# MARINE PHYSICS

# **Improved Theory of Nonlinear Topographic Rossby Waves**

O. E. Polukhina and A. A. Kurkin

Nizhni Novgorod State Technical University, Nizhni Novgorod, Russia Received March 18, 2004; in final form, October 19, 2004

Abstract—A nonlinear dispersion theory of topographic Rossby waves of small finite amplitude is presented. The Coriolis parameter ( $\Omega$ ) and the ocean depth (*h*) are assumed to vary mainly along a single direction, while the wave propagates in the perpendicular direction. The basis of the theory is the asymptotic procedure of expanding the hydrodynamical fields in the equations for an ideal vertically homogeneous incompressible rotating fluid in series with respect to small parameters of nonlinearity and topographic dispersion. The temporal evolution of the wave field and its transformation along the coordinate axis is described by a nonlinear evolution equation of the second order of accuracy with respect to small parameters. When the fluid is homogeneous, the equation obtained is equivalent to the Korteweg–de Vries equation of the second order, which is frequently used for describing waves in weakly nonlinear dispersive fluids.

### **INTRODUCTION**

The important role of Rossby waves was revealed comparatively recently, and the study of these waves was developed irregularly. The Laplace tidal theory contains proper solutions, but important geophysical applications of these waves were only developed one and half decades after Rossby discovered planetary waves in the atmosphere. Two decades later, at the end of the 1960s, corresponding motions in the ocean were recognized. It was found that particular oceanographic "weather", which is significant for a lot of applications (e.g., shipping, fishing, or hydroacoustics), is determined by Rossby waves and nonlinear individual structures, such as the synoptic eddies and rings directly associated with them. In addition, Rossby waves play an important role in the redistribution of energy throughout the World Ocean, since, due to their instability, the current energy is mainly transported by Rossby waves, which, in turn, contribute to the largescale horizontal exchange.

The presence of the planetary waves is associated with the latitudinal variations of the Coriolis parameter. Rossby waves, however, can exist over a sloping bottom even if this parameter is constant. Such waves are called topographic waves. In the general case, both of the factors make contributions-planetary and topographic. The total effect of the factors is characterized by the gradient of the value of  $\Omega/h$  (the external potential vorticity, where h is the ocean depth and  $\Omega$  is the Coriolis parameter). The lines of constant external potential vorticity for the North Atlantic are presented in [4, 5]. These contours indicate the possible directions of the barotropic planetary topographic Rossby waves. The barotropic planetary topographic mode of the Rossby waves can also exist in a stratified fluid when the wavelength is so large that the waves are not affected by the stratification. The results of the laboratory modeling of the topographical Rossby waves in a tank with a linearly sloping bottom are described in [13]. The propagation of the low-frequency Rossby waves is studied in [15] from the point of view of the refraction theory. The fundamentals of the nonlinear dispersion theory of the Rossby waves were established as early as in the 1970s–1980s, e.g., in [1–3, 8, 10, 14].

In the present study, we propose an improved nonlinear dispersion theory of the planetary topographic Rossby waves of small finite amplitude for the case when the Coriolis parameter and depth of the ocean vary mainly along a single direction and the wave propagates in the perpendicular direction. The basis of the theory is the linear asymptotic procedure of the expansion of the hydrodynamical fields in the equations for an ideal vertically homogeneous incompressible rotating fluid in series with respect to small parameters of nonlinearity and topographic dispersion. As a result, the temporal evolution of the wave field and its transformation along the coordinate axis of propagation are described by a nonlinear evolution equation (NEE) of the second order of accuracy with respect to small parameters. This equation contains nonlinear and dispersion terms, nonlinear dispersion terms, and terms responsible for the changes in the wave amplitude caused by the fluid heterogeneity ( $\Omega$  and h) along the axis of propagation. When the fluid is homogeneous, the NEE obtained is equivalent to the Korteweg-de Vries (KDV) equation of the second order, which is frequently used for the description of waves in fluids with a weak nonlinearity and dispersion. The variable coefficients of the NEE are determined by the integrals along the transverse coordinate, which are dependent on  $\Omega$  and h, as well as on the solutions of the boundary problems that describe the structure of the wave field in the direction perpendicular to the direction of the wave propagation. For a particular case of topographical Rossby waves, we consider two examples of bottom profiles, which are uniform in the transverse direction: a channel with a step-wise bottom and a channel with a linearly sloping bottom. For these cases, the coefficients in the KDV equation of the first order were explicitly calculated and the possible shapes of the soliton-like Rossby waves were analyzed.

