

On KP-integrable Hurwitz functions

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ABSTRACT: There is now a renewed interest [1]–[4] to a Hurwitz τ -function, counting the isomorphism classes of Belyi pairs, arising in the study of equilateral triangulations and Grothendieck's *dessins d'enfant*. It is distinguished by belonging to a particular family of Hurwitz τ -functions, possessing conventional Toda/KP integrability properties. We explain how the variety of recent observations about this function fits into the general theory of matrix model τ -functions. All such quantities possess a number of different descriptions, related in a standard way: these include Toda/KP integrability, several kinds of W -representations (we describe four), two kinds of integral (multi-matrix model) descriptions (of Hermitian and Kontsevich types), Virasoro constraints, character expansion, embedding into generic set of Hurwitz τ -functions and relation to knot theory. When approached in this way, the family of models in the literature has a natural extension, and additional integrability with respect to associated new time-variables. Another member of this extended family is the Itsykson-Zuber integral.

KEYWORDS: Matrix Models, Integrable Hierarchies, Topological Field Theories

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1 Introduction

Hurwitz τ -function [5–8] is a new important subject of theoretical physics, which seems relevant to description of non-perturbative phenomena beyond $2d$ conformal field theory, actually beginning from the $3d$ Chern-Simons and knot theory, see [9–11]. In general, Hurwitz τ -functions do not belong [7, 8] to a narrower well-studied class of KP/Toda τ -functions, i.e. are not straightforwardly reducible to free fermions ($\widehat{U(1)}$ Kac-Moody algebras) and Plucker relations (the Universal Grassmannian). However, the special cases, when they *do*, help to establish links between the known and unknown, and are very instructive for development of terminology and research directions. A particular case of the previously known example of this type [12, 13] was recently considered again in [1]–[4]

and finally seems to attract reasonable attention. In the present paper we further extend it and consider from the perspective of the modern τ -function theory, thus slightly broadening the consideration in those papers.

In systematic presentation, the story begins from the celebrated formula [14] for the Hurwitz numbers,

$$\mathcal{N}_{\Delta_1, \dots, \Delta_k} = \sum_R d_R^2 \varphi_R(\Delta_1) \dots \varphi_R(\Delta_k) \tag{1.1}$$

which expresses them through the properly normalized symmetric-group characters $\varphi_R(\Delta)$. Here $\Delta_1, \dots, \Delta_k$ and R are Young diagrams and d_R is the dimension of representation R of the symmetric group $S_{|R|}$ divided by $|R|!$, [15–18]. The ordinary Hurwitz numbers (counting ramified coverings of the Riemann sphere with ramifications of a given type) arise when all $\Delta_1, \dots, \Delta_k$ have the same size (the same number of boxes), then the sum in (1.1) goes over R of the same size. If the size $|\Delta| > |R|$, then $\varphi_R(\Delta) = 0$, if $|\Delta| < |R|$, then

$$\varphi_R(\Delta) = \frac{(|R| - |\Delta| + k)!}{k!(|R| - |\Delta|)!} \varphi_R(\Delta, \mathbf{1}^{|R|-|\Delta|}) \tag{1.2}$$

where at the r.h.s. $|R| - |\Delta|$ lines of unit length is added to the Young diagram Δ , and k is the number of lines of unit length in the diagram Δ . See [5, 6, 19] and especially [7, 8] for more details about all this.

The symmetric group characters $\varphi_R(\Delta)$ are related to the linear group ones (the Schur functions)

$$\chi_R[X] = \chi_R\{p\} \Big|_{p_n = \text{Tr } X^n} \tag{1.3}$$

as follows [15–18]

$$\chi_R\{p\} = \sum_{\Delta} d_R \varphi_R(\Delta) p_{\Delta} \cdot \delta_{|R|, |\Delta|} \tag{1.4}$$

or [19]

$$\chi_R\{p_m + \delta_{m,1}\} = \sum_{\Delta} d_R \varphi_R(\Delta) p_{\Delta} \tag{1.5}$$

The difference between the two is that in (1.4) the sum goes only over $|\Delta|$ of size $|R|$, while in (1.5) there is no restriction. For a Young diagram $\Delta := \delta_1 \geq \delta_2 \geq \dots \geq \delta_{l(\Delta)}$, which is an ordered partition of $|\Delta|$ into a sum of $l(\Delta)$ integers δ_i , associated is the multi-time variable

$$p_{\Delta} = p_{\delta_1} p_{\delta_2} \dots p_{\delta_{l(\Delta)}} \tag{1.6}$$

In the particular case when all p_n are the same, $p_n = N$, i.e. when X is an $N \times N$ unit matrix, $X = I_N$, eq. (1.4) provides φ -decomposition of the dimensions $D_R(N)$ of the irreducible representation R of the Lie algebra $gl(N)$

$$D_R(N) = \chi_R[I_N] = \sum_{\Delta} d_R \varphi_R(\Delta) N^{l(\Delta)} \delta_{|R|, |\Delta|} \tag{1.7}$$

The standard definition of these dimensions is the celebrated hook formula [15–18]

$$\frac{D_R(N)}{d_R} = \prod_{i,j \in R} (N + i - j) = \prod_i \frac{(\lambda_i + N - i)!}{(N - i)!} \tag{1.8}$$

In fact, for study of integrability important is just the fact that all p_n are the same, and in what follows we mostly use the letter u instead of N , to downplay association with the representation dimensions and emphasize that u does not need to be a positive integer.

Combining (1.1) and (1.4), it is natural to consider the generating function¹

$$h_k\{p^{(1)}, \dots, p^{(k)}\} = \sum_{\Delta_1, \dots, \Delta_k} \mathcal{N}_{\Delta_1, \dots, \Delta_k} p_{\Delta_1} \dots p_{\Delta_k} = \sum_R d_R^2 \prod_{i=1}^k \frac{\chi_R\{p^{(i)}\}}{d_R} \quad (1.9)$$

It is well known that for $k = 1$ and $k = 2$ these h -functions are KP and Toda lattice τ -functions respectively; moreover, they are trivial τ -functions:

$$\begin{aligned} h_1\{p\} &= \sum_R d_R \chi_R\{p\} = e^{p_1}, \\ h_2\{\bar{p}, p\} &= \sum_R \chi_R\{\bar{p}\} \chi_R\{p\} = \exp\left(\sum_m \frac{1}{m} \bar{p}_m p_m\right) \end{aligned} \quad (1.10)$$

It is also known [7, 8] that for $k \geq 3$ with generic $p^{(i \geq 3)}$ these h -functions do *not* belong to the KP/Toda family as functions of $\{p^{(1)}\}$ or $\{p^{(1)}, p^{(2)}\}$. However, of course, this *can* happen for particular choices of $\{p^{(i \geq 3)}\}$, and they *do*, provided all $p_m^{(i)} = u^{(i)}$ for all m .

In other words, making use of (1.7) we restrict h -functions to more specific generating functions:

$$Z_{(k,n)}(s, u_1, \dots, u_n | p^{(i)}) = \sum_R s^{|R|} d_R^{2-k-n} \left(\prod_{i=1}^k \chi_R\{p^{(i)}\} \right) \left(\prod_{i=1}^n D_R(u_i) \right) \quad (1.11)$$

at $k = 1, 2$, which, given their origin and properties, we call hypergeometric (following [12, 13]) Hurwitz τ -functions. The formally continued to negative values $(2, -1)$ member of this family $Z_{(2,-1)}$ is the celebrated Itsykson-Zuber integral:

$$Z_{(2,-1)}\{\bar{p}, p\} = \sum_R \frac{d_R \chi_R[X] \chi_R[Y]}{D_R(N)} = J_{IZ}(N) \quad (1.12)$$

with $p_n = \text{tr } X^n$ and $\bar{p}_n = \text{tr } Y^n$ (see eq. (77) in [20]), note that for representations R with $D_R(N) = 0$ the characters in the numerator are also vanishing, and these R do not contribute to the sum. For $(1, 0)$ and $(1, 1)$ we get just the trivial exponentials

$$Z_{(1,0)} = \sum_R s^{|R|} d_R \chi_R\{p\} = e^{s p_1} \quad (1.13)$$

and

$$Z_{(1,1)} = \sum_R s^{|R|} D_R(N) \chi_R\{p\} = \exp\left(N \sum_{m=1}^N \frac{s^m p_m}{m}\right) \quad (1.14)$$

¹This definition could depend slightly on whether one imposes restrictions like $|\Delta_i| = |\Delta_j|$ and $|R| = |\Delta_i|$ in the sums.

The particular case $Z_{(1,2)}$ of generating numbers of isomorphism classes of the Belyi pairs was studied in [2–4].²

In fact, *models $Z_{(1,n)}$ with $n > 2$ are far more interesting.* This becomes obvious already for $N = 1$, when only symmetric diagrams $R = [m]$ contribute, with $D_{[m]}(N = 1) = 1$ and $d_{[m]} = 1/m!$, so that (1.11) turns into a simple series

$$Z_{(1,n)}(\text{all } u_i = 1) = \sum_{m=0}^{\infty} s^m \chi_{[m]}\{p\} d_{[m]}^{1-n} D_{[k]}^n = \sum_{m=0}^{\infty} (m!)^{n-1} s^m \chi_{[m]}\{p\} = \sum_{m=0}^{\infty} \frac{(m!)^{n-2} (sp_1)^m}{m} + O(p_2, \dots) \tag{1.15}$$

The underlined series is nicely convergent for $n = 1$ and $n = 2$, while for $n > 2$ it is asymptotic series, defined up to non-perturbative corrections. For $n = 3$ we get the archetypical example:

$$\sum_m m! \cdot s^m \tag{1.16}$$

where non-perturbative ambiguity is proportional to

$$\oint \frac{e^{-x} dx}{1 - xs} = \frac{e^{-1/s}}{s} \tag{1.17}$$

This example appears in the study of $Z_{(1,3)}$. The usual way to handle the series like (1.15) is the integral transformation:

$$f(s) = \sum_m a_m s^m \longrightarrow F(s) = \sum_m a_m m! \cdot s^m = \frac{1}{s} \int_{x_+} e^{-x/s} f(x) dx \tag{1.18}$$

For generic N this formalism turns into the theory of Kontsevich-like models.

