Strictly strong $(n-1)$-equilibrium in n-person multicriteria games

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Abstract. Using some specific approach to the coalition-consistency analysis in n-person multicriteria games we introduce two refinements of (weak Pareto) equilibria: the strong and strictly strong $(n-1)$-equilibriums. Axiomatization of the strictly strong $(n-1)$-equilibria (on closed families of multicriteria games) is provided in terms of consistency, strong one-person rationality, suitable variants of Pareto optimality and converse consistency axiom and others.

Keywords: multicriteria games; Pareto equilibria; strong equilibrium; consistency; axiomatizations.

1. Introduction

The concept of strictly strong $(n-1)$-equilibria (in n-person strategic games and in multicriteria games) is based on some specific approach to the coalition-consistency analysis, offered in (Kuzyutin D., 1995, Kuzyutin, D., 2000). Namely, we suppose that trying to investigate the coalition-consistency of some acceptable Nash equilibrium x, every player i does not consider the deviations of coalitions $S, i \in S$ with her participance (since player i may be sure in her own strategic choice x_i). This approach allows to make the strong Nash equilibria (Aumann R. J., 1959) requirements slightly weaker.

We show(in section 2) that the strong and strictly strong $(n-1)$-equilibrium differs from other closely related solution concepts: coalition-proof equilibrium (Bernheim B. et al.,1987) and semi-strong Nash equilibrium (Kaplan G., 1992). The axiomatization of strong and strictly strong $(n-1)$-equilibria in n-person strategic games was given in (Kuzyutin, D., 2000).

In section 3 we explore the same approach to coalition-consistency analysis in n-person multicriteria games (or the games with vector payoffs) and offer two refinements of the weak Pareto equilibria (Shapley L., 1959, Voorneveld M. et al., 1999).

The axiomatic characterization of strictly strong $(n-1)$- equilibria (on closed families of multicriteria games) is provided in section 4 using the technique offered in (Peleg B. and Tijs S., 1996, Norde H. et al., 1996, Voorneveld M. et al., 1999). In this axiomatization the suitable variants of Pareto-optimality and converse consistency axioms play a role to distinguish between the strictly strong $(n-1)$-equilibria and other equilibrium solutions in multicriteria games.

2. Strong and strictly strong $(n-1)$-equilibrium in strategic games

Consider a game in strategic form $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}$, where N is a finite set of players $|N| = n$ $A_i \neq \emptyset$ is the set of player's i strategies; and $u_i : \ A = \prod_{j \in N} A_j \rightarrow R^1$ is the payoff function of player $i \in N$. A solution (optimality principle) φ, defined on a class of strategic games Γ, is a function that assigns to each game $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N} \in \Gamma$ a subset $\varphi(G)$ of A. We'll call a strategy profile x the optimal situation, if $x \in \varphi(G)$. Let $S \subset N$, $S \neq \emptyset$, be a coalition; $S \subset N$, $S \neq \emptyset$, N proper coalition; $A_S = \prod_{j \in S} A_j$ — a set of all possible players' $i \in S$ strategy profiles.

The concept of strong (Nash) equilibria was offered by Aumann R. J., 1959.

Definition 1. $x \in A$ is *a strong Nash equilibrium (SNE)*, if $\forall S \subset N$, $S \neq \emptyset$, $\forall y_S \in A_S$, $\exists i \in S$:

$$u_i(x) \geq u_i(y_S, x_{N \setminus S}),$$

where $y_S = (y_j) j \in S$, $x_{N \setminus S} = (x_j)_{j \in N \setminus S}$.

Definition 2. $x \in A$ is *weakly Pareto-optimal (WPO)*, if $\forall y \in A$, $\exists i \in N$:

$$u_i(x) \geq u_i(y).$$

Definition 3. $x \in A$ is *a strictly strong Nash equilibrium (SSNE)*, if there do not exist coalition $S \subset N$ and $y_S \in A_S$ such that:

$$u_i(y_S, x_{N \setminus S}) \geq u_i(x) \ \forall i \in S,$$

$$\exists j \in S : \ u_j(y_S, x_{N \setminus S}) > u_j(x).$$

Notice that the concept of SNE (as well as $SSNE$) deals with a r b i t r a r y deviations of a l l p o s s i b l e coalitions $S \subset N$. We denote by $NE(G)$, $SNE(G)$, $SSNE(G)$ the set of Nash equilibriums (Nash J. F., 1950), strong Nash equilibriums and strictly strong Nash equilibriums of G respectively. The following inclusions hold: $NE(G) \supset SNE(G) \supset SSNE(G)$.

