by Hölder's inequality

$$\overline{\lim}_{t_n\to\infty}\int_A |\nabla\varphi_n|^{\frac{p}{1+t_n}}\,dx \le \mathcal{H}^d(A).$$

Hence, choosing an $L^p(A)$-weakly convergent subsequence $|\nabla\varphi_{n_m}|^{\frac{1}{1+t_{n_m}}} \to f$, one has

$$\int_A f\,dx \le \mathcal{H}^d(A).$$

On the other hand, (2.3) and Fatou's lemma show that $f \ge 1$ a.e., which yields

$$\lim_{t_{n_m}\to\infty}\int_A |\nabla\varphi_{n_m}|^{\frac{p}{1+t_{n_m}}}\,dx = 1.$$

Hence $|\nabla\varphi_{n_m}|^{\frac{1}{1+t_{n_m}}} \to 1$ in the norm of any $L^p(A)$, $p < \infty$. Extracting again an almost everywhere convergent subsequence we get the claim. $\square$

In what follows we set $\varphi_n := \varphi_{t_n}$ and assume that $\varphi_n \to \varphi$ and $|\nabla\varphi_n|^{\frac{1}{1+t_n}} \to 1$ almost everywhere.

Lemma 2.5. *Let $C_n \subset B_r$ be convex sets such that $I_{C_n} \to I_C$ almost everywhere. If C is of positive measure, then* $\mathrm{dist}(\partial C_n, \partial C) \to 0$.

Proof. The set C can be taken convex by letting $I_C := \underline{\lim}_n I_{C_n}$. We may assume that $r = 1$. It is known and readily verified by induction that every convex set $U \subset B_r$ with $\mathcal{H}^d(U) \ge \delta$ contains a ball of volume at least $\kappa_1(d)\delta$, where $\kappa_1(d)$ depends only on d. Let B be a ball of radius $\varepsilon > 0$ centered at some point $x_0 \in \partial U$. Then

$$\mathcal{H}^d(U\cap B) \ge \kappa_2(d)\varepsilon^d\delta,$$

where $\kappa_2(d)$ depends only on d. It follows that, whenever $\mathcal{H}^d(C_n) \ge \mathcal{H}^d(C)/2$, one has $\|I_C - I_{C_n}\|_{L^1} = \mathcal{H}^d(C \triangle C_n) \ge 2^{-1}\kappa_2(d)\mathcal{H}^d(C)\mathrm{dist}(\partial C_n, \partial C)^d$. $\square$

Note that due to convexity one has $\operatorname{dist}(\partial C_n, \partial C) = \operatorname{dist}(C_n, C)$.

Lemma 2.6. *The sequence of potentials φ_{t_n} converges to φ uniformly on A. In particular, φ is continuous and has convex sub-level sets $A_s = \{y\colon \varphi(y) \le s\}$.*

Proof. Clearly, it is sufficient to prove the claim for a subsequence. As noted above, one can assume, in addition, that $|\nabla\varphi_n|^{\frac{1}{1+t_n}} \to 1$ almost everywhere. Let us redefine φ as follows: $\varphi := \underline{\lim}_n \varphi_n$. Then the sub-level sets A_s of φ are convex since the corresponding sub-level sets $A_{s,n} = \{x\colon \varphi_n(x) \le s\}$ of φ_n are convex. Since $|T_n| = \varphi_n |\nabla\varphi_n|^{\frac{1}{1+t_n}}$, we have shown that $|T_n| \to \varphi$ almost everywhere. Then the equality $\nu = \mu \circ T_n^{-1}$ yields that the image of μ under the mapping $x \mapsto \varphi(x) \in \mathbb{R}^+$, denoted by $\mu_\varphi \in \mathcal{P}(\mathbb{R}^+)$, coincides with $\nu_{|x|} \in \mathcal{P}(\mathbb{R}^+)$, where $\nu_{|x|}$ is the image of ν under the mapping $x \mapsto |x|$. Due to our assumptions on ν, this implies that μ_φ has a strictly increasing continuous distribution function, i.e.,

1) $\mu(A_{s_1}) < \mu(A_{s_2})$ whenever $s_1 < s_2$,
2) $\mu(\{\varphi = t\}) = 0$ for all $t \in [0, r]$.