## **BASIC EQUATIONS**

The basic equations that describe the motion of an ideal homogeneous incompressible rotating fluid between a flat lid and the bottom in the hydrostatic approximation can be written in the form [8]

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\left(\frac{\omega + \Omega}{h}\right) = 0, \qquad (1)$$

where x and y are the horizontal coordinates, t is the time, (u, v) are the corresponding components of the horizontal velocity vector, and  $\omega$  is the vorticity. Using

 $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ ,  $uh = -\frac{\partial \psi}{\partial y}$  and  $vh = \frac{\partial \psi}{\partial x}$ , Eq. (1) can be rewritten as

$$\begin{pmatrix} \frac{\partial}{\partial t} - \frac{1}{h} \frac{\partial \Psi}{\partial y} \frac{\partial}{\partial x} + \frac{1}{h} \frac{\partial \Psi}{\partial x} \frac{\partial}{\partial y} \end{pmatrix} \times \left( \frac{1}{h} \begin{bmatrix} \frac{\partial}{\partial x} \left( \frac{1}{h} \frac{\partial \Psi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{1}{h} \frac{\partial \Psi}{\partial y} \right) + \Omega \end{bmatrix} \right) = 0.$$

$$(2)$$

Let us direct the axes of the Cartesian coordinate system so that the Coriolis parameter  $\Omega$  and the ocean depth vary mainly along the y axis. Let us consider a wave, which propagates along the x axis.

Let us reduce Eq. (2) to a dimensionless form by using the characteristic depth H as a depth scale; the characteristic wavelength  $L_x$  as a scale along the x axis;  $L_y$  as a scale along the y axis; and the characteristic current velocity  $U_0$  as a scale for the velocity

$$(x, y, t) = \left(L_x \tilde{x}, L_y \tilde{y}, \frac{L_x}{U_0} \tilde{t}\right), \quad \Psi = U_0 L_y H \tilde{\Psi},$$
$$h = H \tilde{h}, \quad \text{and} \ \Omega = \frac{U_0}{L_y} \tilde{\Omega}.$$

By omitting the  $\sim$  signs, we can write Eq. (2) in the dimensionless variables

$$\begin{pmatrix} h\frac{\partial}{\partial t} - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y} \end{pmatrix} \times \left(\frac{1}{h}\left[\mu\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial\psi}{\partial y}\right) + \Omega\right] \right) = 0,$$

$$(3)$$

where  $\mu = \left(\frac{L_y}{L_x}\right)^2$  is the parameter of the topographic

dispersion. Let us assume that the wavelength  $L_x$  is much greater than the characteristic length scale along

the y axis  $(L_x \ge L_y \text{ or } \mu \ll 1)$  and the wave amplitude is a small but a finite value. Let  $\varepsilon \ll 1$  be a dimensionless parameter that characterizes the wave amplitude (nonlinearity) and express the unknown function in Eq. (3) as

$$\Psi(x, y, t) = \varepsilon \widehat{\Psi}(x, y, t).$$

Then, Eq. (3) takes the form

$$\begin{pmatrix} h\frac{\partial}{\partial t} - \varepsilon \frac{\partial \widehat{\Psi}}{\partial y} \frac{\partial}{\partial x} + \varepsilon \frac{\partial \widehat{\Psi}}{\partial x} \frac{\partial}{\partial y} \end{pmatrix} \times \left( \frac{1}{h} \left[ \mu \varepsilon \frac{\partial}{\partial x} \left( \frac{1}{h} \frac{\partial \widehat{\Psi}}{\partial x} \right) + \varepsilon \frac{\partial}{\partial y} \left( \frac{1}{h} \frac{\partial \widehat{\Psi}}{\partial y} \right) + \Omega \right] \right) = 0.$$

$$(4)$$

Let us introduce a new variable instead of x and t

$$\tilde{t} = t - \int_{0}^{x} \frac{d\xi}{c(\xi)}, \quad \tilde{x} = x,$$

where c is the unknown velocity of the wave propagation. Then, the derivatives in Eq. (4) are defined as

