ISSN 1061-9208, Russian Journal of Mathematical Physics, Vol. 21, No. 2, 2014, pp. 197–218. © Pleiades Publishing, Ltd., 2014.

Adiabatic Approximation via Hodograph Translation and Zero-Curvature Equations

M. V. Karasev

Dept. of Applied Mathematics,
National Research University Higher School of Economics
E-mail: mkarasev@hse.ru, Karasev.Mikhail@gmail.com

Received April 6, 2014

Abstract. For quantum as well classical slow–fast systems, we develop a general method which allows one to compute the adiabatic invariant (approximate integral of motion), its symmetries, the adiabatic guiding center coordinates and the effective scalar Hamiltonian in all orders of a small parameter. The scheme does not exploit eigenvectors or diagonalization, but is based on the ideas of isospectral deformation and zero-curvature equations, where the role of "time" is played by the adiabatic (quantization) parameter. The algorithm includes the construction of the zero-curvature adiabatic connection and its splitting generated by averaging up to an arbitrary order in the small parameter.

DOI 10.1134/S106192081402006X

1. INTRODUCTION

Let us consider a family of "fast" self-adjoint operators $\mathbb{H} = \mathbb{H}(x)$ parametrized by points x running over a "slow" phase space $\mathcal{D} \approx \mathbf{R}^{2n}$ with Poisson brackets

$$\{x^j, x^l\} = J^{jl}, \qquad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{1.1}$$

Assume that, at each x, the spectrum of $\mathbb{H}(x)$ is discrete, the eigenvalues $\lambda_k(x)$ are not degenerate and ordered as $\lambda_0(x) < \lambda_1(x) < \cdots$.

Assume also that the x-dependence of all these objects is smooth and allows us to make the quantization $x \to \hat{x}$ with Heisenberg commutation relations

$$[\hat{x}^j, \hat{x}^l] = -i\varepsilon J^{jl}. \tag{1.2}$$

Here ε is a small (adiabatic) parameter. The presence of ε in (1.2) explains that the x-space is said to be "slow."

Now one can consider the quantized self-adjoint Hamiltonian $\hat{\mathbb{H}} = \mathbb{H}(\hat{x})$, where the components of the vector-operator $\hat{x} = (\hat{x}^1, \dots, \hat{x}^{2n})$ are assumed to be Weyl-symmetrized. This Hamiltonian is the main object of our study.

The well-known Born–Oppenheimer method [1–4] provides the opportunity of replacing the Hamiltonian $\hat{\mathbb{H}}$ by its "terms" $\hat{\lambda}_k = \lambda_k(\hat{x})$ with accuracy $O(\varepsilon)$:

$$\hat{\mathbb{H}} \longrightarrow \hat{\lambda}_k + O(\varepsilon). \tag{1.3}$$

The procedure (1.3) reduces the Hamiltonian with the operator-valued symbol $\mathbb{H}$ to the effective Hamiltonians with scalar-valued symbols λ_k.

The general adiabatic approximation problem is to continue the asymptotic expression (1.3) and to compute higher order corrections to the adiabatic terms

$$\hat{\mathbb{H}} \longrightarrow \hat{\Lambda}_k, \qquad \Lambda_k = \lambda_k + \varepsilon\mu_k + \varepsilon^2\nu_k + \cdots \tag{1.4}$$

Also one needs not only to compute Λ_k, but also to describe the reduction operation (1.4) in order to reconstruct the states of the Hamiltonian $\hat{\mathbb{H}}$ from the states of the reduced effective Hamiltonian $\hat{\Lambda}_k$.

There are known adiabatic approximation schemes [5–9] based on the knowledge of the eigenvectors of the operator-values symbol $\mathbb{H}(x)$. This approach in higher ε-orders is not very explicit if the symbol $\mathbb{H}$ is not a matrix but an infinite-dimensional operator, and does not allow one to correlate the quantum adiabatic approximation with the classical scheme known in Hamiltonian mechanics [10–15].

In the present paper, we avoid both of these difficulties by developing the method of the work [16]. The main steps of our approach are the following ones.

(i) We represent $\mathbb{H}(x) = f_0(\mathbb{S}_0(x), x)$ via an "action" family of operators $\mathbb{S}_0$ whose spectrum is integer. Then we introduce the unitary transform $\mathbb{U}_0^{-1}\mathbb{H}\mathbb{U}_0$, which reduces this operator-valued symbol to $f_0(\mathbb{S}_0(\underline{x}), x)$ with fixed (frozen) point $\underline{x} \in \mathcal{D}$. The phase space *hodograph translation* $x \to \underline{x}$ is a sort of isospectral deformation made by a zero-curvature connection over the phase space $\mathcal{D}$.

(ii) We consider a unitary operator $\hat{\mathbb{U}}$ transforming the quantum Hamiltonian $\hat{\mathbb{H}} \to \hat{\mathbb{U}}^{-1} \cdot \hat{\mathbb{H}} \cdot \hat{\mathbb{U}}$ to the "hodograph form" $f(\mathbb{S}_0(\underline{x}), \hat{x})$ with some ε-deformed scalar function $f = f_\varepsilon(s, x)$. Since the eigenvalues of $\mathbb{S}_0(\underline{x})$ are integers $k = 0, 1, 2, \dots$, we obtain the desirable adiabatic terms (1.4) in this way: $\Lambda_k(x) = f_\varepsilon(k, x)$.

(iii) In order to derive explicit analytic formulas for the ε-expression (1.4) as well as for the quantum "action" $\hat{\mathbb{S}} = \hat{\mathbb{U}} \cdot \mathbb{S}_0(\underline{x}) \cdot \hat{\mathbb{U}}^{-1}$ and for the quantum coordinates $\hat{\mathbb{X}} = \hat{\mathbb{U}} \cdot \hat{x} \cdot \hat{\mathbb{U}}^{-1}$ commuting with $\hat{\mathbb{S}}$, we write out the *basic homological equation* by using the Groenewold–Moyal $*$-product in the phase space.

(iv) Instead of direct computation of the operator-value family $\mathbb{U} = \mathbb{U}_\varepsilon(x)$, we regard it as a parallel section over the (ε, x)-space, introduce the corresponding *zero-curvature connection*, and try to compute the coefficients of this connection. We observe that the basic homological equation involves the *hodograph transform* of this (ε, x)-connection and write out the zero-curvature equation for the hodograph coefficients.

(v) The obtained pair: homological+zero-curvature equations is our final system, which looks like an *ε-dynamical system* (the small parameter ε plays the role of "time"). This system can easily be solved asymptotically up to any order $O(\varepsilon^N)$. This produces explicit asymptotic expansions for the adiabatic terms (1.4), for the quantum adiabatic invariant $\mathbb{S} = \mathbb{S}_\varepsilon(x)$ and for the quantum-deformed slow coordinates $\mathbb{X}^j = \mathbb{X}^j_\varepsilon(x)$.

(vi) The original quantum Hamiltonian, up to $O(\varepsilon^\infty)$, can be finally presented as $\hat{\mathbb{H}} = f(\hat{\mathbb{S}}, \hat{\mathbb{X}})$ with the quantum integral of motion $\hat{\mathbb{S}}$ and new quantum "slow" coordinates $\hat{\mathbb{X}} = ((\hat{\mathbb{X}}^j))$ obeying, up to $O(\varepsilon^\infty)$, the relations

$$[\hat{\mathbb{S}}, \hat{\mathbb{X}}^j] = 0, \qquad [\mathbb{X}^j, \mathbb{X}^l] = -i\varepsilon J^{jl} \qquad (j, l = 1, \dots, 2n).$$

These commutation relations allow one to restrict $\hat{\mathbb{H}}$ onto the kth eigensubspace of $\hat{\mathbb{S}}$ (analogs of the "Landau levels" for a particle moving in a magnetic field). On these subspaces, the effective adiabatic Hamiltonians are $\hat{\Lambda}_k = f(k, \hat{\mathbb{X}})$.

Note that the above-mentioned dynamical system for the adiabatic term (effective slow Hamiltonian) and for the hodograph coefficients is *universal*. It does not depend on the original Hamiltonian or its eigenstates. The information about the concrete physical systems is placed to the "initial" data at the zero "time" level $\varepsilon = 0$ only.

In principle, this system could provide not asymptotic but exact adiabatic transformation, i.e., complete "separation" of fast and slow coordinates. But we have no exact solvability theorem for this dynamical system even for small ε. Its nonsolvability is equivalent to the nonintegrability, in the exact sense, of the original Hamiltonian. And conversely: were this system solvable for a small enough ε, the original Hamiltonian would be exactly integrable.

Note that although our approach is developed for quantum slow–fast Hamiltonian, it can also be applied to matrix equations like the Pauli, Dirac, or Maxwell equations. In this case, we obtain a "hodograph" alternative to the matrix diagonalization method [17–19]. This method does not work in the case of degenerate eigenvalues (multiplicity > 1), but the hodograph alternative still works; see in Section 7 below.

Finally, we stress that the developed method can effectively be used in semiclassical theory and also in classical mechanics. If the commutators between fast coordinates contain a small parameter $\hbar$, and the commutators between slow coordinates are proportional to $\varepsilon\hbar$, then one can compute all objects in our general scheme by applying the semiclassical technique as $\hbar \to 0$.

It is important to note that in the critical zone $\hbar = O(\varepsilon)$ one needs to know not only the first correction μ_k in (1.4), but also ν_k or even higher corrections in order to compute just the zero–order approximation for the semiclassical wave–functions of $\hat{\mathbb{H}}$.

Another important point: from our quantum formulas, by replacing the commutators $\frac{i}{\hbar}[\cdot,\cdot]$ by the Poisson brackets at the limit $\hbar = 0$, we obtain an alternative way to develop the adiabatic approximation up to $O(\varepsilon^\infty)$ in the framework of classical Hamiltonian mechanics (see in Section 8 below). In the case of adiabatic Hamiltonian systems, our algorithm turns out to be essentially different from the action–angle approach [10, 11], the slow manifold approach [12], and the geometric averaging approach [15]. Our scheme makes it possible to compute the asymptotics up to $O(\varepsilon^\infty)$ not only for the adiabatic invariant and the reduced Hamiltonian but also for canonical "guiding center" slow coordinates and non-Abelian fast symmetries. The new geometrical feature which we incorporate into this classical framework is the curvature-free connection providing the opportunity to parallel distribute all the objects from the freezed point $\underline{x}$ over the slow phase space. The new analytical feature is the ε*–dynamics* using the adiabatic parameter for the role of "time."

2. PHASE SPACE HODOGRAPH

Let us replace the set of integers $\{k\}$ numerating the eigenvalues $\lambda_k(x)$ of $\mathbb{H}(x)$ by a family of self-adjoint operators $\mathbb{S}_0(x)$ whose spectrum is integer. Namely, we represent the operator-values symbol $\mathbb{H}$ as follows:

$$\mathbb{H}(x) = f_0(\mathbb{S}_0(x), x), \tag{2.1}$$

where

$$\exp\{2\pi i \mathbb{S}_0(x)\} = \mathbb{I}, \qquad \forall x \in \mathcal{D} \tag{2.2}$$

and $f_0 = f_0(s, x)$ is a smooth real function monotonic in the s coordinate.

The family $\mathbb{S}_0$ can formally be defined as $\mathbb{S}_0 = \sum_{k\geqslant 0} k\mathbb{P}_k$, where $\mathbb{P}_k$ is the kth eigenprojection for $\mathbb{H}$. The spectrum of $\mathbb{S}_0(x)$ consists of the integers $k = 0, 1, 2, \dots$ at each x, and so it follows from (2.1) that

$$\lambda_k(x) = f_0(k, x). \tag{2.3}$$

One has to pay attention to the following: the spectrum of $\mathbb{S}_0$ is discrete and so the choice of the function f_0 is not unique, but the final results do not depend on this choice. The same remark is applied to all our computations, and we do not mention this specially below.

Note that the representation (2.1) is very common in the classical setting (in Hamiltonian mechanics), where an analog of $\mathbb{S}_0$ is called, up to $O(\varepsilon)$, the adiabatic invariant of a slow–fast system. Our goal is to construct the quantum adiabatic invariant for the quantum Hamiltonian $\hat{\mathbb{H}}$ with an arbitrary accuracy $O(\varepsilon^\infty)$.

First of all, let us point out that, in view of the periodicity condition (2.2), we can connect the operator $\mathbb{S}_0(x)$ at any x with the operator $\mathbb{S}_0(\underline{x})$ taken at a fixed point $\underline{x}$ by a unitary transformation

$$\mathbb{S}_0(x) = \mathbb{U}_0(x) \cdot \mathbb{S}_0(\underline{x}) \cdot \mathbb{U}_0(x)^{-1}, \qquad x \in \mathcal{D}. \tag{2.4}$$

This is the so-called "isospectral deformation." The proof of (2.4) follows from general theorems [20, 21]; the construction of $\mathbb{U}_0$ is discussed in Appendix A.

