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ON THE EXISTENCE OF SOLUTIONS OF THE FIRST BOUNDARY VALUE PROBLEM FOR ELLIPTIC EQUATIONS ON UNBOUNDED DOMAINS

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Abstract: In this paper we consider the first boundary value problem for elliptic systems, defined on unbounded domains $\Omega \subset \mathbb{R}^n$, which solutions satisfy a condition of finiteness of the Dirichlet integral, also known as the energy integral

$$\int_{\Omega} |\nabla u|^2 dx < \infty.$$

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1. Introduction

Let Ω be an unbounded open subset of $\mathbb{R}^n, n \geq 2$. As is customary, by $W^1_{2,loc}(\Omega)$ we denote the space of functions which are locally Sobolev, i.e.,

$$W^1_{2,loc}(\Omega) = \{f : f \in W^1_2(\Omega \cap B^x_\rho)\, \forall \rho > 0\,, \forall x \in \mathbb{R}^n\},$$

where B^x_ρ the open ball in $\mathbb{R}^n$ of radius ρ centered at the point x [9]. If $x = 0$, we write B_ρ instead of B^x_ρ. In this case, denote by $\overset{\circ}{W}{}^1_{2,loc}(\Omega)$ the subset of $W^1_{2,loc}(\mathbb{R}^n)$ which is the closure of $C^\infty_0(\Omega)$ in the system of seminorms

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$\|u\|_{W_2^1(\Omega\cap B_\rho)}, \rho > 0$. Further, following [10, Subsec. 1.1], denote by $L_2^1(\Omega)$ the space of distributions ("generalized functions") whose first derivatives belong to $L_2(\Omega)$; in other words,

$$L_2^1(\Omega) = \{f \in \mathcal{D}'(\Omega) : \int\limits_\Omega |\nabla f|^2 dx < \infty\}.$$

Let $\omega \subseteq \mathbb{R}^n$ be an open set and let $\mathcal{K} \subset \omega$ be a compact set. Denote by $\Phi_\varphi(\mathcal{K}, \omega)$ the set of functions $\psi \in C_0^\infty(\omega)$ such that $\psi = \varphi$ in a neighborhood of $\mathcal{K}$, or, in other words, $\psi - \varphi \in \overset{\circ}{W}{}^1_{2,\,loc}(\mathbb{R}^n \setminus \mathcal{K})$. Write $\Psi(\mathcal{K}, \omega) = \{\psi \in C_0^\infty(\omega) : \psi = 1$ in a neighborhood of $\mathcal{K}$.

The quantity

$$\operatorname{cap}_\varphi(\mathcal{K}, \omega) = \inf_{\psi \in \Phi_\varphi(\mathcal{K}, \omega)} \int\limits_\omega |\nabla\psi|^2 dx$$

is referred to as the capacity of the compact set $\mathcal{K}$ with respect to an open set ω [10, Subsec. 7.2]. The capacity of an arbitrary closed subset $E \subset \omega$ of $\mathbb{R}^n$ is defined by the rule

$$\operatorname{cap}_\varphi(E, \omega) = \sup_{\mathcal{K} \subset E} \operatorname{cap}_\varphi(\mathcal{K}, \omega),$$

where the supremum on the right-hand side is taken over all compacta $\mathcal{K} \subset E$. If $\omega = \mathbb{R}^n$, then we write $\operatorname{cap}_\varphi(E)$ instead of $\operatorname{cap}_\varphi(E, \mathbb{R}^n)$.

We also need the following capacity [10, Subsec. 9.1]:

$$\operatorname{Cap}(\mathcal{K}, W_2^1(\omega)) = \inf_{\psi \in \Psi(\mathcal{K}, \omega)} \left(\int\limits_\omega |\nabla\psi|^2 dx + \int\limits_\omega |\psi|^2 \, dx \right).$$

As above, the capacity of an arbitrary set $E \subset \omega$ closed in $\mathbb{R}^n$ is given by the rule

$$\operatorname{Cap}(E, W_2^1(\omega)) = \sup_{\mathcal{K} \subset E} \operatorname{Cap}(\mathcal{K}, W_2^1(\omega)),$$

where the supremum on the right-hand side is taken over all compacta $\mathcal{K} \subset E$.

Finally, denote by W_2^{-1} the space of continuous linear functionals on W_2^1. A set $E \subset \mathbb{R}^n$ is said to be $(2,1)$-*polar* if the only element of W_2^{-1} supported by E is zero [10, Subsec. 9.2].

2. Statement of the Problem

Here and below, L stands for the divergence operator of the form

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right),$$

where measurable bounded coefficients a_{ij} satisfy the uniform ellipticity condition

$$c_1|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\,\xi_j \le c_2|\xi|^2, \quad \xi \in \mathbb{R}^n, \quad c_1, c_2 > 0.$$

By a solution of the Dirichlet problem

$$\begin{cases} Lu &= 0 \ \text{ on } \Omega \\ u|_{\partial\Omega} &= \varphi, \end{cases} \tag{1}$$

where $\varphi \in W^1_{2,\,loc}(\mathbb{R}^n)$, we mean a function $u \in W^1_{2,\,loc}(\Omega)$ such that

1) $u-\varphi \in \overset{\circ}{W}{}^1_{2,\,loc}(\Omega)$, i.e., $(u-\varphi)\mu \in \overset{\circ}{W}{}^1_2(\Omega)$ for any function $\mu \in C_0^\infty(\mathbb{R}^n)$;
2) the function u has the bounded Dirichlet integral

$$\int\limits_{\Omega} |\nabla u|^2 dx < \infty;$$

3)

$$\int\limits_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial \psi}{\partial x_i}\, dx = 0$$

for any function $\psi \in C_0^\infty(\Omega)$.

3. Main Results

Theorem 1. *Let* $\mathrm{cap}_{\varphi-c}(\mathbb{R}^n \setminus \Omega) < \infty$ *for some constant* $c \in \mathbb{R}$. *Then problem* (1) *has a solution.*

Theorem 2. *Let problem* (1) *have a solution, and let*

$$\int\limits_{\mathbb{R}^n\setminus\Omega} |\nabla \varphi|^2 dx < \infty.$$

Then there is a constant $c \in \mathbb{R}$ *such that* $\mathrm{cap}_{\varphi-c}(\mathbb{R}^n \setminus \Omega) < \infty$.

Theorem 3. *For any function $\varphi \in W^1_{2,\,loc}(\mathbb{R}^n)$, the condition $cap_{\varphi-\,c}(\mathbb{R}^n \setminus \Omega) < \infty$ is equivalent to the inequality*

$$\sum_{k=1}^{\infty} cap_{\varphi-\,c}((\overline{B}_{r_{k+1}} \setminus B_{r_{k-1}}) \cap (\mathbb{R}^n \setminus \Omega), B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}) < \infty,$$

where

$$r_k = \begin{cases} 2^k \ , & \text{if } n \geq 3 \\ 2^{2^k} , & \text{if } n = 2. \end{cases}$$

Let $\omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and let μ be a measure on ω such that

$$\sup_{x \in \mathbb{R}^n,\, \rho > 0} \rho^{1-n} \mu(B^x_\rho \cap \omega) < \infty.$$

In this case, for any function $v \in W^1_2(\omega)$, there is a $c \in \mathbb{R}$ such that

$$\sigma(\omega, \mu)\|v - c\|_{L_2(\omega,\mu)} \leq \|\nabla v\|_{L_2(\omega)}, \tag{2}$$

where the constant $\sigma(\omega, \mu) > 0$ does not depend on v [10, Subsec. 1.4.5].

