ISSN 0965-5425, Computational Mathematics and Mathematical Physics, 2009, Vol. 49, No. 2, pp. 363–372. © Pleiades Publishing, Ltd., 2009.
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Some Properties of the Equations Governing a Two-Dimensional Quasi-Gasdynamic Model of Traffic Flows

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Received May 23, 2008

Abstract—A recently proposed two-dimensional quasi-gasdynamic model of traffic flows is considered. Its Petrovskii parabolicity is analyzed, and the stability of small perturbations against a constant background is investigated. In a nonlinear setting, an energy equality is derived and an energy estimate of the solution is obtained.

DOI: 10.1134/S0965542509020146

Keywords: models of traffic flows, quasi-gasdynamic system of equations, second-order parabolic systems, stability of small perturbations, energy equality.

INTRODUCTION

The quasi-gasdynamic system of equations and the associated kinetically consistent difference schemes have been successfully applied in the mathematical modeling of certain complicated fluid dynamics problems (see [1, 2]). A new two-dimensional quasi-gasdynamic system for simulating traffic flows was recently proposed in [3].

In [4–7], the stability of small perturbations for quasi-gasdynamic systems was studied and a classification of these systems was given. Below, these issues are considered for the model described in [3], which, though related to the barotropic quasi-gasdynamic model from [7] in the two-dimensional case, has important specific features (in particular, the former model involves no equation for the second (lateral) velocity component). For a constant lateral velocity v, we establish the nonuniform parabolicity of the model (in the sense of Petrovskii) and give sufficient and necessary conditions for its uniform parabolicity. For arbitrary v, necessary and sufficient conditions for nonuniform parabolicity are presented. For constant v, the stability of small perturbations against a constant background is analyzed and time-uniform estimates are given for relative perturbations in the L^2 norms in the Cauchy problem and the initial–boundary value problem for the corresponding linearized system.

Additionally, in the original nonlinear setting for constant v and spatially periodic solutions, we deduce an energy equality and obtain a global a priori energy estimate of the solution.

1. TWO-DIMENSIONAL QUASI-GASDYNAMIC MODEL OF TRAFFIC FLOWS AND ITS PARABOLICITY

The two-dimensional quasi-gasdynamic system of equations governing traffic flows [3] can be written as

$$\begin{gathered}\partial_t\rho + \partial_x(\rho u) + \partial_y(\rho v) = R_0[\rho, u] \\ \equiv \partial_x\{\tau[\partial_x(\rho u^2 + p) + 2\partial_y(\rho u v)]\} + \partial_y\{\tau[\partial_y(\rho v^2) + \kappa\partial_y\rho]\},\end{gathered} \tag{1}$$

$$\begin{gathered}\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_y(\rho u v) + \partial_x p = R_1[\rho, u] \\ \equiv \partial_x\{\tau[\partial_x(\rho u^3 + pu) + 2\partial_y(\rho u^2 v)]\} + \partial_y\{\tau[\partial_y(\rho u v^2) + \kappa\partial_y(\rho u)]\} + \rho a.\end{gathered} \tag{2}$$

The unknown functions are the traffic density ρ and the streamwise traffic velocity u; they depend on the coordinates x, y, and t (where x and y are the streamwise and spanwise coordinates and t is time). The partial derivatives with respect to x, y, and t are denoted by ∂_x, ∂_y, and ∂_t, respectively. It is assumed that

$$\tau = \tau(\rho, u) > 0, \quad p = p(\rho) > 0, \quad \kappa = \kappa(\rho) > 0, \quad v = v(\rho, u), \quad a = a(\rho, u, x, y, t),$$

where v is the lateral velocity. The details concerning the last functions are of no matter in this paper and, for this reason, are omitted.

Consider classical solutions (ρ, u) to system (1), (2) that satisfy it for the arguments (x, y, t) from a domain Q in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$ with values in a domain $D \subset \mathbb{R}^+ \times \mathbb{R}$. It is assumed that $\tau \in C^1(D)$, $v \in C^2(D)$, $a \in C(D \times Q)$, $p \in C^2(D_0)$, $\kappa \in C^1(D_0)$, and $p' > 0$. Here, the (finite or infinite) interval $D_0 \subset \mathbb{R}^+$ is the projection of D onto the first coordinate axis.