Unfortunately, the sets $SNE(G)$ and $SSNE(G)$ are often empty (see, for instance, Petrosjan L. and Kuzyutin D., 2008) by the reason of "too strong" requirements to the solution used in def. 1, 3. We'll consider an opportunity to make these requirements slightly weaker that leads to new concept of coalition-stable equilibrium.

We guess a game $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ is "of common knowledge", when every player knows all players' strategy sets and payoff functions. Moreover, suppose that trying to investigate the coalition stability of some acceptable strategy profile x, every player i does not consider the deviations of coalitions $S \in i$ with her participance since player i may be sure in her own strategic choice x_i. The related motivation was used early for other purposes in Kuzyutin D., 1995 to define the i-stability property in n-person extensive game.

Definition 4. Let $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma$, $|N| = n$.

1. $n \geq 2$: $x \in A$ *is a strong* $(n-1)$*-equilibrium* (SNE^{n-1}), if for every player $i \in N$ the following condition holds:

$$\forall S \subset N \setminus \ \{i\}, \ \forall \ y_S \in A_S, \exists \ j \in S : \ u_j(x) \geq u_j(y_S, x_{N \setminus S});$$

2. $n = 1$: $x_i \in A_i$ is *a strong* $(n-1)$*-equilibrium* in one-player game $G = (\{i\}, A_i, u_i)$, if

$$u_i(x_i) \geq u_i(y_i) \ \forall y_i \in A_i$$

Definition 5.

1. $n \geq 2$: $x \in A$ is *a strictly strong* $(n-1)$*-equilibrium* ($SSNE^{n-1}$), if for every player $i \in N$ there do not exist a coalition $S \subset N \setminus \{i\}$ and $y_S \in A_S$ such that:

$$u_j(y_S, x_{N\setminus S}) \geq u_j(x) \ \forall \ j \in S,$$

$$\exists \ k \in S : \ u_k(y_S, x_{N\setminus S}) > u_k(x).$$

2. $n = 1$: $SSNE^{n-1}$ coincides with SNE^{n-1}.

Remark 1. Another possible definition of SNE^{n-l} $(n > 2)$ is as follows: x is a SNE^{n-l} if $\forall \ S \subset N, \ |S| \leq n-1, \ \forall \ y_S \in A_S \ \exists \ j \in S$:

$$u_j(x) \geq u_j(y_S, x_{N\setminus S}).$$

However, we guess the def. 4 is more useful to clarify the offered approach every player (independently of others) holds on to check the coalition stability of x. The optimally principles SNE^{n-l} and $SSNE^{n-l}$ deal with a r b i t r a r y deviations of c e r t a i n (c r e d i b 1 e) coalitions.

It is clear that

$$NE(G) \supset SNE^{n-l}(G) \supset SSNE^{n-1}(G).$$

Further we have:

$$NE(G) \supset SNE^{n-1}(G) \supset SNE(G),$$

$$NE(G) \supset SNE^{n-1}(G) \supset SSNE(G).$$

The example shows that these inclusions may be strict.

Example 1. Let the three-person game $G = (N = \{1, 2, 3\}, \ A_1 = \{x_1, y_1\}, \ A_2 = \{x_2, y_2\}, \ A_3 = \{x_3, y_3\}, (u_i)_{i\in N})$, be given by the following normal form:

	x_3			y_3	
	x_2	y_2		x_2	y_2
x_1	$(7, 7, 0)$	$(0, 0, 5)$	x_1	$(4, 4, 9)$	$(0, 0, 5)$
y_1	$(0, 0, 5)$	$(5, 5, 15)$	y_1	$(0, 0, 5)$	$(0, 0, 0)$

For the convenience we'll restrict ourselves to the players' pure strategies (player 1 chooses a row, player 2 a column and player 3 a block of the table). Here: $NE(G) = \{(y_1, y_2, x_3), (x_1, x_2, y_3)\}$; $SNE(G) = SSNE(G) = \emptyset$; $SNE^{n-1}(G) = SSNE^{n-1}(G) = \{x_1, x_2, y_3)\}; WPO(G) = \{(y_1, y_2, x_3), (x_1, x_2, x_3)\}$ is the set of weak Pareto-optimal strategy profiles in G.