From (2.1) and (2.4), we see that

$$\mathbb{H}(x) = \mathbb{U}_0(x) \cdot f_0(\mathbb{S}_0(\underline{x}), x) \cdot \mathbb{U}_0(x)^{-1}. \tag{2.5}$$

Thus, the unitary transformed symbol $\mathbb{U}_0(x)^{-1}\cdot\mathbb{H}(x)\cdot\mathbb{U}_0(x)$ on the kth eigensubspace of $\mathbb{S}_0(\underline{x})$ is reduced to the scalar term $f_0(k,x)=\lambda_k(x)$ for any $x\in\mathcal{D}$.

This is the way to obtain the scalar symbol without using the diagonalization or eigenvectors of $\mathbb{H}(x)$, but just by shifting all phase space points x into a fixed position $\underline{x}$ in the family of operators $\mathbb{S}_0$. This procedure can be referred to as the *phase space hodograph translation*.

Now in order to obtain a quantum analog of (2.5), one needs to replace $\mathbb{U}_0$ by a unitary operator $\hat{\mathbb{U}}=\mathbb{U}(\hat{x})$ with a deformed symbol $\mathbb{U}=\mathbb{U}_\varepsilon(x)$ depending on the parameter ε from (1.2). The scalar function f_0 has to be ε-deformed as well. Thus we replace (2.5) by its quantum analog:

$$\hat{\mathbb{H}}=\hat{\mathbb{U}}\cdot f(\mathbb{S}_0(\underline{x}),\hat{x})\cdot\hat{\mathbb{U}}^{-1}. \tag{2.6}$$

In the limit $\varepsilon=0$, we have

$$\mathbb{U}(x)|_{\varepsilon=0}=\mathbb{U}_0(x),\qquad f|_{\varepsilon=0}=f_0, \tag{2.7}$$

and therefore relation (2.6) is reduced to (2.5) at $\varepsilon=0$.

Let us introduce the operators

$$\hat{\mathbb{S}}=\hat{\mathbb{U}}\cdot\mathbb{S}_0(\underline{x})\cdot\hat{\mathbb{U}}^{-1}, \tag{2.8}$$

$$\hat{\mathbb{X}}=\hat{\mathbb{U}}\cdot\hat{x}\cdot\hat{\mathbb{U}}^{-1}. \tag{2.9}$$

Then (2.6) reads

$$\hat{\mathbb{H}}=f(\hat{\mathbb{S}},\hat{\mathbb{X}}). \tag{2.10}$$

It is also obvious that the Heisenberg commutation relation holds:

$$[\hat{\mathbb{X}}^j,\hat{\mathbb{X}}^l]=-\varepsilon J^{jl}, \tag{2.11}$$

and, additionally,

$$[\hat{\mathbb{S}},\hat{\mathbb{X}}^j]=0\qquad\forall j. \tag{2.12}$$

It follows from (2.12) that the operator $\hat{\mathbb{S}}$ is the *quantum integral of motion* for the Hamiltonian $\hat{\mathbb{H}}$. After restriction of (2.10) to the kth eigensubspace of $\hat{\mathbb{S}}$, we see that

$$\hat{\mathbb{H}}|_{k\text{th eigensubspace}}=f(k,\hat{x}') \tag{2.13}$$

with operators $\hat{x}'\overset{\text{def}}{=}\hat{\mathbb{X}}|_{\text{k-th eigensubspace}}$ satisfying relations $[\hat{x}'^j,\hat{x}'^l]=-i\varepsilon J^{jl}$.

Thus we obtain the reduction (1.4) with

$$\Lambda_k(x)\overset{\text{def}}{=}f(k,x). \tag{2.14}$$

The ε-expansion (1.4) then follows from (2.14):

$$\mu_k(x)=\frac{\partial f}{\partial\varepsilon}(k,x)\bigg|_{\varepsilon=0},\qquad \nu_k(x)=\frac{1}{2}\frac{\partial^2 f}{\partial\varepsilon^2}(k,x)\bigg|_{\varepsilon=0},\quad\dots. \tag{2.15}$$

Conclusion: after the hodograph translation by means of (2.4)–(2.7), the Hamiltonian $\hat{\mathbb{H}}$ is transformed to the form (2.10) with the quantum integral of motion $\hat{\mathbb{S}}$ and can be reduced by (2.13), (2.14) to the scalar-valued Hamiltonian Λ_k.

If one is interested in the asymptotic expansion of Λ_k as $\varepsilon\to 0$, then there is no need to derive the basic relation (2.10) exactly: one can derive it only asymptotically, up to $O(\varepsilon^\infty)$. This is what we want to obtain as the result. But at the same time, until the very end, i.e., until the final equations allowing to compute f, $\mathbb{S}$, $\mathbb{X}$, we do all transformations exactly in ε.

3. BACKWARD TRANSLATION BY ADIABATICITY (QUANTIZATION) PARAMETER

Now let us recall the composition formula for the Weyl-symmetrized functions in $\hat{x}$-operators [22–24]:

$$\hat{g} \cdot \hat{r} = \widehat{g * r}, \qquad (g * r)(x) \stackrel{\text{def}}{=} g(x) \exp\Big\{ -\frac{i\varepsilon}{2} \underset{\leftarrow}{D} \cdot J \cdot \underset{\rightarrow}{D} \Big\} r(x). \tag{3.1}$$

Here we denote $D = \partial/\partial x$ and the left–right arrows indicate the x-coordinate to which the derivatives D are applied.

We shall also use the operation $\odot$ defined by

$$g \odot r \stackrel{\text{def}}{=} \frac{1}{\varepsilon}(g * r - gr). \tag{3.1a}$$

Note that it follows from the definition (3.1) of the Groenewold–Moyal $*$-product that

$$\frac{\partial}{\partial \varepsilon}(g * r) = -\frac{i}{2} Dg * JDr, \qquad D(g * r) = Dg * r + g * Dr. \tag{3.2}$$

The use of $*$ allows one to replace the hat-quantization notation $\widehat{(\dots)}$ in formulas (2.6), (2.8), (2.9) by star-algebra products.

In particular, (2.6) can be rewritten as follows:

$$\mathbb{H} = \mathbb{U} * \underline{\mathbb{F}} * \mathbb{U}^{-1*}, \tag{3.3}$$

where

$$\underline{\mathbb{F}}(x) \stackrel{\text{def}}{=} f(\mathbb{S}_0(\underline{x}), x), \tag{3.4}$$

and the inversion operation $\dots^{-1*}$ in (3.3) is taken with respect to the $*$-product (3.1).

Let us note that the operator-valued family $\mathbb{U} = \mathbb{U}_\varepsilon(x)$ in identity (3.3), as well as the functions $f = f_\varepsilon(s, x)$ from (3.4), depends on ε, but the operator-valued symbol $\mathbb{H} = \mathbb{H}(x)$ does not depend on ε. Thus, identity (3.3) can, in a sense, be regarded as a parallel transport of $\mathbb{F}$ to $\mathbb{H}$ back in the ε-space (i.e., back to the Cauchy data if ε is interpreted as a "time" variable). Thus the phase space hodograph translation is now completed by the ε-space translation.

Since $\mathbb{H}$ does not depend on ε, by applying the ε-derivative to (3.3), one obtains

$$\frac{\partial}{\partial \varepsilon}(\mathbb{U} * \underline{\mathbb{F}} * \mathbb{U}^{-1*}) = 0. \tag{3.5}$$

Using (3.2), we then compute from (3.5)

$$\begin{gathered} \frac{\partial}{\partial \varepsilon}\mathbb{U} * \underline{\mathbb{F}} * \mathbb{U}^{-1*} + \mathbb{U} * \frac{\partial}{\partial \varepsilon}\underline{\mathbb{F}} * \mathbb{U}^{-1*} + \mathbb{U} * \underline{\mathbb{F}} * \frac{\partial}{\partial \varepsilon}(\mathbb{U}^{-1*}) - \frac{i}{2} D\mathbb{U} * JD\underline{\mathbb{F}} * \mathbb{U}^{-1*} \\ - \frac{i}{2} D\mathbb{U} * \underline{\mathbb{F}} * JD(\mathbb{U}^{-1*}) - \frac{i}{2}\mathbb{U} * D\underline{\mathbb{F}} * JD(\mathbb{U}^{-1*}) = 0. \end{gathered} \tag{3.6}$$

From the identity $\mathbb{U}^{-1*} * \mathbb{U} = \mathbb{I}$, by applying the derivatives with respect to ε and x, we derive

$$D(\mathbb{U}^{-1*}) * \mathbb{U} = -\mathbb{U}^{-1*} * D\mathbb{U},$$
$$\frac{\partial}{\partial \varepsilon}(\mathbb{U}^{-1*}) * \mathbb{U} = -\mathbb{U}^{-1*} * \frac{\partial}{\partial \varepsilon}\mathbb{U} - \frac{i}{2}\mathbb{U}^{-1*} * D\mathbb{U} * J\mathbb{U}^{-1*} * D\mathbb{U}.$$

Now let us use these relations in (3.6), introduce the notation

$$i\mathbb{U}^{-1*} * D\mathbb{U} \overset{\text{def}}{=} \underline{\mathbb{A}}, \qquad i\mathbb{U}^{-1*} * \frac{\partial}{\partial\varepsilon}\mathbb{U} \overset{\text{def}}{=} \underline{\mathbb{B}}, \tag{3.7}$$

and also act on (3.6) by $\mathbb{U}^{-1*}*$ and by $*\mathbb{U}$, respectively, from left and right. Then (3.6) implies

$$i[\mathbb{F} \overset{*}{,} \underline{\mathbb{B}}] = \frac{i}{2}J[\underline{\mathbb{F}} \overset{*}{,} \underline{\mathbb{A}}] * \underline{\mathbb{A}} + \langle\!\langle \underline{\mathbb{A}} \overset{*}{,} D\underline{\mathbb{F}} \rangle\!\rangle - \frac{\partial}{\partial\varepsilon}\underline{\mathbb{F}}. \tag{3.8}$$

Here we have introduced the general notation

$$\langle\!\langle \mathbb{Y} \overset{*}{,} \mathbb{Z} \rangle\!\rangle \overset{\text{def}}{=} \frac{1}{2}(\mathbb{Y} * J\mathbb{Z} + J\mathbb{Z} * \mathbb{Y}), \qquad [\mathbb{M} \overset{*}{,} \mathbb{N}] \overset{\text{def}}{=} \mathbb{M} * \mathbb{N} - \mathbb{N} * \mathbb{M}. \tag{3.9}$$

Both of these general operations are skew-symmetric. It follows from (3.1a) that

$$[\mathbb{M} \overset{*}{,} \mathbb{N}] = [\mathbb{M}, \mathbb{N}] + \varepsilon[\mathbb{M} \overset{\circ}{,} \mathbb{N}].$$

Therefore, Equation (3.8) can be written as

$$i[\underline{\mathbb{F}}, \underline{\mathbb{B}}] = \beta(\underline{\mathbb{F}}, \underline{\mathbb{A}}, \underline{\mathbb{B}}) - \frac{\partial}{\partial\varepsilon}\underline{\mathbb{F}}, \tag{3.10}$$

where $\underline{\mathbb{F}}$ is given by (3.4) and the operation β is determined by the $*$-product as follows:

$$\beta(\mathbb{G}, \mathbb{Z}, \mathbb{M}) = \frac{i}{2}J[\mathbb{G} \overset{*}{,} \mathbb{Z}] * \mathbb{Z} + \langle\langle \mathbb{Z} \overset{*}{,} D\mathbb{G} \rangle\rangle + i\varepsilon[\mathbb{M} \overset{\circ}{,} \mathbb{G}]. \tag{3.11}$$

Equation (3.10) will be referred to as the *basic homological equation.*

The computation that we have just made can be extended. Let us consider the transformation

$$\underline{\mathbb{G}} \longrightarrow \mathcal{U}(\underline{\mathbb{G}}) \overset{\text{def}}{=} \mathbb{U} * \underline{\mathbb{G}} * \mathbb{U}^{-1*} \tag{3.12}$$

applied to an arbitrary operator-valued symbol $\underline{\mathbb{G}} = \underline{\mathbb{G}}(x)$, which is assumed to be ε-independent. Then, as above, one can compute the ε-derivative of $\mathcal{U}$ by applying (3.2) and using the notation (3.7). The result is the following one:

$$\frac{\partial}{\partial\varepsilon}\mathcal{U}(\underline{\mathbb{G}}) = \mathcal{U}\Big(i[\underline{\mathbb{G}}, \underline{\mathbb{B}}] - \beta(\underline{\mathbb{G}}, \underline{\mathbb{A}}, \underline{\mathbb{B}})\Big). \tag{3.13}$$

This equation allows us to compute each next $(N+1)$th term in the asymptotic ε-expansion of $\mathcal{U}$ via the previous N terms under the assumption that the N terms of the expansions for $\underline{\mathbb{B}}$ and $\underline{\mathbb{A}}$ are known.