Of course, (1.11) are very special, besides they are τ -functions [1, 12, 13], they actually belong to the class of *matrix model τ -functions* [45–51]. This not-yet-rigourously-defined class is characterized by coexistence of a wide variety of very different representations and properties [52]:

²Belyi pair describes a complex curve as a covering of CP^1 , ramified at just three points $0, 1, \infty$ (the pair is the curve C and the mapping $C \rightarrow CP^1$). According to G.Belyi and A.Grothendieck [21–25], existence of such representation is necessary and sufficient for arithmeticity of the curve and arithmetic curves are in one-to-one correspondence with the equilateral triangulations (*dessins d'enfant*). Thus, enumeration of Belyi pairs is a typical matrix model problem (see more on relations between counting the Belyi maps, Hurwitz numbers and matrix models in [26–28]), though equivalence of matrix model [29, 30] and sum-over-metrics descriptions [31, 32], proved in [33, 34] on the lines of [35–41] remains a big mystery from the point of view of the complicated embedding of moduli space of arithmetic curves into the entire moduli space, see [42] and, for a related consideration, [43]. The Belyi pairs are enumerated by the triple Hurwitz numbers $\mathcal{N}_{\Delta_0, \Delta_1, \Delta_\infty}$, but no adequate language is still found to describe the full generating function $h_3\{p^{(1)}, p^{(2)}, p^{(3)}\} = \sum_R d_R^{-1} \chi_R(p^{(1)}) \chi_R(p^{(2)}) \chi_R(p^{(3)})$, see [7, 8]. The suggestion of [2] was to sacrifice any details about Δ_0 and Δ_1 and keep only information about the numbers $l(\Delta_0)$ and $l(\Delta_1)$ of unglued sheets of the covering over 0 and 1: then such special generating function $Z_{(1,2)}$ is obviously a KP τ -function. In fact, it is enough to do so just at one (not obligatory two) of the three points: $Z_{(2,1)}$ is also a conventional Toda lattice τ -function. Presentation of standard results about these quantities and their multi-point counterparts is the purpose of the present paper. As to triple coverings, enumeration is the simplest, but not the most interesting part of the story. An explicit construction of the Belyi functions is extremely hard: for relatively vast set of examples see [44]. A crucial problem of string theory remains expressing the Mumford measure and its constituents (determinants of $\bar{\partial}$ operators) for arithmetic curves through combinatorial triple $\Delta_1, \Delta_2, \Delta_3$.

- they are KP/Toda τ -functions,
- they possess integral (“matrix-model”) representations of “ordinary” and Kontsevich types,
- they satisfy Virasoro- or W-like constraints (possess a D -module representation and obey the AMM/EO topological recursion [53–61]),
- they possess various W -representations [62–64], including ones via Casimir operators and via cut-and-join operators,
- they possess special linear decompositions into linear- and symmetric-group characters,
- they are Hurwitz τ -functions.

The purpose of this paper is to describe all these properties within the context of the hypergeometric Hurwitz τ -functions (1.11).

For illustrative purposes and to avoid notational confusions we list the simplest examples of dimensions (1.8), linear group characters $\chi_R\{p\}$, and appropriately normalized symmetric group characters $\varphi_R(\Delta)$ from [5, 6]:

R	$D_R(N)/d_R$	$\chi_R\{p\}$	d_R	$\varphi_R(1)$	$\varphi_R(2)$	$\varphi_R(11)$	$\varphi_R(3)$	$\varphi_R(21)$	$\varphi_R(111)$...
[1]	N	p_1	1	1						
[2]	$N(N+1)$	$\frac{p_2+p_1^2}{2}$	$\frac{1}{2}$	2	1	1				
[11]	$N(N-1)$	$\frac{-p_2+p_1^2}{2}$	$\frac{1}{2}$	2	-1	1				
[3]	$N(N+1)(N+2)$	$\frac{2p_3+3p_2p_1+p_1^3}{6}$	$\frac{1}{6}$	3	3	3	2	3	1	
[21]	$(N-1)N(N+1)$	$\frac{-p_3+p_1^3}{3}$	$\frac{1}{3}$	3	0	3	-1	0	1	
[111]	$N(N-1)(N-2)$	$\frac{2p_3-3p_2p_1+p_1^3}{6}$	$\frac{1}{6}$	3	-3	3	2	-3	1	
...										

2 Representation via cut-and-join operators

The linear group characters (Schur functions) $\chi_R\{p\}$ are common eigenfunctions of the set of commuting generalized cut-and-join operators [5, 6], and symmetric group characters $\varphi_R(\Delta)$ are their corresponding eigenvalues:

$$\hat{W}_\Delta \chi_R = \varphi_R(\Delta) \chi_R \tag{2.1}$$

What we need in (1.11) are rather operators with slightly different eigenvalues:

$$\hat{O}(u) \chi_R = \frac{D_R(u)}{d_R} \chi_R \tag{2.2}$$

However, eq. (1.7) allows one to make them easily from \hat{W}_Δ :

$$\hat{O}(u) = \sum_{\Delta} u^{l(\Delta)} \hat{W}_\Delta \hat{P}_{|\Delta|} \tag{2.3}$$

where $\hat{P}_{|\Delta|}$ is a projector, selecting the Young diagrams of the size $|\Delta|$,

$$\hat{P}_{|\Delta|} = \oint \frac{dz}{z} z^{-|\Delta| + \hat{L}_0} \tag{2.4}$$

with

$$\hat{L}_0 = \sum_n n p_n \frac{\partial}{\partial p_n} \tag{2.5}$$

so that

$$\hat{P}_{|\Delta|} \chi_R = \chi_R \delta_{|R|, |\Delta|} \tag{2.6}$$

and \hat{W}_Δ are the general cut-and-join operators from [5, 6].

Thus

$$\begin{aligned} Z_{(1,n)}(s, u_1, \dots, u_n | p) &= \left(\prod_{i=1}^k \hat{O}(u_i) \right) e^{sp_1}, \\ Z_{(2,n)}(s, u_1, \dots, u_n | \bar{p}, p) &= \left(\prod_{i=1}^n \hat{O}(u_i) \right) \exp \left(\sum_m \frac{s^m}{m} p_m \bar{p}_m \right) \end{aligned} \tag{2.7}$$

These are actually the W -representations [62–64] of the τ -functions (1.11), because $\hat{O}(u)$ are, in fact, elements of the integrability-preserving $GL(\infty)$ group. However, this is not quite so obvious: operator (2.3) does not have a form where this property is obvious. In fact, one can make a triangular transformation in (2.3) and get rid of projector operators $\hat{P}_{|\Delta|}$:

$$\hat{O}(u) = u^{\hat{L}_0} \left(1 + \frac{\hat{W}_2}{u} + \frac{\hat{W}_3 + \hat{W}_{22}}{u^2} + \frac{\hat{W}_4 + \hat{W}_{32} + \hat{W}_{222}}{u^3} + \frac{\hat{W}_5 + \hat{W}_{42} + \hat{W}_{33} + \hat{W}_{322} + \hat{W}_{2222}}{u^4} \dots \right) =$$

or

$$\hat{O}(u) = u^{\hat{W}_1} \sum'_{\Delta} u^{l(\Delta) - |\Delta|} \hat{W}_\Delta \tag{2.8}$$

where sum goes over all diagrams containing no lines of unit length (we denote this restriction by prime).

Since, say [5, 6], $\hat{W}_{22} = \frac{1}{2} (\hat{W}_2^2 - 3\hat{W}_3 - \hat{W}_{11})$, this expressions has chances to be exponentiated. In this case, the exponent should contain even less types of operators, to provide an element from $GL(\infty)$: it should actually be [7, 8] a *linear* combination of Casimir operators. We shall now demonstrate this.

3 Representation via Casimir operators

We want to find an exponential representation of the operator $\hat{O}(u)$, and what we know is that the eigenvalues of $\log \hat{O}(u)$ are logarithms of (1.8). More precisely, we need the $1/N$ -expansion of

$$\log \left(\frac{D_R(N)}{N^{|R|} \cdot d_R} \right) = \sum_{(i,j) \in R} \log \left(1 + \frac{i-j}{N} \right) = \sum_{m=1}^{\infty} \frac{(-)^{m+1}}{N^m \cdot m} \tilde{\sigma}_R(m+1) \tag{3.1}$$

where

$$\tilde{\sigma}_R(m+1) = \sum_{(i,j) \in R} (i-j)^m = \sum_{k=0}^m \frac{m!}{k!(m-k)!} \sum_{j=1}^{l(\Delta)} \left((-j)^{m-k} \sum_{i=1}^{r_j} i^k \right) \quad (3.2)$$

In fact, one can easily check that these quantities are linear combinations of the eigenvalues $\sigma(m)$ of the Casimir operators [65],

$$\hat{C}_m \chi_R = \sigma_R(m) \chi_R \quad (3.3)$$

which are given by

$$\sigma_R(m) = \frac{1}{m} \sum_{j=1}^{l(R)} \left((r_j - j + 1/2)^m - (-j + 1/2)^m \right) \quad (3.4)$$

In particular,

$$\begin{aligned} \sigma_R(1) &= \sum_i r_j = \sum_{(i,j) \in R} 1 = \tilde{\sigma}_R(1), \\ \sigma_R(2) &= \frac{1}{2} \sum_{j=1}^{l(R)} r_j (r_j - 2j + 1) = \sum_{j=1}^{l(R)} \left(\frac{r_j(r_j + 1)}{2} - jr_j \right) = \sum_{(i,j) \in R} (i-j) = \tilde{\sigma}_R(2), \end{aligned} \quad (3.5)$$

...

However, for higher m relations are a little more involved:

$$\tilde{\sigma}_R(m) = \sigma_R(m) - \sum_{k=1}^m \frac{(m-1)!}{(2k)!(m-1-2k)!} (1-2^{1-2k}) B_{2k} \cdot \sigma_R(m-2k) \quad (3.6)$$

The sum has finite number of items, $k < \frac{m}{2}$, and B_{2k} are the Bernoulli numbers,

$$\sum_n \frac{B_n t^n}{n!} = \frac{te^t}{e^t - 1}, \quad \text{or} \quad \sum_n \frac{B_{2n} t^{2n}}{(2n)!} = \frac{te^t}{e^t - 1} - 1 - \frac{t}{2} \quad (3.7)$$

$B_1 = \frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$, $B_{10} = \frac{5}{66}$, $B_{12} = -\frac{691}{2730}$, $B_{14} = \frac{7}{6}$, $B_{16} = -\frac{3617}{510}$, ...