Certainly, player 3 can not accept the situation $(y_1, y_2, x_3) \in NE(G) \cap WPO(G)$ since the best player 1 and player 2 joint response to the player 3 strategy x_3 is (x_1, x_2) that leads to the least possible payoff of player 3. At the same time the strategy profile (x_1, x_2, y_3) is free from such danger, and satisfies the requirements of coalition stability from def. 4, 5 although does not satisfy weak Pareto-optimality.

Thus one can use the strong $(n-1)$-equilibrium concept to obtain a unique optimal outcome in strategic game G.

Notice, that the extended analysis of closely related example is offered in Bernheim B. et al.,1987 in connection with the coalition-proof Nash equilibrium concept.

Definition 6. Let $G = (N, (A_i)_{i\in N}, (u_i)_{i\in N})$ be a game, let $x \in A$ and let $\emptyset \neq S \subset N$. *An internally consistent improvement (ICI) of S upon x* is defined by induction on $|S|$. If $|S| = 1$, that is $S = \{i\}$ for some $i \in N$, then $y_i \in A_i$ is an ICI of i upon x if it is an improvement upon x, that is, $u_i(y_i, x_{N\setminus\{i\}}) > u_i(x)$.
If $|S| > 1$ then $y_S \in A_S$ is an ICI of S upon x if:

1. $u_i(y_s, x_{N\setminus S}) > u_i(x)$ for all $i \in S$,
2. no $T \subset S$, $T \neq \emptyset$, S has an ICI upon $(y_S, x_{N\setminus S})$.

x is *a coalition-proof Nash equilibrium (CPNE)* if no $T \subset N$, $T \neq \emptyset$, has an ICI upon x.

The reader is refereed to Bernheim B. et al.,1987 for discussion and motivation.

Definition 7. (Kaplan G., 1992) Let $G = (N, (A_i)_{i\in N}, (u_i)_{i\in N})$ be a game. $x \in A$ is *a semi-strong Nash equilibrium (SMSNE)*, if for every $\emptyset \neq S \subset N$ and every $y_S \in NE(G^{S,x})$ there exists $i \in S$ such that $u_i(x) \geq u_i(y_S, x_{N\setminus S})$.

Notice that the concept of $CPNE$ (as well as $SMSNE$) deals only with c e r t a i n deviations of a l l p o s s i b l e coalitions $S \subset N$.

To clarify the difference between $CPNE$ and $SMSNE$ from the one hand and the strong $(n-1)$-equilibrium concept from the other we consider the following example.

Example 2. $G = (N = \{1, 2, 3\}, A_1 = \{x_1, y_1\}, A_2 = \{x_2, y_2\}, A_3 = \{x_3, y_3\}, (u_i)_{i\in N})$, is the three-person strategic game:

	x_3			y_3	
	x_2	y_2		x_2	y_2
x_1	$(9, 9, 0)$	$(4, 10, 0)$	x_1	$(4, 4, 9)$	$(0, 0, 5)$
y_1	$(0, 0, 5)$	$(5, 5, 10)$	y_1	$(0, 0, 5)$	$(0, 0, 0)$

Here: $NE(G) = \{(y_1, y_2, x_3), (x_1, x_2, y_3)\}$; $WPO(G) = \{(x_1, x_2, x_3), (x_1, y_2, x_3), (y_1, y_2, x_3)\}$; $SNE(G) = SSNE(G) = \emptyset$; $CPNE(G) = SMSNE(G) = NE(G) \cap WPO(G) = \{(y_1, y_2, x_3)\}$, but $SSNE^{n-1}(G) = \{(x_1, x_2, y_3)\}$.

Notice that (as in example 1) player 3 can reject the strategy profile (y_1, y_2, x_3) by the reason of other players have the profitable joint deviation (x_1, x_2) from (y_1, y_2, x_3) that is still possible (although (x_1, x_2) is not ICI of $S = \{1, 2\}$ upon (y_1, y_2, x_3)). If such deviation takes place player 3 will receive the least feasible payoff (independently of further possible deviation y_2 of player 2).