Theorem 4. *Let problem* (1) *have a solution, and let μ_k be a family of measures on ω_k, where ω_k, $k = 1, 2, \ldots$, are pairwise disjoint Lipschitz domains in $\mathbb{R}^n$ such that*

$$\sup_{x \in \mathbb{R}^n,\, \rho > 0} \rho^{1-n} \mu_k(B^x_\rho \cap \omega_k) < \infty$$

and

$$\sum_{k=1}^{\infty} \int_{\omega_k \setminus \Omega} |\nabla \varphi|^2 dx < \infty. \tag{3}$$

Write

$$m_k(\varphi) = \inf_{c \in \mathbb{R}} \|\varphi - c\|_{L_2(\omega_k \setminus \Omega, \mu_k)}.$$

Then

$$\sum_{k=1}^{\infty} \sigma^2(\omega_k, \mu_k) m^2_k(\varphi) < \infty, \tag{4}$$

where $\sigma(\omega_k, \mu_k)$ stands for the coefficient in inequality (2).

To prove the Theorems 1 – 4 we need a number of auxiliary results.

An inequality from the following lemma is fairly well-known [e.g., 5 p. 288, p. 398] and occurs in various forms. However, for the sake of completeness, we give a detailed proof of this inequality.

Lemma 1 (Special Hardy inequality). *Let $\psi \in C_0^\infty(\mathbb{R}^n)$ and $n \geq 3$. Then*

$$\int\limits_{\mathbb{R}^n} |\nabla\psi|^2\, dx \geq k \int\limits_{\mathbb{R}^n} \frac{|\psi|^2}{|x|^2}\, dx,$$

where constant k doesn't depend on u.

Proof. Let's pass to the polar coordinates. Hence, the integral in the right-hand side takes the form

$$\int dS \int\limits_0^\infty \frac{|\psi|^2}{r^2}\, r^{n-1}\, dr,$$

where first integral is taken over all angular coordinates. Let's fix angular coordinates and obtain a chain of transformations

$$\int\limits_0^\infty \frac{|\psi|^2}{r^2}\, r^{n-1}\, dr = \int\limits_0^\infty |\psi^2|\, r^{n-3}\, dr = \frac{1}{n-2} \int\limits_0^\infty |\psi|^2\, (r^{n-2})'\, dr =$$

$$\frac{1}{n-2} \left(r^{n-2}\, |\psi|^2 \Big|_{r=0}^{\infty} - \int\limits_0^\infty 2\, |\psi|\, |\psi|'\, r^{n-2}\, dr \right).$$

The first term in the final bracket, obviously, equals zero, as ψ is a sampling function. Let's estimate the modulus of the second term, using the inequality $ab \leq \varepsilon\, a^2 + \frac{1}{\varepsilon}\, b^2$, considering $a = |\psi|\, r^{\frac{n-3}{2}}$, $b = r^{\frac{n-1}{2}} |\psi\,'|$.

$$\left| \int\limits_0^\infty 2\, |\psi|\, |\psi|'\, r^{n-2}\, dr \right| \leq 2 \int\limits_0^\infty |\psi|\, |\psi'|\, r^{n-2}\, dr \leq$$

$$2 \left(\varepsilon \int\limits_0^\infty |\psi|^2\, r^{n-3}\, dr + \frac{1}{\varepsilon} \int\limits_0^\infty |\psi'|^2\, r^{n-1}\, dr \right).$$

Thus, we obtain a chain of inequalities

$$\int\limits_0^\infty |\psi|^2\, r^{n-3}\, dr \leq \frac{2}{n-2} \int\limits_0^\infty |\psi|\, |\psi'|\, r^{n-2}\, dr \leq$$

$$\frac{2\,\varepsilon}{n-2} \int\limits_0^\infty |\psi|^2\, r^{n-3}\, dr + \frac{2}{\varepsilon(n-2)} \int\limits_0^\infty |\psi'|^2\, r^{n-1}\, dr.$$

Consequently, by transferring of the first term to the left-hand side, we obtain following inequality

$$(1-\frac{2\,\varepsilon}{n-2})\int\limits_0^\infty |\psi|^2\, r^{n-3}\, dr \le \frac{2}{\varepsilon(n-2)}\int\limits_0^\infty |\psi'|^2\, r^{n-1}\, dr.$$

Given that $|\psi'|^2 \le |\nabla\psi|^2$ and that r^{n-1} represents the Jacobian of the transformation to the polar coordinates, after returning to the initial coordinates, we obtain

$$\int\limits_{\mathbb{R}^n} |\nabla\psi|^2\, dx \ge k \int\limits_{\mathbb{R}^n} \frac{|\psi|^2}{|x|^2}\, dx.$$

$\square$

Remark. Taking a sequence $\{\psi_k\} \in C_0^\infty(\mathbb{R}^n)$, which is fundamental in L_2^1, i.e. in seminorm

$$\|\cdot\|_{L_2^1(\mathbb{R}^n)} = \left(\int\limits_{\mathbb{R}^n} |\nabla\psi|^2\, dx\right)^{\frac{1}{2}},$$

by the special Hardy inequality, we immediately obtain a fundamentality of this sequence in the metric

$$\|\cdot\| = \left(\int\limits_{\mathbb{R}^n} |\nabla\psi|^2\, dx + \int\limits_{\mathbb{R}^n} \frac{|\psi|^2}{|x|^2}\, dx\right)^{\frac{1}{2}}.$$

Therefore, the special Hardy inequality is also fair for $\psi \in \overset{\circ}{L}{}_2^1(\mathbb{R}^n)$.

Lemma 2 (General Hardy inequality)**.** *Let $u \in L_2^1(\mathbb{R}^n)$ and $n \ge 3$. Then there is a constant c such that the following inequality is fair*

$$\int\limits_{\mathbb{R}^n} |\nabla u|^2\, dx \ge k \int\limits_{\mathbb{R}^n} \frac{|u-c|^2}{|x|^2}\, dx,$$

where constant k doesn't depend on u.