What is important about the Casimir operators is that they contain single sums over j , and this property guarantees integrability [7, 8]. It is of course preserved by *linear* combinations, i.e. $\hat{\tilde{C}}_n$ with the eigenvalues $\tilde{\sigma}(n)$ are as good from this point of view as \hat{C}_n with the eigenvalues $\sigma(n)$.

Thus we obtained the desired exponential representation of the operators

$$\hat{O}(u) = u^{\hat{L}_0} \exp \left\{ \sum_{m=1}^{\infty} \frac{(-)^{m+1}}{u^m \cdot m} \hat{\tilde{C}}_{m+1} \right\} \quad (3.8)$$

Moreover, when there are many u variables, one can simply consider them as the Miwa-like reparametrisation of a new type of variables,

$$\eta_m = \frac{(-)^{m+1}}{m} \sum_{i=1}^n u_i^{-m}, \quad \eta_0 = \sum_{i=1}^n \log u_i \quad (3.9)$$

and the function (2.7) becomes also a function of these additional time-variables η :

$$Z_{(1,n)}(s, u_1, \dots, u_n | p) = \left(\prod_{i=1}^n \hat{O}(u_i) \right) e^{sp_1} = \exp \left(\sum_{m=0}^{\infty} \eta_m \hat{C}_{m+1} \right) \cdot e^{sp_1} \quad (3.10)$$

This function, as a function of the variables η_k , is very similar to the τ -function [7, 8]:

$$Z = \exp \left(\sum_{m=0}^{\infty} \bar{\eta}_m \hat{C}_{m+1} \right) \cdot e^{sp_1} \quad (3.11)$$

where the variables

$$\bar{\eta}_m = \sum_{i=1}^n u_i^{-m}, \quad (3.12)$$

related to $\sigma_R(m)$, are the linearly transformed variables η_m . In spite of this similarity, two functions are not connected with each other by a relation, describing equivalent integrable hierarchies [66–68]. In particular, change of the basis (3.6), which relates the operators \hat{C}_k with $\hat{\tilde{C}}_k$, is not given by a change of the spectral parameter, see e.g. [69] for more details.

Explicit relation between (3.8) and (2.8) is an interesting exercise, concerning commutative algebra of cut-and-join operators and their relation to the Casimir operators. It can be easily checked in the lowest orders of the u^{-1} -expansion with the help of multiplication table from [5, 6].

4 $Z_{(2,n)}$ as a τ -function of Toda lattice

Eq. (3.8) immediately implies that $Z_{(2,n)}$ is a Toda lattice τ -function (thus, $Z_{(1,n)}$ is a KP τ -function). Indeed, according to [7, 8] the exponential of *linear* combinations of the Casimir operators belongs to $GL(\infty)$ which preserves the KP/Toda integrability. In case of $Z_{(2,n)}$ the product of the $GL(\infty)$ operators (3.8) acts on the trivial τ -function $\exp \left(\sum_m \frac{s^m p_m \bar{p}_m}{m} \right)$.

Still, there are many other ways to demonstrate that $Z_{(2,n)}$ is a τ -function of the Toda lattice hierarchy. The most important is the free-fermion approach of [70] and closely related determinant formulas, see [45–51, 71]. From the point of view of Hurwitz theory, the basic well-known fact is that the character expansion

$$\sum_R g_R \chi_R(p) \quad (4.1)$$

is a KP τ -function *iff* coefficients g_R satisfy the Plücker relations, of which the generic solution is

$$g_R = \det_{ij} \left(F(r_i - i, j) \right) \quad (4.2)$$

with arbitrary function F of two variables.

Likewise, according to [72]

$$\tau_n(p, \bar{p} | f) = \sum_{R, R'} f_{R, R'}(n) \chi_R(p) \chi_{R'}(\bar{p}) \quad (4.3)$$

is a Toda lattice τ -function, iff

$$f_{R,R'}(n) = \det_{ij \leq n} \left(F(r_i - i, r'_j - j) \right) \tag{4.4}$$

Parameter n here plays a role of the Toda zero-time p_0 .

A particular class of solutions of this type is provided by a much simpler diagonal coefficients $f_{R,R'}(n)$ [1, 12, 13]

$$f_{R,R'}(u) = \delta_{R,R'} \prod_{i,j \in R} f(u + i - j) \tag{4.5}$$

where $f(x)$ is an arbitrary function of a single variable. This class of τ -functions of the Toda lattice hierarchy explicitly given by the free-fermion average

$$\tau_n(p, \bar{p} | f) = \left\langle n \left| e^{H(p)} e^{\sum T_m \cdot \psi_m^* \psi_m} e^{\bar{H}(\bar{p})} \right| n \right\rangle \tag{4.6}$$

where the normal ordering is defined w.r.t. the zero vacuum: $:\psi_m^* \psi_m := \psi_m^* \psi_m - \langle 0 | \psi_m^* \psi_m | 0 \rangle$ and the coefficients T_k are introduced via $f(k) = e^{T_{k-1} - T_k}$ with $T_{-1} = 0$. More explanations of the notation see in [12, 13, 70, 71]. This τ -function was named *hypergeometric* in [12, 13]. In particular, from (3.6) it follows that the operators $\hat{O}(u)$ yield the coefficients precisely of the this form, thus the functions $Z_{(2,n)}$ belong to this class.

In fact, one can even restrict the sum in (4.3) to the diagrams with no more than n lines, where n is the zero-time:

$$\tilde{\tau}_n(p, \bar{p} | f) = \sum_{R: l(R) \leq n} f_R(n) \chi_R(p) \chi_R(\bar{p}) \tag{4.7}$$

it is still a Toda lattice τ -function [7, 8].

The generic Hurwitz τ -function

$$h(p^{(1)}, \dots, p^{(k)} | \beta) = \exp \left(\sum_{\Delta} \beta_{\Delta} \hat{W}_{\Delta} \right) \sum_R d_R^{2-k} \chi_R\{p^{(1)}\} \dots \chi_R\{p^{(k)}\} \tag{4.8}$$

does not satisfy criteria (4.2) and (4.4) as a function of any time or time pairs, see [7, 8] for a detailed consideration (it is not even clear if it fits into the wide class of the non-Abelian τ -functions of [73–76]). Notable exceptions are the cases when $k = 1, 2$ and when β_{Δ} are adjusted to provide any *linear* combination of the standard Casimir operators (3.4), which are nicknamed as *complete cycles* in [77, 78]. The functions (1.11) use additional freedom (4.5) to enlarge k , but keeping $p^{(3)}, \dots, p^{(k)}$ very special: constant. This corresponds to choosing $f(x) = \prod_{i=1}^k (x + u_i)$ in (4.5) while the s -dependence is introduced by the rescaling $p_k \rightarrow s^k p_k$.

Of course, this $Z_{(2,n)}(u_1, \dots, u_n | p, \bar{p})$ is a very special kind of a lattice τ -function. In particular, it possesses a simple integral representation in the form of eigenvalue matrix model (as foreseen already in [12, 13]). We construct such representations in the generic case in the next section, and then consider particular more explicit examples.

5 Matrix model representations

Making use of orthogonality condition [79, eq. (3.1)],³

$$\int \int_{N \times N} \chi_R[X] \chi_Q[Y] e^{i \text{Tr} XY} dX dY = \frac{D_R(N)}{d_R} \delta_{R,Q} \quad (5.6)$$

one can easily rewrite (1.11) in the form of multi-matrix models. Indeed, from (5.6) it follows that

$$\begin{aligned} Z_{(2,1)}(N|p, \bar{p}) &= \sum_R \frac{D_R(N)}{d_R} \chi_R\{p\} \chi_R\{\bar{p}\} \\ &= \int \int_{N \times N} \left(\sum_R \chi_R[X] \chi_R\{p\} \right) \left(\sum_Q \chi_Q[Y] \chi_Q\{\bar{p}\} \right) e^{i \text{Tr} XY} dX dY = \\ &= \int \int_{N \times N} e^{\sum_n \frac{1}{n} p_n \text{Tr} X^n + \sum_n \frac{1}{n} \bar{p}_n \text{Tr} Y^n} e^{i \text{Tr} XY} dX dY \end{aligned} \quad (5.7)$$

what is just the conventional 2-matrix model, as was already noted in [12, 13].

³The simplest way to prove (5.6) is to make use of formula from Fourier theory

$$\int dx dy f(x) g(y) e^{-xy} = f\left(\frac{\partial}{\partial x}\right) g(x) \Big|_{x=0} \quad (5.1)$$

where the x -integral goes over the real axis, and the y -integral runs over the imaginary one. Now after performing the integration over angular variables and using the Itzykson-Zuber formula, one obtains the multiple eigenvalue integral

$$\int dX dY \chi_R(X) \chi_Q(Y) e^{-\text{tr} XY} \sim \int \prod_i \det_{ij} x_i^{N+R_j-j} \det_{ij} y_i^{N+Q_j-j} e^{-\sum_i x_i y_i} \quad (5.2)$$

where we used the Weyl formula for the characters of linear groups

$$\chi_R = \frac{\det_{ij} x_i^{N+R_j-j}}{\Delta(x)} \quad (5.3)$$

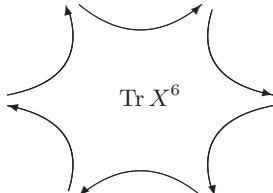
and $\Delta(x)$ is the Van-der-Monde determinant. Using now formula (5.1) and

$$\int \det_{ij} f_i(x_j) \det_{ij} g_i(y_j) \prod_i K(x_i, y_i) = \det_{ij} \int f_i(x) g_j(y) K(x, y) \quad (5.4)$$

one immediately obtains (5.6).