4. ZERO-CURVATURE EQUATIONS

Now let us look at the operator-valued family $\mathbb{U} = \mathbb{U}_\varepsilon(x)$ as at a parallel section of the operator bundle over the (ε, x)-space. The corresponding zero-curvature connection is given by the x-coefficients $\mathbb{A}$ and ε-coefficients $\mathbb{B}$ as

$$iD\mathbb{U} * \mathbb{U}^{-1*} \overset{\text{def}}{=} \mathbb{A}, \qquad i\frac{\partial}{\partial\varepsilon}\mathbb{U} * \mathbb{U}^{-1*} \overset{\text{def}}{=} \mathbb{B}. \tag{4.1}$$

Both $\mathbb{A}$ and $\mathbb{B}$ smoothly depend on x and ε. They are related to the operator-valued families $\underline{\mathbb{A}}$ and $\underline{\mathbb{B}}$ (3.7) by means of the unitary transformations (3.12), namely,

$$\mathbb{A} = \mathcal{U}(\underline{\mathbb{A}}), \qquad \mathbb{B} = \mathcal{U}(\underline{\mathbb{B}}). \tag{4.2}$$

The pair $(\underline{\mathbb{A}}, \underline{\mathbb{B}})$ can be referred to as the *hodograph* of the connection, since it represents the connection coefficients $(\mathbb{A}, \mathbb{B})$ parallel transported to the fixed point $\underline{x} \in \mathcal{D}$. Below we use such a phase-space hodograph representation, because the above homological equation (3.10) can easily be written via $\underline{\mathbb{A}}$ and $\underline{\mathbb{B}}$.

Now let us consider the zero-curvature equations for our connection. They can be expressed in terms of the hodograph coefficients $\underline{\mathbb{A}}$ and $\underline{\mathbb{B}}$ just as the integrability conditions for the parallel transport equations (3.7):

$$iD_j \mathbb{U} = \mathbb{U} * \underline{\mathbb{A}}_j, \qquad i\frac{\partial}{\partial \varepsilon}\mathbb{U} = \mathbb{U} * \underline{\mathbb{B}}. \tag{3.7a}$$

By using (3.2), we derive

$$D_j \underline{\mathbb{A}}_l - D_l \underline{\mathbb{A}}_j - i[\underline{\mathbb{A}}_j \overset{*}{,} \underline{\mathbb{A}}_l] = 0, \tag{4.3}$$

$$\frac{\partial}{\partial \varepsilon}\underline{\mathbb{A}}_j = D_j \underline{\mathbb{B}} - i[\underline{\mathbb{A}}_j \overset{*}{,} \underline{\mathbb{B}}] + \frac{1}{2}\underline{\mathbb{A}} * JD\underline{\mathbb{A}}_j \tag{4.4}$$

for any $j, l = 1, \dots, 2n$.

Note that, at $\varepsilon = 0$, the family $\mathbb{U}_0(x)$ is related to the "action" family $\mathbb{S}_0(x)$ and so the initial data $\underline{\mathbb{A}}|_{\varepsilon=0} = \underline{\mathbb{A}}_0$ can be computed via $\mathbb{S}_0$ (see Appendix A). For this initial family $\underline{\mathbb{A}}_0(x)$, the integrability conditions (4.3) in the x-coordinates are of course valid.

Lemma 4.1. *It follows from Equation* (4.4) *that identities* (4.3) *hold for any* $\varepsilon > 0$ *if they hold at* $\varepsilon = 0$.

Let us pay attention to the fact that, although the operator $\hat{\mathbb{U}}$ is assumed to be unitary, its operator-valued symbol $\mathbb{U}$ is not unitary, since $\mathbb{U}^{-1*} \neq \mathbb{U}^{-1}$. That is why the x-coefficients $\mathbb{A}$ and $\underline{\mathbb{A}}$ of the connection and the hodograph are self-adjoint operators, but the ε-coefficients $\mathbb{B}$ and $\underline{\mathbb{B}}$ are not. The latter can be separated into Hermitian and anti-Hermitian parts:

$$\mathbb{B} = \mathbb{B}_\vee + i\mathbb{B}_\wedge, \qquad \underline{\mathbb{B}} = \underline{\mathbb{B}}_\vee + i\underline{\mathbb{B}}_\wedge. \tag{4.5}$$

The operators $\mathbb{B}_\wedge$ and $\underline{\mathbb{B}}_\wedge$ are explicitly determined by the x-coefficients as follows:

$$\mathbb{B}_\wedge = \frac{i}{4}\mathbb{A} * J\mathbb{A}, \qquad \underline{\mathbb{B}}_\wedge = \frac{i}{4}\underline{\mathbb{A}} * J\underline{\mathbb{A}}. \tag{4.6}$$

The Hermitian parts $\mathbb{B}_\vee$ and $\underline{\mathbb{B}}_\vee$ are related to the x-coefficients via the zero-curvature equation like (4.4). For instance, (4.4) implies the following equation

$$\frac{\partial}{\partial \varepsilon}\underline{\mathbb{A}}_j = D_j \underline{\mathbb{B}}_\vee - i[\underline{\mathbb{A}}_j \overset{*}{,} \underline{\mathbb{B}}_\vee] + \frac{1}{2}\langle\!\langle \underline{\mathbb{A}} \overset{*}{,} JD\underline{\mathbb{A}}_j \rangle\!\rangle. \tag{4.7}$$

Here we use notation (3.9).

Thus the zero-curvature equations are reduced to the ε-evolution system (4.7) for $\underline{\mathbb{A}}$ via $\underline{\mathbb{B}}_\vee$, and to the algebraic identity (4.6) for $\underline{\mathbb{B}}_\wedge$.

Recall that, for the whole $\underline{\mathbb{B}}$ (4.5) we have the basic homological equation (3.8). The anti-Hermitian part of this equation holds automatically in view of (4.6). The Hermitian part yields the equation for $\underline{\mathbb{B}}_\vee$ as in (3.10):

$$i[\mathbb{F}, \underline{\mathbb{B}}_\vee] = \beta_{\mathrm{sym}}(\underline{\mathbb{F}}, \underline{\mathbb{A}}, \underline{\mathbb{B}}_\vee) - \frac{\partial}{\partial \varepsilon}\underline{\mathbb{F}}, \tag{4.8}$$

where $\beta_{\mathrm{sym}} \overset{\mathrm{def}}{=} (\beta + \beta^*)/2$.

Thus we conclude that the complete ε-evolution system actually consists of two equations, (4.7)+(4.8).

5. INTEGRABILITY VIA ε-DYNAMICS

First, we resolve the homological equation (4.8) and separate the equation for $\underline{\mathbb{B}}_\vee$ from the equation for $\underline{\mathbb{F}}$ by using the following statement.

Lemma 5.1. *For any "fast" operator-valued family $\mathbb{Q}(x)$, the solutions $\mathbb{Q}^{\underline{\perp}}$ and $\mathbb{Q}^{\underline{\&}}$ of the homological equations*

$$i[f(\mathbb{S}_0(\underline{x}), x), \mathbb{Q}^{\underline{\perp}}(x)] = \mathbb{Q}(x) - \mathbb{Q}^{\underline{\&}}(x), \tag{5.1}$$

$$[\mathbb{S}_0(\underline{x}), \mathbb{Q}^{\underline{\&}}(x)] = 0 \tag{5.2}$$

are given by the formulas

$$\mathbb{Q}^{\underline{\&}}(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{it\mathbb{S}_0(\underline{x})}\mathbb{Q}(x)e^{-it\mathbb{S}_0(\underline{x})}\,dt, \tag{5.3}$$

$$\mathbb{Q}^{\underline{\perp}}(x) = \frac{1}{2\pi}\int_0^{2\pi} dt\sum_{m\neq 0} e^{-imt}e^{it\mathbb{S}_0(\underline{x})}\mathbb{Q}(x)e^{-it\mathbb{S}_0(\underline{x})}\times\Big(f(\mathbb{S}_0(\underline{x})+m, x) - f(\mathbb{S}_0(\underline{x}), x)\Big)^{-1}. \tag{5.4}$$

The operator $\mathbb{Q}^{\underline{\perp}}$ has an additional property:

$$(\mathbb{Q}^{\underline{\perp}})^{\underline{\&}} = 0. \tag{5.5}$$

This statement is an analog of Lemma A.1 in Appendix A and can be proved just by direct substitution of (5.3), (5.4) into (5.1), (5.2).

By applying Lemma 5.1 to Equation (4.8), we obtain two relations

$$\frac{\partial}{\partial\varepsilon}\underline{\mathbb{F}} = \beta_{\rm sym}(\underline{\mathbb{F}}, \underline{\mathbb{A}}, \underline{\mathbb{B}}_\vee)^{\underline{\&}}, \qquad \underline{\mathbb{B}}_\vee = \beta_{\rm sym}(\underline{\mathbb{F}}, \underline{\mathbb{A}}, \underline{\mathbb{B}}_\vee)^{\underline{\perp}}. \tag{5.6}$$

Now let us represent the right-hand sides of (5.6) in a more convenient form. Note that

$$\beta_{\rm sym}(\mathbb{G}, \mathbb{Z}, \mathbb{M}) = \underline{\alpha}(\mathbb{G}, \mathbb{Z}) + i\varepsilon[\mathbb{M} \,\text{⚲}\, \mathbb{G}],$$

where we denote

$$\underline{\alpha}(\mathbb{G}, \mathbb{Z}) \stackrel{\rm def}{=} \langle\!\langle \mathbb{Z} \,{}^*_{;}\, D\mathbb{G} + \frac{i}{2}[\mathbb{G} \,{}^*_{;}\, \mathbb{Z}]\rangle\!\rangle \tag{5.7}$$

Then property (5.5) reduces the first relation (5.6) to

$$\frac{\partial f}{\partial\varepsilon}(\mathbb{S}_0(\underline{x}), x) = \underline{\alpha}\big(f(\mathbb{S}_0(\underline{x}), x), \underline{\mathbb{A}}(x)\big)^{\underline{\&}}. \tag{5.8}$$

This is a dynamical equation for the function f. The role of "time" in this equation is played by the quantization (adiabaticity) parameter ε from the commutation relations (1.2).

The second relation in (5.6) can be written as follows:

$$\underline{\mathbb{B}}_\vee = \underline{\alpha}\big(f(\mathbb{S}_0(\underline{x}), x), \underline{\mathbb{A}}(x)\big)^{\underline{\perp}} + i\varepsilon[\underline{\mathbb{B}}_\vee(x)^{\underline{\perp}} \,\text{⚲}\, f(\mathbb{S}_0(\underline{x}), x)]. \tag{5.9}$$

Claim. The complete dynamical system consists of three equations (5.8), (5.9), and (4.7) for the unknowns f, $\underline{\mathbb{A}}$ and $\underline{\mathbb{B}}_\vee$. This system is *universal* and does not depend on the Hamiltonian or its eigenstates, etc.

The proof of the solvability of this system is equivalent to the proof of integrability of the original Hamiltonian $\hat{\mathbb{H}}$. Thus, we have reformulated the integrability problem in terms of dynamics in the ε-space.

The exact solvability of the obtained system is generally a very difficult question even for small "time" ε. Here we are interested only in the formal asymptotics as $\varepsilon \to 0$.

Theorem 5.1. *System* (5.8), (5.9), (4.7) *plus identity* (4.6) *explicitly generates the formal asymptotic expansion for the scalar adiabatic Hamiltonian in* (2.10)

$$f = f_0 + \varepsilon f' + \frac{\varepsilon^2}{2} f'' + \cdots , \tag{5.10}$$

as well as for the hodograph of the zero-curvature connection

$$\underline{\mathbb{A}} = \underline{\mathbb{A}}_0 + \varepsilon \underline{\mathbb{A}}' + \frac{\varepsilon^2}{2} \underline{\mathbb{A}}'' + \cdots , \qquad \underline{\mathbb{B}} = \underline{\mathbb{B}}_0 + \varepsilon \underline{\mathbb{B}}' + \frac{\varepsilon^2}{2} \underline{\mathbb{B}}'' + \cdots , \tag{5.10a}$$

starting from the "initial" data at $\varepsilon = 0$. *The "initial" data are taken from the information about the concrete Hamiltonian* $\mathbb{H}$. *Namely, the function* f_0 *is taken from* (2.1), *the* x*-component* $\underline{\mathbb{A}}_0$ *of the hodograph is determined by* $\mathbb{S}_0$ *via the parallel translation procedure as in Appendix A, and the* ε*-component* $\underline{\mathbb{B}}_0$ *of the hodograph is derived from* (5.9), (4.6) *as follows*:

$$\underline{\mathbb{B}}_0(x) = \underline{\alpha}_0\big(f_0(\mathbb{S}_0(\underline{x}), x), \underline{\mathbb{A}}_0(x)\big)^{\perp} + \frac{i}{4} \underline{\mathbb{A}}_0 J \underline{\mathbb{A}}_0. \tag{5.11}$$

Here, on the right-hand side, we use the $\varepsilon = 0$ *limit of* (5.7), *i.e.,*

$$\underline{\alpha}_0(\mathbb{G}, \mathbb{Z}) \stackrel{\text{def}}{=} \langle\!\langle \mathbb{Z}, D\mathbb{G} + \frac{i}{2}[\mathbb{G}, \mathbb{Z}] \rangle\!\rangle_0 \tag{5.12}$$

and the skew-product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_0$ *is the limit of* (3.9) *at* $\varepsilon = 0$:

$$\langle\!\langle \mathbb{Z}, \mathbb{Z}' \rangle\!\rangle_0 = \frac{1}{2}(\mathbb{Z} \cdot J\mathbb{Z}' + J\mathbb{Z}' \cdot \mathbb{Z}). \tag{5.13}$$

Now let us take into account that actually Eq. (3.13) for the operation $\mathcal{U}$ does not contain the anti-Hermitian part $\underline{\mathbb{B}}_\wedge$ and is reduced to

$$\frac{\partial}{\partial \varepsilon} \mathcal{U}(\underline{\mathbb{G}}) = \mathcal{U}\big(i[\underline{\mathbb{G}} \stackrel{*}{,} \underline{\mathbb{B}}_\vee] - \underline{\alpha}(\underline{\mathbb{G}}, \underline{\mathbb{A}})\big). \tag{5.14}$$

Once the asymptotics of the hodograph coefficients $\underline{\mathbb{A}}$ and $\underline{\mathbb{B}}$ are known, Equation (5.14) implies the asymptotics $\mathcal{U} = \mathcal{U}_0 + \varepsilon \mathcal{U}' + \frac{\varepsilon^2}{2} \mathcal{U}'' + \cdots$ very easily and explicitly.