Proof. The fact that u belongs to the space $L_2^1(\mathbb{R}^n)$ is equivalent to the following condition

$$\int\limits_{\mathbb{R}^n} |\nabla u|^2\, dx < \infty.$$

Let's decompose the space $L_2^1(\mathbb{R}^n)$ into a direct product of the space $\overset{\circ}{L}{}_2^1(\mathbb{R}^n)$ and its orthogonal complement. Let u_0 be a projection of u on $\overset{\circ}{L}{}_2^1(\mathbb{R}^n)$, and h is a component from the orthogonal complement. Considering that the space $\overset{\circ}{L}{}_2^1(\mathbb{R}^n)$ is Hilbert and separable, we find out that for any $v \in \overset{\circ}{L}{}_2^1(\mathbb{R}^n)$ it is true that

$$\int\limits_{\mathbb{R}^n} \nabla v \nabla h \, dx = 0.$$

Hence, $\triangle h = 0$ in $\mathbb{R}^n$. From the Parseval's identity, we obtain that

$$\int\limits_{\mathbb{R}^n} |\nabla h|^2 \, dx + \int\limits_{\mathbb{R}^n} |\nabla u_0|^2 \, dx = \int\limits_{\mathbb{R}^n} |\nabla u|^2 \, dx.$$

Due to the finiteness of the right-hand side, we have the finiteness of each term on the left-hand side. In particular, we obtain that

$$\int\limits_{\mathbb{R}^n} |\nabla h|^2 \, dx < \infty.$$

Recalling the ellipticity of h, we obtain that h is constant. Then, using the special Hardy inequality with respect to $u_0 = u - h = u - c$, we obtain the general Hardy inequality. $\square$

Lemma 3. *In case of $n = 2$, the general Hardy inequality takes the form*

$$\int\limits_{\mathbb{R}^2} |\nabla u|^2 \, dx \geq k \int\limits_{|x| \geq 2\delta} \frac{|u|^2}{|x|^2 \ln^2 \frac{|x|}{\delta}} \, dx,$$

for any function $u \in L_2^1(\mathbb{R}^2)$ and for any constant $\delta > 0$, where constant k doesn't depend on u, which is equivalent to the inequality

$$\int\limits_{\mathbb{R}^2} |\nabla u|^2 \, dx \geq k \int\limits_{\mathbb{R}^2} \frac{|u|^2}{|x|^2 \ln^2 |x|} \, dx,$$

for any function $u \in L_2^1(\mathbb{R}^2)$ such that $u = 0$ almost everywhere in a neighborhood of zero, and where constant k doesn't depend on u.

Proof. At first let's prove this proposition for a function $u \in C^\infty(\mathbb{R}^2)$.

Let's pass to the polar coordinates. Hence, the integral in the right-hand side takes the form

$$\int_0^{2\pi} d\phi \int_0^{\infty} \frac{r|u|^2}{r^2 \ln^2 r}\, dr.$$

Let's fix angular coordinates and obtain a chain of transformations

$$\int_0^{\infty} \frac{r|u|^2}{r^2 \ln^2 r}\, dr = \int_0^{\infty} \frac{|u|^2}{r \ln^2 r}\, dr = -\int_0^{\infty} \left(\frac{1}{\ln r}\right)' |u|^2\, dr =$$

$$-\frac{1}{\ln r}\, |u|^2 \Big|_{r=0}^{\infty} + \int_0^{\infty} \frac{1}{\ln r}\, 2\, |u||u|' dr.$$

The first term in the final bracket, obviously, is zero, as u vanishes in a neighborhood of zero. Let's estimate the modulus of the second term, using the inequality $ab \le \varepsilon\, a^2 + \frac{1}{\varepsilon}\, b^2$, considering $a = \frac{|u|}{r^{\frac{1}{2}} \ln r}$, $b = r^{\frac{1}{2}} |u'|$.

$$\left| \int_0^{\infty} \frac{1}{\ln r} 2|u||u|' dr \right| = \left| \int_0^{\infty} \frac{2r^{\frac{1}{2}} |u||u|'}{r^{\frac{1}{2}} \ln r}\, dr \right| \le 2 \int_0^{\infty} \frac{r^{\frac{1}{2}} |u||u'|}{r^{\frac{1}{2}} \ln r} dr \le$$

$$2 \left(\varepsilon \int_0^{\infty} \frac{|u|^2}{r \ln^2 r}\, dr + \frac{1}{\varepsilon} \int_0^{\infty} r|u'|^2\, dr \right).$$

Thus, we obtain a chain of inequalities

$$\int_0^{\infty} \frac{|u|^2}{r \ln^2 r}\, dr \le 2\, \varepsilon \int_0^{\infty} \frac{|u|^2}{r \ln^2 r}\, dr + \frac{2}{\varepsilon} \int_0^{\infty} r|u'|^2\, dr.$$

Consequently, by transferring of the first term to the left-hand side, we obtain the following inequality

$$(1 - 2\, \varepsilon) \int_0^{\infty} \frac{|u|^2}{r \ln^2 r}\, dr \le \frac{2}{\varepsilon} \int_0^{\infty} r|u'|^2\, dr.$$

Given that $|u'|^2 \le |\nabla u|^2$ and that r represents the Jacobian of the transformation to the polar coordinates, after returning to the initial coordinates, we obtain

$$\int_{\mathbb{R}^2} |\nabla u|^2\, dx \ge k \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2 \ln^2 |x|}\, dx.$$

Then, using the passage to the limit and the fact that C^∞ is dense in L_2^1 [10, p. 18], we obtain the proof of the Lemma. ☐

Lemma 4. *Let E be a $(2,1)$-polar set. Then $u\big|_E = 0$ for any function $u \in W^1_{2,\,loc}(\mathbb{R}^n)$, i.e. $\mu u \in \overset{\circ}{W}{}^1_2(\mathbb{R}^n \setminus E)$ for any function $\mu \in C_0^\infty(\mathbb{R}^n)$.*

Proof. It is known [10, p. 331, Theorem 1] that the space $\mathcal{D}(\Omega)$ is dense in W_2^1 if and only if $\mathbb{R}^n \setminus \Omega$ is a $(2,1)$-polar set. That implies the statement of the Lemma. ☐

Lemma 5. *Let $Cap((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_0}, W_2^1(\mathbb{R}^n)) > 0$ for some r_0. Then*

$$\|\varphi\|_{L_2(B_r)} \leq A\,\|\nabla\varphi\|_{L_2(B_r)}$$

for any $r > 2\,r_0$ and for any $\varphi \in W^1_{2,\,loc}(\mathbb{R}^n)$ such that

$$\varphi\Big|_{(\mathbb{R}^n\setminus\Omega)\cap\overline{B}_{r_0}} = 0,$$

where constant A doesn't depend on φ.

Proof. Let's suppose the contrary. Then for any constant A there is $r > 2\,r_0$ and a function $\varphi \in W^1_{2,\,loc}(\mathbb{R}^n)$ such that

$$\varphi\Big|_{(\mathbb{R}^n\setminus\Omega)\cap\overline{B}_{r_0}} = 0,$$

and besides it is true that

$$\|\varphi\|_{L_2(B_r)} > A\,\|\nabla\varphi\|_{L_2(B_r)}.$$

Let's choose a sequence $A_s = s,\ s = 1, 2, \ldots$ There is a sequence φ_s such that $\|\varphi_s\|_{L_2(B_r)} > s\,\|\nabla\varphi_s\|_{L_2(B_r)}$. Denote

$$\psi_s = \frac{\varphi_s}{\|\varphi_s\|_{L_2(B_r)}}.$$

It is obvious that $\|\psi_s\|_{L_2(B_r)} = 1$, while

$$\|\nabla\psi_s\|_{L_2(B_r)} \to 0 \ \text{ as } s \to \infty.$$

Consequently, $\|k-\psi_s\|_{W_2^1(B_r)}$ tends to zero as $s \to \infty$ for some constant k. Thus, taking the function $(k - \psi_s)\eta$, where $\eta \in C_0^\infty(B_{2r_0})$, $\eta \equiv 1$ in a neighborhood of B_{r_0}, we obtain that

$$\mathrm{Cap}((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_0}, W_2^1(\mathbb{R}^n)) \leq \int\limits_{\mathbb{R}^n} |\nabla((k-\psi_s)\eta)|^2 dx \leq \mathrm{const}\, \|k-\psi_s\|_{W_2^1(B_r)}.$$