Remark 2. $SSNE^{n-l}$ does not coincide with $CPNE$ (as soon as with $SMSNE$) in general case.

Remark 3. Let φ be one of the optimality principles: $CPNE$ or $SMSNE$. $SSNE^{n-1}$ is not a refinement of φ, and φ is not a refinement of $SSNE^{n-1}$.

3. Coalition stable equilibriums in multicriteria games

Now let us turn to so-called multicriteria games (or the games with vector payoffs) when every player may take several criteria into account. Formally, let $G = (N, (A_i)_{i\in N}, (u_i)_{i\in N})$ be a finite multicriteria game, were N is a finite set of players, $|N| = n$, $A_i \neq \emptyset$ is the finite set of pure strategies of player $i \in N$, and for each player $i \in N$ the function $u_i : \prod_{j \in N} A_j \to R^{r(i)}$ maps each strategy profile to a point in $r(i)$-dimensional Euclidean space. Note that player i in multicriteria game G tries to maximaize $r(i)$ scalar criteria (i.e. all the components of her vector valued payoff function $u_i(x_i, x_{-i})$).

The concept of equilibrium point for multicriteria games was proposed by **??** as a natural generalization of the Nash equilibrium concept for unicriterium games.

Let $a,\ b \in R^t$, and $a > b$ means that $a_i > b_i$ for all $i = 1, \ldots, t$; $a \geq b$ means that $a_i \geq b_i$ for all $i = 1, \ldots, t$, and $a \neq b$.

The vector $a \in M \subseteq R^t$ is weak Pareto efficient (or undominated) in M iff $\{b \in R^t :\ b > a\} \cap M = \emptyset$. In this case we'll use the following notation: $a \in WPO(M)$.

Given strategy profile $x = (x_i, x_{-i})$ in the finite multicriteria game G denote by

$$M_i(G, x_{-i}) = \{u_i(y_i, x_{-i}), y_i \in A_i\}$$

the set of all player's i attainable vector payoffs (due to arbitrary choice of his strategy $y_i \in A_i$).

Definition 8. The strategy profile $x = (x_1, \ldots, x_n) \in \prod_{j \in N} A_j$ is called *(weak Pareto) equilibrium* in multicriteria game G iff for each player $i \in N$ there does not exist a strategy $y_i \in A_i$ such that:

$$u_i(y_i, x_{-i}) > u_i(x_i, x_{-i}) \tag{1}$$

Note that (1) is equivalent to the following condition:

$$u_i(x_i, x_{-i}) \in WPO(M_i(G, x_{-i}))\ \forall i \in N. \tag{2}$$

Let $E(G)$ be the set off all (weak Pareto) equilibriums in multicriteria game G.

Definition 9. The strategy profile $x = (x_1, \ldots, x_n)$ is called *strong equilibrium* (in a sense of) in multicriteria game G iff

$$\forall S \subset N, S \neq \emptyset \quad y_S \in \prod_{j \in S} A_j :\ \begin{cases} u_i(y_S, x_{-S}) > u_i(x_S, x_{-S}) \\ i \in S \end{cases}.$$

Definition 10. The strategy profile $x = (x_1, \ldots, x_n)$ is called *strictly strong equilibrium* in multicriteria game G iff

$$\forall S \subset N, S \neq \emptyset \quad y_S \in \prod_{j \in S} A_j :\ \begin{cases} u_i(y_S, x_{-S}) \geq u_i(x_S, x_{-S}) \\ i \in S \end{cases}.$$

Definition 11. Let $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ be a finite multicriteria game with n players, $|N| = n$.

1. $n \geq 2$: $x = (x_1, \ldots, x_n) \in \prod_{j \in N} A_j$ is a *strong* $(n-1)$*-equilibrium* if for each player $i \in N$ the following condition holds:

$$\forall S \subset N \backslash \{i\}, \quad y_S \in A_S :\ \begin{cases} u_j(y_S, x_{-S}) > u_i(x_S, x_{-S}) \\ j \in S \end{cases}. \tag{3}$$

2. $n = 1$: $x_i \in A_i$ is a *strong* $(n-1)$*-equilibrium* in one player multicriteria game $G = (\{i\}, A_i, u_i)$ if $\ y_i \in A_i :\ u_i(y_i) > u_i(x_i)$.