Note that 2) implies that $I_{A_{s,n}} \to I_{A_s}$ almost everywhere for each $s > 0$. By Lemma 2.5 we have

$$\operatorname{dist}(\partial A_{s,n}, \partial A_s) \to 0, \quad s > 0.$$

Now, given $\varepsilon > 0$, we divide $[0, r]$ by points $s_1, \ldots, s_N$ with $|s_{i+1} - s_i| < \varepsilon$ and take $\delta = \max_{i \le N} \operatorname{dist}(\partial A_{s_i}, \partial A_{s_{i+1}})$. There exists M such that

$$\operatorname{dist}(\partial A_{s_i,n}, \partial A_{s_i}) < \delta/2$$

for every $i = 1, \ldots, N$ and every $n > M$. This implies that $\sup_{x\in A} |\varphi_n(x) - \varphi(x)| \le 2\varepsilon$ for all $n \ge M$. Hence $\varphi_n \to \varphi$ uniformly. Since φ_n are continuous as powers of convex functions, φ is continuous as well. The proof is complete. $\square$

Lemma 2.7. *Let $N_x := N_{\partial A_{\varphi(x)}, x}$, where $A_{\varphi(x)} = \{y\colon \varphi(y) \le \varphi(x)\}$ and*

$$S := \{x \in A\colon N_x \text{ contains more than one element}\},$$

i.e. S is the set of all the points x such that the boundary of the sub-level set containing x is not differentiable at x. Then $\mathcal{H}^d(S) = 0$.

Proof. First we consider the case $d = 2$. Fix an orthonormal basis $\{e_1, e_2\}$ and identify every unit vector n with $\alpha \in [0, 2\pi)$, where α is the angle between e_1 and n. We write

$$\mathrm{n} := \mathrm{n}_\alpha := \cos\alpha \cdot e_1 + \sin\alpha \cdot e_2.$$

The set S is a countable union of the sets

$$S_{p,q} := \big\{x\colon \ [p-q, p+q] \subset N_x\big\}, \quad p, q \in \mathbb{Q} \cap [0, 2\pi).$$

If S has a positive measure, then $\mathcal{H}^d(S_{p,q}) > 0$ for some p, q. If $x \in S_{p,q}$, then we have $A_{\varphi(x)} \subset \{z\colon \langle z - x, \mathrm{n}_p\rangle \le 0\}$. Note that the line

$$l_{x,p}(z) = \{z\colon \langle z - x, \mathrm{n}_p\rangle = 0\}$$

intersects $S_{p,q}$ exactly at x. Indeed, otherwise we get two points x, y such that the sub-level sets $A_{\varphi(x)}$ and $A_{\varphi(y)}$ both intersect $l_{x,p}$ at two different points and belong to the same half-plane P with $\partial P = l_{x,p}$. Hence neither $A_{\varphi(x)} \subset A_{\varphi(y)}$ nor $A_{\varphi(y)} \subset A_{\varphi(x)}$ hold, which is impossible. Thus we obtain that $l_{x,p} \cap S_{p,q} = \{x\}$. Finally, applying Fubini's theorem and disintegrating Lebesgue measure along the lines parallel to $l_{0,p}$, we obtain $\mathcal{H}^2(S_{p,q}) = 0$, which is a contradiction. So the lemma is proved for $d = 2$. The multi-dimensional case follows by induction and Fubini's theorem.

Indeed, fix an orthonormal basis $\{e_1, \dots, e_d\}$. Note that all sections of a convex body are convex. Disintegrating S along e_i and applying the result for $d-1$, we obtain that

$$S_i = \{\text{projection of } N_x \text{ on } x_i = 0 \text{ has more than one element}\}$$

has measure zero. Since $S = \bigcup_{i=1}^d S_i$, the proof is complete. $\square$

Proof of Theorem 2.1: According to Lemma 2.6 and Lemma 2.4, we have $\varphi_n \to \varphi$ uniformly and $|\nabla \varphi_n|^{\frac{1}{1+t_n}} \to 1$ almost everywhere. It remains to prove that

$$\nabla \varphi_n / |\nabla \varphi_n| \to \mathrm{n}$$

almost everywhere. Let us fix $x \in A$. Since $\varphi_n(x) \to \varphi(x)$, one has $I_{A_{\varphi_n(x),n}} \to I_{A_{\varphi(x)}}$ almost everywhere, where

$$A_{\varphi_n(x),n} = \{y\colon \ \varphi_n(y) \le \varphi_n(x)\}, \ A_{\varphi(x)} = \{y\colon \ \varphi(y) \le \varphi(x)\}.$$

According to Lemma 2.7, $\mathrm{n}(x)$ is well-defined for almost all x. The same holds for every $\nabla \varphi_n(x)/|\nabla \varphi_n(x)|$. So, without loss of generality we can fix x in the interior of A such that $\mathrm{n}(x)$ and $\nabla \varphi_n(x)/|\nabla \varphi_n(x)|$ are well-defined. If the vectors $\nabla \varphi_n(x)/|\nabla \varphi_n(x)|$ do not converge to $\mathrm{n}(x)$, then, extracting a convergent subsequence from a sequence of unit vectors $\{\nabla \varphi_n(x)/|\nabla \varphi_n(x)|\}$, we obtain a unit vector $\eta \ne \mathrm{n}(x)$. By using convergence $I_{A_{\varphi_n(x),n}} \to I_{A_{\varphi(x)}}$, one can show that $\langle \eta, z - x\rangle \le 0$ for all $z \in A$, i.e., $\eta \in N_x$, which contradicts the choice of x.