Taking the limit as $s \to \infty$, it follows that $\mathrm{Cap}((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_0}, W_2^1(\mathbb{R}^n)) = 0$. This contradiction proves the Lemma. □

Proof of the Theorem 1. $\{r_i\}_{i=1}^n$ and $\{\rho_i\}_{i=1}^n$ are infinitely increasing sequences of real numbers. Let $r_i < \rho_i$ for all i, and

$$\mathrm{cap}_{\varphi - c}((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_i}, B_{\rho_i}) < \mathrm{cap}_{\varphi - c}(\mathbb{R}^n \setminus \Omega) + \frac{1}{2^i}, \; i = 1, 2, \ldots$$

It is obvious [11] that capacity $\mathrm{cap}_{\varphi - c}((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_i}, B_{\rho_i})$ is achieved by the function $v_i \in \overset{\circ}{W}{}_2^1(B_{\rho_i})$ such that

$$\begin{cases} \triangle v_i = 0 \quad B_{\rho_i} \setminus ((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_i}) \\ v_i \Big|_{(\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_i}} = \varphi - c, \end{cases} \tag{5}$$

where the last equality means that $(v_i - (\varphi - c))\mu \in \overset{\circ}{W}{}_2^1(B_{\rho_i} \setminus ((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_i}))$ for any $\mu \in C_0^\infty(B_{\rho_i})$. Along with the problem (5), let's consider another problem:

$$\begin{cases} Lu_i = 0 \;\text{ on } B_{\rho_i} \setminus ((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_i}) \\ u_i \Big|_{(\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_i}} = \varphi - c, \end{cases} \tag{6}$$

where $u_i \in \overset{\circ}{W}{}_2^1(B_{\rho_i})$.

The following statement takes place: let function u_i be a solution of the problem (6), and function v_i is a solution of the problem (5). Then

$$\int\limits_{B_{\rho_i}} |\nabla v_i|^2 dx \leq \int\limits_{B_{\rho_i}} |\nabla u_i|^2 dx \leq c \int\limits_{B_{\rho_i}} |\nabla v_i|^2 dx\,, \tag{7}$$

where c is a non-negative constant, which doesn't depend on u_i and v_i. Let's prove this fact. The left-hand inequality, obviously, follows from the definition of capacity. Let's prove the right-hand inequality. Given that the function u_i is a solution of the problem (6), it is true that

$$\int\limits_{B_{\rho_i}} \sum_{l,m=1}^{n} a_{lm}(x) \frac{\partial u_i}{\partial x_m} \frac{\partial \psi}{\partial x_l}\, dx = 0$$

for any function $\psi \in \overset{\circ}{W}{}_2^1(\Omega)$. In particular, taking $\psi = u - v$, we obtain

$$\int\limits_{B_{\rho_i}} \sum_{l,m=1}^{n} a_{lm}(x) \frac{\partial u_i}{\partial x_m} \frac{\partial u_i}{\partial x_l}\, dx - \int\limits_{B_{\rho_i}} \sum_{l,m=1}^{n} a_{lm}(x) \frac{\partial u_i}{\partial x_m} \frac{\partial v_i}{\partial x_l}\, dx = 0,$$

from where the following estimates are obtained

$$\gamma \int\limits_{B_{\rho_i}} |\nabla u_i|^2 dx \le \int\limits_{B_{\rho_i}} \sum_{l,m=1}^{n} a_{lm}(x) \frac{\partial u_i}{\partial x_m} \frac{\partial u_i}{\partial x_l}\, dx = \int\limits_{B_{\rho_i}} \sum_{l,m=1}^{n} a_{lm}(x) \frac{\partial u_i}{\partial x_m} \frac{\partial v_i}{\partial x_l}\, dx \le$$

$$const \left(\int\limits_{B_{\rho_i}} |\nabla u_i|^2 dx \right)^{\frac{1}{2}} \left(\int\limits_{\Omega} |\nabla v_i|^2 dx \right)^{\frac{1}{2}},$$

that prove the right-hand inequality in (7).

It is obvious that

$$\mathrm{cap}_{\varphi - c}((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_i}) \le \int\limits_{B_{\rho_i}} |\nabla v_i|^2 dx < \mathrm{cap}_{\varphi - c}(\mathbb{R}^n \setminus \Omega) + \frac{1}{2^i}.$$

At the same time, from the inequality (7), it follows that

$$\int\limits_{B_{\rho_i}} |\nabla u_i|^2 dx < c \left(\mathrm{cap}_{\varphi - c}(\mathbb{R}^n \setminus \Omega) + \frac{1}{2^i} \right).$$

If $\mathrm{Cap}(\mathbb{R}^n \setminus \Omega, W_2^1(\mathbb{R}^n)) = 0$ then the set $\mathbb{R}^n \setminus \Omega$ is $(2,1)$-polar [10, p. 331], what means, according to the Lemma 4, that the function $1 - \varphi$ is zero on $\mathbb{R}^n \setminus \Omega$. Thus, taking the unit function, we obtain the required solution of the problem (1).

Now let $\mathrm{Cap}((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_i}, W_2^1(\mathbb{R}^n)) > 0$ for some r_i. Then, from the Lemma 5, we obtain that the sequence $\{u_i\}_{i=1}^n$ is bounded both in $L_2(B_r)$ and $W_2^1(B_r)$ for any r. Indeed, for sufficiently large i, j it is true that

$$u_i - u_j \in \overset{\circ}{W}{}^1_{2,\,loc}(\mathbb{R}^n \setminus ((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_r)), \quad i, j > i_0.$$

Thus, fixing j, we have

$$\|u_i\|^2_{L_2^1} + \|u_i\|^2_{W_2^1(K)} \le \alpha \left(\|u_j\|^2_{W_2^1(K)} + cap_{\varphi-c}(\mathbb{R}^n \setminus \Omega) + \frac{1}{2^i} + \frac{1}{2^j} \right)$$

for any compacta $K \subset \mathbb{R}^n$, where constant $\alpha > 0$ doesn't depend on u_i.