This formula can be also described in the pure combinatorics terms using the Feynman diagrams. The role of propagator here is played by $\langle X_{ij} Y_{kl} \rangle = \delta_{il} \delta_{jk}$. Therefore, the formula reduces to trivial combinatorics: connecting the free ends of multi-linear combinations of trace operators. For example,

$$\langle \text{Tr} X \text{Tr} Y \rangle = \delta^{ij} \delta^{kl} \delta_{il} \delta_{jk} = N,$$

$$\begin{aligned} \langle \text{Tr} X^2 \text{Tr} Y^2 \rangle &= 2N^2, \\ \langle \text{Tr} X^2 (\text{Tr} Y)^2 \rangle &= 2N, \\ \langle (\text{Tr} X)^2 (\text{Tr} Y)^2 \rangle &= 2N^2, \end{aligned} \quad \begin{array}{c} \text{Tr } X^6 \end{array} \quad (5.5)$$


...

Here we used the relation

$$\sum_Q \chi_Q[Y] \chi_Q\{\bar{p}\} = e^{\sum_n \frac{1}{n} \bar{p}_n \text{Tr} Y^n} \quad (5.8)$$

which we also need below in the form

$$\sum_S \chi_S[Y_1] \chi_S[X_2] = e^{\sum_n \frac{1}{n} \text{Tr} Y_1^n \text{Tr} X_2^n} = \text{Det}\left(I \otimes I - Y_1 \otimes X_2\right)^{-1} \quad (5.9)$$

Similarly to (5.7),

$$\begin{aligned} Z_{(2,2)}(N_1, N_2 | p, \bar{p}) &= \sum_R \frac{D_R(N_1) D_R(N_2)}{d_R^2} \chi_R\{p\} \chi_R\{\bar{p}\} = \\ &= \sum_S \int \int_{N_1 \times N_1} \left(\sum_R \chi_R\{p\} \chi_R[X_1] \right) \chi_S[Y_1] e^{i \text{Tr} X_1 Y_1} dX_1 dY_1 \\ &\quad \times \int \int_{N_2 \times N_2} \chi_S[X_2] \left(\sum_Q \chi_Q[Y_2] \chi_Q\{p\} \right) e^{i \text{Tr} X_2 Y_2} dX_2 dY_2 = \\ &= \int \int_{N_1 \times N_1} e^{\sum_n \frac{1}{n} p_n \text{Tr} X_1^n} e^{i \text{Tr} X_1 Y_1} dX_1 dY_1 \\ &\quad \times \int \int_{N_2 \times N_2} e^{\sum_n \frac{1}{n} \bar{p}_n \text{Tr} Y_2^n} e^{i \text{Tr} X_2 Y_2} dX_2 dY_2 \frac{1}{\text{Det}\left(I_{N_1} \otimes I_{N_2} - Y_1 \otimes X_2\right)} = \\ &= \int_{N_1 \times N_1} \int_{N_2 \times N_2} d\mathcal{K}_{N_1}(Y_1 | p) \frac{1}{\text{Det}\left(I_{N_1} \otimes I_{N_2} - Y_1 \otimes X_2\right)} d\mathcal{K}_{N_2}(X_2 | \bar{p}) \end{aligned} \quad (5.10)$$

where generalized Kontsevich measure is defined as [80–82]

$$d\mathcal{K}_N(Y | p) = dY \int_{N \times N} e^{\sum_n \frac{1}{n} p_n \text{Tr} X^n} e^{i \text{Tr} X Y} dX \quad (5.11)$$

Further,

$$\begin{aligned} Z_{(2,3)}(N_1, N_2, N_3 | p, \bar{p}) &= \quad (5.12) \\ &= \int d\mathcal{K}_{N_1}(Y_1 | p) \frac{1}{\text{Det}\left(I_{N_1} \otimes I_{N_2} - Y_1 \otimes X_2\right)} e^{i \text{Tr} X_2 Y_2} dX_2 dY_2 \frac{1}{\text{Det}\left(I_{N_2} \otimes I_{N_3} - Y_2 \otimes X_3\right)} d\mathcal{K}_{N_3}(X_3 | \bar{p}) \end{aligned}$$

and for generic k we have:

$$Z_{(2,n)}(N_1, \dots, N_n | p, \bar{p}) = \int d\mathcal{K}_{N_1}(Y_1 | p) \frac{\prod_{i=2}^{n-1} e^{i \text{Tr} X_i Y_i} dX_i dY_i}{\prod_{i=1}^{n-1} \text{Det}\left(I_{N_i} \otimes I_{N_{i+1}} - Y_i \otimes X_{i+1}\right)} d\mathcal{K}_{N_n}(X_n | \bar{p}) \quad (5.13)$$

One can observe amusing parallels with the conformal matrix models [83–85], which already have a number of other interesting applications [86–95].

One can make the Miwa transformation of time variables $p_m = \text{Tr} \Lambda^{-m}$ in order to transform these matrix integrals to an equivalent form depending on the external matrix Λ . Sometimes it turns out very convenient as we shall see below.

6 Miwa transformation to Kontsevich matrix models

Now we make the Miwa transformation of one set of the time variables in $Z_{(2,k)}$ in order to obtain matrix integrals of the Kontsevich type. This kind of integrals are sometimes more convenient. In particular, the Virasoro constraints for $Z_{(1,2)}$ are evident in this representation.

For the sake of simplicity, we consider only $Z_{(2,1)}$ case, a generic case is treated in full analogy. Thus, we make the Miwa transformation of times $\bar{p}_m = \text{Tr } \Lambda^{-m}$ in the formula (5.7), so that

$$\exp\left(\sum_n \frac{\bar{p}_m}{m} \text{Tr } Y^m\right) = \frac{(\det \Lambda)^N}{\det(\Lambda \otimes I - I \otimes Y)} \quad (6.1)$$

Then, the integral becomes

$$\begin{aligned} Z_{(2,1)}(N|p, \bar{p}) &= \int dX dY e^{i \text{Tr } XY} e^{\sum_m \frac{p_m}{m} \text{Tr } X^m} e^{\sum_m \frac{\bar{p}_m}{m} \text{Tr } Y^m} \\ &= \int dX dY e^{i \text{Tr } XY} e^{\sum_m \frac{p_m}{m} \text{Tr } X^m} \frac{(\det \Lambda)^N}{\det(\Lambda \otimes I - I \otimes Y)} \end{aligned} \quad (6.2)$$

The integral over matrix Y can be easily calculated (to this end, one has first to perform integration over the angular variables and then make Fourier transform w.r.t. to the eigenvalues of Y), the result reads

$$Z_{(2,1)}(N|p, \bar{p}) = (-\det \Lambda)^N \int_{X_+} dX_{N \times N} e^{-\text{Tr } X \Lambda + \sum_m \frac{p_m}{m} \text{Tr } X^m} \quad (6.3)$$

where integral runs over $N \times N$ positive-definite matrices, that is matrices with *positive eigenvalues*. This follows from the standard Fourier transform:

$$\int \frac{e^{ixy}}{y - i0} dy = 2\pi i \theta(x) \quad (6.4)$$

Integral (6.3) is not yet quite of Kontsevich type: it *essentially* depends on the matrix size N and one can not reach an arbitrary point in the space of time variables. In order to lift this restriction, one can add the logarithmic term which makes the parameter u and the number of integrations independent variables:

$$Z_{(2,1)}(u|p, \bar{p}) = (-\det \Lambda)^u \int_{X_+} dX_{N \times N} e^{-\text{Tr } X \Lambda + (u-N) \text{Tr } \log X + \sum_m \frac{p_m}{m} \text{Tr } X^m} \quad (6.5)$$

One can easily check for concrete N that expansion of this integral into p_k -series coincides with $Z_{(2,1)}(u|p, \bar{p})$ from (1.11). Note also that this integral, if considered as a function of

time variables $\bar{p}_m = \text{Tr } \Lambda^{-m}$, does not depend⁴ on N , which is the necessary property of Kontsevich integrals [80–82].

Integral (6.5) was obtained in [3] within a different approach. From this formula one immediately obtains a one-matrix model describing $Z_{(1,2)}(u, v|p)$ at integer points $v = N$. This can be done in different ways.

One possibility is to put $p_m = v$, then we obtain the double-logarithm model of [3]:

$$(-\det \Lambda)^u \int_{X_+} dX_{N \times N} e^{-\text{Tr } X \Lambda + (u-N)\text{Tr } \log X - v \log(1-X)} \quad (6.6)$$

This kind of models were thoroughly investigated in [96], still in a moment we will see that (6.6) is equivalent to an even better studied theory.

Another possibility just to put $\Lambda = 1$. Then the result is

$$Z_{(1,2)}(u, v|p) \Big|_{v=N} = \int_{X_+} dX_{N \times N} e^{-\text{Tr } X + (u-N)\text{Tr } \log X + \sum_n \frac{p_n}{n} \text{Tr } X^n} \quad (6.7)$$

Since this integral goes over only the positive X_+ , it is equivalent to the model of complex matrices where X is an obviously positive-definite matrix product HH^\dagger [97–99]:

$$Z^{\mathbf{C}} = \int dH dH^\dagger e^{\text{Tr } V(HH^\dagger)} \sim \int \prod_i dh_i^2 \Delta^2(h_i^2) e^{\sum_i V(h_i^2)} \quad (6.8)$$

where $V(X)$ is arbitrary potential of the matrix model, h_i^2 are eigenvalues of HH^\dagger and $\Delta(h)$ is the Van-der-Monde determinant. Thus, $Z_{(1,2)}(u, v|p)$ from [2–4], and, hence, the double-logarithm model (6.6) is nothing but the well-known complex matrix model. Among other things, this immediately implies the Virasoro constraints for $Z_{(1,2)}(u, v|p)$.