It remains to verify the evolution equation for a smooth potential φ. Indeed, let us choose an orthonormal basis $\{e_i\}$ at x such that $e_1 = \mathrm{n}$ and every vector e_i, $2 \le i \le d$, belongs to the tangent space of ∂A_t at x. Let us write the change of variables formula for $T = \varphi \cdot \mathrm{n}$. Differentiating along n we find

$$\partial_{\mathrm{n}} T = \partial_{\mathrm{n}} \varphi \cdot \mathrm{n} + \varphi \cdot \partial_{\mathrm{n}} \mathrm{n}.$$

Differentiating the identity $\langle \mathrm{n}, \mathrm{n}\rangle = 1$, we see that $\partial_{\mathrm{n}} \mathrm{n}$ belongs to the tangent space of ∂A_t at x. In addition, $\partial_{\mathrm{n}} \varphi = |\nabla \varphi|$. Next we note that

$$\partial_{e_i} T = \varphi \cdot \partial_{e_i} \mathrm{n}, \quad \langle \partial_{e_i} \mathrm{n}, \mathrm{n}\rangle = 0, \quad 1 \le i \le d.$$

Hence

$$\det DT = |\nabla \varphi| \varphi^{d-1} \det\bigl(\langle \partial_{e_i} n, e_j\rangle\bigr).$$

Since $K = \det\bigl(\langle \partial_{e_i} \mathrm{n}, e_j\rangle$, we have $\det DT = |\nabla \varphi| \varphi^{d-1} K$. Thus one obtains the following change of variables formula (the Monge–Ampère equation):

$$\varrho_0 = \varrho_1(\varphi \cdot \mathrm{n}) |\nabla \varphi| \varphi^{d-1} K.$$

It remains to note that the level sets ∂A_s are shrinking with the velocity $1/|\nabla \varphi|$ in the direction of $-\mathrm{n}$. Hence (2.1) follows from the change of variables formula. The proof is complete.

Example 2.8. Let A be a convex compact set. Set

$$\varrho_1(x) := \frac{C_{d,r}}{|x|^{d-1}}, \ \varrho_0(x) := \frac{1}{\mathcal{H}^d(A)},$$

where $C_{d,r} = \Bigl(\int_{B_r} \frac{dx}{|x|^{d-1}}\Bigr)^{-1}$. Varying r we can show the existence of a weak solution (in the "transportation sense") to the classical Gauss curvature flow which starts from ∂A and satisfies the equation

$$\dot{x}(s) = -c \ K(x) \cdot \mathrm{n}(x),$$

where c can be chosen arbitrarily.

Certainly, a rigorous justification of this formula requires some additional work, since we have not proved that φ is differentiable.

3. Injectivity and uniqueness

In this section, we prove that T is invertible and essentially unique.

Recall that the Legendre transform of a convex function W on a convex set A is defined by

$$W^*(y) = \sup_{x\in A}\bigl(\langle x, y\rangle - W(x)\bigr).$$

Let $\partial W(x)$ denote the subdifferential of W at x. Recall also the following known fact from the theory of convex functions (see, e.g., [20, Theorem 23.5]).

Lemma 3.1. *Let $v \in \partial W(x)$ for every $x \in [x_1, x_2]$, where $x_1 \neq x_2$ and $[x_1, x_2] = \{tx_1 + (1-t)x_2,\ t \in [0,1]\}$. Then $[x_1, x_2] \subset \partial W^*(v)$. In particular, W^* is not differentiable at v.*

In addition to the singular set $S \subset A$ of all points x such that $N_{\partial A_t, x}$, where $t = \varphi(x)$, contains more than one element, we introduce another set of degeneracy of n defined by

$$U = \{x \in A\backslash S\colon \text{ there is } x' \in \partial A_t,\ t = \varphi(x), \text{ such that } x' \neq x \text{ and } \mathrm{n}(x) \in N_{\partial A_t, x'}\}.$$

Proposition 3.2. (i) *Consider the set $C = \partial A_t$ for some fixed t. Then the set $\mathrm{n}(U \cap C)$ in S^{d-1} has $\mathcal{H}^{d-1}$-measure zero.*

(ii) *The sets $T(U)$ and*

$$\widetilde{T}(S) := \bigcup_{x\in S} \varphi(x)\cdot N_{\partial A_{\varphi(x)}, x}$$

have ν-measure zero.