By using the mass balance equation (1), momentum equation (2) is rearranged into

$$\partial_t u + u\partial_x u + v\partial_y u + \frac{1}{\rho}\partial_x p = \frac{1}{\rho}(R_1[\rho, u] - R_0[\rho, u]u). \tag{3}$$

First, let $v \equiv \text{const}$ (this is the basic case treated in [3]). Then the right-hand sides of Eqs. (1) and (2) can be rewritten in the nonconservative form

$$R_0[\rho, u] = \tilde{R}_{0c}[\rho, u] + b_{0c}, \quad R_1[\rho, u] = \tilde{R}_{1c}[\rho, u] + b_{1c}, \tag{4}$$

where

$$\begin{gathered}\tilde{R}_{0c}[\rho, u] = \tau[(p' + u^2)\partial_x^2\rho + 2uv\partial_x\partial_y\rho + (\kappa + v^2)\partial_y^2\rho + 2\rho u\partial_x^2 u + 2\rho v\partial_x\partial_y u], \\ \tilde{R}_{1c}[\rho, u] = \tau[(p' + u^2)u\partial_x^2\rho + 2u^2 v\partial_x\partial_y\rho + (\kappa + v^2)u\partial_y^2\rho \\ + (p + 3\rho u^2)\partial_x^2 u + 4\rho uv\partial_x\partial_y u + \rho(\kappa + v^2)\partial_y^2 u].\end{gathered} \tag{5}$$

Here, b_{ic} (and b_i below), $i = 0, 1, 2$, depend on ρ, u, $\nabla\rho$, and ∇u, where $\nabla = (\partial_x, \partial_y)$, and are lower order terms. Therefore, the right-hand side of Eq. (3) can be transformed into the nonconservative form

$$\frac{1}{\rho}(R_1[\rho, u] - R_0[\rho, u]u) = \tau\left[\left(\frac{p}{\rho} + u^2\right)\partial_x^2 u + 2uv\partial_x\partial_y u + (\kappa + v^2)\partial_y^2 u\right] + b_2, \tag{6}$$

where there are no second derivatives of ρ.

Let $\lambda \in \mathbb{C}$; $\xi, \eta \in \mathbb{R}$; and $\xi^2 + \eta^2 = 1$. According to Eqs. (1) and (3) and formulas (4)–(6), the principal symbol of the system is the matrix $P_c^{(0)}(\rho, u, \lambda, \xi, \eta) = \lambda I + \tau A_c(\rho, u, \xi, \eta)$, where I is the second-order identity matrix and A_c is given by

$$A_c(\rho, u, \xi, \eta) = \begin{bmatrix} (p' + u^2)\xi^2 + 2uv\xi\eta + (\kappa + v^2)\eta^2 & 2\rho u\xi^2 + 2\rho v\xi\eta \\ 0 & \left(\frac{p}{\rho} + u^2\right)\xi^2 + 2uv\xi\eta + (\kappa + v^2)\eta^2 \end{bmatrix}. \tag{7}$$

System (1), (3) is nonuniformly or uniformly Petrovskii parabolic in D [8, 9] if

$$\inf_{\xi^2 + \eta^2 = 1}\lambda[A_c] > 0 \quad \text{in} \quad D,$$
$$\inf_D(\tau \inf_{\xi^2 + \eta^2 = 1}\lambda[A_c]) > 0,$$

respectively. Here, $\lambda[A_c]$ are the eigenvalues of A_c (which coincide with its diagonal elements).

Define $\Gamma(\rho) := \rho p'(\rho)/p(\rho)$ is the first adiabatic exponent of $p(\rho)$.

Proposition 1. *Let* $v \equiv \text{const}$. *Then the following assertions hold*:

1. *The two-dimensional quasi-gasdynamic model is nonuniformly parabolic in any domain D.*

2. *A sufficient condition for its uniform parabolicity in D is that*

$$\inf_D\left(\tau\min\left\{\min\{\Gamma, 1\}\frac{p}{\rho}, \kappa\right\}\right) > 0.$$

3. *For its uniform parabolicity in D, it is necessary and, if, for some* $\delta > 1$, *at least one of the two conditions*

$$|u| \le \delta\sqrt{\min\{\Gamma, 1\}\frac{p}{\rho}}, \quad |v| \le \delta\sqrt{\kappa}, \tag{8}$$

is satisfied at each point of D, it is also sufficient that

$$\inf_D\left(\tau\min\left\{\min\{\Gamma, 1\}\frac{p}{\rho} + u^2, \kappa + v^2\right\}\right) > 0.$$

Proof. Obviously,

$$A_c(\rho, u, \xi, \eta) = \begin{bmatrix} \Gamma\frac{p}{\rho}\xi^2 + \kappa\eta^2 + (u\xi + v\eta)^2 & 2\rho u\xi(u\xi + v\eta) \\ 0 & \frac{p}{\rho}\xi^2 + \kappa\eta^2 + (u\xi + v\eta)^2 \end{bmatrix},$$

which directly implies items 1 and 2.