In a similar way one can make the Miwa transformation of one set of times and perform integration like (6.4) in order to obtain from (5.12) a two-matrix model representation of $Z_{1,3}$:

$$Z_{1,3}(u, v, w|p) \Big|_{u=N} \sim \int_{X_+} dX_{N \times N} \int_{Y_+} dY_{N \times N} \exp \left(-\text{Tr } Y^{-1} - i \text{Tr } XY + \sum_m \frac{p_m}{m} \text{Tr } X^m + (v-N)\text{Tr } \log X + (v-w-N)\text{Tr } \log Y \right) \quad (6.9)$$

⁴This integral is independent of N in the following sense. Calculate the coefficient in front of, say, $p_1 p_2$ at different values of N :

$$\begin{aligned} N = 1: & \quad \frac{u(u+1)(u+2)}{2\lambda^3} \\ N = 2: & \quad \frac{u(u+1)(u+2)}{2} \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) \left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} \right) \\ N = 3: & \quad u^2 \left(\frac{1}{\lambda_1^3} + \frac{1}{\lambda_2^3} + \frac{1}{\lambda_3^3} \right) + \frac{u(u^2+2)}{2} \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \right) \left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} \right) + \frac{u^2}{2} \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \right)^3 \\ & \quad \dots \end{aligned}$$

All these expressions look different and depending on N , but in fact are all equal to the independent on N polynomial

$$u^2 \bar{p}_3 + \frac{u(u^2+2)}{2} \bar{p}_1 \bar{p}_2 + \frac{u^2}{2} \bar{p}_1^3.$$

where we assume that $N \leq v \leq w$ (hence the asymmetry of the integral w.r.t. interchanging v and w), otherwise the integrals diverge. From experience in [100, 101] and [102] it comes with no surprise that this $Z_{(1,3)}$ satisfies $\widetilde{W}^{(3)}$ constraints. In these constraints the values of u, v and w are arbitrary, and the symmetry is restored (see (7.7)).

7 Virasoro/ \widetilde{W} constraints

The simplest way to obtain Virasoro/ \widetilde{W} constraints for $Z_{(k,n)}$ is to construct the loop equations (Ward identities) of the corresponding matrix models, which are associated with arbitrary changes of integration variables in the matrix integral. The Ward identities for the two-matrix model describing $Z_{(2,n)}$ are quite involved and are expressed in terms of the \widetilde{W}_∞ -algebra of refs. [100, 101]. However, when one set of times is eliminated things simplify a lot. In particular, when only l first $\bar{p}_i, i \leq l$, are non-vanishing, the constraints imposed on p -dependence involve only $\widetilde{W}^{(i)}$ -operators with $i \leq l$ [100, 101]. As we now see, the same seems true for $Z_{(1,n)}$ models, where all \bar{p} are non-vanishing, but the same. This result can imply additional kinds of matrix-model representations for $Z_{(1,n)}$.

To begin with, $Z_{(1,1)}(u|p) = \exp\left(\sum_{m=1}^{\infty} \frac{us^m p_m}{m}\right)$ satisfies

$$\left(\hat{J}_m^{\mathbf{C}} - \frac{m+1}{s} \frac{\partial}{\partial p_{m+1}}\right) Z_{(1,1)}(u|p) = 0, \quad m \geq 0 \tag{7.1}$$

with

$$\hat{J}_m^{\mathbf{C}} = m \frac{\partial}{\partial p_m} \tag{7.2}$$

The next model $Z_{(1,2)}$ is equivalent to the complex one-matrix model (6.7), for which the Ward identities are just the Virasoro constraints, derived in [99]:

$$\left(\hat{L}_m^{\mathbf{C}} - \frac{m+1}{s} \frac{\partial}{\partial p_{m+1}}\right) Z_{(1,2)}(u, v|p) = 0, \quad m \geq 0 \tag{7.3}$$

where

$$\hat{L}_m^{\mathbf{C}} = \sum_{k=1}^{\infty} (m+k)p_k \frac{\partial}{\partial p_{m+k}} + \sum_{a=1}^{n-1} a(n-a) \frac{\partial^2}{\partial p_a \partial p_{m-a}} + (u+v)m \frac{\partial}{\partial p_m} + uv\delta_{m,0} \tag{7.4}$$

One can easily check that these constraints are indeed satisfied by (1.11) at $k=1, n=2$. Note that integration domain $x > 0$ is preserved by the transformation $\delta x = x^{m+1}$ only for $m \geq 0$, thus *there is no $\hat{L}_{-1}^{\mathbf{C}}$ constraint* — this seems not to match the claim of [3]. Let us stress that in case of (7.3) the second term in the brackets can be interpreted as the shift of the p_1 -variable, but this is no longer so for more general \widetilde{W} -constraints, see (7.1) and [100–102]. Note also that we do not include $\partial/\partial p_0$ terms in the sum, and give the corresponding contributions explicitly. Usually they would be proportional to the matrix size N , but in Virasoro constraints this size does not need to be integer. Moreover, the would be N^2 is substituted by uv , while $2N$ by $(u+v)$.

Likewise, the $Z_{(1,3)}$ function (6.9) satisfies the $\widetilde{W}^{(3)}$ constraint:

$$\left(\hat{M}_m^{\mathbf{C}} - \frac{m+1}{s} \frac{\partial}{\partial p_{m+1}} \right) Z_{(1,3)}(u, v, w | p) = 0, \quad m \geq 0 \quad (7.5)$$

where

$$\hat{M}_0^{\mathbf{C}} = \sum_{a,b=1}^{\infty} \left((a+b)p_a p_b \frac{\partial}{\partial p_{a+b}} + ab p_{a+b} \frac{\partial^2}{\partial p_a \partial p_b} \right) + (u+v+w) \sum_{a=1}^{\infty} a p_a \frac{\partial}{\partial p_a} + uvw \quad (7.6)$$

and, more generally,

$$\begin{aligned} \hat{M}_m^{\mathbf{C}} = & \sum_{k,l=1}^{\infty} (k+l+m)p_k p_l \frac{\partial}{\partial p_{k+l+m}} + \sum_{k=1}^{\infty} \left(\sum_{a=1}^{k+m-1} + \sum_{a=1}^m \right) a(k+m-a)p_k \frac{\partial^2}{\partial p_a \partial p_{k+m-a}} + \\ & + \sum_{a+b+c=m} abc \frac{\partial^3}{\partial p_a \partial p_b \partial p_c} + uvw \delta_{m,0} + \frac{m^2(m+1)}{2} \frac{\partial}{\partial p_m} + (uv+vw+wu)m \frac{\partial}{\partial p_m} + \\ & + (u+v+w) \left(\sum_{k=1}^{\infty} (k+m)p_k \frac{\partial}{\partial p_{k+m}} + \sum_{a+b=m} \frac{\partial^2}{\partial p_a \partial p_b} \right) \end{aligned} \quad (7.7)$$

Clearly, this time $N^3 \rightarrow uvw$, $3N^2 \rightarrow (uv+vw+wu)$ and $3N \rightarrow (u+v+w)$. We keep the same label \mathbf{C} for these operators, to emphasize similarity with (7.4). In fact they belong to the class of the \widetilde{W} -operators [45–51, 100–102], appearing in description of Kontsevich and multi-matrix models and mnemonically are powers of the current $\hat{J}^{\mathbf{C}}$ defined by (9.22), subjected to peculiar normal ordering, when all the $\hat{J}_-^{\mathbf{C}}$ operators on the right are simply thrown away, see [100, 101] for a detailed description.

Similarly, one can treat the models $Z_{(1,n)}$ with higher $n > 3$. They satisfy similar $\widetilde{W}^{(n)}$ -constraints. In principle, they can be derived either from multi-matrix models or from any of the W -representations, described in the present paper.

For illustrative purposes we provide just one more example:

$$\left(\hat{N}_m^{\mathbf{C}} - \frac{m+1}{s} \frac{\partial}{\partial p_{m+1}} \right) Z_{(1,4)}(u, v, w, x | p) = 0, \quad m \geq 0 \quad (7.8)$$

and the simplest of operators $\widetilde{W}^{(4)}$ is

$$\begin{aligned} \hat{N}_0^{\mathbf{C}} = & \sum_{a,b,c=1}^{\infty} \left((a+b+c)p_a p_b p_c \frac{\partial}{\partial p_{a+b+c}} + abc p_{a+b+c} \frac{\partial^2}{\partial p_a \partial p_b \partial p_c} \right) + \\ & + \frac{3}{2} \sum_{a+b=c+d} cd p_a p_b \frac{\partial^2}{\partial p_c \partial p_d} + \frac{1}{2} \sum_{a,b=1}^{\infty} ab p_a p_b \frac{\partial^2}{\partial p_a \partial p_b} + \\ & + (u+v+w+x) \sum_{a,b=1}^{\infty} \left((a+b)p_a p_b \frac{\partial}{\partial p_{a+b}} + ab p_{a+b} \frac{\partial^2}{\partial p_a \partial p_b} \right) \\ & + (uv+uw+ux+vw+vx+wx) \sum_{k=1}^{\infty} k p_k \frac{\partial}{\partial p_k} + \\ & + \sum_{k=1}^{\infty} \frac{k^2(k+1)}{2} p_k \frac{\partial}{\partial p_k} + uvwx \end{aligned} \quad (7.9)$$

8 Naive W -representations

In addition to W -representation (3.8) in terms of the Casimir operators, which immediately implies integrability, one can rewrite the generating functions (2.7) as an exponential in a more straightforward way, which also provides nice expressions manifestly belonging to integrability-preserving $GL(\infty)$ group [70, 103, 104].

8.1 The case of $Z_{(1,1)}$

From (2.7) and from the fact that the operator $\hat{O}(u)$ in (2.8) preserves unity, $\hat{O}(u) \cdot 1 = 1$, it follows that

$$Z_{(1,1)}(s, u) = \hat{O}(u) \circ e^{sp_1} \cdot 1 = \exp\left(\hat{O}(u) \circ sp_1 \circ \hat{O}(u)^{-1}\right) \cdot 1 \quad (8.1)$$

(the last equality holds for any function, not obligatory exponential, but $Z_{(1,1)}(s, u)$ is expressed via exponential). Note that to use these kind of formulas one needs to rewrite (2.1) and (2.7) as some operator relations using composition \circ instead of action of operators, i.e. e^{sp_1} in (8.1) is treated not as a function, but as an operator (of multiplication by e^{sp_1}). For example, for $\hat{W}_{[1]} = \hat{L}_0 = \sum_n np_n \frac{\partial}{\partial p_n}$ and $\chi_{[1]} = p_1$ one has

$$\hat{W}_{[1]} \circ \chi_{[1]} = \chi_{[1]} + \chi_{[1]} \circ \hat{W}_{[1]} \quad (8.2)$$

and (2.1) is reproduced if we apply this identity to unity, which is annihilated by \hat{W}_Δ :

$$\hat{W}_{[1]} \circ \chi_{[1]} \cdot 1 = \chi_{[1]} \cdot 1 + \chi_{[1]} \circ \hat{W}_{[1]} \cdot 1 = \chi_{[1]} \cdot 1 = p_1 \quad (8.3)$$

For the sake of brevity, we omit the sign of composition \circ throughout this section, since it is implied at any operator expressions here.

