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The threshold aggregation

Fuad Aleskerov [a,b,*], Vyacheslav V. Chistyakov [c], Valery Kalyagin [c]

[a] Department of Mathematics for Economics, State University Higher School of Economics, 20 Myasnitskaya Street, Moscow 101000, Russian Federation
[b] Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russian Federation
[c] Department of Applied Mathematics and Informatics, State University Higher School of Economics, 25/12 Bol'shaya Pechërskaya Street, Nizhny Novgorod 603155, Russian Federation

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ABSTRACT

For a social decision problem we define a new aggregation procedure—the threshold rule—for the construction of an output ranking from the individual m-graded rankings with an arbitrary integer $m \geq 2$. An axiomatic characterization of the procedure is given.

1. Introduction

In many journals, articles for publication are sent to two referees, and the decision for the acceptance is based on the referees' reports. Very often when one of the referees gives a positive report, and the other one gives a negative report, the article is rejected. What kind of rule should be used in order to describe this decision?

The aim of this article is to investigate the following problem of construction of a social decision function. A society of n agents evaluates alternatives from a finite set X using complete and transitive preferences (rankings). A social decision is sought for among complete and transitive social preferences over the alternatives. This kind of aggregation has been considered in many publications, beginning with the seminal work by Arrow (1963).

In a recent series of three articles by (Aleskerov and Yakuba, 2003), (Aleskerov & Yakuba, 2007) and (Aleskerov et al., 2007) an axiomatic construction of a new aggregation procedure, called the *threshold rule*, has been presented for three-graded rankings, i.e., when evaluations of alternatives are made by grades 1, 2 and 3 meaning 'bad', 'average' and 'good', respectively. The axioms used in these papers are Pairwise Compensation, Pareto Domination, Noncompensatory Threshold and Contraction.

The Pairwise Compensation axiom means that if all agents, but two, evaluate two alternatives equally, and the two agents put 'mutually inverse' grades, then the two alternatives have the same rank in the social decision (which may also be interpreted as 'anonymity of grades').

The Pareto Domination axiom states that if the grades of all agents for the first alternative are not less than for the second alternative and the grade of at least one agent for the first alternative is strictly greater than that for the second one, then in the social ranking the first alternative has a higher rank than the second alternative.

The Noncompensatory Threshold axiom reveals the main idea of the threshold aggregation: if at least one agent evaluates an alternative as 'bad', then, no matter how many 'good' grades it admits, in the social ranking this alternative is ranked lower than any alternative evaluated as 'average' by all agents.

In this context the Contraction axiom means that if for two alternatives the evaluations of some agent are equal, then the agent may be 'excluded' from the consideration when the social ranking is constructed, and the social decision is achieved by the remaining agents' evaluations.

It was shown by Aleskerov et al. (2007) that the threshold rule is the only rule that satisfies the above axioms. In the context of three-graded rankings the threshold rule aggregates individual preferences in the following way: if the number of 'bad' evaluations of the first alternative is less than the number of 'bad' evaluations of the second alternative, then the first alternative has higher rank in the social ranking, and if the

* Corresponding author. Department of Mathematics for Economics, State University Higher School of Economics, 20 Myasnitskaya Street, Moscow 101000, Russian Federation. Tel./fax: +7 495 6211342.
E-mail addresses: fuad_aleskerov2001@yahoo.com (F. Aleskerov), czeslaw@mail.ru (V.V. Chistyakov), vkalyagin@hse.nnov.ru (V. Kalyagin).

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numbers of 'bads' for both alternatives are equal and the number of 'average' evaluations of the first alternative is less than that of the second alternative, then the first alternative is socially more preferable.

In this paper we extend the notion of the threshold rule to the case when the agents' evaluations are represented by the m-valued grades with an arbitrary integer $m\geq 2$ and show that the threshold rule is the only rule that satisfies the abovementioned appropriately interpreted axioms.

2. Basic concepts

Let X be a finite set of alternatives of cardinality $|X|\geq 2$. The set $M=\{1,2,\ldots,m\}$ with integer $m\geq 2$ is the set of (ordered) grades $1<2<\ldots<m$, which will also be denoted, for convenience, as the interval $[1,m]$. The set $N=\{1,\ldots,n\}$, with integer $n\geq 1$, is the set of agents, also denoted by $[1,n]$. Each alternative $x\in X$ is evaluated by means of n grades $x_1,\ldots,x_n$ from $[1,m]$ via the map $x\longmapsto\hat{x}=(x_1,\ldots,x_n)\in[1,m]^n$, where the last set is the Cartesian product of n sets $[1,m]$, i. e., the set of all n-dimensional vectors with components from $[1,m]$.

We study the problem of ranking the elements from X using the set $\hat{X}=\{\hat{x}: x\in X\}$, which contains vectors of m-graded evaluations of the elements from X. By a *ranking* of X we mean a *weak order* P on X, i.e., a binary relation on X which is irreflexive ($(x,x)\notin P$ for all $x\in X$), transitive ($(x,y)\in P$ and $(y,z)\in P$ imply $(x,z)\in P$ for all $x,y,z\in X$) and negatively transitive ($(x,y)\notin P$ and $(y,z)\notin P$ imply $(x,z)\notin P$ for all x,y,z from X). If P is a weak order on X, the *indifference* relation I on X is defined as follows: given $x,y\in X$, $(x,y)\in I$ iff $(x,y)\notin P$ and $(y,x)\notin P$. Clearly, I is an equivalence relation on X, i.e., a reflexive, symmetric and transitive binary relation.