It is also true that $\min_{\xi^2+\eta^2=1}\lambda[A_c] = \lambda_{\min}[S_0]$, where $\lambda_{\min}[S_0]$ is the minimum eigenvalue of the matrix

$$S_0 = \begin{bmatrix} \alpha + u^2 & uv \\ uv & \kappa + v^2 \end{bmatrix},$$

and the parameter α takes the values $\Gamma p/\rho$ and p/ρ. Moreover, $\lambda_{\min}[S_0]$ satisfies the two-sided estimate

$$\frac{\det S_0}{\operatorname{tr} S_0} \le \lambda_{\min}[S_0] \le 2\frac{\det S_0}{\operatorname{tr} S_0}.$$

Since

$$\det S_0 \le (\alpha + u^2)(\kappa + v^2) \le \min\{\alpha + u^2, \kappa + v^2\}\operatorname{tr} S_0,$$

and, under of one of the conditions in (8), we have

$$\det S_0 \ge \frac{1}{1+\delta^2}(\alpha + u^2)(\kappa + v^2) \ge \frac{1}{2(1+\delta^2)}\min\{\alpha + u^2, \kappa + v^2\}\operatorname{tr} S_0;$$

thus, item 3 also holds.

The necessary and sufficient conditions for uniform parabolicity in D are also easy to write in the general case. They are omitted since they are rather cumbersome. Recall that uniform parabolicity allows us to formulate an existence and uniqueness theorem for a local-in-time classical solution to the Cauchy problem for the system of equations under study. This theorem is similar to that presented in [7] and, for this reason, is omitted. Note that, in contrast to [7], there are no constraints on Γ from above.

Now, consider the general case $v = v(\rho, u)$. Then

$$R_0[\rho, u] = \tilde{R}_0[\rho, u] + b_0, \quad R_1[\rho, u] = \tilde{R}_1[\rho, u] + b_1,$$

where

$$\tilde{R}_0[\rho, u] = \tilde{R}_{0c}[\rho, u] + 2\tau(\rho u v'_\rho\partial_x\partial_y\rho + \rho v v'_\rho\partial_y^2\rho + \rho u v'_u\partial_x\partial_y u + \rho v v'_u\partial_y^2 u),$$

$$\tilde{R}_1[\rho, u] = \tilde{R}_{1c}[\rho, u] + 2\tau(\rho u^2 v'_\rho\partial_x\partial_y\rho + \rho u v v'_\rho\partial_y^2\rho + \rho u^2 v'_u\partial_x\partial_y u + \rho u v v'_u\partial_y^2 u),$$

and v'_ρ and v'_u denote the partial derivatives of v. Therefore, formula (6) remains the same (with another b_2). The principal symbol of the system is $P^{(0)}(\rho, u, \lambda, \xi, \eta) = \lambda I + \tau A(\rho, u, \xi, \eta)$, where

$$A = A_c + \begin{bmatrix} 2\rho u v'_\rho \xi\eta + 2\rho v v'_\rho \eta^2 & 2\rho u v'_u \xi\eta + 2\rho v v'_u \eta^2 \\ 0 & 0 \end{bmatrix}.$$

Define the dimensionless variables $\hat{u} := u/\sqrt{p'}$ and $\hat{v} := v/\sqrt{\kappa}$.

Proposition 2. *The two-dimensional quasi-gasdynamic model is nonuniformly parabolic in the domain D if and only if*

$$\left|\frac{\rho v'_\rho}{\sqrt{\kappa}} - \frac{\hat{v}}{\hat{u}^2}\right| < \sqrt{\left(\frac{\hat{v}}{\hat{u}^2}\right)^2 + 1 + \frac{1+\hat{v}^2}{\hat{u}^2}} \quad \textit{at points} \quad (\rho, u) \in D \quad \textit{with} \quad u \neq 0, \tag{9}$$

$$\operatorname{sgn} v \frac{\rho v'_\rho}{\sqrt{\kappa}} > -\frac{1}{2|\hat{v}|}(1 + \hat{v}^2) \quad \textit{at points} \quad (\rho, u) \in D \quad \textit{with} \quad u = 0, \quad v \neq 0. \tag{10}$$

Proof. The nonuniform parabolicity of the system is reduced to the condition that the quadratic form

$$a_{11}(\rho, u, \xi, \eta) = (p' + u^2)\xi^2 + 2(uv + \rho u v'_\rho)\xi\eta + (\kappa + v^2 + 2\rho v v'_\rho)\eta^2$$

is positive definite at each point of D. The matrix

$$Q = \begin{bmatrix} p' + u^2 & uv + \rho u v'_\rho \\ uv + \rho u v'_\rho & \kappa + v^2 + 2\rho v v'_\rho \end{bmatrix}$$

of this quadratic form is positive definite if and only if $\det Q > 0$. Defining $w := \rho v'_\rho/\sqrt{\kappa}$, we obtain

$$\det Q = -p'\kappa[\hat{u}^2 w^2 - 2\hat{v}w - (1 + \hat{u}^2 + \hat{v}^2)].$$

For $u \neq 0$, it is convenient to write

$$\det Q = -p'\kappa\hat{u}^2\left[w^2 - 2\frac{\hat{v}}{\hat{u}^2}w - \left(1 + \frac{1+\hat{v}^2}{\hat{u}^2}\right)\right],$$

and the condition $\det Q > 0$ leads to (9).