Due to the compactness of the embedding $W_2^1(B_r)$ in $L_2(B_r)$, we can choose a subsequence of the sequence $\{u_i\}_{i=1}^n$ which is fundamental in $L_2(B_r)$. In order not to overload the indexes, we denote this subsequence also as $\{u_i\}_{i=1}^n$. Let's take a function $\eta \in C_0^\infty(B_r)$ such that $\eta \equiv 1$ in an open neighborhood of the set $\overline{B}_{r/2}$. Due to the fact that u_i satisfies (6), for the difference $u_i - u_j$ we obtain that

$$\int\limits_{B_r} \sum_{l,m=1}^n a_{lm}(x) \frac{\partial(u_i - u_j)}{\partial x_m} \frac{\partial \psi}{\partial x_l}\, dx = 0,$$

where $\psi = \eta^2\,(u_i - u_j)$. In other words,

$$\int\limits_{B_r} \sum_{l,m=1}^n a_{lm}(x) \frac{\partial(u_i - u_j)}{\partial x_m} \frac{\partial \eta^2}{\partial x_l} (u_i - u_j)\, dx +$$

$$\int\limits_{B_r} \sum_{l,m=1}^n a_{lm}(x) \eta^2\, \frac{\partial(u_i - u_j)}{\partial x_m}\, \frac{\partial(u_i - u_j)}{\partial x_l}\, dx = 0.$$

Let's rewrite the last relation in the form

$$\gamma \int\limits_{B_r} \eta^2 |\nabla(u_i - u_j)|^2 dx \le -2 \int\limits_{B_r} \sum_{k,l=1}^n a_{kl}(x) \frac{\partial(u_i - u_j)}{\partial x_l} \frac{\partial \eta}{\partial x_k}\, \eta\,(u_i - u_j)\, dx,$$

whence, in view of the inequality $ab \le \frac{1}{2}\,a^2 + \frac{1}{2}\,b^2$, we obtain that

$$\int\limits_{B_r} \eta^2 |\nabla(u_i - u_j)|^2 dx \le c_1 \int\limits_{B_r} |\nabla(u_i - u_j)|^2 \eta^2 dx + c_2 \int\limits_{B_r} |\nabla \eta|^2 (u_i - u_j)^2\, dx,$$

where c_1, c_2 are non-negative constants, which don't depend on u_i. Thus, we have

$$\int\limits_{B_{r/2}} |\nabla(u_i - u_j)|^2 dx \le \beta \int\limits_{B_r} (u_i - u_j)^2\, dx,$$

where constant $\beta > 0$ doesn't depend on u_i. Last inequality proves that the sequence $\{u_i\}_{i=1}^n$ is fundamental in $W_2^1(B_{r/2})$ for any $r > 0$. Therefore, there is a function $u \in W^1_{2,loc}(\mathbb{R}^n)$ such that for any $r > 0$ the sequence $\{u_i\}_{i=1}^n$ tends to u in $W_2^1(B_r)$. It is obvious that the function u is the desired solution of the problem (1). □

Proof of the Theorem 2. Let's suppose that the function u is a solution of the problem (1). Let's extend u on $\mathbb{R}^n \setminus \Omega$ with value φ. Let $n \geq 3$, then there is a constant $c \in \mathbb{R}$ such that for the function u the general Hardy inequality takes place. Denote

$$\nu_R = \eta\left(\frac{|x|}{R}\right)(u-c),$$

where $\eta \in C_0^\infty(B_2)$ and $\eta \equiv 1$ in an open neighborhood of the set $\overline{B}_1$. Hence, we obtain

$$\nu_R\big|_{(\mathbb{R}^n\setminus\Omega)\cap\overline{B}_R} = \varphi - c.$$

The Dirichlet integral for the function ν_R can be estimated

$$\int\limits_{B_{2R}} \left|\nabla\left(\eta\left(\frac{|x|}{R}\right)(u-c)\right)\right|^2 dx \leq$$

$$2\left(\int\limits_{B_{2R}} \left|\nabla\eta\left(\frac{|x|}{R}\right)(u-c)\right|^2 dx + \int\limits_{B_{2R}} \left|\eta\left(\frac{|x|}{R}\right)\nabla(u-c)\right|^2 dx\right).$$

Let's notice that

$$\left|\nabla\eta\left(\frac{|x|}{R}\right)\right| \leq \frac{p}{R} \text{ and } \frac{1}{R^2} \leq \frac{4}{|x|^2} \text{ for } x \in B_{2R},$$

where $p > 0$ is a constant. Then, considering the Hardy inequality, we obtain

$$\int\limits_{B_{2R}} \left|\nabla\eta\left(\frac{|x|}{R}\right)(u-c)\right|^2 dx \leq \frac{p^2}{R^2}\int\limits_{B_{2R}\setminus B_R} |u-c|^2\, dx \leq$$

$$4p^2 \int\limits_{B_{2R}\setminus B_R} \frac{|u-c|^2}{|x|^2} dx \leq \frac{4p^2}{k}\int\limits_{B_{2R}} |\nabla u|^2 dx.$$

Thus,

$$\operatorname{cap}_{\varphi - c}((\mathbb{R}^n\setminus\Omega)\cap\overline{B}_R) \leq \int\limits_{B_{2R}} |\nabla\nu_R|^2 dx \leq \gamma \int\limits_{B_{2R}} |\nabla u|^2 dx < \infty,$$

where $\gamma > 0$ is a constant, which doesn't depend on ν_R. Proceeding to the limit as $R \to \infty$, we obtain

$$\operatorname{cap}_{\varphi - c}(\mathbb{R}^n \setminus \Omega) \le \gamma \int_{\mathbb{R}^n} |\nabla u|^2 dx < \infty,$$

that proves the Theorem 2 for $n \ge 3$.

In case of $n = 2$, denoting

$$\nu_R = \eta\left(\frac{\ln \frac{|x|}{R}}{\ln R}\right) u,$$

where $\eta \in C_0^\infty(\mathbb{R}^2)$, $\eta = 0$ in a neighborhood of zero and $\eta \equiv 1$ in a open neighborhood of B_1, we obtain

$$\nu_R\Big|_{(\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{R^2}} = \varphi.$$

The Dirichlet integral for the function ν_R can be estimated

$$\int_{B_{2R^2}} \left|\nabla\left(\eta\left(\frac{\ln \frac{|x|}{R}}{\ln R}\right) u\right)\right|^2 dx \le$$

$$2\left(\int_{B_{2R^2}} \left|\nabla\eta\left(\frac{\ln \frac{|x|}{R}}{\ln R}\right) u\right|^2 dx + \int_{B_{2R^2}} \left|\eta\left(\frac{\ln \frac{|x|}{R}}{\ln R}\right) \nabla u\right|^2 dx\right).$$

Let's notice that

$$\left|\nabla\eta\left(\frac{\ln \frac{|x|}{R}}{\ln R}\right)\right| \le \frac{2q}{|x| \ln R^2} \text{ and } \frac{1}{\ln^2 R^2} \le \frac{m}{\ln^2 |x|} \text{ for } x \in B_{2R^2},$$

where $q, m > 0$ are some constants. Then, considering the Hardy inequality, we obtain

$$\int_{B_{2R^2}} \left|\nabla\eta\left(\frac{\ln \frac{|x|}{R}}{\ln R}\right) u\right|^2 dx \le \int_{B_{2R^2} \setminus B_{R^2}} \frac{4\,q^2}{|x|^2 \ln^2 R^2}\, |u|^2\, dx \le$$

$$4\,q^2 m \int_{B_{2R^2} \setminus B_{R^2}} \frac{|u|^2}{|x|^2 \ln^2 |x|}\, dx \le \frac{4q^2 m}{k} \int_{B_{2R^2}} |\nabla u|^2 dx.$$

Thus,

$$\operatorname{cap}_{\varphi}((\mathbb{R}^2 \setminus \Omega) \cap \overline{B}_{R^2}) \leq \int\limits_{B_{2R^2}} |\nabla \nu_R|^2 dx \leq \gamma \int\limits_{B_{2R^2}} |\nabla u|^2 dx < \infty,$$

where $\gamma > 0$ is a constant, which doesn't depend on ν_R. Proceeding to the limit as $R \to \infty$, we obtain

$$\operatorname{cap}_{\varphi}(\mathbb{R}^2 \setminus \Omega) \leq \gamma \int\limits_{\mathbb{R}^2} |\nabla u|^2 dx < \infty.$$