Then the connection coefficients $\mathbb{A}$ and $\mathbb{B}$ can be obtained up to $O(\varepsilon^\infty)$ from (4.2), (5.10a) as well as the quantum action (2.8) and the quantum slow coordinates (2.9):

$$\mathbb{S}(x) = \mathcal{U}(\mathbb{S}_0(\underline{x})) = \mathbb{S}_0(x) + \varepsilon \mathbb{S}'(x) + \frac{\varepsilon^2}{2} \mathbb{S}''(x) + \cdots , \tag{5.15}$$

$$\mathbb{X}(x) = \mathcal{U}(x) = x + \varepsilon \mathbb{X}'(x) + \frac{\varepsilon^2}{2} \mathbb{X}''(x) + \cdots . \tag{5.16}$$

6. FIRST TERMS OF THE ASYMPTOTIC EXPANSIONS

Let us explicitly write out the two leading terms of the asymptotic expansions for all our objects.

At the limit $\varepsilon = 0$, the connection x-coefficients $\mathbb{A}_0$ are derived from $\mathbb{S}_0$ as it is shown in Appendix A, just by solving the homological equation

$$i[\mathbb{S}_0, \mathbb{A}_0] = D\mathbb{S}_0. \tag{6.0}$$

This can be done in the class of zero-curvature self-adjoint connections which is equivalent to the existence of the unitary operator family $\mathbb{U}_0$ (over the x-phase space) and of the unitary transformation $\mathbb{G} \longrightarrow \mathcal{U}_0(\mathbb{G}) = \mathbb{U}_0 \cdot \mathbb{G} \cdot \mathbb{U}_0^{-1}$ under the fast operator algebra (see Appendix A).

The ε-coefficient $\mathbb{B}_0 = \mathcal{U}_0(\underline{B}_0)$ can be obtained from (5.11). Let us note that if $\mathbb{G} = \mathcal{U}_0(\underline{\mathbb{G}})$, then

$$\mathcal{U}_0(\underline{\mathbb{G}}^{\&}) = \mathbb{G}^{\&}, \qquad \mathcal{U}_0(\underline{\mathbb{G}}^{\perp}) = \mathbb{G}^{\perp}, \tag{6.1}$$

where the operations & and $\perp$ are defined in the same way as in (5.3), (5.4), but using the operator $\mathbb{S}_0(x)$ instead of $\mathbb{S}_0(\underline{x})$. Also note that

$$\mathcal{U}_0(D\underline{\mathbb{G}}) = D\mathbb{G} + i[\mathbb{A}_0, \mathbb{G}], \tag{6.2}$$

and so

$$\alpha_0(\mathbb{G}) \stackrel{\text{def}}{=} \mathcal{U}_0(\underline{\alpha}_0(\underline{G}, \underline{A}_0)) = \langle\!\langle \mathbb{A}_0, D\mathbb{G} \rangle\!\rangle_0 + \frac{i}{2} \langle\!\langle \mathbb{A}_0, [\mathbb{A}_0, \mathbb{G}] \rangle\!\rangle_0. \tag{6.3}$$

Thus the transformation $\mathcal{U}_0$ produces, in fact, the inverse operation for hodograph translation from the fixed point $\underline{x}$ back to x at the "time" level $\varepsilon = 0$.

In particular, from (5.11), taking into account (2.1), we derive

$$\mathbb{B}_0 = \alpha_0(\mathbb{H})^{\perp} + \frac{i}{4} \mathbb{A}_0 J \mathbb{A}_0. \tag{6.4}$$

Now from Equation (5.14) for $\mathcal{U} = \mathcal{U}_0 + \varepsilon \mathcal{U}' + \cdots$, we obtain the first ε-correction to the leading unitary transformation $\mathcal{U}_0$ as follows:

$$\mathcal{U}'(\underline{\mathbb{G}}) = i[\mathbb{G}, \mathbb{B}_{0\vee}] - \alpha_0(\mathbb{G}). \tag{6.5}$$

In particular, in the expansion $\mathbb{A} = \mathcal{U}(\underline{\mathbb{A}}) = \mathbb{A}_0 + \varepsilon \mathbb{A}' + \frac{\varepsilon^2}{2} \mathbb{A}'' + \cdots$, from (6.5) we derive

$$\mathbb{A}' = i[\mathbb{A}_0, \mathbb{B}_{0\vee}] - \alpha_0(\mathbb{A}_0). \tag{6.6}$$

Similarly, in the expansion $\mathbb{B} = \mathcal{U}(\underline{\mathbb{B}}) = \mathbb{B}_0 + \varepsilon \mathbb{B}' + \frac{\varepsilon^2}{2} \mathbb{B}'' + \cdots$, we derive from (6.5)

$$\mathbb{B}' = i[\underline{\mathbb{B}}_0, \underline{\mathbb{B}}_{0\vee}] - \alpha_0(\mathbb{B}_0).$$

About the quantum-deformed slow coordinates $\mathbb{X}^j = \mathcal{U}(x^j)$ we first note that, in view of the commutation relations (1.2), there is an exact formula (see also Appendix B):

$$\mathbb{X}(x) = x + \varepsilon J \mathbb{A}(x). \tag{6.7}$$

Thus the asymptotic expansion (5.16) is derived just from the ε-expansion for the x-coefficients $\mathbb{A}$ of the connection as follows:

$$\mathbb{X}(x) = x + \varepsilon J \mathbb{A}_0(x) + \varepsilon^2 J \mathbb{A}'(x) + \cdots , \tag{6.7a}$$

where $\mathbb{A}'$ is given by (6.6).

The first ε-correction in the expansion (5.15) of the quantum adiabatic invariant is given by

$$\mathbb{S}' = \mathcal{U}'\big(\mathbb{S}_0(\underline{x})\big) = i[\mathbb{S}_0, \mathbb{B}_{0\vee}] + \frac{1}{2} \langle\!\langle D\mathbb{S}_0, \mathbb{A}_0 \rangle\!\rangle_0. \tag{6.8}$$

In the expansion (1.4) for the adiabatic terms $\Lambda_k(x)$, we have $\lambda_k(x) = f_0(k, x)$. The first ε-correction $\mu_k(x) = f'(k, x)$ can be obtained from (5.8), (5.10) at $\varepsilon = 0$:

$$\mu_k(x) = \underline{\alpha}_0\big(f_0(\mathbb{S}_0(\underline{x}), x), \underline{\mathbb{A}}_0(x)\big)^{\underline{\&}}\Big|_{k\text{th eigenspace of } \mathbb{S}_0(\underline{x})}. \tag{6.9}$$

Relation (5.8) for f' at $\varepsilon = 0$ can also be rewritten by applying the transformation $\mathcal{U}_0$ to both of its sides and by using the identities (6.1)–(6.3). The resulting formula is the following one:

$$\mu_k = \alpha_0(\mathbb{H})^{\&}_k, \tag{6.10}$$

where the lower index k means the restriction onto the kth eigenspace of $\mathbb{H}$.

Now let us take into account relation (6.0), which implies $D\mathbb{H} = Df_0(\mathbb{S}_0, \cdot) + i[\mathbb{H}, \mathbb{A}_0]$. Then (6.10) is transformed to

$$\mu_k = -\langle \mathbb{A}^{\&}_{0k}, \mathrm{ad}(\lambda_k)\rangle - \frac{i}{2}\big([\mathbb{H}, \mathbb{A}_0]J\mathbb{A}_0\big)^{\&}_k. \tag{6.11}$$

Here by $\mathrm{ad}(\lambda_k)$ we denote the Hamiltonian vector field corresponding to the leading adiabatic term λ_k on the slow phase space.

The first summand on the right-hand side of (6.11) represents the Berry contribution generating the "geometric phase" factor

$$\exp\Big\{i\int_{x(0)}^{x(t)} \mathbb{A}^{\&}_{0k}\, dx\Big\}$$

along the trajectory $x(t)$ of the vector field $\mathrm{ad}(\lambda_k)$. This first summand in the effective slow Hamiltonian uses only the commutant part $\mathbb{A}^{\&}_0$ from the splitting (A.6) of the zero-curvature connection. The second summand in (6.11) actually depends on the part $\tilde{\mathbb{A}}_0$ from the splitting (which is orthogonal to the commutant of $\mathbb{S}_0$) since

$$\big([\mathbb{H}, \mathbb{A}_0]J\mathbb{A}_0\big)^{\&} = \big([\mathbb{H}, \tilde{\mathbb{A}}_0]J\tilde{\mathbb{A}}_0\big)^{\&}. \tag{6.12}$$

Some formulas for this summand (called the "no name" term) were obtained in different situations in [16, 25–27].

7. SPECTRAL DEGENERACY AND ADIABATIC DEFORMATION OF FAST SYMMETRIES

Now one can omit the nondegeneracy condition for the eigenvalues of the operator-valued symbol $\mathbb{H} = \mathbb{H}(x)$. Namely, let us assume that $\mathbb{H}$ is *super-integrable*, i.e., it is represented in the form (2.1) and the commutant of the "action" operator $\mathbb{S}_0$ has a noncommutative[1] algebra whose generators $\mathbb{L}^\gamma_0 = \mathbb{L}^\gamma_0(x)$ obey the relations

$$[\mathbb{S}_0(x), \mathbb{L}^\gamma_0(x)] = 0, \qquad [\mathbb{L}^\gamma_0(x), \mathbb{L}^\rho_0(x)] = -i\Psi^{\gamma\rho}(\mathbb{L}_0(x)). \tag{7.1}$$

In the last relations, in the right-hand side, the components of the set $\mathbb{L}_0$ are assumed to be Weyl-symmetrized.

The nonzero rank of the tensor $((\Psi^{\gamma\rho}))$ in (7.1) provides a nontrivial degeneracy $M_k > 1$ of the eigenvalue $\lambda_k(x)$.

Of course, the concrete realization of the symmetry algebra (7.1) is defined up to a gauge transformation only. Therefore, one can choose the following special gauge (see also Appendix A):

$$\mathbb{L}^\gamma_0 = \mathcal{U}_0(\mathbb{L}^\gamma_0(\underline{x})) \quad \text{or} \quad \mathbb{L}^\gamma_0(x) = \mathbb{U}_0(x)\cdot\mathbb{L}^\gamma_0(\underline{x})\cdot\mathbb{U}_0(x)^{-1}, \quad x \in \mathcal{D}. \tag{7.2}$$

In order to produce the quantization $x \to \hat{x}$, we construct, up to $O(\varepsilon^\infty)$, the adiabatically deformed symmetries $\mathbb{L}^\gamma = \mathbb{L}^\gamma_\varepsilon(x)$ by using the transformation $\mathcal{U}$ (3.12), i.e., via the following formula

$$\mathbb{L}^\gamma = \mathcal{U}(\mathbb{L}^\gamma_0(\underline{x})). \tag{7.3}$$

[1] For this reason, the diagonalization method used in the matrix approximation scheme [18, 19] does not work in the case of degenerate eigenvalues λ_k: the commutant of a diagonal matrix contains not only diagonal matrices in this case.

Thus $\hat{\mathbb{L}}^\gamma = \hat{\mathbb{U}} \cdot \mathbb{L}_0^\gamma(\underline{x}) \cdot \hat{\mathbb{U}}^{-1}$, and in view of (2.6), (2.8), (2.9), we have the commutation relations

$$[\hat{\mathbb{H}}, \hat{\mathbb{L}}^\gamma] = 0, \qquad [\hat{\mathbb{S}}, \hat{\mathbb{L}}^\gamma] = 0, \qquad [\hat{\mathbb{X}}, \hat{\mathbb{L}}^\gamma] = 0. \tag{7.4}$$

Since one knows the complete asymptotic expansion as $\varepsilon \to 0$ for the transformation $\mathcal{U} = \mathcal{U}_\varepsilon$, formula (7.3) provides the asymptotic expansion for $\mathbb{L}^\gamma$.

Then we can try to apply the same "dynamical" scheme of the adiabatic approximation as in the above sections. The goal is to obtain, up to $O(\varepsilon^\infty)$, a representation like (2.10) and commutation relations (2.11), (2.12).

The critical place in this approach is the equation for $\underline{\mathbb{F}}$ in (5.6). The right-hand side of this equation (see it also in Eq. (5.8)) is an operator commuting with $\mathbb{S}_0(\underline{x})$. The last fact is due to the averaging operation $\underline{\&}$ (5.3) staying in this right-hand side. But now, in the presence of the symmetry algebra (7.1), one cannot claim that if an operator commutes with $\mathbb{S}_0(\underline{x})$, then it must be a function in $\mathbb{S}_0(\underline{x})$. Now such an operator may be a function in the generators $\mathbb{L}_0^\gamma(\underline{x})$ as well.