For $u = 0$, the condition $\det Q > 0$ leads to (10).

It is easy to verify that, as $u \longrightarrow 0$, condition (9) transforms into the nonstrict variant of condition (10).

2. STABILITY OF SMALL PERTURBATIONS

Let $v \equiv \text{const}$ and $a = 0$. Consider solutions to system (1), (3) of the form $\rho = \bar{\rho} + \delta\rho$ and $u = \bar{u} + \delta u$, where $\bar{\rho} > 0$ and $\bar{u}$ are constant background values of the unknowns and $\delta\rho$ and δu are their small perturbations. Substituting these solutions into the system, we proceed as in the derivation of formulas (4)–(6), taking into account that $\bar{\rho} > 0$ and $\bar{u}$ are constants and discarding the terms whose orders are higher than the first with respect to the perturbations (and their derivatives). As a result, we derive the following linearized system for the perturbations:

$$\partial_t\delta\rho + \bar{u}\partial_x\delta\rho + \bar{\rho}\partial_x\delta u + v\partial_y\delta\rho$$
$$= \tau(\bar{\rho}, \bar{u})[(p'(\bar{\rho}) + \bar{u}^2)\partial_x^2\delta\rho + 2\bar{u}v\partial_x\partial_y\delta\rho + (\kappa(\bar{\rho}) + v^2)\partial_y^2\delta\rho + 2\bar{\rho}\bar{u}\partial_x^2\delta u + 2\bar{\rho}v\partial_x\partial_y\delta u],$$
$$\partial_t\delta u + \bar{u}\partial_x\delta u + v\partial_y\delta u + \frac{p'(\bar{\rho})}{\bar{\rho}}\partial_x\delta\rho$$
$$= \tau(\bar{\rho}, \bar{u})\left[\left(\frac{p(\bar{\rho})}{\bar{\rho}} + \bar{u}^2\right)\partial_x^2\delta u + 2\bar{u}v\partial_x\partial_y\delta u + (\kappa(\bar{\rho}) + v^2)\partial_y^2\delta u\right].$$

For notational simplicity, the bar over the background values and, as before, the function arguments ρ and u are omitted.

By introducing the column vector of perturbations $\delta\mathbf{z} = (\delta\rho, \delta u)$, the last system is rewritten in the vector form

$$\partial_t\delta\mathbf{z} + \sqrt{p'}B^{(1)}\partial_x\delta\mathbf{z} + \sqrt{\kappa}\hat{v}\partial_y\delta\mathbf{z} = \tau[p'A^{(11)}\partial_x^2\delta\mathbf{z} + 2\sqrt{p'\kappa}A^{(12)}\partial_x\partial_y\delta\mathbf{z} + \kappa(1 + \hat{v}^2)\partial_y^2\delta\mathbf{z}] \tag{11}$$

with the matrices

$$A^{(11)} = \begin{bmatrix} 1 + \hat{u}^2 & 2\dfrac{\rho}{\sqrt{p'}}\hat{u} \\ 0 & \dfrac{1}{\Gamma} + \hat{u}^2 \end{bmatrix}, \quad A^{(12)} = \begin{bmatrix} \hat{u}\hat{v} & \dfrac{\rho}{\sqrt{p'}}\hat{v} \\ 0 & \hat{u}\hat{v} \end{bmatrix}, \quad B^{(1)} = \begin{bmatrix} \hat{u} & \dfrac{\rho}{\sqrt{p'}} \\ \dfrac{\sqrt{p'}}{\rho} & \hat{u} \end{bmatrix}.$$

First, consider the Cauchy problem for linearized system (11) in the half-space $\mathbb{R}^2 \times \mathbb{R}^+$ with the initial conditions

$$\delta\mathbf{z}|_{t=0} = \delta\mathbf{z}^0 := (\delta\rho^0, \delta u^0). \tag{12}$$

The functions $\delta\mathbf{z}^0$ and $\delta\mathbf{z}$ can be assumed to be complex-valued. Define the nondimensionalized vectors

$$\delta\hat{\mathbf{z}} := \left(\frac{\delta\rho}{\rho}, \frac{\delta u}{\sqrt{p'}}\right), \quad \delta\hat{\mathbf{z}}^0 := \left(\frac{\delta\rho^0}{\rho}, \frac{\delta u^0}{\sqrt{p'}}\right).$$

The vector $\delta\hat{\mathbf{z}}$ consists of the relative perturbation in the density and the perturbation in the velocity divided by the background speed of sound.