Thus, the Theorem 2 is completely proved. □

Proof of the Theorem 3. Let $\operatorname{cap}_{\varphi - c}(\mathbb{R}^n \setminus \Omega) < \infty$. Then, by the Theorem 2, there is a function u, which is a solution of the problem (1). Let $n \geq 3$. Let's consider a shearing function $\eta_k \in C_0^\infty(\mathbb{R}^n)$ such that $\eta_k(x) = 1$ on $B_{2^{k+1}} \setminus B_{2^{k-1}}$ and $\operatorname{supp} \eta_k(x) \subset B_{2^{k+2}} \setminus B_{2^{k-2}}$, $k = 1, 2, \ldots$, which is constructed as follows. Let $\eta(x)$ is a monotone non-decreasing function from $C^\infty(\mathbb{R}^n)$, which is equal to zero on the interval $[-\infty, \frac{1}{4}]$ and is equal to one on the interval $[\frac{3}{4}, +\infty]$. Further, we denote by $\eta_k(x)$ the following function

$$\eta_k(x) = \begin{cases} \eta\left(\dfrac{|x| - r_{k-2}}{r_{k-1} - r_{k-2}}\right), & \text{if } x \in \overline{B}_{r_{k-1}} \setminus B_{r_{k-2}} \\ 1, & \text{if } x \in \overline{B}_{r_{k+1}} \setminus B_{r_{k-1}} \\ \eta\left(\dfrac{r_{k+2} - |x|}{r_{k+2} - r_{k+1}}\right), & \text{if } x \in \overline{B}_{r_{k+2}} \setminus B_{r_{k+1}}. \end{cases}$$

We have the estimate

$$|\nabla \eta_k(x)|^2 \leq \frac{c}{|x|^2},$$

where c doesn't depend on k. Then, considering the Hardy inequality, we obtain a chain of inequalities

$$\operatorname{cap}_{\varphi - c}((\overline{B}_{r_{k+1}} \setminus B_{r_{k-1}}) \cap (\mathbb{R}^n \setminus \Omega), B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}) \leq$$

$$\int\limits_{B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}} |\nabla (\eta_k(x) u(x))|^2 \, dx \leq 2 \int\limits_{B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}} |\nabla \eta_k(x) u(x)|^2 \, dx +$$

$$2 \int\limits_{B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}} |\eta_k(x) \nabla u(x)|^2 \, dx \leq 2c \int\limits_{B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}} \frac{|u(x)|^2}{|x|^2} \, dx +$$

$$b_1 \int\limits_{B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}} |\nabla u(x)|^2 \, dx,$$

where b_1 is a positive constant, which doesn't depend on u. Thus,

$$\sum_{k=1}^{\infty} \operatorname{cap}_{\varphi - c}((\overline{B}_{r_{k+1}} \setminus B_{r_{k-1}}) \cap (\mathbb{R}^n \setminus \Omega), B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}) \leq$$
$$\sum_{k=1}^{\infty} 2 \int\limits_{B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}} \frac{|u(x)|^2}{|x|^2} \, dx + \sum_{k=1}^{\infty} b_1 \int\limits_{B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}} |\nabla u(x)|^2 \, dx.$$

Due to the fact that each point $x \in \mathbb{R}^n$ belongs to no more than three areas $B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}$, we obtain

$$\sum_{k=1}^{\infty} 2 \int\limits_{B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}} \frac{|u(x)|^2}{|x|^2} \, dx + \sum_{k=1}^{\infty} b_1 \int\limits_{B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}} |\nabla u(x)|^2 \, dx \leq$$
$$b_2 \int\limits_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^2} \, dx + b_3 \int\limits_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx \leq b_4 \int\limits_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx < \infty,$$

where b_2, b_3, b_4 are positive constants, which don't depend on u.

Now let $n = 2$. Denote

$$\eta_k(x) = \begin{cases} \eta \left(\dfrac{\ln \frac{|x|}{r_{k-2}}}{\ln \frac{r_{k-1}}{r_{k-2}}} \right), & \text{if } x \in \overline{B}_{r_{k-1}} \setminus B_{r_{k-2}} \\ 1\,, & \text{if } x \in \overline{B}_{r_{k+1}} \setminus B_{r_{k-1}} \\ \eta \left(\dfrac{\ln \frac{r_{k+2}}{|x|}}{\ln \frac{r_{k+2}}{r_{k+1}}} \right), & \text{if } x \in \overline{B}_{r_{k+2}} \setminus B_{r_{k+1}}, \end{cases}$$

where $\eta(x)$ is a monotone non-decreasing function from $C^\infty(\mathbb{R}^n)$, which is equal to zero on the interval $[-\infty, \frac{1}{4}]$ and is equal to one on the interval $[\frac{3}{4}, +\infty]$. We have the estimate

$$|\nabla \eta_k(x)|^2 \leq \frac{c}{|x|^2 \ln^2 |x|},$$

where c doesn't depend on k. Then, considering the Hardy inequality, we obtain a chain of inequalities

$$\operatorname{cap}_{\varphi - c}((\overline{B}_{r_{k+1}} \setminus B_{r_{k-1}}) \cap (\mathbb{R}^n \setminus \Omega), B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}) \leq$$

$$\int\limits_{B_{r_{k+2}}\setminus\overline{B}_{r_{k-2}}} |\nabla\left(\eta_k(x)u(x)\right)|^2\,dx \le 2 \int\limits_{B_{r_{k+2}}\setminus\overline{B}_{r_{k-2}}} |\nabla\eta_k(x)u(x)|^2\,dx +$$
$$2 \int\limits_{B_{r_{k+2}}\setminus\overline{B}_{r_{k-2}}} |\eta_k(x)\nabla u(x)|^2\,dx \le 2c \int\limits_{B_{r_{k+2}}\setminus\overline{B}_{r_{k-2}}} \frac{|u(x)|^2}{|x|^2\ln^2|x|}\,dx +$$
$$b_1 \int\limits_{B_{r_{k+2}}\setminus\overline{B}_{r_{k-2}}} |\nabla u(x)|^2\,dx,$$

where b_1 is a positive constant, which doesn't depend on u. Thus,

$$\sum_{k=1}^{\infty} \operatorname{cap}_{\varphi-c}((\overline{B}_{r_{k+1}}\setminus B_{r_{k-1}})\cap(\mathbb{R}^n\setminus\Omega), B_{r_{k+2}}\setminus\overline{B}_{r_{k-2}}) \le$$
$$\sum_{k=1}^{\infty} 2 \int\limits_{B_{r_{k+2}}\setminus\overline{B}_{r_{k-2}}} \frac{|u(x)|^2}{|x|^2\ln^2|x|}\,dx + \sum_{k=1}^{\infty} b_1 \int\limits_{B_{r_{k+2}}\setminus\overline{B}_{r_{k-2}}} |\nabla u(x)|^2\,dx.$$

Due to the fact that each point $x\in\mathbb{R}^n$ belongs to no more than three areas $B_{r_{k+2}}\setminus\overline{B}_{r_{k-2}}$, we obtain

$$\sum_{k=1}^{\infty} 2 \int\limits_{B_{r_{k+2}}\setminus\overline{B}_{r_{k-2}}} \frac{|u(x)|^2}{|x|^2\ln^2|x|}\,dx + \sum_{k=1}^{\infty} b_1 \int\limits_{B_{r_{k+2}}\setminus\overline{B}_{r_{k-2}}} |\nabla u(x)|^2\,dx \le$$
$$b_2 \int\limits_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^2\ln^2|x|}\,dx + b_3 \int\limits_{\mathbb{R}^n} |\nabla u(x)|^2\,dx \le b_4 \int\limits_{\mathbb{R}^n} |\nabla u(x)|^2\,dx < \infty,$$

where b_2, b_3, b_4 are positive constants, which don't depend on u.