Thus from the first equation in (5.6), one can reconstruct $\underline{\mathbb{F}}$ only as a function

$$\underline{\mathbb{F}} = f\big(\mathbb{S}_0(\underline{x}), \mathbb{L}_0(\underline{x}), x\big), \qquad f = f_\varepsilon(s, l, x). \tag{7.5}$$

Therefore, Eq. (5.8) is now replaced by the following one:

$$\frac{\partial f}{\partial \varepsilon}\big(\mathbb{S}_0(\underline{x}), \mathbb{L}_0(\underline{x}), x\big) = \underline{\alpha}\Big(f\big(\mathbb{S}_0(\underline{x}), \mathbb{L}_0(\underline{x}), x\big), \underline{\mathbb{A}}_0(x)\Big)^{\underline{\&}}. \tag{7.6}$$

But the initial data for f is still $f|_{\varepsilon=0} = f_0(s, x)$, i.e., at $\varepsilon = 0$, as follows from (2.5), it is a function in s, x but not in the symmetry algebra arguments l^γ.

In the same way, we now have to change the arguments of f in (5.9), (5.10).

After solving up to $O(\varepsilon^\infty)$ the ε-dynamical system (5.8), (5.9), (4.7) with new arguments of f, we obtain not (2.10), but

$$\hat{\mathbb{H}} = f(\hat{\mathbb{S}}, \hat{\mathbb{L}}, \hat{\mathbb{X}}), \qquad f = f_0(s, x) + \varepsilon f'(s, l, x) + \cdots \tag{7.7}$$

plus relations (2.11), (2.12). The first ε-correction f' in (7.7) is derived from the identity

$$f'\big(\mathbb{S}_0(\underline{x}), \mathbb{L}_0(\underline{x}), x\big) = \underline{\alpha}\big(f_0(\mathbb{S}_0(\underline{x}), x), \underline{\mathbb{A}}_0(x)\big)^{\underline{\&}}. \tag{7.8}$$

By applying the transformation $\mathcal{U}_0$ to both sides of (7.8), one can rewrite this relation as follows:

$$f'(\mathbb{S}_0(x), \mathbb{L}_0(x), x) = \mathbb{A}_0^{\&}(x) \cdot JDf_0\big(\mathbb{S}_0(x), x\big) - \frac{i}{2}\big([\mathbb{H}(x), \mathbb{A}_0(x)]J\mathbb{A}_0(x)\big)^{\&}. \tag{7.9}$$

Conclusion. In the presence of a noncommutative symmetry algebra of the fast operator-valued symbol $\mathbb{H}$, we have the asymptotic representation (7.7) for the Hamiltonian $\hat{\mathbb{H}}$. This representation reduces the Hamiltonian to the scalar term at the zero order in ε. But in higher ε-orders, there still remains the dependence in fast symmetry generators:

$$\hat{\mathbb{H}} = f_0(\hat{\mathbb{S}}, \hat{\mathbb{X}}) + \varepsilon f'(\hat{\mathbb{S}}, \hat{\mathbb{L}}, \hat{\mathbb{X}}) + O(\varepsilon^2). \tag{7.10}$$

After restricting (7.10) to the kth eigensubspace of $\hat{\mathbb{S}}$, we obtain

$$\hat{\mathbb{H}}\Big|_{k\text{th subspace}} = \lambda_k(\hat{x}') + \varepsilon\mu_k(\hat{\mathbb{L}}', \hat{x}') + O(\varepsilon^2), \tag{7.11}$$

where $\mu_k(l, x) \stackrel{\text{def}}{=} f'(k, l, x)$, $\hat{x}' \stackrel{\text{def}}{=} \hat{\mathbb{X}}\big|_{k\text{th subspace}}$, $\hat{\mathbb{L}}' \stackrel{\text{def}}{=} \hat{\mathbb{L}}\Big|_{k\text{th subspace}}$, and the commutations relations repeat (7.1), (7.4), (2.11):

$$[\hat{\mathbb{L}}'^{\gamma}, \hat{\mathbb{L}}'^{\rho}] = -i\Psi^{\gamma\rho}(\hat{\mathbb{L}}'), \qquad [\hat{\mathbb{L}}'^{\gamma}, \hat{x}'^{j}] = 0, \qquad [\hat{x}'^{j}, \hat{x}'^{l}] = -i\varepsilon J^{jl}. \tag{7.12}$$

In order to eliminate the fast-coordinate dependence from the Hamiltonian (7.11), at least up to $O(\varepsilon^2)$, one needs information about the trajectories of the vector field $\mathrm{ad}(\lambda_k)$ on the slow phase space $\mathcal{D}$.

For instance, let us assume that all these trajectories $x(t)$ are periodic. Then the fast operator corresponding to (7.11),

$$-i\frac{d}{dt} + \mu_k\big(\mathbb{L}_0(\underline{x}), x(t)\big), \tag{7.13}$$

will act in the space of "fast" eigenvectors of $\mathbb{S}_0(\underline{x})$ periodic in the t-coordinate. Let us consider the monodromy operator for (7.13) and compute its eigenvalues, i.e., determine the Floquet multipliers $m_k^{(j)}$, $j = 1, \dots, M_k$. These multipliers will be functions in the Hamiltonian λ_k, as well as in its symmetries $\varkappa^1, \dots, \varkappa^{2N-1}$, i.e.,

$$m_k^{(j)} = m_k^{(j)}(\lambda_k, \varkappa), \qquad \{\lambda_k, \varkappa^{(l)}\} = 0, \qquad \{\varkappa^{(l)}, \varkappa^{(m)}\} = \Phi^{lm}(\varkappa). \tag{7.14}$$

In this way, we reduce the operator (7.11) to the series of "slow" Hamiltonians

$$\hat{\lambda}_k + \varepsilon m_k^{(j)}(\hat{\lambda}_k, \hat{\varkappa}) + O(\varepsilon^2) \tag{7.15}$$

over the algebra with commutation relations

$$[\hat{\varkappa}^{(l)}, \hat{\varkappa}^{(m)}] = -i\varepsilon\Phi^{lm}(\hat{\varkappa}) + O(\varepsilon^3), \qquad [\hat{\lambda}_k, \hat{\varkappa}^{(l)}] = O(\varepsilon^3). \tag{7.16}$$

Thus, in the degenerate case, the *first ε-correction in the effective slow Hamiltonian* (7.15) *is determined by the Floquet multipliers of the periodic operator* (7.13) *related to* (7.11).

8. ADIABATIC APPROXIMATION IN SEMICLASSICAL AND CLASSICAL MECHANICS

We can extend the above scheme to the case where the fast operator algebra is represented by functions in generators $\hat{y}^\alpha$ whose commutation relations contain a small semiclassical parameter $\hbar$, namely,

$$[\hat{y}^\alpha, \hat{y}^\beta] = -i\hbar\mathcal{J}^{\alpha\beta}, \qquad \mathcal{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{8.1}$$

At the same time, the commutation relations (1.2) between slow coordinates are replaced by

$$[\hat{x}^j, \hat{x}^l] = -i\hbar\varepsilon J^{jl}. \tag{8.2}$$

Let us consider the Hamiltonian $\hat{\mathbb{H}} = \mathbb{H}(\hat{x})$ with $\hat{x}^j$ obeying the slow relations (8.2) and with the operator-valued symbol $\mathbb{H}(x) = H(\hat{y}; x)$, where the fast coordinates $\hat{y}^\alpha$ obey relations (8.1). Thus, the function $H = H(y; x)$ is a scalar Hamiltonian on the phase space product $\mathbb{R}^{2N} \times \mathcal{D}$ with the Poisson brackets

$$\{\cdot,\cdot\}_\varepsilon = \{\cdot,\cdot\}_0 + \varepsilon\{\cdot,\cdot\}, \tag{8.3}$$

where the brackets $\{\cdot,\cdot\}_0$ over $\mathbb{R}^{2N}$ are generated by the tensor $\mathcal{J}$ from (8.1) and $\{\cdot,\cdot\}$ are brackets (1.1) over $\mathcal{D} \subset \mathbb{R}^{2n}$.

We assume that, for any fixed x, the Hamiltonian $H(\hat{y}; x)$ is super-integrable and can be represented via an action operator $\mathbb{S}_0(x) \stackrel{\text{def}}{=} S_0(\hat{y}; x)$ as in (2.1):

$$H(\hat{y}; x) = f_0(\mathbb{S}_0(x), x). \tag{8.4}$$

Then the semiclassical adiabatic approximation problem is formulated as follows: to find a symbol

$$f = f_\varepsilon(s,l,x) = f_0(s,x) + \varepsilon f'(s,l,x) + \cdots, \tag{8.5}$$

as well as the symbols $S = S_\varepsilon(y;x)$, $L = L_\varepsilon(y;x)$, $X = X_\varepsilon(y;x)$ up to $O(\varepsilon^\infty)$, in such a way that the operators

$$\hat{\mathbb{S}} = S_\varepsilon(\hat{y};\hat{x}), \qquad \hat{\mathbb{L}} = L_\varepsilon(\hat{y};\hat{x}), \qquad \hat{\mathbb{X}} = X_\varepsilon(\hat{y};\hat{x}) \tag{8.6}$$

satisfy the relations

$$\begin{aligned} &[\hat{\mathbb{S}}, \hat{\mathbb{L}}^\gamma] = 0, \qquad [\hat{\mathbb{L}}^\gamma, \hat{\mathbb{L}}^\rho] = -i\hbar\Psi^{\gamma\rho}(\hat{\mathbb{L}}),\\ &[\hat{\mathbb{S}}, \hat{\mathbb{X}}^j] = [\hat{\mathbb{L}}^\gamma, \hat{\mathbb{X}}^j] = 0, \qquad [\hat{\mathbb{X}}^j, \hat{\mathbb{X}}^l] = -i\hbar J^{jl} \end{aligned} \tag{8.7}$$

and represent the Hamiltonian $\hat{H} = H(\hat{y};\hat{x})$ up to $O(\varepsilon^\infty)$ by the formula

$$\hat{H} = f(\hat{\mathbb{S}}, \hat{\mathbb{L}}, \hat{\mathbb{X}}). \tag{8.8}$$

The solution of this $\hbar$-scaled adiabatic problem is described as in the above sections by making some obvious renotation.

Now in the obtained $\hbar$-scaled formulas, let us pass to the limit $\hbar = 0$ taking into account that the commutator $\frac{i}{\hbar}[\cdot,\cdot]$ becomes the Poisson brackets (8.3) at $\hbar = 0$. In this way, one can derive an *algorithm for adiabatic approximation in the classical mechanics setting.*

As we saw above, this algorithm is based on three basic ideas:

(i) hodograph translation over x-phase space and over ε-parameter space;

(ii) zero-curvature equations over (ε, x)-space;

(iii) dynamics by ε as "time" coordinate.

The introduced combination of these ideas probably is new, although each of them, if taking separately, looks somewhat familiar in the theory of integrable and nearly integrable equations [28–30].

Let us formulate our final algorithm for the adiabatic approximation in Hamiltonian mechanics.

Over the fast–slow space with brackets (8.3), one has a Hamiltonian $\mathcal{H} = \mathcal{H}(y;x)$ possessing the zero-order adiabatic invariant $\mathcal{S}_0$. This means that

$$\mathcal{H}(y;x) = f_0(\mathcal{S}_0(y;x);x), \tag{8.9}$$

and for any fixed x, all trajectories of $\mathcal{S}_0(y;x)$ in the fast y-space are 2π-periodic. For some frozen $\underline{x}$, denote by $((\underline{\mathcal{L}}_0^\gamma))$ a basis of symmetries of $\underline{\mathcal{S}}_0 \overset{\text{def}}{=} \mathcal{S}(\cdot;\underline{x})$ over the y-space, i.e.,

$$\{\underline{\mathcal{S}}_0, \underline{\mathcal{L}}_0^\gamma\}_0 = 0, \qquad \{\underline{\mathcal{L}}_0^\gamma, \underline{\mathcal{L}}_0^\rho\}_0 = \Psi^{\gamma\rho}(\underline{\mathcal{L}}_0). \tag{8.10}$$

The problem is the following: *to construct f, $\mathcal{S}$, $\mathcal{L}^\gamma$, $\mathcal{X}^j$ as power series in ε in such a way that*

$$\mathcal{H} = f(\mathcal{S}, \mathcal{L}, \mathcal{X}), \tag{8.11}$$

and so that the following relations be valid, up to $O(\varepsilon^\infty)$:

$$\begin{aligned} &\{\mathcal{X}^j\mathcal{X}^l\}_\varepsilon = \varepsilon J^{jl}, \qquad \{\mathcal{S}, \mathcal{X}^j\}_\varepsilon = \{\mathcal{L}^\gamma, \mathcal{X}^j\}_\varepsilon = 0,\\ &\{\mathcal{S}, \mathcal{L}^\gamma\}_\varepsilon = 0, \qquad \{\mathcal{L}^\gamma, \mathcal{L}^\rho\}_\varepsilon = \Psi^{\gamma\rho}(\mathcal{L}). \end{aligned} \tag{8.12}$$

In particular, it follows from (8.11), (8.12) that the function $\mathcal{S}$ is in involution with the Hamiltonian:

$$\{\mathcal{H}, \mathcal{S}\}_\varepsilon = 0, \tag{8.13}$$

and thus $\mathcal{S}$ is the *adiabatic invariant for* $\mathcal{H}$ *up to* $O(\varepsilon^\infty)$.