Proof. (i) It is sufficient to prove our claim locally on C in a small neighborhood $\mathcal{O}$ of a point x_0 where $\mathrm{n}(x_0)$ is unique. We may assume that $\mathrm{n}(x_0) = -e_d$, the surface $C \cap \mathcal{O}$ is the graph of a convex function $W\colon B \subset \mathbb{R}^{d-1} \to \mathbb{R}$, where B is an open ball containing 0, and that W attains minimum at 0. In addition, we may assume that $\partial W(B)$ is a bounded set. We parameterize $C \cap \mathcal{O}$ in the following way:

$$B \ni (x_1, \dots, x_{d-1}) \mapsto (x_1, \dots, x_{d-1}, W(x)).$$

Since W is Lipschitzian on B, the surface measure $\mathcal{H}^{d-1}$ on $C \cap \mathcal{O}$ corresponds to the measure $(1 + |\nabla W|^2)^{\frac{1}{2}}\mathcal{H}^{d-1}$ on $\mathbb{R}^{d-1}$. The Gauss map n is given by

$$\mathrm{n} = \frac{1}{\sqrt{1 + |\nabla W|^2}}(-\partial_{x_1}W, \dots, -\partial_{x_{d-1}}W, 1).$$

This holds for every $(x_1, \dots, x_{d-1}) \in B$ such that $(x_1, \dots, x_{d-1}, W(x)) \notin S \cap C \cap \mathcal{O}$. The projection of $S \cap C \cap \mathcal{O}$ on B coincides with the points of non-differentiability of W.

It is convenient to identify the half-sphere $S^{d-1} \cap \{y_d \leq 0\}$ with its projection Π^{d-1} on $\mathbb{R}^{d-1}$ and n with the mapping $\widetilde{\mathrm{n}}\colon\ -\frac{\nabla W}{\sqrt{1+|\nabla W|^2}}$ taking values in Π^{d-1}. Note that the surface measure on S^{d-1} has the form $m_{d-1} := \frac{1}{\sqrt{1-|y|^2}}\mathcal{H}^{d-1}$ in the local chart (on the set where $y_1^2 + \cdots + y_{d-1}^2 < 1$)

$$(y_1, \dots, y_{d-1}) \mapsto \Bigl(y_1, \dots, y_{d-1}, \sqrt{1 - y_1^2 - \cdots - y_{d-1}^2}\Bigr).$$

Hence we have to show that

$$m_{d-1}(F \circ \nabla W(U')) = 0, \tag{3.1}$$

where U' is the corresponding projection of $U \cap C$ and

$$F(x) = -\frac{x}{\sqrt{1+|x|^2}}.$$

The mapping F is smooth and nondegenerate everywhere. Hence in order to prove (3.1) it suffices to show that $\mathcal{H}^{d-1}(\nabla W(U')) = 0$. Let us set $W := +\infty$ outside of B. The Legendre transform W^* is finite everywhere. By Lemma 3.1, the set $\nabla W(U')$ is contained in the set of nondifferentiability of W^*, hence has $\mathcal{H}^{d-1}$-measure zero.

(ii) By Fubini's theorem, it suffices to show that for each t the intersection of the set $T(U)$ with the sphere of radius t has zero $\mathcal{H}^{d-1}$-measure. By construction, these intersection coincide with the sets $T(\partial A_t \cap U)$ defined similarly. Therefore, the claim for $T(U)$ follows by assertion (i).

In order to see that the set $\widetilde{T}(S)$ has ν-measure zero, we observe that its intersection with the set $T(A \backslash S)$ of full ν-measure belongs to $T(U)$, which is clear from the definition of U. □

Now we can show that T is invertible.