The converse. Let

$$\sum_{k=1}^{\infty} \operatorname{cap}_{\varphi-c}((\overline{B}_{r_{k+1}}\setminus B_{r_{k-1}})\cap(\mathbb{R}^n\setminus\Omega), B_{r_{k+2}}\setminus\overline{B}_{r_{k-2}}) < \infty$$

and $n\ge 3$. Let's consider a shearing function

$$\widetilde{\psi}_k(x) = \begin{cases} \eta\left(\dfrac{|x|-r_{k-1}}{r_k-r_{k-1}}\right), & \text{if } |x|\le r_k \\ \eta\left(\dfrac{r_{k+1}-|x|}{r_{k+1}-r_k}\right), & \text{if } |x|\ge r_k. \end{cases}$$

Denote

$$\psi_k(x) = \frac{\widetilde{\psi}_k(x)}{\sum\limits_{i=0}^{\infty} \widetilde{\psi}_i(x)}.$$

Obviously,

$$\sum_{k=1}^{\infty} \psi_k(x) = 1.$$

From the condition on the capacity, we have functions $u_k(x)$, which implement the capacity and equal to $\varphi - c$ on $B_{r_{k+1}} \setminus B_{r_{k-1}}$ and with supports from $B_{r_{k+2}} \setminus B_{r_{k-2}}$. Let's notice that

$$\sum_{k=N_1}^{N_2} u_k(x)\psi_k(x) = \varphi - c,$$

if x is from a neighborhood of the set $(\overline{B}_{r_{N_2-1}} \setminus B_{r_{N_1+1}}) \cap (\mathbb{R}^n \setminus \Omega)$. Then we obtain

$$\left| \nabla \sum_{k=N_1}^{N_2} u_k(x)\psi_k(x) \right|^2 = \left| \sum_{k=N_1}^{N_2} \nabla u_k(x)\psi_k(x) + \sum_{k=N_1}^{N_2} u_k(x)\nabla\psi_k(x) \right|^2 \leq$$

$$2 \left| \sum_{k=N_1}^{N_2} \nabla u_k(x)\psi_k(x) \right|^2 + 2 \left| \sum_{k=N_1}^{N_2} u_k(x)\nabla\psi_k(x) \right|^2 .$$

Since for each $x \in \mathbb{R}^n$ there are no more than three natural numbers $k \in \{N_1, \ldots, N_2\}$ such that $\psi_k(x) \neq 0$, then we obtain

$$\left| \sum_{k=N_1}^{N_2} \nabla u_k(x)\psi_k(x) \right|^2 \leq 9 \sum_{k=N_1}^{N_2} |\nabla u_k(x)|^2 |\psi_k(x)|^2.$$

Similarly,

$$\left| \sum_{k=N_1}^{N_2} u_k(x)\nabla\psi_k(x) \right|^2 \leq 9 \sum_{k=N_1}^{N_2} |u_k(x)|^2 |\nabla\psi_k(x)|^2.$$

As a result, we obtain

$$\left| \nabla \sum_{k=N_1}^{N_2} u_k(x)\psi_k(x) \right|^2 \leq 18 \sum_{k=N_1}^{N_2} |\nabla u_k(x)|^2 |\psi_k(x)|^2 + 18 \sum_{k=N_1}^{N_2} |\nabla u_k(x)|^2 |\psi_k(x)|^2.$$

Therefore,

$$\int\limits_{\mathbb{R}^n} \left| \nabla \left(\sum_{k=N_1}^{N_2} u_k(x)\psi_k(x) \right) \right|^2 dx \le 18 \left(\sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |\nabla u_k(x)\psi_k(x)|^2 dx + \sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |u_k(x)\nabla\psi_k(x)|^2 dx \right).$$

The first term in the last expression can be estimated as follows

$$\sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |\nabla u_k(x)\psi_k(x)|^2 dx \le \sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |\nabla u_k(x)|^2 dx,$$

since $|\psi_k(x)| \le 1$. Using the fact that

$$|\nabla\psi_k(x)| \le \frac{r_{k+1} - r_k}{2^k}$$

and Friedrichs' inequality, we estimate the second term as follows

$$\sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |u_k(x)\nabla\psi_k(x)|^2 dx \le \sum_{k=N_1}^{N_2} \frac{(r_{k+1} - r_k)^2}{4^k} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |u_k(x)|^2 dx \le c_1 \sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |\nabla u_k(x)|^2 dx,$$

where c_1 is a positive constant, which doesn't depend on u_k and ψ_k. We obtain a chain of inequalities

$$\operatorname{cap}_{\varphi-c}((\overline{B}_{r_{N_2}} \setminus B_{r_{N_1}}) \cap (\mathbb{R}^n \setminus \Omega)) \le \int\limits_{\mathbb{R}^n} \left| \nabla \left(\sum_{k=N_1}^{N_2} u_k(x)\psi_k(x) \right) \right|^2 dx \le c_2 \sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |\nabla u_k(x)|^2 dx =$$

$$\sum_{k=N_1}^{N_2} \operatorname{cap}_{\varphi-c}((\overline{B}_{r_{k+1}} \setminus B_{r_{k-1}}) \cap (\mathbb{R}^n \setminus \Omega), B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}),$$

where c_2 is a positive constant, which doesn't depend on u_k and ψ_k. As N_2 tending to infinity, we obtain

$$\operatorname{cap}_{\varphi-c}((\mathbb{R}^n \setminus \Omega) \setminus B_{r_{N_1}}) \le$$
$$\sum_{k=N_1}^{\infty} \operatorname{cap}_{\varphi-c}((\overline{B}_{r_{k+1}} \setminus B_{r_{k-1}}) \cap (\mathbb{R}^n \setminus \Omega), B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}) < \infty,$$

what implies that

$$\operatorname{cap}_{\varphi-c}(\mathbb{R}^n \setminus \Omega) < \infty.$$

Now let $n = 2$. Then let's consider a shearing function

$$\widetilde{\psi}_k(x) = \begin{cases} \eta\left(\dfrac{\ln \frac{|x|}{r_{k-1}}}{\ln \frac{r_k}{r_{k-1}}}\right), & \text{if } |x| \le r_k \\ \eta\left(\dfrac{\ln \frac{r_{k+1}}{|x|}}{\ln \frac{r_{k+1}}{r_k}}\right), & \text{if } |x| \ge r_k, \end{cases}$$

and let

$$\psi_k(x) = \frac{\widetilde{\psi}_k(x)}{\sum\limits_{i=0}^{\infty} \widetilde{\psi}_i(x)}.$$