Since $\hat{X}\subset[1,m]^n$ and each alternative $x\in X$ is characterized by its vector $\hat{x}$, with no loss of generality we assume throughout the paper that the set $\hat{X}$ fills the whole 'evaluation space', i. e., $X=\hat{X}=[1,m]^n$, and so, $x\in X=[1,m]^n$ iff $x=\hat{x}=(x_1,\ldots,x_n)$ with $x_i\in[1,m]$.

Note that in the real life situation the cardinality of X might be rather small. But in this model we construct an ordering on the whole set of alternatives $[1,m]^n$, and each particular set of alternatives and preferences among them are embedded into our general weak order.

In what follows it will be convenient to have more general notation

$$[k,l]=\{i\in\{0\}\cup N: k\leq i\leq l\}=\{k,k+1,\ldots,l-1,l\}$$

for an interval with nonnegative integer endpoints k and l with $k\leq l$.

Given $j\in[1,m]$ and $x\in X$, we denote by $v_j(x)$ the number of criteria, in which the alternative x has grade j, $v_j(x)=|\{i\in[1,n]:x_i=j\}|$, and set

$$V_j(x)=\sum_{k=1}^{j} v_k(x) \text{ with } V_0(x)=0.$$

Clearly, $0\leq v_j(x)\leq n$, $0\leq V_{j-1}(x)\leq V_j(x)\leq n$ for all $j\in[1,m]$ and

$$\sum_{j=1}^{m} v_j(x)=n \text{ or } V_m(x)=n \text{ for all } x\in X.$$

A binary relation $P=P_{m-1}$ on $X=[1,m]^n$ is said to be *generated by the threshold rule* provided, given $x,y\in X$, we have: if $m=2$, then $(x,y)\in P_1$ iff $v_1(x)<v_1(y)$, and if $m\geq 3$, then $(x,y)\in P_{m-1}$ iff $v_1(x)<v_1(y)$ or there exists a $k\in[2,m-1]$ such that $v_j(x)=v_j(y)$ for all $j\in[1,k-1]$ and $v_k(x)<v_k(y)$. Cf. also (Chistyakov and Kalyagin, 2008).

Thus, a vector x is more preferable than a vector y if x has less grades 1 than y; if both of these vectors have the same number of grades 1, then the numbers of grades 2 are compared, and so on.

It is to be noted that the value $v_m(x)$ is automatically determined if the values $v_1(x),\ldots,v_{m-1}(x)$ are already known.

The relation P generated by the threshold rule is a weak order on X. The inclusion $(x,y)\in P$ can be interpreted in the sense that the alternative x is strictly better than the alternative y. The indifference $(x,y)\in I$ is equivalent to the condition $v_j(x)=v_j(y)$ for all $j\in[1,m]$. This equality means that the two alternatives have the same bunch of grades placed in different positions.

3. Axioms and the representation theorem

We look for a function $\varphi:X\rightarrow N$, which will represent the threshold rule defined above and satisfy the following three axioms.

Axiom (A.1) describes the fact that the criteria are equally important. The second Axiom (A.2) represents the Pareto principle and the last Axiom (A.3) formalizes the non-compensatory principle.

Axiom (A.1). If $v_j(x)=v_j(y)$ for all $j\in[1,m-1]$, then $\varphi(x)=\varphi(y)$.

Axiom (A.2). If $x\succ y$ in X, then $\varphi(x)>\varphi(y)$, where $x\succ y$ means that $x_i\geq y_i$ for all $i\in[1,n]$ and there is an $i_0\in[1,n]$ such that $x_{i_0}>y_{i_0}$.

Axiom (A.3). For each $k\in[3,m]$ the following condition (A.3.k) holds: if $v_j(x)=v_j(y)$ for all $j\in[1,m-k]$ (no assumption if $k=m$), $v_{m-k+1}(x)+1=v_{m-k+1}(y)\neq n-V_{m-k}(y)$, $V_{m-k+2}(x)=n$ and $V_{m-k+1}(y)+v_m(y)=n$, then $\varphi(x)>\varphi(y)$.

Note that Axioms (A.1), (A.2) and (A.3) are independent, but their formulation depends on the number of grades m. It is known that the function φ, given by $\varphi(x)=\sum_{j=1}^{m} jv_j(x)=\sum_{i=1}^{m} x_i$ for $x\in X$, satisfies (A.1) and (A.2); however, it does not satisfy (A.3).

Our main result is the following representation theorem:

Theorem 1. A weak order P on the set of alternatives X is generated by the threshold rule iff it is generated by a function φ satisfying Axioms (A.1)–(A.3) in the sense that $P=\{(x,y)\in X\times X:\varphi(x)>\varphi(y)\}$.

A dual model to that considered above can be formulated.

A binary relation $P^d=P^d_{m-1}$ on $X=[1,m]^n$ is said to be *generated by the dual threshold rule* provided, given $x,y\in X$, we have: if $m=2$, then $(x,y)\in P^d_1$ iff $v_2(y)<v_2(x)$, and if $m\geq 3$, then $(x,y)\in P^d_{m-1}$ iff $v_m(y)<v_m(x)$ or there exists a $k\in[2,m-1]$ such that $v_j(x)=v_j(y)$ for all $j\in[k+1,m]$ and $v_k(y)<v_k(x)$.

The idea behind this is as follows. If we are interested in the utmost perfection (or quality) of alternatives from X, then we apply the binary relation P generated by the threshold rule to rank the alternatives from the set X. Now, if at least one good feature of alternatives is of main concern, then in order to rank the set of alternatives X, we employ the binary relation P^d generated by the dual threshold rule. The relation P^d is a weak order on X, for which a representation theorem similar to Theorem 1 holds.

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