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tilde{t}}, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{x}} - \frac{1}{c(\tilde{x})}\frac{\partial}{\partial \tilde{t}}$$

By omitting ~ again, we have

$$\begin{pmatrix} h\frac{\partial}{\partial t} - \varepsilon \frac{\partial \widehat{\Psi}}{\partial y} \left( \frac{\partial}{\partial x} - \frac{1}{c(x)} \frac{\partial}{\partial t} \right) + \varepsilon \left( \frac{\partial \widehat{\Psi}}{\partial x} - \frac{1}{c(x)} \frac{\partial \widehat{\Psi}}{\partial t} \right) \frac{\partial}{\partial y} \end{pmatrix}$$

$$\times \left( \frac{1}{h} \left[ \mu \varepsilon \left( \frac{\partial}{\partial x} - \frac{1}{c(x)} \frac{\partial}{\partial t} \right) \left( \frac{1}{h} \left( \frac{\partial \widehat{\Psi}}{\partial x} - \frac{1}{c(x)} \frac{\partial \widehat{\Psi}}{\partial t} \right) \right)$$

$$+ \varepsilon \frac{\partial}{\partial y} \left( \frac{1}{h} \frac{\partial \widehat{\Psi}}{\partial y} \right) + \Omega \right] \right) = 0.$$

$$(5)$$

In the approximation of long waves of small amplitude ( $\mu \ll 1$ ,  $\varepsilon \ll 1$ ), the dispersion and nonlinearity parameters are assumed to be of the same order of smallness ( $\mu \sim \varepsilon$ ). By assuming  $\mu = \varepsilon$  in Eq. (5), let us seek the solution as an expansion with respect to powers of the small parameter  $\varepsilon$ 

$$\widehat{\Psi}(x, y, t) = \sum_{i=0}^{n} \varepsilon^{i} \Psi_{i}(x, y, t).$$
(6)

Using the method of multiple scales [9], we introduce a set of slow variables

$$x_i = \varepsilon^l x \quad (i \ge 1). \tag{7}$$

Then,

$$\frac{\partial}{\partial x} = \sum_{i=1}^{\infty} \varepsilon^{i} \frac{\partial}{\partial x_{i}}.$$
(8)

Rewriting Eq. (5) and taking into account asymptotic expansion (6) and Eqs. (7) and (8) in the lowest order

with respect to  $\varepsilon$  (i.e., for  $\varepsilon^1$ ), we obtain a linear problem for finding the function  $\psi_0$ 

$$\frac{\partial}{\partial y} \left[ \frac{1}{h} \frac{\partial^2 \Psi_0}{\partial y \partial t} \right] - \frac{1}{c} \frac{\partial}{\partial y} \left[ \frac{\Omega}{h} \right] \frac{\partial \Psi_0}{\partial t} = 0$$
(9)

with the boundary conditions

$$\Psi_0 = 0$$
 for  $y = 0$ ,  $l$  or  $\Psi_0 \longrightarrow 0$  for  $y \longrightarrow \pm \infty$ . (10)

Separating the variables in Eq. (9) and substituting  $\Psi_0$  as

$$\Psi_0 = A(x,t)F(y,x), \tag{11}$$

we have a linear eigenvalue problem from which the function F(y, x) and the velocity of long waves c(x) can be obtained

$$LF = \frac{\partial}{\partial y} \left[ \frac{1}{h} \frac{\partial F}{\partial y} \right] - \frac{1}{c} \frac{\partial}{\partial y} \left[ \frac{\Omega}{h} \right] F = 0, \quad F = 0 \text{ for}$$
(12)  
$$y = 0, l \text{ or } F \longrightarrow 0 \text{ for } y \longrightarrow \pm \infty.$$

Let us assume that the waves of only one mode are excited, so that F and c are uniquely defined.