The following two propositions involve the complex Lebesgue $L^2(G)$ and Sobolev $H^1(G)$ spaces, where G is a domain in $\mathbb{R}^m$ (including $G = \mathbb{R}^m$). For a vector function $\mathbf{w} = (w_1, \ldots, w_l) \in L^2(G)$, we define $\|\mathbf{w}\|_{L^2(G)} := \||\mathbf{w}|\|_{L^2(G)}$.

Proposition 3. *Let* $\delta\mathbf{z}^0 \in L^2(\mathbb{R}^2)$. *Then the Fourier solution to Cauchy problem* (11), (12) *satisfies the t-global estimate*

$$\max\{\sup_{t\geq 0}\|\delta\hat{\mathbf{z}}(\cdot, t)\|_{L^2(\mathbb{R}^2)}, \sqrt{2\tau p'\underline{\lambda}}\|\nabla\delta\hat{\mathbf{z}}\|_{L^2(\mathbb{R}^2\times\mathbb{R}^+)}\} \leq \|\delta\hat{\mathbf{z}}^0\|_{L^2(\mathbb{R}^2)}, \tag{13}$$

where $|\nabla\varphi| := (|\partial_x\varphi|^2 + |\partial_y\varphi|^2)^{1/2}$ *and*

$$\underline{\lambda} := C(\Gamma)\min\left\{1 + \frac{1}{\Gamma}, 2\frac{\kappa}{p'}\right\}, \quad C(\Gamma) := \frac{1}{\Gamma + 1}\min\left\{\frac{\Gamma}{\Gamma + 1}, \frac{1}{4}\right\}.$$

If $\delta\mathbf{z}^0 \in H^1(\mathbb{R}^2)$, *then we also have the t-global estimate*

$$\max\{\sup_{t\geq 0}\|\nabla\delta\hat{\mathbf{z}}(\cdot, t)\|_{L^2(\mathbb{R}^2)}, \sqrt{2\tau p'\underline{\lambda}}\|\partial^2\delta\hat{\mathbf{z}}\|_{L^2(\mathbb{R}^2\times\mathbb{R}^+)}\} \leq \|\nabla\delta\hat{\mathbf{z}}^0\|_{L^2(\mathbb{R}^2)},$$

where $|\partial^2\varphi| := (|\partial_x^2\varphi|^2 + 2|\partial_x\partial_y\varphi|^2 + |\partial_y^2\varphi|^2)^{1/2}$.

Proof. Following [7], we perform the change of variables $\delta\hat{\mathbf{z}} = P^{-1}\delta\mathbf{z}$, where $P := \mathrm{diag}\{\rho, \sqrt{p'}\}$ (i.e., P is a diagonal matrix of order 2 with the indicated diagonal elements). Multiplying system (11) by P^{-1} on the left, we transform it into

$$\partial_t\delta\hat{\mathbf{z}} + \sqrt{p'}\hat{B}^{(1)}\partial_x\delta\hat{\mathbf{z}} + \sqrt{\kappa}\hat{v}\partial_y\delta\hat{\mathbf{z}} = \tau[p'\hat{A}^{(11)}\partial_x^2\delta\hat{\mathbf{z}} + 2\sqrt{p'\kappa}\hat{A}^{(12)}\partial_x\partial_y\delta\hat{\mathbf{z}} + \kappa(1+\hat{v}^2)\partial_y^2\delta\hat{\mathbf{z}}] \tag{14}$$

with the nondimensionalized matrices

$$\hat{A}^{(11)} = \begin{bmatrix} 1+\hat{u}^2 & 2\hat{u} \\ 0 & \frac{1}{\Gamma}+\hat{u}^2 \end{bmatrix}, \quad \hat{A}^{(12)} = \begin{bmatrix} \hat{u}\hat{v} & \hat{v} \\ 0 & \hat{u}\hat{v} \end{bmatrix}, \quad \hat{B}^{(1)} = \begin{bmatrix} \hat{u} & 1 \\ 1 & \hat{u} \end{bmatrix},$$

which are obtained from the previous ones by the similarity transformations $\hat{A}^{(11)} := P^{-1}A^{(11)}P$, $\hat{A}^{(12)} := P^{-1}A^{(12)}P$, and $\hat{B}^{(1)} := P^{-1}B^{(1)}P$.