Corollary 3.3. *The mapping T is injective on a set of full μ-measure. Hence there exists a measurable mapping $T^{-1}\colon B_r \to A$ such that $T(T^{-1}(y)) = y$ for ν-almost all y and $T^{-1}(T(x)) = x$ for μ-almost all x.*

Proof. Since the equality $T(x_1) = T(x_2)$ may only happen if $\varphi(x_1) = \varphi(x_2)$, i.e., x_1 and x_2 belong to the same level set ∂A_t, it follows from our previous considerations that T is injective outside the set $T^{-1}(\widetilde{T}(S) \cup T(U))$. This set has μ-measure zero because the set $\widetilde{T}(S) \cup T(U)$ has ν-measure zero by the above proposition. □

Theorem 3.4. *The mapping T constructed above is unique in the following sense: if a measurable mapping $T_0\colon A \to B_r$ is such that $\nu = \mu \circ T_0^{-1}$ and $T_0 = \varphi_0 \cdot \mathrm{n}_0$, where $\varphi_0\colon A \to [0, r]$ is a continuous function with convex sub-level sets $A_{t,0} := \{\varphi_0 \le t\}$ and n_0 is the corresponding Gauss map, then $T = T_0$ μ-a.e.*

Proof. Let us show that $\varphi_0(x) \le \varphi(x)$ for all $x \in A$. This will yield the equality $\varphi_0 = \varphi$ because otherwise there is t such that $\mu(\{\varphi_0 \le t\}) > \mu(\{\varphi \le t\})$, which is impossible since both sides equal $\nu(B_t)$. Set

$$C_t := \Big\{x \in A\colon\ x \in \partial A_{t,0} \cap A_t\Big\}, \quad D_t := \Big\{x \in A\colon\ x \in \partial A_t \backslash \mathrm{Int}(A_{t,0})\Big\},$$

$$U_\tau := \bigcup_{t \ge \tau} C_t, \quad V_\tau := \bigcup_{t \ge \tau} D_t.$$

We observe that for every $x \in \partial A_t \backslash \mathrm{Int}(A_{t,0})$ there exists $t' \ge t$ depending on x such that $x \in C_{t'}$. Indeed, if $x \in \partial A_{t,0} \cap \partial A_t$, then $x \in C_t$. Otherwise one has $x \in \partial A_{t',0}$ for some $t' = t'(x) > t$. Since $x \in \partial A_t$, we have $x \in \mathrm{Int}(A_{t'})$. Hence

$$V_\tau \subset U_\tau. \tag{3.2}$$

For every Borel set $C \subset A$, set

$$\widetilde{T}(C) := \bigcup_{x \in C} \varphi(x) \cdot N_{\partial A_{\varphi(x)}, x}.$$

Let us show that

$$T_0(C_t) \subset \widetilde{T}(D_t). \tag{3.3}$$

Suppose that $x_0 \in \partial A_{t,0} \cap A_t$, $t = \varphi_0(x_0)$, $v = \mathrm{n}_{x_0,0} \in N_{\partial A_{t,0},x_0}$. We show that $v \in T(\partial A_t \backslash \mathrm{Int}(A_{t,0}))$. Let us consider the support hyperplane $L_{x_0,v} \perp v$ to $A_{t,0}$ at x_0. If $x_0 \in \partial A_t$ and $v \in N_{\partial A_t,x_0}$, the claim is obvious. Otherwise $L_{x_0,v}$ splits A_t in two convex parts A'_t and A''_t. Since $L_{x_0,v}$ is a support hyperplane to $A_{t,0}$, one of these parts, say, A''_t, and $A_{t,0}$ are separated by $L_{x_0,v}$. There exists a hyperplane L parallel to $L_{x_0,v}$ that is supporting to A''_t and passes through a point $x_1 \in \partial A''_t$. Then $v \in N_{\partial A_t,x_1}$. This proves (3.3). Hence we have

$$T_0(U_t) \subset \widetilde{T}(V_t), \quad 0 \le t \le r. \tag{3.4}$$

Suppose now that there exists x_0 such that $\varphi_0(x_0) > \varphi(x_0)$. Then, by the continuity of φ and φ_0, there is $\tau > 0$ for which the inclusion in (3.2) is strict and there is a neighborhood in U_τ not intersecting V_τ. Therefore, $\mu(U_\tau) > \mu(V_\tau)$. Taking into account that T is injective on a full measure set, we obtain

$$\nu(T_0(U_\tau)) = \mu(T_0^{-1}(T_0(U_\tau))) \ge \mu(U_\tau) > \mu(V_\tau) = \nu(T(V_\tau)),$$

which contradicts (3.4) because $\nu(T(V_\tau)) = \nu(\widetilde{T}(V_\tau))$ according to Corollary 3.2. $\square$

4. Duality

Now we consider certain duality properties of the potential φ. The duality principle of Kantorovich is a powerful tool for investigating the Monge–Kantorovich problem. In our case we also have a kind of the duality formula which relates the potential φ to some function ψ that can be considered as the support function of the family of level sets A_t. Note that some interesting duality results for the solution of the Monge–Kantorovich problem on a sphere with applications to the prescribed Gauss curvature problem have been obtained in [18].