Our algorithm consists of the following steps:

(1) to find the "initial data" $\underline{\mathcal{A}}_0$, $\underline{\mathcal{B}}_0$ at $\varepsilon = 0$ for hodograph coefficients of a zero-curvature connection over (ε, x)-space;

(2) to write out and asymptotically solve the ε-dynamical system for the function f of the type (8.5), as well as for the hodograph coefficients $\underline{\mathcal{A}}$, $\underline{\mathcal{B}}$;

(3) by using $\underline{\mathcal{A}}$, $\underline{\mathcal{B}}$ to write out and asymptotically solve the ε-dynamical system for the canonical transformation $g = g_\varepsilon$ which generates the functions $\mathcal{S}$, $\mathcal{L}^\gamma$, $\mathcal{X}^j$ from their "initial data" at $\varepsilon = 0$ and $x = \underline{x}$:

$$\mathcal{S} = g^* \underline{\mathcal{S}}_0, \qquad \mathcal{L}^\gamma = g^* \underline{\mathcal{L}}_0^\gamma, \qquad \mathcal{X}^j = g^* x^j. \tag{8.14}$$

Let us carry out these steps.

Denote by $\underline{Y}(y, \tau)$ the 2π-periodic in τ trajectories of $\underline{\mathcal{S}}_0$, and introduce the averaging operation

$$\varphi(y) \to \langle \varphi \rangle (y) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} \varphi\big(\underline{Y}(y, \tau)\big)\, d\tau, \tag{8.15}$$

as well as the "integration" operation

$$\varphi(y) \to \mathbf{I}\big(\varphi(y)\big) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} \varphi\big(\underline{Y}(y, \tau)\big)(\tau - \pi)\, d\tau. \tag{8.16}$$

These two operations are the classical analogs of the operations & and # used in Appendix A and in Lemma 5.1.

By solving the equation $\{\mathcal{S}_0, \mathcal{A}_0\}_0 = D\mathcal{S}_0$, we find the "free" connection form $\mathcal{A}_0 = \mathcal{A}_{0j}\, dx^j$ in the zero-curvature class:

$$D_j \mathcal{A}_{0l} - D_l \mathcal{A}_{0j} + \{\mathcal{A}_{0j}, \mathcal{A}_{0l}\}_0 = 0, \qquad j, l = 1, \dots, 2n. \tag{8.17}$$

This can be done by the splitting $\mathcal{A}_0 = \tilde{\mathcal{A}}_0 + \langle \mathcal{A}_0 \rangle$ in which $\tilde{\mathcal{A}}_0 = \mathbf{I}(D\mathcal{S}_0)$ and $\langle \mathcal{A}_0 \rangle = a_0(\mathcal{S}_0, \mathcal{L}_0, \cdot)$, where a_0 is a solution of $\delta a_0 = -\tilde{c}_0$ with $\tilde{c}_0(\mathcal{S}_0, \mathcal{L}_0, \cdot) = \langle \{\tilde{\mathcal{A}}_0, \tilde{\mathcal{A}}_0\}_0 \rangle$ (for details, see Appendix A).

Note that in this splitting the first summand $\tilde{\mathcal{A}}_0$ and its curvature $\tilde{c}_0$ are related to the classical Hannay angle [8, 13], while the second summand a_0 is related to the quantum Berry phase in (6.12),since the classical limit of $\mathbb{A}_0^{\&}$ is just $\langle \mathcal{A}_0 \rangle$. The mentioned above condition $\delta a_0 = -\tilde{c}_0$ represents the relationship between Hannay and Berry contributions (see also in [7]), but the fact that this condition actually establishes *vanishing of the curvature for the complete "free" connection* $\mathcal{A}_0$ probably was missed.

In view of (8.17) one can correctly define the multi–Hamiltonian flow $y \to Y_0$ generated by $\mathcal{A}_0(\cdot, x)$ in which the slow coordinate x plays the role of "multi–time" with the "initial data" $Y_0\big|_{x=\underline{x}} = y$. We index this flow by $Y_0 \stackrel{\text{def}}{=} Y_0^{x,\underline{x}}$ and refer to it as to *free translation*. This family of mappings translates the fast y–fiber over a fixed $\underline{x}$ onto fast y–fibers over all x, and obeys the relation $Y_0^{x,\underline{x}} = \big(Y_0^{\underline{x},x}\big)^{-1}$. Any function $\underline{f}$ over $\underline{x}$ can be distributed to fast fibers over all x by means of $f \stackrel{\text{def}}{=} \big(Y_0^{\underline{x},x}\big)^* \underline{f}$ which determines a parallel section with respect to our "free" connection: $Df + \mathcal{A}_0, f_0 = 0$.

Lemma. *The zero–order adiabatic invariant as well as its symmetries are consistent with free translations, i.e.,*

$$\mathcal{S}_0\big(Y_0^{x,\underline{x}}, x\big) = \mathcal{S}_0(y, \underline{x}), \mathcal{L}_0\big(Y_0^{x,\underline{x}}, x\big) = \mathcal{L}_0(y, \underline{x}). \tag{8.18}$$

Free translations acting as symplectic mappings between fast fibers transform the family of energy levels and the fast flow of $\mathcal{S}_0$ *over a fixed* $\underline{x}$ *to the family of energy levels and the fast flow of* $\mathcal{S}_0(\cdot, x)$ *over each* $x \in \mathcal{D}$.

Now one can define the mapping of the whole slow–fast bundle

$$g_0 \colon (y, x) \to (\mathcal{Y}_0(y, x), x), \qquad \mathcal{Y}_0(y, x) \stackrel{\text{def}}{=} Y_0^{\underline{x},x}(y). \tag{8.19}$$

This mapping is canonical with respect to brackets (8.3) at the "time" level $\varepsilon = 0$. This is the classical analog of the quantum mapping $\mathcal{U}_0$. Then we choose $\underline{\mathcal{A}}_0 \stackrel{\text{def}}{=} g_0^{-1*}\mathcal{A}_0$ or $\underline{\mathcal{A}}_0(y,x) \stackrel{\text{def}}{=} CalA_0(Y_0^{x,\underline{x}}(y),x)$. This is the classical analog of the quantum hodograph $\underline{\mathbb{A}}_0 = \mathcal{U}_0^{-1}(\mathbb{A}_0)$ (see Appendix A).

Another hodograph coefficient at $\varepsilon = 0$ is determined by a formula similar to (5.11):

$$\underline{\mathcal{B}}_0 = \underline{\omega}_0^{-1}\mathbf{\underline{I}}\Big(\underline{\mathcal{A}}_0 JD\underline{f}_0 + \frac{1}{2}\underline{\mathcal{A}}_0 J\{\underline{f}_0, \underline{\mathcal{A}}_0\}_0\Big), \tag{8.20}$$

where $\underline{f}_0$ and $\underline{\omega}_0$ are related to f_0 as follows:

$$\underline{f}_0 \stackrel{\text{def}}{=} f_0\big(\underline{\mathcal{S}}_0(y),x\big), \qquad \underline{\omega}_0 \stackrel{\text{def}}{=} \partial f_0\big(\underline{\mathcal{S}}_0(y),x\big). \tag{8.21}$$

At the second step, let us write out the ε-dynamical system of f, $\underline{\mathcal{B}}$, $\underline{\mathcal{A}}$ (an analog of (5.8), (5.9), (4.7)):

$$\begin{gathered}\frac{\partial}{\partial\varepsilon}\underline{f} = \Big\langle \underline{\mathcal{A}}JD\underline{f} + \frac{1}{2}\underline{\mathcal{A}}J\{\underline{f},\underline{\mathcal{A}}\}_\varepsilon\Big\rangle,\\ \underline{\mathcal{B}} = \underline{\omega}^{-1}\mathbf{\underline{I}}\big(\underline{\mathcal{A}}JD\underline{f} + \frac{1}{2}\underline{\mathcal{A}}J\{\underline{f},\underline{\mathcal{A}}\}_\varepsilon\big) - \varepsilon\underline{\omega}^{-1}\{\underline{f},\mathbf{\underline{I}}(\underline{\mathcal{B}})\},\\ \frac{\partial}{\partial\varepsilon}\underline{\mathcal{A}}_j = D_j\underline{\mathcal{B}} - \{\underline{\mathcal{A}}_j,\underline{\mathcal{B}}\}_\varepsilon + \frac{1}{2}\underline{\mathcal{A}}JD\underline{\mathcal{A}}_j.\end{gathered} \tag{8.22}$$

Here $\underline{f}$ and $\underline{\omega}$ are related to the unknown f similarly to (8.21), i.e., $\underline{f} \stackrel{\text{def}}{=} f(\underline{\mathcal{S}}_0(y),x)$ and $\underline{\omega} \stackrel{\text{def}}{=} \partial f(\underline{\mathcal{S}}_0(y),x)$.

Starting from the "initial" data at $\varepsilon = 0$,

$$f|_{\varepsilon=0} = f_0 \quad (8.21), \qquad \underline{\mathcal{B}}|_{\varepsilon=0} = \underline{\mathcal{B}}_0 \quad (8.20), \qquad \underline{\mathcal{A}}|_{\varepsilon=0} = \underline{\mathcal{A}}_0,$$

we compute from (8.22) the complete asymptotic expansions for f, $\underline{\mathcal{B}}$, $\underline{\mathcal{A}}$ as $\varepsilon \to 0$.

At the third step, we write out the ε-dynamical equation for the canonical mapping g which is the classical analog of the transformation $\mathcal{U}$ (3.12) at $\hbar = 0$. The "initial" data for this mapping is $g|_{\varepsilon=0} = g_0$ (8.19). From Eq. (3.13), we derive

$$g(y,x) = \big(\mathcal{Y}(y,x), \mathcal{X}(y,x)\big), \tag{8.23}$$

where

$$\frac{\partial}{\partial\varepsilon}\mathcal{Y} = \mathcal{J}\varkappa(\mathcal{Y},\mathcal{X}), \qquad \mathcal{Y}\Big|_{\varepsilon=0} = \mathcal{Y}_0(y,x), \quad \frac{\partial}{\partial\varepsilon}\mathcal{X} = J\theta(\mathcal{Y},\mathcal{X}), \qquad \mathcal{X}\Big|_{\varepsilon=0} = x, \tag{8.24}$$

and the 1-forms $\varkappa$, θ are defined by

$$\varkappa \stackrel{\text{def}}{=} \frac{1}{2}\partial\underline{\mathcal{A}}_j/\partial y \cdot J^{jl}\underline{\mathcal{A}}_l + \partial\underline{\mathcal{B}}/\partial y, \quad \theta \stackrel{\text{def}}{=} \underline{\mathcal{A}} + \frac{\varepsilon}{2}D\underline{\mathcal{A}}_j J^{jl}\underline{\mathcal{A}}_l + \varepsilon D\underline{\mathcal{B}}. \tag{8.25}$$

After solving (8.24) asymptotically as $\varepsilon \to 0$, we then apply (8.14) and find

$$\begin{gathered}\mathcal{S} = \underline{\mathcal{S}}_0(\mathcal{Y}) \quad \text{or} \quad \mathcal{S}(y;x) = \mathcal{S}_0(\mathcal{Y}(y,x);\underline{x}),\\ \mathcal{L}^\gamma = \underline{\mathcal{L}}_0^\gamma(\mathcal{Y}) \quad \text{or} \quad \mathcal{L}^\gamma(y;x) = \mathcal{L}_0(\mathcal{Y}(y,x);\underline{x}).\end{gathered} \tag{8.26}$$

This completes the algorithm.

Remark. Formulas (8.26) use only the $\mathcal{Y}$–part of the solution of the system (8.24). Another, i.e.,$\mathcal{X}$–part of this solution generates the guiding center coordinates $\mathcal{X}^j$ which meet canonical relations (8.12). Note that new fast coordinates $\mathcal{Y}$ are also canonical:

$$\{\mathcal{Y}^j\mathcal{Y}^l\}_\varepsilon = \varepsilon\mathcal{J}^{jl}, \qquad \{\mathcal{Y},\mathcal{X}\}_\varepsilon = 0.$$

We can determine the *curvature-free adiabatic connection* in the classical framework analogously to its quantum version (4.2), namely: $\mathcal{A} = \underline{\mathcal{A}}(\mathcal{Y},\mathcal{X})$. It is related to the slow (guiding center) coordinates and to new fast coordinates as follows:

$$\mathcal{X} = x + \varepsilon J\mathcal{A}, \qquad D\mathcal{Y} + \{\mathcal{A},\mathcal{Y}\}_\varepsilon = 0.$$

The connection $\mathcal{A}$ generates free translations $Y^{x,\underline{x}}$ by the same way as our “free” connection $\mathcal{A}_0$ generates the translations $Y_0^{x,\underline{x}}$ at the level $\varepsilon = 0$ (see above).