We can now use (2.8) to calculate the operator $\hat{O}(u)sp_1\hat{O}(u)^{-1}$, which stands in the exponent in (8.1). For this we need the explicit formulas for \hat{W}_Δ from [5, 6]. For $\Delta = \delta_1 \geq \delta_2 \geq \dots \geq \delta_{l(\Delta)} \geq 0 = \{\dots, \underbrace{2, \dots, 2}_{m_2}, \underbrace{1, \dots, 1}_{m_1}\}$

$$\hat{W}_\Delta = \prod_k \frac{1}{m_k! k^{m_k}} : \hat{D}_k^{m_k} : \quad (8.4)$$

where \hat{D} are defined in terms of the Miwa matrix X from $p_k = \text{Tr } X^k$:

$$\hat{D}_k = \text{Tr} \left(X \frac{\partial}{\partial X^{tr}} \right)^k = \text{Tr} (X \partial_X)^k \quad (8.5)$$

and the double dots denote normal ordering: all the X -derivatives stand to the right of all X 's, e.g.

$$: \text{Tr} (X \partial_X)^2 : = : X_{ij} \frac{\partial}{\partial X_{kj}} X_{kl} \frac{\partial}{\partial X_{il}} : = X_{ij} X_{kl} \frac{\partial^2}{\partial X_{kj} \partial X_{il}} \quad (8.6)$$

(this example illustrates also the meaning of the transposition superscript X^{tr}). It is because of the normal ordering that \hat{W}_Δ annihilates unity.

Now we can act with \hat{W}_Δ on p_1 . The commutator

$$[:\hat{D}_k :, p_1] = k : \text{Tr } X^2 \partial_X (X \partial_X)^{k-2} : \quad (8.7)$$

This implies that

$$\begin{aligned} \hat{W}_2 p_1 &= \frac{1}{2} : \hat{D}_2 : p_1 = p_1 \hat{W}_2 + \underline{\text{Tr } X^2 \partial_X} \\ \hat{W}_3 p_1 &= \frac{1}{3} : \hat{D}_3 : p_1 = p_1 \hat{W}_3 + : \text{Tr } X^2 \partial_X X \partial_X : \\ \hat{W}_{22} p_1 &= \frac{1}{8} : (\hat{D}_2)^2 : p_1 = p_1 \hat{W}_{22} + \frac{1}{2} : \hat{D}_2 \text{Tr } X^2 \partial_X : \\ &\dots \end{aligned} \quad (8.8)$$

Now, add the two last lines:

$$\begin{aligned} [(\hat{W}_3 + \hat{W}_{22}), p_1] &= \frac{1}{2} \left(: \hat{D}_2 \text{Tr } X^2 \partial_X : + 2 : \text{Tr } X^2 \partial_X X \partial_X : \right) \\ &= \frac{1}{2} \text{Tr } X^2 \partial_X : \hat{D}_2 := \left(\underline{\text{Tr } X^2 \partial_X} \right) \hat{W}_2 \end{aligned} \quad (8.9)$$

where the underlined operator is just the same as in the first line of (8.8).

Coming back to (8.1), we see that

$$\begin{aligned} \hat{O}(u) p_1 &= \left(1 + \frac{\hat{W}_2}{u} + \frac{\hat{W}_3 + \hat{W}_{22}}{u^2} + \dots \right) u^{\hat{L}_0} p_1 = \left(u + \hat{W}_2 + \frac{\hat{W}_3 + \hat{W}_{22}}{u} + \dots \right) p_1 u^{\hat{L}_0} = \\ &= \left\{ p_1 \left(u + \hat{W}_2 + \frac{\hat{W}_3 + \hat{W}_{22}}{u} + \dots \right) + \underline{\text{Tr } X^2 \partial_X} + \underline{\text{Tr } X^2 \partial_X} \frac{\hat{W}_2}{u} + \dots \right\} u^{\hat{L}_0} = \\ &= u p_1 \hat{O}(u) + \left(\underline{\text{Tr } X^2 \partial_X} \right) \hat{O}(u) = (u p_1 + \hat{L}_{-1}) \hat{O}(u) \end{aligned} \quad (8.10)$$

where

$$\begin{aligned} \hat{L}_0 &= \hat{W}_{[1]} = \text{Tr } X \partial_X = \sum_m m p_m \frac{\partial}{\partial p_m}, \\ \hat{L}_{-1} &= \underline{\text{Tr } X^2 \partial_X} = \sum_m m p_{m+1} \frac{\partial}{\partial p_m} \end{aligned} \quad (8.11)$$

Thus we obtain from (8.1) a W -representation

$$Z_{(1,1)}(s, u|p) = e^{s(\hat{L}_{-1} + u p_1)} \cdot 1 \quad (8.12)$$

alternative to (3.8).

8.2 Direct check of (8.12)

In fact, $Z_{(1,1)}(s, u|p)$ is known explicitly, see (1.14). The relation

$$e^{s(\hat{L}_{-1} + u p_1)} \cdot 1 = \exp \left(u \sum_m \frac{s^m p_m}{m} \right) = Z_{(1,1)}(s, u|p) = \sum_R s^{|R|} D_R(u) \chi_R\{p\} \quad (8.13)$$

implied by (8.12), follows from the Campbell-Hausdorff formula, if it is written in the form

$$\exp\left(\frac{[B, A]}{2} - \frac{[A, [A, B]]}{3} + \frac{[[A, B], B]}{6} + \dots\right) \cdot e^A \cdot e^B = e^{A+B} \quad (8.14)$$

We choose $A = sup_1$ and $B = s\hat{L}_{-1}$, since in this case $e^B \cdot 1 = 1$. Then only the first and the third terms at the very left exponential contributes giving $us^2p_2/2$ and $us^3p_3/3$. More generally, only the terms of the form

$$\sum_m \frac{\text{ad}_B^m(A)}{m(m+1)} \quad (8.15)$$

contribute. Since clearly $\text{ad}_B^m(A) = mp_{m+1}$, while all other commutators (like $\sum_m \frac{\text{ad}_A^m(B)}{m+1}$) are vanishing,

$$\begin{aligned} e^{s(\hat{L}_{-1} + up_1)} \cdot 1 &= e^{A+B} \cdot 1 = \exp\left(\sum_{m=1} \frac{\text{ad}_B^m(A)}{m(m+1)}\right) e^A \\ &= \exp\left(\sum_{m=1} \frac{s^{m+1}up_{m+1}}{m+1}\right) \cdot e^{sup_1} = \exp\left(u \sum_{m=1} \frac{s^m p_m}{m}\right) \end{aligned} \quad (8.16)$$

which is exactly (8.13).

8.3 The case of $Z_{(1,2)}$

This time instead of (8.1) one needs

$$Z_{(1,2)}(s, u, v) = \hat{O}(v)\hat{O}(u) e^{sp_1} \cdot 1 = \exp\left(\hat{O}(v)\hat{O}(u)sp_1\hat{O}(u)^{-1}\hat{O}(v)^{-1}\right) \cdot 1 \quad (8.17)$$

and thus an appropriate modification of (8.10):

$$\begin{aligned} \hat{O}(v)\hat{O}(u)p_1 &= \hat{O}(v) \left(up_1\hat{O}(u) + \left(\underline{\text{Tr } X^2 \partial_X}\right) \hat{O}(u) \right) = \\ &= uv p_1 \hat{O}(v)\hat{O}(u) + u \left(\underline{\text{Tr } X^2 \partial_X}\right) \hat{O}(v)\hat{O}(u) + \hat{O}(v) \left(\underline{\text{Tr } X^2 \partial_X}\right) \hat{O}(u) = \\ &= uv p_1 \hat{O}(v)\hat{O}(u) + (u+v) \left(\underline{\text{Tr } X^2 \partial_X}\right) \hat{O}(v)\hat{O}(u) \\ &\quad - \left[\underline{\text{Tr } X^2 \partial_X}, \left(v + \hat{W}_2 + \frac{\hat{W}_3 + \hat{W}_{22}}{v} + \dots \right) \right] \hat{O}(u) = \\ &= uv p_1 \hat{O}(v)\hat{O}(u) + (u+v) \left(\underline{\text{Tr } X^2 \partial_X}\right) \hat{O}(v)\hat{O}(u) \\ &\quad + \left(\underline{\underline{\text{Tr } X^2 \partial_X X \partial_X}} \right) \hat{O}(v)\hat{O}(u) = \\ &= \left(uv p_1 + (u+v)\hat{L}_{-1} + \hat{M}_{-1} \right) \hat{O}(v)\hat{O}(u) \end{aligned} \quad (8.18)$$

with

$$\begin{aligned} \hat{L}_{-1} &= \underline{\text{Tr } X^2 \partial_X} = \sum_m mp_{m+1} \frac{\partial}{\partial p_m}, \\ \hat{M}_{-1} &= \underline{\underline{\text{Tr } X^2 \partial_X X \partial_X}} = \sum_{a,b} (a+b-1) p_a p_b \frac{\partial}{\partial p_{a+b-1}} + ab p_{a+b+1} \frac{\partial^2}{\partial p_a \partial b} \end{aligned} \quad (8.19)$$

Combining this with (8.17) we immediately reproduce the result of [2]:

$$Z_{(1,2)}(s, u, v) = \exp \left\{ s \left(uv p_1 + (u + v) \hat{L}_{-1} + \hat{M}_{-1} \right) \right\} \cdot 1 \quad (8.20)$$