We make up the matrix

$$p'\xi^2\hat{A}^{(11)} + 2\sqrt{p'\kappa}\xi\eta\hat{A}^{(12)} + \kappa(1+\hat{v}^2)\eta^2 I = p'\hat{A},$$

where, as before, $\xi^2 + \eta^2 = 1$ and

$$\hat{A} = \begin{bmatrix} (1+\hat{u}^2)\xi^2 + 2\beta\hat{u}\hat{v}\xi\eta + \beta^2(1+\hat{v}^2)\eta^2 & 2(\hat{u}\xi^2 + \beta\hat{v}\xi\eta) \\ 0 & \left(\frac{1}{\Gamma}+\hat{u}^2\right)\xi^2 + 2\beta\hat{u}\hat{v}\xi\eta + \beta^2(1+\hat{v}^2)\eta^2 \end{bmatrix}$$

with $\beta := \sqrt{\kappa/p'}$. Clearly, $p'\hat{A} = P^{-1}A_cP$ (see (7)). Define

$$\hat{A}_s := \frac{1}{2}(\hat{A} + \hat{A}^T) = \begin{bmatrix} \xi^2 + \beta^2\eta^2 + (\hat{u}\xi + \beta\hat{v}\eta)^2 & (\hat{u}\xi + \beta\hat{v}\eta)\xi \\ (\hat{u}\xi + \beta\hat{v}\eta)\xi & \frac{1}{\Gamma}\xi^2 + \beta^2\eta^2 + (\hat{u}\xi + \beta\hat{v}\eta)^2 \end{bmatrix},$$

which is the symmetric part of the matrix $\hat{A}$. Let

$$c_0 := \det\hat{A}_s = \frac{1}{\Gamma}\xi^4 + \left(1+\frac{1}{\Gamma}\right)\beta^2\xi^2\eta^2 + \beta^4\eta^4 + \left(\frac{1}{\Gamma}\xi^2 + 2\beta^2\eta^2\right)(\hat{u}\xi + \beta\hat{v}\eta)^2 + (\hat{u}\xi + \beta\hat{v}\eta)^4,$$

$$c_1 := \mathrm{tr}\,\hat{A}_s = \left(1+\frac{1}{\Gamma}\right)\xi^2 + 2\beta^2\eta^2 + 2(\hat{u}\xi + \beta\hat{v}\eta)^2.$$

The eigenvalues of $\hat{A}_s$ are positive and satisfy the lower estimate $\lambda[\hat{A}_s] \ge c_0/c_1$. In the case under study, it is easy to see that $c_0 \ge C(\Gamma)c_1^2$; hence, we have the lower estimate

$$\lambda[\hat{A}_s] \ge \underline{\lambda} \tag{15}$$

with $C(\Gamma)$ and $\underline{\lambda}$ defined above.

The subsequent argument is similar to that presented in [7] in the analysis of linearized parabolic quasi-gasdynamic systems and, for this reason, is omitted. Note only that it makes use of the Fourier transform and the energy method and is based on the symmetry of $\hat{B}^{(1)}$ and on estimate (15). Although the matrix $\hat{A}$ is not symmetric (in contrast to [7]), the inequality

$$\mathrm{Re}(\hat{A}\psi, \psi)_{\mathbb{C}^2} = (\hat{A}_s\psi, \psi)_{\mathbb{C}^2} \ge \underline{\lambda}|\psi|^2 \quad \forall\psi \in \mathbb{C}^2 \tag{16}$$

is sufficient for the proof.

Remark 1. If $v \not\equiv \text{const}$, then the multiplier $\hat{v}$ on the left-hand side of system (11) is replaced by the matrix

$$B^{(2)} := \begin{bmatrix} \hat{v} + \frac{\rho}{\sqrt{p'}} v'_\rho & \frac{\rho}{\sqrt{p'}} v'_u \\ 0 & \hat{v} \end{bmatrix},$$

which is no longer symmetrizable by the similarity transformation with the matrix P.

Let Ω be a bounded two-dimensional domain. Consider linearized system (11) in the cylinder $Q_T := \Omega \times (0, T)$ (where $T > 0$) with the initial and boundary conditions

$$\delta \mathbf{z}|_{t=0} = \delta \mathbf{z}^0, \quad \delta \mathbf{z}|_{\partial\Omega \times (0, T)} = 0. \tag{17}$$

Weak solutions from the energy class of this initial–boundary value problem are defined according to [10].

Proposition 4. *Let* $\delta \mathbf{z}^0 \in L^2(\Omega)$. *Then the initial–boundary value problem* (11), (17) *has a unique weak solution from the energy class for any* $T > 0$ *and that solution satisfies a t-global estimate of form* (13) *with* Ω *substituted for* $\mathbb{R}^2$.