For every $y \in B_r$ we set

$$\psi(y) = \sup_{x\colon\ \varphi(x) \le |y|} \langle x, y\rangle. \tag{4.1}$$

Note that the restriction of ψ to $\partial B_{|y|}$ coincides with the support function $S_{A_{|y|}}$ of $A_{|y|} = \{x\colon\ \varphi(x) \le |y|\}$, where the support function is defined by

$$S_{A_{|y|}}(v) := \sup_{x \in A_{|y|}} \langle v, x\rangle.$$

Lemma 4.1. *For ν-almost all y one has*

$$\psi(y) = \langle T^{-1}(y), y\rangle. \tag{4.2}$$

Proof. It is clear that the supremum on the right-hand side of (4.1) is attained at a point p such that $y \in N_{\partial A_{\varphi(p)},p}$. This implies that p coincides with $T^{-1}(y)$ for ν-almost all y, hence is ν-almost everywhere well-defined, which yields our claim. $\square$

Now we show how to describe ψ as a limit of certain functions depending on prelimit potentials φ_t. Recall that the Legendre transform W_t^* satisfies the inequality

$$W_t(x) + W_t^*(y) \ge \langle x, y\rangle. \tag{4.3}$$

An equality holds if and only if $y \in \partial W_t(x)$ and $x \in \partial W_t^*(y)$. Moreover, W_t and W_t^* satisfy the identities

$$\nabla W_t^* \circ \nabla W_t(x) = x, \quad \nabla W_t \circ \nabla W_t^*(y) = y$$

almost everywhere on the sets A and $\nabla W_t(A)$. Since

$$\nabla W_t = |T_t|^t T_t,$$

one has

$$T_t^{-1}(y) = \nabla W_t^*(|y|^t y).$$

In what follows we denote by I the identity matrix and by I_z the orthogonal projector on the one-dimensional vector subspace generated by z, i.e.,

$$I_z v = \frac{\langle v, z\rangle}{|z|}\frac{z}{|z|}.$$

We have found a sequence $t_n \to +\infty$ for which the mappings T_{t_n} converge to T almost everywhere on A, hence converges in measure μ. For this sequence, the following holds.

Lemma 4.2. *The mappings $T_{t_n}^{-1}$ converge to T^{-1} in measure ν. Hence there exists a subsequence $t'_n \to \infty$ such that $T_{t'_n}^{-1} \to T^{-1}$ ν-almost everywhere.*

Proof. Since $\mu \circ T^{-1} = \mu \circ T_{t_n}^{-1} = \nu$, for any ν-measurable function f, the functions $f \circ T_{t_n}$ converge to $f \circ T$ in measure μ (see, e.g., [4, Corollary 9.9.11]). Hence the mappings $T^{-1} \circ T_{t_n}$ converge to $T^{-1} \circ T = I$ in measure μ. Therefore, for every $c > 0$ one has

$$\nu\Bigl(y\colon\ |T_{t_n}^{-1}(y) - T^{-1}(y)| \geq c\Bigr) = \mu\Bigl(x \in A\colon\ |x - T^{-1}T_{t_n}(x)| \geq c\Bigr) \to 0$$

as $n \to \infty$, which completes the proof. □

Theorem 4.3. *Let a function ψ_t be defined by the relation*

$$W_t^*(z) = |z|^{\frac{t}{1+t}} \psi_t\bigl(z|z|^{-\frac{t}{1+t}}\bigr).$$

Equivalently,

$$\psi_t(y) = \frac{W_t^*(y|y|^t)}{|y|^t}.$$

Then one has $\psi = \lim\limits_{t_n\to\infty} \psi_{t_n}$ almost everywhere for some sequence $\{t_n\}$.

Proof. Note that it is consistent with our previous choice of W_t to assume that $W_t^*(0) = 0$. Indeed, $W_t(x) \geq \langle x, y\rangle - W_t^*(y)$, hence taking $y = 0$ we find $W_t(x) \geq 0$. Taking any $x_0 \in \partial W_t^*(0)$ we easily obtain $W_t(x_0) = 0$. Indeed, for $(x_0, 0)$ inequality (4.3) becomes an equality, hence $W_t(x_0) + W_t^*(0) = \langle x_0, 0\rangle = 0$.