8.4 Operators $\hat{\mathcal{O}}(u_1, \dots, u_n)$

Now generalizing (8.17), one can define the operator $\hat{\mathcal{O}}(u_1, \dots, u_n)$

$$Z_{(1,n)}(s, u_1, \dots, u_n) = \exp \left(\prod_{i=1}^k \hat{\mathcal{O}}(u_i) s p_1 \prod_{I=1}^n \hat{\mathcal{O}}(u_i)^{-1} \right) \cdot 1 = \hat{\mathcal{O}}(u_1, \dots, u_n) \cdot 1 \quad (8.21)$$

The sequence of underlined operators is evidently

$$\text{ad}_{\hat{W}_2}^k p_1 = : \text{Tr } X^2 \partial_X (X \partial_X)^{k-1} : \quad (8.22)$$

in particular,

$$\begin{aligned} \hat{L}_{-1} &= [\hat{W}_2, p_1] = \text{Tr } X^2 \partial_X, \\ \hat{M}_{-1} &= [\hat{W}_2, \text{Tr } X^2 \partial_X] = : \text{Tr } X^2 \partial_X X \partial_X : \\ \hat{N}_{-1} &= [\hat{W}_2, : \text{Tr } X^2 \partial_X X \partial_X :] = : \text{Tr } X^2 \partial_X (X \partial_X)^2 : \\ &\dots \end{aligned} \quad (8.23)$$

Therefore the naive W -representations of the functions Z_k look as follows:

$$Z_{(1,k)}(\vec{u}) = \hat{\mathcal{O}}_k(\vec{u}) \cdot 1 \quad (8.24)$$

where

$$\begin{aligned} \hat{\mathcal{O}}_1 &= e^{s p_1}, \\ \hat{\mathcal{O}}_2(u) &= e^{s(\hat{L}_{-1} + u p_1)}, \\ \hat{\mathcal{O}}_3(u, v) &\stackrel{[2]}{=} e^{s(\hat{M}_{-1} + (u+v)\hat{L}_{-1} + uv p_1)}, \\ \hat{\mathcal{O}}_4(u, v, w) &= e^{s(\hat{N}_{-1} + (u+v+w)\hat{M}_{-1} + (uv+vw+wu)\hat{L}_{-1} + uvw p_1)}, \\ &\dots \end{aligned} \quad (8.25)$$

and

$$\begin{aligned} \hat{L}_{-1} &= \sum_m m p_{m+1} \frac{\partial}{\partial p_m}, \\ \hat{M}_{-1} &= \sum_{a,b} (a+b-1) p_a p_b \frac{\partial}{\partial p_{a+b-1}} + ab p_{a+b+1} \frac{\partial^2}{\partial p_a \partial p_b}, \\ \hat{N}_{-1} &= \sum_{a,b,c=1}^{\infty} \left((a+b+c-1) p_a p_b p_c \frac{\partial}{\partial p_{a+b+c-1}} + abc p_{a+b+c+1} \frac{\partial^3}{\partial p_a \partial p_b \partial p_c} \right) + \\ &\quad + \frac{3}{2} \sum_{a,b=1}^{\infty} \sum_{c=1}^{a+b} ab p_c p_{a+b+1-c} \frac{\partial^2}{\partial p_a \partial p_b} + \frac{1}{2} \sum_{a=1}^{\infty} a^2 (a+1) p_{a+1} \frac{\partial}{\partial p_a}, \\ &\dots \end{aligned} \quad (8.26)$$

Formula (8.20) for $Z_{(1,2)}$ appeared in [2].

Note that this representation of the operators $\hat{\mathcal{O}}_k(\vec{u})$ also makes manifest that they are elements of $GL(\infty)$ [70, 103, 104] which gives yet another proof of integrability: this property guarantees that $Z_{(1,n)}(\vec{u})$ is a τ -function of the KP hierarchy.

8.5 Hierarchy in n

Operators (8.25) form a clear hierarchy in n , and one can easily move in n in both directions. Let us look at the simpler one: the decrease of n .

Since $D_R(v) = d_R v^{|R|} (1 + O(v^{-1}))$, one has

$$\lim_{v \rightarrow \infty} Z_{(1,n+1)} \left(\frac{s}{v}, \vec{u}, v \right) = Z_{(1,n)}(s, \vec{u}) \tag{8.27}$$

For example, for $n = 0$,

$$\lim_{v \rightarrow \infty} \exp \left(\sum_m \frac{(s/v)^m \cdot v}{m} p_m \right) = e^{sp_1} \tag{8.28}$$

Thus

$$\hat{O}_n(\vec{u}) = \lim_{v \rightarrow \infty} \hat{O}_{n+1}(v, \vec{u})^{1/v} \tag{8.29}$$

In particular, taking \hat{O}_2 from [2], we immediately get:

$$\dots \rightarrow \exp \left\{ s \left(\hat{M}_{-1} + (u+v)\hat{L}_{-1} + uv p_1 \right) \right\} \rightarrow \exp \left\{ s \left(\hat{L}_{-1} + u p_1 \right) \right\} \rightarrow e^{sp_1} \tag{8.30}$$

It now looks rather obvious that the previous term on the left is

$$\exp \left\{ s \left(\hat{N}_{-1} + (u+v+w)\hat{M}_{-1} + (uv + vw + wu)\hat{L}_{-1} + uvw p_1 \right) \right\} \tag{8.31}$$

and so on.

9 Description in terms of the w_∞ -algebra

The W -representation (3.8) can be further transformed and simplified. Since it is expressed through the Casimir operators (3.4), which belong to the W_∞ algebra, and no central extensions are relevant for our considerations, one can make use of its alternative representation in terms of ordinary differential operators [105]. This is a very powerful technique, see [69] for the recent review, and this also turns to be the case in application to our problem.

9.1 Combined Casimir operators \hat{C} as distinguished $\hat{W}_0^{(m)}$

In this approach operators from w_∞ are represented by polynomial of z and $D = z\partial_z$. In most considerations D can be considered just as an integer number. In particular, the standard Casimir operators (3.4) are mapped [69, 105] into

$$\hat{C}(n) \rightarrow \frac{\left(D - \frac{1}{2}\right)^n - \left(-\frac{1}{2}\right)^n}{n} \tag{9.1}$$

Substituting this into the sums in (3.6), we obtain that combined Casimir operators, given by this seemingly complicated formula, are in fact mapped into something clearly distinguished:

$$\hat{C}(n+1) \rightarrow \sum_{i=1}^{D-1} i^n \tag{9.2}$$

and then, from (3.8)

$$\hat{O}(u) = u^{\hat{C}_1} \exp \left\{ \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{u^n \cdot n} \hat{C}(n+1) \right\} \rightarrow u^D \exp \left(\sum_{i=0}^{D-1} \log \left(1 + \frac{i}{u} \right) \right) = \frac{\Gamma(u+D)}{\Gamma(u)} \tag{9.3}$$

i.e. as an element of the w_∞ algebra, operator $\hat{O}(u)$ is just an ordinary Γ -function! In fact, Bernoulli numbers naturally arise in the coefficients of the large- u asymptotics of $\log \Gamma(u)$.

Moreover, the sums at the r.h.s. (9.2) are also associated with the very special operators, what provides a spectacular interpretation of $\hat{C}(n)$. Namely, monomials zD^n are images of

$$\begin{aligned} p_1 &\rightarrow z \cdot 1, \\ \hat{L}_{-1} = \sum_n n p_{n+1} \frac{\partial}{\partial p_n} &\rightarrow z \cdot D, \\ \hat{M}_{-1} = \sum_{a,b} \left((a+b-1) p_a p_b \frac{\partial}{\partial p_{a+b-1}} + a b p_{a+b+1} \frac{\partial^2}{\partial p_a \partial p_b} \right) &\rightarrow z \cdot D^2, \\ &\dots \end{aligned} \tag{9.4}$$

and the sums in (9.2) are the zeroth harmonics of the same operators:

$$\begin{aligned} \hat{L}_0 = \sum_n n p_{n+1} \frac{\partial}{\partial p_n} &\rightarrow D = \sum_{i=1}^{D-1} 1, \\ \hat{M}_0 = \sum_{a,b} \left((a+b) p_a p_b \frac{\partial}{\partial p_{a+b}} + a b p_{a+b} \frac{\partial^2}{\partial p_a \partial p_b} \right) &\rightarrow D(D-1) = 2 \sum_{i=1}^{D-1} i, \\ \hat{N}_0 = \sum_{a,b,c=1}^{\infty} \left((a+b+c) p_a p_b p_c \frac{\partial}{\partial p_{a+b+c}} + a b c p_{a+b+c} \frac{\partial^3}{\partial p_a \partial p_b \partial p_c} \right) + \\ + \frac{3}{2} \sum_{a,b=1}^{\infty} \sum_{c=1}^{a+b-1} a b p_c p_{a+b-c} \frac{\partial^2}{\partial p_a \partial p_b} + \frac{1}{2} \sum_{a=1}^{\infty} a(a^2-1) p_a \frac{\partial}{\partial p_a} &\rightarrow \frac{1}{2} D(D-1)(2D-1) = 3 \sum_{i=1}^{D-1} i^2 \\ &\dots \end{aligned} \tag{9.5}$$

Let us introduce a unified notation $\hat{W}_n^{(m)}$ for all these W -operators:

$$p_k = \hat{W}_k^{(1)}, \quad \hat{L}_k = \hat{W}_k^{(2)}, \quad \hat{M}_k = \hat{W}_k^{(3)}, \quad \hat{N}_k = \hat{W}_k^{(4)}, \dots \tag{9.6}$$