Proof. It is similar to that presented in [7] and makes use of the energy method for system (14); the symmetry of $\hat{B}^{(1)}$; and the inequality

$$\mathrm{Re}\int_\Omega [(p'\hat{A}^{(11)}\partial_x \mathbf{w}, \partial_x \mathbf{w})_{\mathbb{C}^2} + 2(\sqrt{p'}\kappa\hat{A}^{(12)}\partial_x \mathbf{w}, \partial_y \mathbf{w})_{\mathbb{C}^2} + (\kappa(1+\hat{v}^2)\partial_y \mathbf{w}, \partial_y \mathbf{w})_{\mathbb{C}^2}]dxdy$$

$$\geq \underline{\lambda}\|\nabla \mathbf{w}\|^2_{L^2(\Omega)} \quad \forall \mathbf{w} \in H^1_0(\Omega),$$

which follows (see [7]) from (16). Here, $H^1_0(\Omega)$ is the closure of smooth compactly supported functions in Ω with respect to the $H^1(\Omega)$ norm.

Remark 2. All the assertions remain valid if the terms with mixed derivatives in Eqs. (1) and (2) are written in symmetrized form:

$$\partial_x\{\tau\partial_y(\rho u v)\} + \partial_y\{\tau\partial_x(\rho u v)\}, \quad \partial_x\{\tau\partial_y(\rho u^2 v)\} + \partial_y\{\tau\partial_x(\rho u^2 v)\}.$$

3. ENERGY EQUALITY AND ESTIMATE

Returning to the original system (1), (2), we consider its solution (ρ, u) in $\mathbb{R}^2 \times [0, T]$ (where $T > 0$) that are periodic with a period of $X > 0$ with respect to x and with a period of $Y > 0$ with respect to y with the initial conditions

$$(\rho, u)|_{t=0} = (\rho^0(x, y), u^0(x, y)), \quad (x, y) \in \mathbb{R}^2, \tag{18}$$

where the initial data (ρ^0, u^0) are periodic in the same sense and $\rho^0 > 0$.

Let $t \in [0, T]$. Integrating Eqs. (1) and (2) over $\Omega := (0, X) \times (0, Y)$ and taking into account the periodicity conditions, we obtain

$$\partial_t \int_\Omega \rho dxdy = 0, \quad \partial_t \int_\Omega \rho u dxdy = \int_\Omega \rho a dxdy. \tag{19}$$

Equation (2) is multiplied by u and is integrated over Ω. Integration by parts with the periodicity conditions taken into account yields

$$\int_\Omega\Big[\frac{1}{2}\partial_t(\rho u^2)+\frac{1}{2}(\partial_t\rho)u^2-\rho u^2\partial_x u-\rho u v\partial_y u+p'(\partial_x\rho)u\Big]dxdy$$
$$+\int_\Omega\tau\{[\partial_x(\rho u^3+pu)+2\partial_y(\rho u^2 v)]\partial_x u+[\partial_y(\rho u v^2)+\kappa\partial_y(\rho u)]\partial_y u\}dxdy = \int_\Omega\rho a u dxdy. \tag{20}$$

Define the known function

$$P_0(r) := \int_{r_0}^{r}\Big(\frac{r}{s}-1\Big)p'(s)ds,\quad r>0,$$

with some $r_0>0$ such that

$$P_0'(r) = \int_{r_0}^{r}\frac{p'(s)}{s}ds,\quad P_0''(r) = \frac{p'(r)}{r},\quad r>0. \tag{21}$$

Clearly, $P_0(r)>0$ for $r>0$, where $r\neq r_0$, and $P_0(r_0)=0$.

Equation (1) is multiplied by $P_0'(\rho)-(1/2)u^2$ and is integrated over Ω. Integration by parts with the periodicity conditions taken into account gives

$$\int_\Omega\Big[\partial_t P_0(\rho)-\frac{1}{2}(\partial_t\rho)u^2-\rho u P_0''(\rho)\partial_x\rho+\rho u^2\partial_x u+(\partial_y(\rho v))P_0'(\rho)+\rho v u\partial_y u\Big]dxdy$$
$$+\int_\Omega\tau\{[\partial_x(\rho u^2+p)+2\partial_y(\rho u v)](P_0''(\rho)\partial_y\rho-u\partial_x u)$$
$$+[\partial_y(\rho v^2)+\kappa\partial_y\rho](P_0''(\rho)\partial_y\rho-u\partial_y u)\}dxdy = 0. \tag{22}$$

Using (21) and adding (20) to (22), we derive the energy equality

$$\partial_t\int_\Omega\Big(\frac{1}{2}\rho u^2+P_0(\rho)\Big)dxdy+\int_\Omega(\partial_y(\rho v))P_0'(\rho)dxdy+\int_\Omega\tau\Psi(\rho,u)dxdy = \int_\Omega\rho a u dxdy, \tag{23}$$

where

$$\Psi(\rho,u) := [\partial_x(\rho u^2+p)+2\partial_y(\rho u v)]\frac{1}{\rho}\partial_x p+[\partial_y(\rho v^2)+\kappa\partial_y\rho]\frac{1}{\rho}\partial_y p$$
$$+[(\rho u^2+p)\partial_x u+2\rho u v\partial_y u]\partial_x u+[(\rho v^2+\kappa\rho)\partial_y u]\partial_y u.$$