The inequality $W_t^*(a) - W_t^*(b) \leq \langle a - b, \nabla W^*(a)\rangle$ yields, by substituting $b = 0$ and $a = y|y|^t$, that

$$\psi_t(y) \leq \langle \nabla W_t^*(y|y|^t), y\rangle = \langle y, T_t^{-1}(y)\rangle.$$

Similarly, if $a = 0$ and $b = y|y|^t$, one has $\psi_t(y) \geq \langle v, y\rangle$ for any $v \in \partial W_t^*(0)$. In particular,

$$|\psi_t(y)| \leq \mathrm{diam}(A)|y|.$$

One has

$$\nabla W_t^*(z) = \frac{t}{1+t}|z|^{-\frac{1}{1+t}}\frac{z}{|z|} \cdot \psi_t\bigl(z|z|^{-\frac{t}{1+t}}\bigr) + \Bigl(I - \frac{t}{1+t}I_z\Bigr)\nabla\psi_t\bigl(z|z|^{-\frac{t}{1+t}}\bigr).$$

Substituting $z = |y|^t y$ we obtain

$$T_t^{-1}(y) = \frac{t}{1+t}\psi_t(y)\frac{y}{|y|^2} + \Bigl(I - \frac{t}{1+t}I_y\Bigr)\nabla\psi_t(y). \tag{4.4}$$

Taking the inner product with y we find

$$\langle T_t^{-1}(y), y\rangle = \frac{t}{1+t}\psi_t(y) + \frac{1}{1+t}\langle \nabla\psi_t(y), y\rangle. \tag{4.5}$$

In view of Lemma 4.2 and equality (4.2) it suffices to show that $\frac{1}{1+t_n}\langle \nabla\psi_{t_n}(y), y\rangle \to 0$ almost everywhere for some $\{t_n\}$. Indeed, since $\psi_t \le \langle T_t^{-1}(y), y\rangle$, we obtain from (4.5) that

$$\langle T_t^{-1}(y), y\rangle \le \frac{t}{1+t}\langle T_t^{-1}(y), y\rangle + \frac{1}{1+t}\langle \nabla\psi_t(y), y\rangle.$$

Hence

$$\psi_t \le \langle T_t^{-1}(y), y\rangle \le \langle \nabla\psi_t(y), y\rangle.$$

Taking into account that $\psi_t \ge -\mathrm{diam}(A)|y|$, we see that $\{\langle \nabla\psi_t(y), y\rangle\}$ is uniformly bounded from below. The integration by parts formula yields

$$\int_{B_r} \langle \nabla\psi_t(y), y\rangle\, dx = -d\int_{B_r} \psi_t\, dx + \int_{\partial B_r} |y|\psi_t\, d\mathcal{H}^{d-1}.$$

Applying again the estimate $|\psi_t| \le \mathrm{diam}(A)|y|$ we obtain

$$\sup_t \int_{B_r} \langle \nabla\psi_t(y), y\rangle\, dx < \infty,$$

hence $\sup_t \|\langle \nabla\psi_t(y), y\rangle\|_{L^1(B_r)} < \infty$. Therefore,

$$\lim_{t\to\infty} \frac{1}{1+t}\|\langle \nabla\psi_t(y), y\rangle\|_{L^1(B_r)} = 0.$$

Extracting a subsequence we complete the proof. $\square$

Remark 4.4. Taking a scalar product of (4.4) with any vector $v \perp y$ we obtain the equality $\partial_v\psi_n(y) = \langle T_n^{-1}(y), v\rangle$. Let us set

$$\partial_v\psi(y) := \lim_{t_n\to\infty} \partial_v\psi_n(y).$$

In view of convergence $T_n \to T$ this definition makes sense. Moreover, we have

$$\partial_v\psi(y) = \langle T^{-1}(y), v\rangle, \tag{4.6}$$

for any $v \perp y$.

Taking into account (4.2) we obtain the following remarkable relation:

$$T^{-1}(y) = \frac{\psi(y)}{|y|}e_1(y) + \sum_{i=2}^{d} \partial_{e_i(y)}\psi(y)\ e_i(y),$$

where $\{e_i(y)\}$ is an orthonormal system of unit vectors chosen in such a way that $e_1(y) = y/|y|$ and $e_i(y) \perp y$, $2 \le i \le d$.

Remark 4.5. Let us see what happens in the limit with the duality formula

$$W_t(x) + W_t^*(z) \ge \langle x, z\rangle.$$

It can be rewritten as

$$\frac{1}{t+2}\varphi_t^{t+2}(x) + |y|^t\psi_t(y) \ge \langle x, y\rangle|y|^t$$

by letting $z := y|y|^t$. If $y = T_t(x) = \varphi_t \frac{\nabla\varphi_t}{|\nabla\varphi_t|}|\nabla\varphi_t|^{\frac{1}{1+t}}$, then an equality holds. It is known that this is possible only if the pair (x, y) belongs to the graph of T_t. Hence we obtain the following duality relation:

$$\frac{1}{t+2}\varphi_t^2(x)|\nabla\varphi_t(x)|^{-\frac{t}{t+1}} + \psi_t(T_t(x)) = \langle x, T_t(x)\rangle.$$

In the limit $t \to \infty$ we find

$$\psi(T(x)) = \langle x, T(x)\rangle.$$

It is worth noting that for the constructed transformation T one can establish a change of variables formula involving certain analogs of Alexandroff's determinants. Such formulas which neglect singular components are known for optimal transformations and triangular transformations (concerning the latter, see [4, Ch. 10] and [5]).