Let $SE(G), SSE(G)$ and $SE^{n-1}(G)$ be the sets S of all strong equilibriums, strictly strong equilibriums and strong $(n-1)$-equilibriums in multicriteria game G correspondly.
Definition 12.

1. $n \geq 2$: $x = (x_1, \ldots, x_n) \in A$ is a *strictly strong* $(n-1)$*-equilibrium* if for every player $i \in N$ the following condition holds:

$$\forall S \subset N \backslash \{i\}, \quad y_S \in A_S : \begin{cases} u_j(y_S, x_{-S}) \geq u_j(x_S, x_{-S}) \\ j \in S \end{cases} . \tag{4}$$

2. $n = 1$: $x_i \in A_i$ is a *strictly strong* $(n-1)$*-equilibrium* in one-person multicriteria game $G = (\{i\}, A_i, u_i)$, if $y_i \in A_i$: $u_i(y_i) \geq u_i(x_i)$.

The set of all strictly strong $(n-1)$-equilibriums in G denote by $SSE^{n-1}(G)$.
Definition 13. The strategy profile $x = (x_1, \ldots, x_n) \in \prod_{j \in N} A_j$ is called *Pareto efficient* in a multicriteria game G iff

$$y \in \prod_{j \in N} A_j : \begin{cases} u_i(y) \geq u_i(x) \\ i \in N \end{cases} . \tag{5}$$

The set of all Pareto efficient strategy profiles in G denote by $PO_{MG}(G)$.
It is clear that

$$E(G) \supset SE^{n-1}(G) \supset SSE^{n-1}(G),$$

$$SSE^{n-1}(G) \supset SSE(G),$$

$$PO_{MG}(G) \supset SSE(G).$$

4. Axiomatization of strictly strong $(n-1)$- equilibria in multicriteria games

In this section we give axiomatization of SSE^{n-1} correspondence on closed classes of multicriteria games in terms on consistency, one-person rationality, suitable variants of converse consistency and Pareto-optimality axiom and others.
Let Γ be a set of muliticriteria games G and let φ be a solution on Γ.
Definition 14. φ satisfies *strong one-person rationality (SOPR)* if for every one-person game $G = (\{i\}, A_i, u_i) \in \Gamma$

$$\varphi(G) = \{x_i \in A_i | \quad y_i \in A_i : \ u_i(y_i) \geq u_i(x_i)\}$$

Let $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ be a game, $n =| N | \geq 2$, let $S \subset N$ be a proper coalition, i.e. $S \neq \emptyset, N$.
Definition 15. The proper *reduced game* $G^{S,x}$ of G (with respect to S and x) is the multicriteria game $G^{S,x} = (S, (A_i)_{i \in S}, (u_i^x)_{i \in S})$, where

$$u_i^x(y_S) = u_i(y_S, x_{N \backslash S}) \ \forall \ y_S \in A_S, \ \forall \ i \in S.$$

A family Γ of multicriteria games is r-closed, if $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma$, $S \subset N$, $S \neq \emptyset, N$ and $x \in A$ imply that $G^{S,x} \in \Gamma$.

Definition 16. Let Γ be a r-closed family of strategic games. A solution φ on Γ satisfies *consistency* $(CONS)$, if for every $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma$, $\forall\, S \subset N$, $S \neq \emptyset, N$, $\forall\, x \in \varphi(G)$ the following condition holds:

$$x_S \in \varphi(G^{S,x}).$$

The $CONS$ property means the restriction x_S of the optimal strategy profile $x \in \varphi(G)$ still satisfies the optimality principle φ in every reduced game $G^{S,x}$. If $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma$, and $n \geq 2$, then we denote:

$$\widetilde{\varphi}(G) = \{x \in A \mid \forall\, S \subset N,\ S \neq \emptyset, N,\ x_S \in \varphi(G^{S,x})\} \tag{6}$$

Taking (6) into account one can notice that $CONS$ property means $\varphi(G) \subset \widetilde{\varphi}(G)$ for every $G \in \Gamma$.