In the higher orders (i > 0), we obtain nonhomogeneous boundary problems in the form

$$L\frac{\partial \Psi_i}{\partial t} = R_i \tag{13}$$

with boundary conditions imposed on  $\psi_i$  similar to (10). The term  $R_i$  on the right-hand side of the boundary problem implies various functional dependencies on the function  $\psi_k$  with k < i and unknown derivatives with respect to slow times  $\partial \psi_i / \partial x_i$ . In order to separate the variables in (13), it is necessary to impose corresponding conditions on these derivatives.

Thus, in the higher order for  $\varepsilon^2$ , we have

$$\frac{\partial A}{\partial x} + \varepsilon \left(\beta \frac{\partial^3 A}{\partial t^3} + \alpha A \frac{\partial A}{\partial t} + qA\right) = O(\varepsilon^2)$$
(14)

(see the derivation of Eq. (14) in the Appendix). This equation is a generalized Korteweg–de Vries equation, and it was obtained for the nonlinear topographic Rossby waves in [8]. The KDV equation with constant coefficients was also used for describing the evolution of the atmospheric Rossby waves with account for the horizontal shear zonal flow [14] and baroclinic Rossby waves in the equatorial region of the ocean [10]. In these studies, it was pointed out that, for selected types of the density stratification in the atmosphere or ocean, the coefficient of quadratic nonlinearity  $\alpha$  became equal to zero and the nonlinear term of the next order should be taken into account.

It needs to be pointed out that, in the case under study, when the external conditions ( $\Omega$  and *h*) slightly change in the direction of the wave propagation, i.e., depend on *x*, the coefficients  $\alpha$ ,  $\beta$ , and *q* in Eq. (14) also are variables (depend on *x*). On the contrary, when  $\Omega$  and *h* depend only on *y*, the coefficients  $\alpha$ ,  $\beta$ , and *q* are constants. In so doing, the coefficient *q* vanishes and Eq. (14) becomes the classical Korteweg–de Vries equation.

On subsequent expansion and taking into account Eqs. (5)–(8), we obtain a nonlinear evolution equation

$$\frac{\partial A}{\partial x} + \varepsilon \left(\beta \frac{\partial^3 A}{\partial t^3} + \alpha A \frac{\partial A}{\partial t} + qA\right) + \varepsilon^2 \left(s_2 \frac{\partial^3 A}{\partial t^5} + s_3 A^2 \frac{\partial A}{\partial t} + s_4 A \frac{\partial^3 A}{\partial t^3} + s_5 \frac{\partial A}{\partial t} \frac{\partial^2 A}{\partial t^2}\right)$$
(15)

$$+ s_6 \frac{\partial^2 A}{\partial t^2} - s_7 A^2 + s_8 \int A dt + s_9 \frac{\partial A}{\partial t} \int A dt = O(\varepsilon^3).$$

(for the derivation of this equation, see in the Appendix). This equation represents a generalized KDV equation of the second order for nonlinear topographic Rossby waves that propagate in a slightly inhomogeneous fluid. The coefficients of the equation obtained are variable along the path of the wave propagation.

It should also be noted that such laws of  $\Omega$  and *h* variations are possible, that the coefficient  $\alpha$  at the term of the nonlinearity of the first order of smallness (quadratic nonlinearity) vanishes and the nonlinearity of the second order of smallness (cubic nonlinearity) should be taken into account. In the future, we plan to study the effect of the second order of smallness within the frameworks of the equation obtained for these particular cases.

Thus, the theory of the unsteady nonlinear topographic Rossby waves allows us to more precisely describe the large-scale processes in the ocean.

#### QUALITATIVE ESTIMATES

Equation (15) can be used for describing the topographic Rossby waves with an accuracy of the second order with respect to a small amplitude. If  $\Omega$  and *h* are independent of x, then Eq. (15) is reduced to a KDV equation of the second order. This simplification of the equation is completely integrableonly for definite relations between the coefficients [7, 12]. In the general case, it can be integrated asymptotically [6, 11]; this means that, using a special substitution, it can be reduced to the classical Korteweg-de Vries equation with the same accuracy with respect to a small amplitude. The asymptotic integrability indicates that the solution of the nonintegrable equation of the second order differs from that of its integrable asymptotically closed analog by a local transformation of the wave field and substitution of the time variable. Therefore, the evolution equation of the first order is sufficient for an adequate description of the wave dynamics, at least, up to the second order inclusively.