This work was supported by the RFBR projects 07-01-00536, 08-01-91205-JF, GFEN-06-01-39003, RF Prezident Grant MD-764.2008.1, DFG Grant 436 RUS 113/343/0(R), ARC Discovery Grant DP0663153, and the SFB 701 at the University of Bielefeld. The second-named author thanks Franck Barthe for his hospitality and fruitful discussions during the author's visit to the University of Paul Sabatier in Toulouse, where this work was partially done.

References

[1] Alexandroff A.D. Existence and uniqueness of a convex surface with a given integral curvature. Dokl. Akad. Nauk USSR, 35 (1942), 131–134 (in Russian).

[2] Ambrosio L. Lecture notes on optimal transport problems. In: Mathematical aspects of evolving interfaces (Funchal, 2000), Lecture Notes in Math. V. 1812. P. 1–52. Springer, Berlin, 2003.

[3] Andrews B. Motion of hypersurfaces by Gauss curvature. Pacific J. Math. 195 (2000), N 1, 1–34.

[4] Bogachev V.I. Measure theory. V. 1,2. Springer, Berlin – New York, 2007.

[5] Bogachev V.I., Kolesnikov A.V., Medvedev K.V. Triangular transformations of measures. Matem. Sbornik 196 (2005), N 3, 309–335 (in Russian); English transl.: Sbornik Math. 196 (2005), N 3, 3–30.

[6] Brenier Y. Polar factorization and monotone rearrangement of vector valued functions. Comm. Pure Appl. Math., 44 (1991), 375–417.

[7] Chen Y.-G., Giga Yo., Goto S. Uniqueness and existence of viscosity solutions of generalized mean curvature flows. J. Diff. Geom., 33 (1991), 749–786.

[8] Evans L.C. Partial differential equations and Monge–Kantorovich mass transfer. In: "Current developments in mathematics", p. 65–126. Int. Press, Boston, MA, 1999.

[9] Evans L.C., Spruck J. Motion of level-sets by mean curvature. I. J. Diff. Geom., 33 (1991), 635–681.

[10] Gage M.E., Hamilton R.S. The heat equation shrinking convex plane curves. J. Diff. Geom., 23 (1986), 69–96.

[11] Giga Y. Surface evolution equations. A level sets approach. Birkhäuser Verlag, Basel, 2006.

[12] Grayson M. The heat equation shrinks embedded closed curves to round points. J. Diff. Geom., 26 (1987), 285–314.

[13] Huisken G. Flow by mean curvature of convex surfaces into spheres. J. Diff. Geom., 20 (1984), 237–266.

[14] Kantorovich L.V. On the translocation of masses. C.R. (Doklady) Acad. Sci. Nauk URSS, 37 (1942), N 7-8, 199–201.

[15] Lott J., Villani C. Ricci curvature for metric-measure spaces via optimal transport. Preprint http://www.math.lsa.umich.edu/~lott/

[16] McCann R.J. Existence and uniqueness of monotone measure-preserving maps. Duke Math. J., 80 (1995), 309–323.

[17] McCann R.J., Topping P. Ricci flows, entropy and optimal transportation, 2007 (preprint).

[18] Oliker V. Embeddings of S^n into $\mathbb{R}^{n+1}$ with given integral Gauss curvature and optimal mass transport on S^n. Adv. Math., 213 (2007), 600–620.

[19] Rachev S.T., Rüschendorf L. Mass transportation problems. V. I, II. Springer, New York, 1998.

[20] Rockafellar R.T. Convex analysis. Princeton, Princeton Univ. Press, 1970.

[21] Sturm K.-T. On the geometry of metric measure spaces. I, II. Acta Math. 196 (2006), N 1, 65–131, 133–177.

[22] Topping P. L-optimal transporation for Ricci flows. Preprint. 2007.

[23] Tso K. Deforming a hypersurfaces by Gauss–Kronecker curvature. Comm. Pure Appl Math., 38 (1985), 876–882.

[24] Villani C. Topics in optimal transportation. Amer. Math. Soc., Providence, Rhode Island, 2003.