Definition 17. A solution φ on Γ satisfies $(n-1)$-*Pareto optimality for multicriteria games* (PO^{n-1}_{MG}), if for every $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma$ with at least two players $(n \geq 2)$, for every $x \in \varphi(G)$ the following conditions holds:

$$\forall\, i \in N \quad y_{-i} \in A_{-i} : \begin{cases} u_j(y_{-i}, x_i) \geq u_j(x_{-i}, x_i) \\ j \in N \setminus \{i\}. \end{cases} \tag{7}$$

Notice that φ satisfies PO^{n-1}_{MG} iff $\forall\, x \in \varphi(G), \forall i \in N$

$$x_{N \setminus \{i\}} \in PO_{MG}(G^{N \setminus \{i\}, x}).$$

Let PO^{n-1}_{MG} be the set of all strategy profiles $x \in \Pi_{i \in N} A_i$, satisfying (7).

Definition 18. Let Γ be a r-closed family of strategic games. A solution φ satisfies $COCONS^{n-1}_{*}$ (the appropriate version of converse for SSE^{n-1}), if for every $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma$, $n \geq 2$, it is true that:

$$\left[x \in \widetilde{\varphi}(G) \text{ and } x \in PO^{n-1}_{MG}(G)\right] \Rightarrow x \in \varphi(G) \tag{8}$$

In accordance with $COCONS^{n-1}_{*}$ property if the restrictions x_S of some strategy profile $x \in A$ satisfy optimality principle φ in every reduced game $G^{S,x}$, and $x \in PO^{n-1}_{MG}(G)$ then x is the optimal strategy profile in the original game G.

Theorem 1. *A solution φ on a r-closed family of multicriteria games Γ satisfies $CONS$, $SOPR$, PO^{n-1}_{MG} and $COCONS^{n-1}_{*}$, if and only if $\varphi = SSE^{n-1}$ (i.e. $\varphi(G) = SSE^{n-1}(G)$ for every $G \in \Gamma$).*

Proof. 1. It is not difficult to verify that SSE^{n-1} satisfies $CONS$, $SOPR$, PO^{n-1}_{MG} and $COCONS^{n-1}_{*}$. Let us check here that SSE^{n-1} satisfies $CONS$ (for instance). If $x \in SSE^{n-1}$ then $\forall\, S \subset N \backslash \{i\}\ \ y_S \in A_S$:

$$\begin{cases} u_j(y_S, x_{-S}) \geq u_j(x_S, x_{-S}) = u_j(x) \\ j \in S. \end{cases} \tag{9}$$

Consider an arbitrary coalition $S_1 \subset N$, $S_1 \neq \emptyset, N$ and reduced game $G^{S_1,x}$. Let $S \subset S_1 \backslash \{i\} \subset S_1 \subset N$. Using (9) we have that

$$\forall\, S \subset S_1 \backslash \{i\}\ \ y_S \in A_S : \begin{cases} u_j(y_S, x_{S_1 \backslash S}, x_{N \backslash S_1}) = u_j(y_S, x_{N \backslash S}) \geq u_j(x) \\ j \in S. \end{cases}$$

This means that $x_{S_1} \in SSE^{n-1}(G^{S_1,x})$, i.e. SSE^{n-1} satisfies $CONS$.

2. Now let φ be a solution on Γ that satisfies the foregoing four axioms. We prove by induction (on the number of players n) that $\varphi(G) = SSE^{n-1}(G)$ for every $G \in \Gamma$.
By $SOPR$ $\varphi(G) = SSE^{n-1}(G)$ for every one-person multicriteria game $G \in \Gamma$. Now assume that

$$\varphi(G) = SSE^{n-1}(G) \ \forall \ G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma, \tag{10}$$

where $1 \leq |N| \leq k$, $k \geq 1$, and consider an arbitrary $(k+1)$-person multicriteria game $G \in \Gamma$. Let $x \in \varphi(G)$. From the $CONS$ of φ it follows that

$$x \in \widetilde{\varphi}(G). \tag{11}$$

Using the induction hypothesis and the notation (6) we obtain:

$$\widetilde{\varphi}(G) = \widetilde{SSE}^{n-1}(G).$$

Moreover, by PO_{MG}^{n-1} of φ,

$$x \in PO_{MG}^{n-1}(G). \tag{12}$$

Taking into account (10), (11), and (12), and $COCONS_*^{n-1}$ property of SSE^{n-1}, we obtain that $x \in SSE^{n-1}(G)$, and, hence, $\varphi(G) \subset SSE^{n-1}(G)$.
Similarly, we may prove that $SSE^{n-1}(G) \subset \varphi(G)$ for every $(k+1)$-person multicriteria game $G \in \Gamma$. The inductive conclusion completes the proof.