Note that (8.11),(8.14) imply $g^{-1*}\mathcal{H} = f(\underline{\mathcal{S}}_0, \underline{\mathcal{L}}_0^\gamma, x)$. Thus the dynamics on the slow phase space is generated by the vector field $JDf = g^{-1*}J\nabla^{cl}\mathcal{H}$. Here $\nabla^{cl} \overset{\text{def}}{=} D + \{\mathcal{A},\cdot\}_\varepsilon$ is the classical version of the quantum covariant derivative ∇ (see Appendix A). Therefore one can say that the curvature-free adiabatic connection $\mathcal{A}$ controls the *adiabatic inertia* of the system. Indeed: the condition of total inertial behavior $JDf = 0$ is equivalent to the condition $\nabla^{cl}\mathcal{H} = 0$.

1. APPENDIX A: SPLITTING OF FREE ADIABATIC CONNECTION

Let us introduce two operators & and # in the following way:

$$\begin{aligned} \mathbb{Q}^{\&} &\overset{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} e^{it\mathbb{S}_0}\cdot\mathbb{Q}\cdot e^{-it\mathbb{S}_0}\,dt,\\ \mathbb{Q}^{\#} &\overset{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} e^{it\mathbb{S}_0}\cdot\mathbb{Q}\cdot e^{-it\mathbb{S}_0}(t-\pi)\,dt. \end{aligned} \tag{A.1}$$

Lemma A.1. *Formulas* (A.1) *generate the solution of the homological equations*

$$i[\mathbb{S}_0,\mathbb{Q}^{\#}] = \mathbb{Q} - \mathbb{Q}^{\&}, \qquad [\mathbb{S}_0,\mathbb{Q}^{\&}] = 0 \tag{A.2}$$

with the property

$$(\mathbb{Q}^{\#})^{\&} = 0. \tag{A.3}$$

Lemma A.2. *The periodicity condition* (2.2) *implies*

$$(D\mathbb{S}_0)^{\&} = 0. \tag{A.4}$$

Corollary A.1. *The homological equation* (6.0), *i.e.,*

$$iD\mathbb{S}_0 = [\mathbb{A}_0,\mathbb{S}_0] \tag{A.5}$$

can be resolved as follows:

$$\mathbb{A}_0 = \mathbb{A}_0^{\&} + \tilde{\mathbb{A}}_0, \tag{A.6}$$

where

$$\tilde{\mathbb{A}}_0 = (D\mathbb{S}_0)^{\#} \tag{A.7}$$

and $\mathbb{A}_0^{\&}$ *commutes with* $\mathbb{S}_0$, *i.e.,*

$$[\mathbb{S}_0,\mathbb{A}_0^{\&}] = 0. \tag{A.8}$$

A splitting (A.6) corresponds to the decomposition of the fast operator algebra into the commutant of $\mathbb{S}_0$ and its orthogonal complement (with respect to the trace scalar product). The operation & is the projection onto the commutant which coincides with the symmetry algebra of the operator-valued symbol $\mathbb{H}$.

Note that, since $\mathbb{A}_0^{\&}$ belongs to the commutant, it can be represented as a function in $\mathbb{S}_0$ and its symmetries $\mathbb{L}_0$ (7.1), i.e.,

$$\mathbb{A}_0^{\&}(x) = a_0(\mathbb{S}_0(x), \mathbb{L}_0(x)). \tag{A.9}$$

Now let $\tilde{\mathbb{C}}_0$ denote the curvature tensor corresponding to the second summand in (A.6):

$$\tilde{\mathbb{C}}_{0jl} \stackrel{\text{def}}{=} D_l\tilde{\mathbb{A}}_{0j} - D_j\tilde{\mathbb{A}}_{0l} - i[\tilde{\mathbb{A}}_{0j}, \tilde{\mathbb{A}}_{0l}] = \tilde{\nabla}_l\tilde{\mathbb{A}}_{0j} - \tilde{\nabla}_j\tilde{\mathbb{A}}_{0l} + i[\tilde{\mathbb{A}}_{0j}, \tilde{\mathbb{A}}_{0l}], \tag{A.10}$$

where

$$\tilde{\nabla}_l \stackrel{\text{def}}{=} D_l + i[\tilde{\mathbb{A}}_{0l}, \cdot], \qquad D_l \equiv \partial/\partial x^l. \tag{A.11}$$

From (A.5), by taking the derivatives with respect to x, it is easy to see that $[\tilde{\mathbb{C}}_{0jl}, \mathbb{S}_0] = 0$ and, therefore, $\tilde{\mathbb{C}}_{0jl} = (\tilde{\mathbb{C}}_{0jl})^{\&}$. Note that the operations & and # (A.1) commute with the covariant derivatives (A.11). Thus, by applying the averaging operation & to (A.10), we obtain

$$\tilde{\mathbb{C}}_{0jl} = i[\tilde{\mathbb{A}}_{0j}, \tilde{\mathbb{A}}_{0l}]^{\&}. \tag{A.12}$$

And since there operators commute with $\mathbb{S}_0$, they can be written in a form similar to (A.9):

$$\tilde{\mathbb{C}}_{0jl}(x) = \tilde{c}_{0jl}(\mathbb{S}_0(x), \mathbb{L}_0(x), x) \tag{A.13}$$

for some scalar smooth functions $c_{0jl} = c_{0jl}(s, l, x)$.

Now we choose the family $\mathbb{A}_0^{\&}$ in (A.6) in such a way that the whole sum $\mathbb{A}_0$ is the connection determined by the unitary family $\mathbb{U}_0$ from (2.4), i.e.,

$$iD\mathbb{U}_0 = \mathbb{A}_0\mathbb{U}_0, \qquad \mathbb{U}_0|_{x=\underline{x}} = \mathbb{I}. \tag{A.14}$$

In this notation, relation (2.4) is just equivalent to (A.5) plus the zero curvature condition for the connection $\mathbb{A}_0$.

The generators $\mathbb{L}_0^\gamma$ of the commutant of $\mathbb{S}_0$ are chosen to be parallel with respect to the connection (A.6), i.e.,

$$iD\mathbb{L}_0^\gamma = [\mathbb{A}_0, \mathbb{L}_0^\gamma] \qquad \text{or} \qquad \nabla\mathbb{L}_0^\gamma = 0, \tag{A.5a}$$

where ∇ corresponds to $\mathbb{A}_0$ similarly to (A.11). Note that ∇ and $\tilde{\nabla}$ are related to each other as $\tilde{\nabla} = \nabla - i[\mathbb{A}_0^{\&}, \cdot]$.

It follows from (A.6), (A.9) and Eqs. (A.5), (A.5a) that the curvature $\mathbb{C}_0$ of $\mathbb{A}_0$ is split into the following sum:

$$\mathbb{C}_0(x) = \tilde{\mathbb{C}}_0(x) + \delta a_0(\mathbb{S}_0(x), \mathbb{L}_0(x), x), \qquad x \in \mathcal{D}. \tag{A.15}$$

Here δ is the covariant derivative in the commutant bundle; it is determined by

$$(\delta a)_{jl} \stackrel{\text{def}}{=} D_l a_j - D_j a_l + i[\![a_l, a_j]\!], \tag{A.16}$$

where the commutator $[\![\cdot, \cdot]\!]$ is generated by relations (7.1) between generators of the commutant. The zero-curvature condition $\mathbb{C}_0 = 0$, in view of (A.15), implies the equation for a_0

$$\delta a_0 = -\tilde{c}_0, \tag{A.17}$$

where $\tilde{c}_0 = ((\tilde{c}_{0jl}))$ are given by (A.12), (A.13).

The solvability of the zero-curvature equation (A.17) is guaranteed by the existence of the unitary family $\mathbb{U}_0$. Of course, one would like to see a formula for a_0 without referring to $\mathbb{U}_0$ in the general non-Abelian case. In the case of nondegenerate eigenvalues λ_k (the multiplicity equals 1), the commutant of $\mathbb{S}_0$ is Abelian, there are no generators $\mathbb{L}_0$ in (A.13), (A.15) and no commutator $[\![\cdot,\cdot]\!]$ in (A.16). In this case, Eq. (A.17) reads

$$da_0 = -\tilde{c} \tag{A.18}$$

in the usual sense of differential forms on $\mathcal{D}$. The closedness of $\tilde{c}_0$, i.e., $D_m\tilde{c}_{0lj}+D_l\tilde{c}_{0jm}+D_j\tilde{c}_{0ml}=0$, easily follows just from the Bianchi identities for the curvature $\tilde{\mathbb{C}}_0$. Therefore, *in the nondegenerate case, the connection* $\mathbb{A}_0$ *is explicitly derived via* (A.6), (A.7), (A.9) *using the integrator # and by taking the primitive of the closed differential* 2*-form* $\tilde{c} = \frac{1}{2}\tilde{c}_{jl}dx^l \wedge dx^j$ *in* (A.18).

Thus, the construction of the curvature-free adiabatic connection and of its splitting (A.6) associated with the projection & onto the symmetry algebra of $\mathbb{H}$ becomes an important and generally nontrivial point of the quantum adiabatic algorithm.

Remark A.1. The transformation $\mathcal{U}_0\colon \underline{\mathbb{G}} \to \mathbb{G} = \mathbb{U}_0\underline{\mathbb{G}}\mathbb{U}_0$ can be computed by solving the Heisenberg type "multi-time" Cauchy problem

$$iD\mathbb{G}/\partial x = [\mathbb{A}_0(x), \mathbb{G}], \qquad \mathbb{G}|_{x=\underline{x}} = \underline{\mathbb{G}}(\underline{x}). \tag{A.19}$$

In order to find the inverse operator $\mathbb{U}_0^{-1}$ or the inverse transformation $\mathcal{U}_0^{-1}$, one just needs to replace x by $\underline{x}$ in Eqs. (A.14), (A.19). For instance, the operator-valued family $\underline{\mathbb{F}} = \mathcal{U}_0^{-1}(\mathbb{F})$ can be computed by solving the Cauchy problem

$$i\partial\underline{\mathbb{F}}/\partial\underline{x}^j = [\mathbb{A}_{0j}(\underline{x}), \underline{\mathbb{F}}], \qquad \underline{\mathbb{F}}|_{\underline{x}=x} = \mathbb{F}(x).$$

In particular, the hodograph coefficients $\underline{\mathbb{A}}_{0l} = \mathcal{U}_0^{-1}(\mathbb{A}_{0l})$ are obtained via the system

$$i\partial\underline{\mathbb{A}}_{0l}/\partial\underline{x}^j = [\mathbb{A}_{0j}(\underline{x}), \underline{\mathbb{A}}_{0l}], \qquad \underline{\mathbb{A}}_{0l}|_{\underline{x}=x} = \mathbb{A}_{0l}(x),$$

where $j,l = 1,\dots,2n$.

2. APPENDIX B: ADIABATIC GUIDING CENTER

The adiabatic invariant $\mathbb{S}$ is the approximate integral of motion for the given Hamiltonian, i.e., $[\mathbb{H} \overset{*}{,} \mathbb{S}] = 0$ up to $O(\varepsilon^\infty)$. Besides $\mathbb{S}$, in the description of the adiabatic system, a significant role is played by the adiabatically deformed slow coordinates $\mathbb{X}^j$ which commute with $\mathbb{S}$, that is, $[\mathbb{S} \overset{*}{,} \mathbb{X}^j] = 0$ up to $O(\varepsilon^\infty)$. These are coordinates of a "guiding center" analogous to the Landau–Peierls coordinates of the Larmor vortex in a magnetic field.

Let us introduce the *guiding center* coordinates as in (6.7)

$$\mathbb{X}(x) = x + \varepsilon J\mathbb{A}(x) \tag{B.1}$$

being based on the slow Darboux coordinates x (2.1).

Note that coordinates (B.1) are taken with respect to the same fixed (laboratory) basis as the original coordinates x. Despite the "affine" structure of (B.1), this definition is consistent with the changing of the slow coordinates. Namely, let $\varphi\colon x \to x'$ be a transformation of one quantum Darboux coordinate set (1.2) to another one, and let $\mathbb{X}$ or $\mathbb{X}'$ be defined by (B.1) in local charts x or x'. Then $\varphi_* \cdot \mathbb{X}_* = \mathbb{X}'_* \cdot \varphi_*$, which allows one *to match one-chart expressions like* (B.1) *to a global map of a quantum slow manifold.*

Now let us analyze formula (B.1). The identity $[\mathbb{S} \overset{*}{,} x] = i\varepsilon JD\mathbb{S}$ implies that the relation $[\mathbb{S} \overset{*}{,} \mathbb{X}] = 0$ is equivalent to the homological equation

$$[\mathbb{A} \overset{*}{,} \mathbb{S}] = iD\mathbb{S}. \tag{B.2}$$

This is the complete ε-analog of Eq. (6.0) or (A.5).