Differentiating and rearranging gives

$$\Psi(\rho,u) = \frac{p'}{\rho}(u\partial_x\rho+v\partial_y\rho)^2+2\rho(u\partial_x u+v\partial_y u)\partial_x p+\frac{1}{\rho}(\partial_x p)^2+2(\partial_y v)(u\partial_x p+v\partial_y p)$$
$$+\kappa\frac{1}{\rho}(\partial_y\rho)\partial_y\rho+\rho(u\partial_x u+v\partial_y u)^2+p(\partial_x u)^2+\kappa\rho(\partial_y u)^2$$
$$=\frac{p'}{\rho}(u\partial_x\rho+v\partial_y\rho)^2+\frac{1}{\rho}[\partial_x p+\rho(u\partial_x+v\partial_y u)]^2+\kappa\frac{p'}{\rho}(\partial_y\rho)^2+p(\partial_x u)^2+\kappa\rho(\partial_y u)^2$$
$$+2(\partial_y v)(u\partial_x p+v\partial_y p).$$

If $v \equiv \text{const}$, we have $(\partial_y(\rho v))P_0'(\rho) = v\partial_y P_0(\rho)$. In view of the periodicity of ρ with respect to y, energy equality (23) becomes

$$\partial_t\int_\Omega\left(\frac{1}{2}\rho u^2 + P_0(\rho)\right)dxdy + \int_\Omega\tau\left\{\frac{p'}{\rho}(u\partial_x\rho + v\partial_y\rho)^2 + \frac{1}{\rho}[\partial_x p + \rho(u\partial_x u + v\partial_y u)]^2\right.$$

$$\left.+ \kappa\frac{p'}{\rho}(\partial_y\rho)^2 + p(\partial_x u)^2 + \kappa\rho(\partial_y u)^2\right\}dxdy = \int_\Omega\rho a u dxdy.$$

Let $|a| \le \bar{a}$, where $\bar{a} = \bar{a}(x, t)$. The last equality is integrated with respect to t. Applying the estimate

$$\left|\int_{Q_T}\rho a u dxdydt\right| \le \sqrt{2}\max_{[0, T]}\left[\int_\Omega\rho dxdy\right]^{1/2}\int_0^T\|\bar{a}\|_{L^\infty(\Omega)}dt\max_{[0, T]}\left[\int_\Omega\frac{1}{2}\rho u^2 dxdy\right]^{1/2},$$

the first equality in (19), and initial conditions (18), by means of the standard argument, we derive the following time-uniform $L^1(\Omega)$-estimate of the solution:

$$\max_{[0, T]}\left\|\frac{1}{2}\rho u^2 + P_0(\rho)\right\|_{L^1(\Omega)}^{1/2} \le \left\|\frac{1}{2}\rho^0(u^0)^2 + P_0(\rho^0)\right\|_{L^1(\Omega)}^{1/2} + \sqrt{2}\|\rho^0\|_{L^1(\Omega)}^{1/2}\int_0^T\|\bar{a}\|_{L^\infty(\Omega)}dt.$$

Using this estimate, we deduce the corresponding $L^2(Q_T)$-estimate for the derivatives of the solution:

$$\left\{\left\|\sqrt{\frac{\tau\kappa p'}{\rho}}\partial_y\rho\right\|_{L^2(Q_T)}^2 + \|\sqrt{\tau p}\partial_x u\|_{L^2(Q_T)}^2 + \|\sqrt{\tau\kappa\rho}\partial_y u\|_{L^2(Q_T)}^2\right.$$

$$\left.+ \left\|\sqrt{\frac{\tau p'}{\rho}}(u\partial_x\rho + v\partial_y\rho)\right\|_{L^2(Q_T)}^2 + \left\|\sqrt{\frac{\tau}{\rho}}[\partial_x p + \rho(u\partial_x u + v\partial_y u)]\right\|_{L^2(Q_T)}^2\right\}^{1/2}$$

$$\le \left\|\frac{1}{2}\rho^0(u^0)^2 + P_0(\rho^0)\right\|_{L^1(\Omega)}^{1/2} + \sqrt{2}\|\rho^0\|_{L^1(\Omega)}^{1/2}\int_0^T\|\bar{a}\|_{L^\infty(\Omega)}dt.$$

This completes the derivation of the global a priori energy estimate for the solution.

ACKNOWLEDGMENTS

This work was supported in part by the Russian Foundation for Basic Research, project nos. 06-01-00187, 08-01-90009-Bel, and 08-06-00302.

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