Corollary 1. Let φ be a solution on r-closed family of games Γ, that satisfies $CONS$ and PO_{MG}^{n-1}. Then

$$\varphi(G) \subset SSE^{n-1}(G) \ \forall \ G \in \Gamma, \ \mid N \mid = n \geq 2.$$

Proof. Let $x \in \varphi(G)$, $n \geq 2$. To prove that $x \in SSE^{n-1}(G)$ we need to verify that for every possible proper coalition

$$S \subset N, \ S \neq N, \emptyset, \quad y_S \in A_S : \ \begin{cases} u_j(y_S, x_{-S}) \geq u_j(x), \\ j \in S. \end{cases}$$

i.e.

$$x_S \in PO_{MG}(G^{S,x}) \ \forall \ S \subset N : \ s = \mid S \mid = 1, 2, \ldots, n-1 \tag{13}$$

By PO_{MG}^{n-1} of φ

$$x_S \in PO_{MG}(G^{S,x}) \ \forall \ S \subset N : \ s = n-1.$$

If $n = 2$ we have already established (13) for all possible proper coalitions. Otherwise (if $n \geq 3$), consider a proper reduced game $G^{S,x}$, where $s = n-1$. By $CONS$ of φ $x_S \in \varphi(G^{S,x})$, and by PO_{MG}^{n-1} $x_S \in PO_{MG}^{s-1}(G^{S,x})$, i.e.

$$x_T \in PO_{MG}(G^{T,x}) \ \forall \ T \subset S, t = |T| = s-1 = n-2.$$

Using the same approach we can establish (13) for every proper coalition $S \subset N$, $s = \mid S \mid = n-1, n-2, \ldots, 1$.

Another axiomatic characterization of the SSE^{n-1} correspondence involves the following axioms (Peleg B. and Tijs S., 1996).

Definition 19. Let Γ be a set of multicriteria games, and φ be a solution Γ. φ satisfies *independence of irrelevant strategies (IIS)* if the following condition holds: if $G = (N, (A_i)_{i\in N}, (u_i)_{i\in N}) \in \Gamma$, $x \in \varphi(G)$, $x_i \in B_i \subset A_i$ for all $i \in N$, and $G^* = (N, (B_i)_{i\in N}, (u_i)_{i\in N}) \in \Gamma$, then $x \in \varphi(G^*)$.

A family of games Γ is called s-closed, if for every game $G = (N, (A_i)_{i\in N}, (u_i)_{i\in N}) \in \Gamma$, and $B_i \subset A_i$, $B_i \neq \emptyset$, $i \in N$, the game $G^* = (N, (B_i)_{i\in N}, (u_i)_{i\in N}) \in \Gamma$. Further, Γ is called closed, if it is both r-closed and s-closed. For example, the set of all finite multicriteria games is closed.

Definition 20. A solution φ on r-closed family of games Γ satisfies the *dummy axiom (DUM)*, if for every game $G = (N, (A_i)_{i\in N}, (u_i)_{i\in N}) \in \Gamma$ and every "dummy player" d in G (i.e. player $d \in N$ such that $\mid A_d \mid = 1$), the following condition holds: $\varphi(G) = A_d \times \varphi(G^{N\setminus\{d\}}, x)$, where x is an arbitrary strategy profile from A.

Note, that SSE^{n-1} satisfies IIS and DUM.

Proposition 1. (Peleg B., Tijs S. [1996]) If a solution φ on closed family of games Γ satisfies IIS and DUM, then φ also satisfies $CONS$.

The next axiomatic characterization of SSE^{n-1} correspondence follows from the theorem 1 and proposition 1.

Theorem 2. *Let Γ be a closed family of multicriteria games. The SSE^{n-1} correspondence is the unique solution on Γ that satisfies $SOPR$, PO^{n-1}_{MG}, $COCONS^{n-1}_*$, IIS and DUM.*

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