Pairwise commutation relations between the guiding center coordinates can be represented in the form

$$\frac{i}{\varepsilon}[\mathbb{X}^j \overset{*}{,} \mathbb{X}^l] = J^{jl} + \varepsilon(J\mathbb{C}J)^{jl}, \tag{B.3}$$

where by $\mathbb{C}$ we denote the curvature corresponding to the connection $\mathbb{A}$, namely,

$$\mathbb{C}_{jl} \overset{\text{def}}{=} D_l\mathbb{A}_j - D_j\mathbb{A}_l - i[\mathbb{A}_j \overset{*}{,} \mathbb{A}_l]. \tag{B.4}$$

By taking the derivatives with respect to x of (B.2), it is easy to see that, for any given solution $\mathbb{A}$ of (B.2), the curvature commutes with the adiabatic invariant $[\mathbb{C}_{jl} \overset{*}{,} \mathbb{S}] = 0$.

There is an internal approach to deal with (B.2), (B.3). Namely, one could choose a particular solution $\tilde{\mathbb{A}}$ of (B.2) with zero average value

$$\tilde{\mathbb{A}} \overset{\text{def}}{=} (DS)^{\#*}, \qquad \tilde{\mathbb{A}}^{\&*} = 0. \tag{B.5}$$

Here the operations $\#*$ and $\&*$ are defined by the $*$-product and $*$-functions following the same lines as in (A.1):

$$\begin{aligned} \mathbb{M}^{\#*} &\overset{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} e_*^{it\mathbb{S}} * \mathbb{M} * e_*^{-it\mathbb{S}}(t-\pi)\,dt, \\ \mathbb{M}^{\&*} &\overset{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} e_*^{it\mathbb{S}} * \mathbb{M} * e_*^{-it\mathbb{S}}\,dt. \end{aligned} \tag{B.6}$$

Let us denote by $\tilde{\mathbb{C}}$ the curvature (B.4) related to the special solution (B.5), and let $\tilde{\mathbb{X}} = x + \varepsilon J\tilde{\mathbb{A}}$. Then $\tilde{\mathbb{C}}$ can be represented as an $*$-function in symmetries of $\mathbb{S}$, i.e., $\tilde{\mathbb{C}}_{jl} = \tilde{c}_*(\mathbb{S}, \mathbb{L}, \tilde{\mathbb{X}})_{jk}$. Relations (B.3) read

$$\frac{i}{\varepsilon}[\tilde{\mathbb{X}} \overset{*}{,} \tilde{\mathbb{X}}] = J + \varepsilon J\tilde{c}_*(\mathbb{S}, \mathbb{L}, \tilde{\mathbb{X}})J. \tag{B.7}$$

In a sense, this internal way of description of the fast–slow system follows the Poinsot approach (see in [31]) from the rigid body mechanics. Here we need not worry about the zero-curvature condition, but the price is the loss of the opportunity to separate the $\mathbb{X}$-coordinate from the $\mathbb{L}$-coordinate and to make the transform $x \to \mathbb{X}$ unitary.

The external way of description (which can be associated with the Lagrange approach) is to postulate that the transformation $x \to \mathbb{X}$ must be canonical, i.e., produced by unitary operators. Then (B.3) must take the form $[\mathbb{X}^j \overset{*}{,} \mathbb{X}^l] = -i\varepsilon J^{jl}$. Therefore, the curvature (B.4) has to be zero.

The zero-curvature condition

$$\mathbb{C} = 0 \tag{B.8}$$

can be achieved if one takes not a special but general solution of (B.2)

$$\mathbb{A} = \tilde{\mathbb{A}} + \mathbb{A}^{\&*}, \tag{B.9}$$

where the second summand on the right-hand side commutes with the adiabatic invariant: $[\mathbb{A}^{\&*} \overset{*}{,} \mathbb{S}] = 0$.

One can represent this additional summand via the symmetries of $\mathbb{S}$:

$$\mathbb{A}_j^{\&*} = a_*(\mathbb{S}, \mathbb{L}, \mathbb{X})_j \tag{B.10}$$

using some $*$-functions a_{*j}. The equation for a_{*j} follows from (B.8):

$$\delta a_* = -\tilde{c}_*. \tag{B.11}$$

Here the non-Abelian differential δ is determined by (A.16), and $\tilde{c}_*$ is given by the averaging of $\tilde{\mathbb{C}}$, i.e.,

$$\tilde{c}_*(\mathbb{S}, \mathbb{L}, \mathbb{X})_{jl} = i[\tilde{\mathbb{A}}_j \overset{*}{,} \tilde{\mathbb{A}}_l]^{\&*}. \tag{B.12}$$

These relations are obtained in the same way as in (A.10), (A.12).

Note that, by having $\mathbb{A}$ with zero curvature $\mathbb{C} = 0$, one obtains the parallel sections $\mathbb{L}^\gamma$ with values in the commutant of $\mathbb{S}$ as in (A.5a):

$$iD\mathbb{L}^\gamma = [\mathbb{A} \overset{*}{,} \mathbb{L}^\gamma], \qquad [\mathbb{S} \overset{*}{,} \mathbb{L}^\gamma] = 0. \tag{B.13}$$

Thus, the guiding center coordinates (B.1) commute with all these symmetries:

$$[\mathbb{X}^j \overset{*}{,} \mathbb{L}^\gamma] = 0.$$

Finally, the splitting of the free connection (B.9) generates the splitting of the guiding center coordinates

$$\mathbb{X} = x + \varepsilon J \tilde{\mathbb{A}} + \varepsilon J \mathbb{A}^{\&*}. \tag{B.14}$$

The first correction $\varepsilon J \tilde{\mathbb{A}}$ in (B.12) has the zero average (see in (B.5)), while *the second correction* $\varepsilon J \mathbb{A}^{\&*}$ *represents the averaged adiabatic deformation of the original slow coordinates* x.

Remark B.1. *All the relations written above must be understood up to* $O(\varepsilon^\infty)$*, i.e., as formal asymptotic series in* ε.

All the objects used in this Appendix B have their $\varepsilon = 0$ *analogs described in Appendix A. The complete* ε*-version can be obtained from the* $\varepsilon = 0$ *version by the transformation* $\mathcal{U} = \mathcal{U}_\varepsilon$*, see* (4.2), (5.15), (5.16), (7.3).

In particular, it follows from $(B.10)$, $(B.12)$ *that* $a_*(\mathbb{S}, \mathbb{L}, \mathbb{X}) = \mathcal{U}\big(a_0(\mathbb{S}_0(\underline{x}), \mathbb{L}_0(\underline{x}), x)\big)$ *and* $\tilde{c}_*(\mathbb{S}, \mathbb{L}, \mathbb{X}) = \mathcal{U}\big(\tilde{c}_0(\mathbb{S}_0(\underline{x}), \mathbb{L}_0(\underline{x}), x)\big)$. *Thus, the solvability of* $(B.11)$ *up to* $O(\varepsilon^\infty)$ *is equivalent to the solvability of* $(A.17)$.

ACKNOWLEDGMENTS

I heartily thank Yu. M. Vorobjev for many helpful discussions and appreciate useful consultations with A. I. Neishtadt. This study (research grant no. 12-01-0140) was supported by The National Research University - Higher School of Economics; Academic Fund Program.

REFERENCES

1. M. Born, *Lectures in Atomic Mechanics* (ONTI, Moscow, 1934).
2. L. Shiff, *Quantum Mechanics* (Inostr. Lit., Moscow, 1957).
3. V. P. Maslov, *Perturbation Theory and Asymptotic Methods* (Izdat. MGU, Moscow, 1965) [in Russian].
4. V. P. Maslov and M. V. Fedoriuk, *Semiclassical Approximation in Quantum Mechanics* (D. Reidel, Dordrecht, 1981).
5. B. Simon, “Holonomy, the Quantum Adiabatic Theorem and Berry’s Phase,” Phys. Rev. Lett. **5** (1), 2167–2170, 1983.
6. M. V. Berry, “Quantal Phase Factors Accompanying Adiabatic Changes,” Proc. R. Soc. London Ser. A **392**, 45–57 (1984).
7. M. V. Berry, “Classical Adiabatic Angles and Quantal Adiabatic Phase,” J. Phys. A **18**, 15–27 (1985)); A. Weinstein, “Connections of Berry and Hannay Type for Moving Lagrangian Submanifolds,” Adv. Math. **82**, 133–159 (1990)..
8. J. H. Hannay, “Angle Variable Holonomy in Adiabatic Excursion of an Integrable Hamiltonian,” J. Phys. A **18**, 221–230 (1985).
9. *Geometric Phases in Physics* (World Scientific, Singapore, 1989).
10. A. I. Neishtadt, “The Separation of Motions in Systems with Rapidly Rotating Phase,” J. Appl. Math. Mech. **48** (2), 133–139 (1984)); A. I. Neishtadt, “Averaging Method and Adiabatic Invariants,” in *Hamiltonian Dynamical Systems and Applications*, W. Graig, Ed. (Springer Verlag, 2008), pp. 53–66.

11. D. V. Treschev, “The Continuous Averaging Method in the Problem of Separation of Fast and Slow Motions,” Reg. Chaotic Dyn. **2** (3/4), 9–20 (1997).
12. M. V. Berry and P. Shukla, “High-Order Classical Adiabatic Reaction Forces: Slow Manifold for a Spin Model,” J. Phys. A: Math. Theor. **43**, 045102 (27pp)(2010); J. Vanneste, “Asymptotics of a Slow Manifold,” SIAM J. Appl. Dynam. Syst. **7**, 1163–1190 (2008)..
13. R. Montgomery, “The Connection Whose Holonomy Is the Classical Adiabatic Angles of Hannay and Berry and Its Generalization to the Non-Integrable Case,” Commun. Math. Phys. **120**, 269–294 (1988).
14. J. E. Marsden, R. Montgomery, and T. Ratiu, “Reduction, Symmetry and Phases in Mechanics,” Mem. Am. Math. Soc. **88** (436), 1–110 (1990).
15. Yu. Vorobiev, “The Averaging in Hamiltonian Systems on Slow-Fast Phase Spaces with S1-Symmetry,” Phys. At. Nucl. **74** (7), 1–5 (2011), M. Avendano-Camacho, J. A. Vallejo, and Yu. Vorobiev, “Higher Order Corrections to Adiabatic Invariants of Generalized Slow-Fast Hamiltonian Systems,” J. Math. Phys. **54**, 082704 (2013).
16. M. V. Karasev, “New Global Asymptotics and Anomalies for the Problem of Quantization of the Adiabatic Invariant,” Funct. Anal. Appl. **24**, 104–114 (1990).
17. K. Yu. Bliokh and Yu. P. Bliokh, “Spin Gauge Fields: from Berry Phase to Topological Spin Transport and Hall Effects,” Ann Physics **319** (1), 13–47 (2005).
18. P. Gosselin, A. Bérard, and H. Mohrbach, “Semiclassical Diagonalization of Quantum Hamiltonian and Equations of Motion with Berry Phase Corrections,” Eur. Phys. B **58**, 137 (2007).
19. P. Gosselin, J. Hanssen, and H. Mohrbach, “Recursive Diagonalization of Quantum Hamiltonians to All Orders in h,” Phys. Rev. D **77**, 085008 (2008).
20. T. Kato, “On the Adiabatic Theorem of Quantum Mechanics,” J. Phys. Soc. Japan **5**, 435–439 (1950).
21. T. Kato, *Perturbation Theory for Linear Operators* (Springer-Verlag, 1966), Sec. 4, Chap. II.
22. H. J. Groenewold, “On the Principles of Elementary Quantum Mechanics,” Physica **12**, 405–60 (1946).
23. J. E. Moyal, “Quantum Mechanics as a Statistical Theory,” Proc. Camb. Phil. Soc. **45**, 99–124 (1949).
24. F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, “Deformation Theory and Quantization,” Ann. Phys. NY **111**, 61–151 (1978).
25. R. G. Littlejohn and W. G. Flynn, “Geometric Phases in the Asymptotic Theory of Coupled Wave Equations,” Phys. Rev. A **44**, 5239–5256 (1991)); R. G. Littlejohn and S. Weigert, “Adiabatic Motion of a Neutral Spinning Particle in an Inhomogeneous Magnetic Field,” Phys. Rev. **A 48**, 924–40 (1993).
26. C. Emmrich and A. Weinstein. “Geometry of the Transport Equation in Multicomponent WKB Approximations,” Comm. Math. Phys. **176**, 701–711 (1996).
27. L. V. Berland and S. Yu. Dobrokhotov, “Operator Separation of Variables in Problems of Short-Wave Asymptotics for Differential Equations with Rapidly Oscillating Coefficients,” Dokl. Akad. Nauk SSSR **297** (1), 80–84 (1987).
28. L. D. Faddeev and S. L. Shatashvili, “Algebraic and Hamiltonian Methods in the Theory of Non-Abelian Anomalies,” Teoret. Mat. Fiz. **60** (2), 206–217 (1984); L. D. Faddeev and L. A. Takhtajan, *Hamiltonian Methods in Theory of Solitons* (Springer-Verlag, Berlin, 1987).
29. S. P. Tsarev, “Geometry of Hamiltonian Systems of Hydrodynamic Type. Generalized Hodograph Method,” Izv. Akad. Nauk SSSR Ser. Mat. **54** (5), 1048–1068 (1990).
30. B. A. Dubrovin and S. P. Novikov, “Hydrodynamics of Weakly Deformed Lattices. Differential Geometry and Hamiltonian Theory,” Russian Math. Surveys **44**, 35–124 (1989).
31. V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Nauka, Moscow, 1989).