Obviously,

$$\sum_{k=1}^{\infty} \psi_k(x) = 1.$$

From the condition on the capacity, we have functions $u_k(x)$, which implement the capacity and equal to $\varphi - c$ on $B_{r_{k+1}} \setminus B_{r_{k-1}}$ and with supports from $B_{r_{k+2}} \setminus B_{r_{k-2}}$. Let's notice that

$$\sum_{k=N_1}^{N_2} u_k(x)\psi_k(x) = \varphi - c,$$

if x is from a neighborhood of the set $(\overline{B}_{r_{N_2-1}} \setminus B_{r_{N_1+1}}) \cap (\mathbb{R}^n \setminus \Omega)$. It is easy to see that the functions ψ_k again satisfy the following relations

$$\int\limits_{\mathbb{R}^n} \left|\nabla\left(\sum_{k=N_1}^{N_2} u_k(x)\psi_k(x)\right)\right|^2 dx \le 18\left(\sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |\nabla u_k(x)\psi_k(x)|^2 dx + \right.$$

$$\left. \sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |u_k(x)\nabla\psi_k(x)|^2 dx \right).$$

The first term in the last expression can be estimated as follows

$$\sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |\nabla u_k(x)\psi_k(x)|^2 dx \le \sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |\nabla u_k(x)|^2 dx,$$

since $|\psi_k(x)| \le 1$. Then, using the fact that $|\nabla\psi_k(x)| \le \dfrac{\mathrm{const}}{|x| \ln |x|}$ and the Hardy inequality, we estimate the second term as follows

$$\sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |u_k(x)\nabla\psi_k(x)|^2 dx \le \sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} \frac{|u_k(x)|^2}{|x|^2 \ln^2 |x|} dx \le$$
$$c_1 \sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |\nabla u_k(x)|^2 dx,$$

where c_1 is a positive constant, which doesn't depend on u_k and ψ_k. We obtain a chain of inequalities

$$\mathrm{cap}_{\,\varphi-\,c}((\overline{B}_{r_{N_2}} \setminus B_{r_{N_1}}) \cap (\mathbb{R}^n \setminus \Omega)) \le \int\limits_{\mathbb{R}^n} \left| \nabla \left(\sum_{k=N_1}^{N_2} u_k(x)\psi_k(x) \right) \right|^2 dx \le$$
$$c_2 \sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |\nabla u_k(x)|^2 dx =$$
$$\sum_{k=N_1}^{N_2} \mathrm{cap}_{\,\varphi-\,c}((\overline{B}_{r_{k+1}} \setminus B_{r_{k-1}}) \cap (\mathbb{R}^n \setminus \Omega), B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}),$$

where c_2 is a positive constant, which doesn't depend on u_k and ψ_k. As N_2 tending to infinity, we obtain

$$\mathrm{cap}_{\,\varphi-\,c}((\mathbb{R}^n \setminus \Omega) \setminus B_{r_{N_1}}) \le$$
$$\sum_{k=N_1}^{\infty} \mathrm{cap}_{\,\varphi-\,c}((\overline{B}_{r_{k+1}} \setminus B_{r_{k-1}}) \cap (\mathbb{R}^n \setminus \Omega), B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}) < \infty,$$

what implies that

$$\operatorname{cap}_{\varphi-c}(\mathbb{R}^n \setminus \Omega) < \infty.$$

$\square$

Proof of the Theorem 4. Let u is a solution of the problem (1). Let's extend u on $\mathbb{R}^n \setminus \Omega$ with value φ. Then, by the inequality (2), we obtain that

$$\sigma(\omega_k, \mu_k)\|u - c_k\|_{L_2(\omega_k,\mu_k)} \le \|\nabla u\|_{L_2(\omega_k)},$$

what implies that

$$\sigma^2(\omega_k, \mu_k)\|\varphi - c_k\|^2_{L_2(\omega_k\setminus\Omega,\mu_k)} = \sigma^2(\omega_k, \mu_k)\|u - c_k\|^2_{L_2(\omega_k\setminus\Omega,\mu_k)} \le \|\nabla u\|^2_{L_2(\omega_k)}.$$

Summing this relation, we obtain

$$\sum_{k=1}^{\infty} \sigma^2(\omega_k, \mu_k)\|\varphi - c_k\|^2_{L_2(\omega_k\setminus\Omega,\mu_k)} \le \sum_{k=1}^{\infty} \|\nabla u\|^2_{L_2(\omega_k)} =$$
$$\sum_{k=1}^{\infty} \int_{\omega_k\cap\Omega} |\nabla u|^2 dx + \sum_{k=1}^{\infty} \int_{\omega_k\setminus\Omega} |\nabla u|^2 dx.$$

Let's notice that

$$\sum_{k=1}^{\infty} \int_{\omega_k\cap\Omega} |\nabla u|^2 dx \le \int_{\Omega} |\nabla u|^2 dx < \infty,$$

and

$$\sum_{k=1}^{\infty} \int_{\omega_k\setminus\Omega} |\nabla u|^2 dx = \sum_{k=1}^{\infty} \int_{\omega_k\setminus\Omega} |\nabla \varphi|^2 dx < \infty.$$

Thus, we have

$$\sum_{k=1}^{\infty} \sigma^2(\omega_k, \mu_k)\|\varphi - c_k\|^2_{L_2(\omega_k\setminus\Omega,\mu_k)} < \infty,$$

which immediately implies (4). The Theorem is completely proved. $\square$

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References

[1] A.L. Beklaryan, Existence theorems for elliptic equations in unbounded domains, *Proceedings of III International conference "Optimization and applications" (OPTIMA-2012)*, Portugal (2012), 47-50.

[2] A.L. Beklaryan, On the Existence of Solutions of the First Boundary Value Problem for Elliptic Equations on Unbounded Domains, *Russian Journal of Mathematical Physics*, **19**, No. 4 (2012), 508-511, DOI: 10.1134/S1061920812040115.

[3] L. Bers, F. John, M. Schechter, *Partial differential equations*, American Mathematical Soc., USA (1964).

[4] I.M. Gel'fand, G.E. Shilov, *Generalized functions. Vol. I: Properties and operations*, MA: Academic Press, Boston (1964).

[5] G.H. Hardy, J.E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, New York-London-Tokyo (1952).

[6] A.N. Kolmogorov, S.V. Fomin, *Elements of the Theory of Functions and Functional Analysis*, Dover Publications, USA (1999).

[7] V.A. Kondrat'ev, The Solvability of the First Boundary Value Problem for Strongly Elliptic Equations, *Tr. Mosk. Mat. Obs.*, **16** (1967), 209-292 [in Russian].

[8] A.A. Kon'kov, On the Dimension of the Solution Space of Elliptic Systems in Unbounded Domains, *Russ. Acad. Sci. Sb. Math.*, **80**, No. 2 (1995), 411-434, DOI: 10.1070/SM1995v080n02ABEH003531.

[9] O.A. Ladyzhenskaya, N.N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York-London (1968).

[10] V.G. Maz'ya, *Sobolev Spaces*, Springer-Verlag, Berlin-New York (1985).

[11] S.G. Mihlin, *Linear partial differential equations*, Vysshaya shkola, Russia (1977) [in Russian].

[12] V.S. Vladimirov, *Equations of mathematical physics*, Mir, Moscow (1984).