Comparing (9.2) with (9.5) we see that

$$\hat{C}(n) = \frac{1}{n} \hat{W}_0^{(n+1)} \tag{9.7}$$

In terms of these operators one can rewrite (3.8) and (9.3) as

$$\hat{O}(u) = \exp \left(\log u \hat{L}_0 + \frac{1}{2u} \hat{M}_0 - \frac{1}{6u^2} \hat{N}_0 + \dots \right) = \exp \left(\sum_{m=2}^{\infty} \frac{(-)^m \hat{W}_0^{(m+1)}}{(m-1)m u^{m-1}} \right) u^{\hat{W}_0^{(2)}} \tag{9.8}$$

so that

$$Z_{(1,n)}(s, \vec{u}) = \prod_{i=1}^n \hat{O}(u_i) \cdot e^{s p_1} = \exp \left(\sum_{m=2}^{\infty} \eta_m \hat{W}_0^{(m+1)} \right) \cdot \exp \left(s p_1 \prod_{i=1}^n u_i \right) \tag{9.9}$$

with

$$\eta_m = \frac{(-)^m}{(m-1)m} \sum_{i=1}^n \frac{1}{u_i^{m-1}} \tag{9.10}$$

9.2 Relation between the two W -representations

At the same time, from (8.25) the same function is given by

$$\begin{aligned} Z_{1,n}(s, \vec{u}) &= \exp \left(s u_1 \dots u_n \left\{ p_1 + \sum_{i=1}^n \frac{1}{u_i} \hat{L}_{-1} + \sum_{i<j}^n \frac{1}{u_i u_j} \hat{M}_{-1} + \sum_{i<j<k}^n \frac{1}{u_i u_j u_k} \hat{N}_{-1} + \dots \right\} \right) \cdot 1 = \\ &= \exp \left(s \left(\prod_{i=1}^n u_i \right) \left(\sum_{m=0}^{\infty} \xi_m \hat{W}_{-1}^{(m+1)} \right) \right) \cdot 1 \end{aligned} \tag{9.11}$$

with

$$\xi_m = \sum_{i_1 \leq i_2 \leq \dots \leq i_m} \frac{1}{u_{i_1} u_{i_2} \dots u_{i_m}} \tag{9.12}$$

In this form there are two differences between (9.9) and (9.11): the grading of \hat{W} -operators (0 and -1 respectively) and the time variables η and ξ , given respectively by power sum and elementary symmetric polynomials of variables u_i^{-1} .

These two W -representations are of course related by the Campbell-Hausdorff formula, this time in the form

$$e^{\hat{B}} e^{\hat{A}} = e^{\hat{A} + [\hat{B}, \hat{A}] + \frac{1}{2!} [\hat{B}, [\hat{B}, \hat{A}]] + \dots} e^{\hat{B}} \tag{9.13}$$

when exponent in the boxed operator is just

$$\hat{C} = \sum_{m=0}^{\infty} \frac{1}{m!} \text{ad}_{\hat{B}}^m \hat{A} \tag{9.14}$$

where we need to substitute $\hat{A} = p_1$ and $\hat{B} = \sum_m \eta_m \hat{W}_0^{(m)}$. Since (9.14) is linear in \hat{A} , the common factor $s \prod u_i$ can be omitted and restored at the very end. Then, if applied to unity, the l.h.s. of (9.13) gives (9.9), and the r.h.s. will provide (9.11), because $e^{\hat{B}} \cdot 1 = 1$. To calculate \hat{C} we need a commutation relation

$$[\hat{W}_0^{(m+1)}, \hat{W}_{-1}^{(n+1)}] = m \hat{W}_{-1}^{m+n} \tag{9.15}$$

which provides \hat{C} in the following form:

$$\hat{C} = \underbrace{\hat{W}_{-1}^{(1)}}_{p_1} + \sum_{m=2}^{\infty} m \eta_m \hat{W}_{-1}^{(m)} + \frac{1}{2!} \sum_{m,n=2}^{\infty} mn \eta_m \eta_n \hat{W}_{-1}^{(m+n)} + \frac{1}{2!} \sum_{l,m,n=2}^{\infty} lmn \eta_l \eta_m \eta_n \hat{W}_{-1}^{(l+m+n)} + \dots \tag{9.16}$$

We want this to be equal to $\sum_{k=0}^{\infty} \xi_k \hat{W}_{-1}^{(k+1)}$. Clearly, each ξ_k is a *finite* multi-linear combination of η_m , for example,

$$\begin{aligned}
 \xi_0 &= 1, \\
 \xi_1 &= 2\eta_2 = \sum_i \frac{1}{u_i}, \\
 \xi_2 &= 3\eta_3 + 2\eta_2^2 = -\frac{1}{2} \sum_i \frac{1}{u_i^2} + \frac{1}{2} \left(\sum_i \frac{1}{u_i} \right)^2 = \sum_{i<j} \frac{1}{u_i u_j}, \\
 \xi_3 &= 4\eta_4 + 6\eta_2 \eta_3 + \frac{4}{3} \eta_2^3 = \sum_{i<j<k} \frac{1}{u_i u_j u_k}, \\
 &\dots
 \end{aligned} \tag{9.17}$$

Thus (9.9) and (9.11) — and thus (3.8) and (8.25) — are indeed related by the simplest of all Campbell-Hausdorff formulas (9.13).

9.3 More details from the w_∞ dictionary

Higher harmonics of the simplest operators $\hat{W}^{(m)}$ are mapped into the following polynomials of z and $D = z\partial_z$:

$$\begin{aligned}
 \hat{J}_k &= \text{res}_z(z^k \hat{J}(z)) \longrightarrow j_k = z^{-k}, \quad k \neq 1, \\
 \hat{L}_k &= \frac{1}{2} \text{res}_z \left(z^{1+k} : \hat{J}(z)^2 : \right) \longrightarrow l_k = z^{-k} \left(z\partial_z - \frac{k+1}{2} \right), \\
 \hat{M}_k &= \frac{1}{3} \text{res}_z \left(z^{2+k} : \hat{J}(z)^3 : \right) \longrightarrow m_k = z^{-k} \left(z^2 \partial_z^2 - kz\partial_z + \frac{(1+k)(2+k)}{6} \right), \\
 \hat{N}_0 &\longrightarrow \frac{1}{2} (2z\partial_z - 1)(z\partial_z - 1)z\partial_z, \quad \hat{N}_{-1} \longrightarrow z(z\partial_z)^3
 \end{aligned} \tag{9.18}$$

(polynomials at the r.h.s. are defined up to constant terms, which do not affect commutators — expressions in (9.5) make use of this freedom). In general, for peculiar operators, which are made from the current

$$\hat{J}(x) = \sum_m \frac{\hat{J}_m}{x^{m+1}} = \sum_{m=1}^{\infty} \left(p_m x^{m-1} + \frac{m}{x^{m+1}} \frac{\partial}{\partial p_m} \right) \tag{9.19}$$

and its derivatives — and at the same time belong to the W_∞ algebra — the mapping rule is:

$$\text{res}_z \left(z^{-k} : \frac{(\hat{J}(z) + \partial_z)^{m+1}}{m+1} : 1 \right) \longrightarrow (z^2 \partial_z)^m z^k \tag{9.20}$$

It is easy to check that above examples fit into this scheme, with

$$\begin{aligned}
 \hat{L}(x) &= \sum_m \frac{\hat{L}_m}{x^{m+2}} = : \hat{J}(x)^2 : \\
 \hat{M}(x) &= \sum_m \frac{\hat{M}_m}{x^{m+3}} = : \hat{J}(x)^3 : \\
 \hat{N}(x) &= \sum_m \frac{\hat{N}_m}{x^{m+4}} = : \hat{J}(x)^4 - (\partial_x \hat{J}(x))^2 : \\
 &\dots
 \end{aligned}
 \tag{9.21}$$

Note, that this formalism is applicable only to operators from W_∞ algebra, i.e. those made from the current (9.19) and its derivatives in a very special way — as linear combinations of those at the l.h.s. of (9.20). Already the forth power of the current, $: \hat{J}^4 :$, does *not* belong to this algebra — this is the reason for the $(\partial \hat{J})^2$ subtraction in $\hat{N} \in W_\infty$. Another typical example are Virasoro operators \hat{L}_n^C in (7.4). They are actually made from the square of *another* current,

$$\hat{j}^C(x) = \sum_{m=1}^{\infty} \left(\frac{1}{2} p_m x^{m-1} + \frac{m}{x^{m+1}} \frac{\partial}{\partial p_m} \right)
 \tag{9.22}$$

with additional factor 1/2 in the positive harmonics. Because of this the w_∞ technique, described in this section, can *not* be used to prove and even check the Virasoro constraints (7.4): it does *not* adequately describe commutation relations between $\hat{L}_n^C \notin W_\infty$ and $\hat{L}_0, \hat{M}_0, \hat{N}_0, \dots \in W_\infty$. However, there are two amusing exceptions: the zero harmonics \hat{L}_0^C and \hat{M}_0^C do belong to W_∞ , this is no longer true neither for \hat{N}_0^C , nor for higher harmonics of \hat{L}^C and \hat{M}^C .

10 Conclusion

This paper gives a brief summary of existing knowledge about the simple family (1.11) with $k = 1, 2$. This family consists of Hurwitz τ -functions which are integrable in the simplest KP/Toda sense. A number of facts are already present in the literature, not only we presented them in a systematic way revealing all the relations between these facts, but we naturally made a number of new claims:

- In addition to the naive W -representation in section 8 we described two others: in terms of the generalized cut-and-join operators, (2.8) and of the Casimir operators, (3.8), providing a direct relation to the Hurwitz theory *a la* [5, 6] and to the KP/Toda integrability respectively. One more version, (9.8), provides a bridge between naive and Casimir W -representations.
- We put together the two-matrix and Kontsevich like models from [3, 12, 13] and pointed out an intriguing relation of higher $Z_{(2,m)}$ to the conformal like matrix models.

- We provided a description of the most studied $Z_{(1,2)}$ model in terms of complex matrix model which directly provides the Virasoro constraints, (7.4). Similarly, the $Z_{(1,3)}$ model is described by the asymmetric two-matrix model with $1/Y$ potential and satisfies the $\widetilde{W}^{(3)}$ -constraints, etc.
- We interpreted (-1)-modes of W -operators which enter the naive W -representation of [2] and its generalizations as multiple commutators of the basic pair: the cut-and-join operator $\hat{W}_{[2]} = \frac{1}{2} : \text{Tr}(X\partial_X)^2 :$ and $\hat{L}_{-1} = : \text{Tr}(X^2\partial_X) :$
- We explained in section 9 how the mapping to the differential operators can be used to drastically simplify derivation of these and many other similar results (note, however, that this approach is directly applicable only to the KP/Toda, but not to general Hurwitz τ -functions, and is thus restricted to models (1.11)).

There are still a lot of formulas to derive, especially for $Z_{(2,n)}$ models with $n > 1$.

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