The qualitative structure of the wave process within the frameworks of the first order approximation and



Fig. 1. Step-wise bottom.



Fig. 2. Cross section of the wave field.

oceanological estimates of the scales of nonlinear and unsteady waves are described in detail in [8]. The existence of the asymptotic integrability makes these estimates valid excluding the case when the coefficient of the quadratic nonlinearity becomes small. In the latter case, the cubic nonlinearity needs to be taken into account and the wave dynamics change. In particular, generation of breezers and "wide" solitons is possible, which are limited in amplitude (such solitons correspond to single eddies extended along the *x* axis). To solve Eq. (15) in the most general case, approximate and numerical calculation methods may be applied.

The examples of topography considered below—a channel with a step-wise bottom and a channel with a linearly sloping bottom—can be used for studying and approximate analysis of the large-scale oceanic wave processes over such kinds of bottom irregularities as the edge of the continental shelf, underwater ridges, and depressions.

#### Topographical Rossby Waves over a Step-wise Bottom

Let us consider an infinite channel with a width l, whose depth does not vary in the longitudinal direction and abruptly changes in the transverse direction. The

function which describes the bottom topography has the form (Fig. 1)

$$h(y) = \begin{cases} h_1, & 0 \le y \le l_1, \\ h_2, & l_1 < y \le l. \end{cases}$$
(16)

In so doing, without a loss of generality, we can assume that  $h_1 > h_2$  (the inverse case can be obtained by the corresponding linear change of the *y* variable). The Coriolis parameter is assumed to be constant

$$\Omega = \Omega_0 = \text{const.} \tag{17}$$

The transverse structure of the wave field and phase velocity of long waves under such conditions can be easily found from the eigenvalue problem (12)

$$F(y) = \begin{cases} y/l_1, & 0 \le y \le l_1, \\ (y-l)/(l_1-l), & l_1 < y \ge l, \end{cases}$$
$$c = \frac{\Omega_0 l_1 (l_1-l)(h_1-h_2)}{l_1 (h_1-h_2) + lh_2}.$$

The form of function F(y) is shown in Fig. 2, and the dimensionless phase velocity  $c^* = \frac{c}{\Omega_0 l}$  as a function of the dimensionless parameters  $\chi = h_2/h_1$  and  $\delta = l_1/l$   $(0 \le \chi \le 1, 0 \le \delta \le 1)$  is presented in Fig. 3. The negative values of the phase velocity indicate that the wave propagates along the *x* axis from the domain of positive values to that of negative ones. With the increase in the depth increment (i.e., with the decrease of  $\chi$ ), the absolute value of the phase velocity increases, reaching its

maximum value 
$$c^*$$
 for a fixed value of  $\chi$  at  $\delta = \frac{\chi - \sqrt{\chi}}{\chi - 1}$ 

The coefficients  $\alpha$ ,  $\beta$ , and q in the KDV equation (14) can also be easily obtained for given depth profiles (16) and the Coriolis parameter value (17)

$$\alpha = -\frac{15}{2} \frac{l_1(h_2^2 - h_1^2) - lh_2^2}{c^2 l_1(l_1 - l)h_1 h_2(h_2 - h_1)}.$$

$$\beta = -\frac{1}{3} \frac{l_1(h_2 - h_1) + lh_1}{\Omega_0 c^2(h_2 - h_1)}.$$
(18)

The coefficient of quadratic nonlinearity  $\alpha$  and the coefficient of dispersion  $\beta$  are constants, and the coefficient *q* at the term responsible for the irregularity of the external conditions along the coordinate of the wave propagation in the case studied is equal to zero, as neither the channel depth nor the Coriolis parameter depends on the *x* coordinate.

The dimensionless coefficients  $\alpha^* = \alpha \Omega_0^2 l^3 h_1$  and  $\beta^* = \beta \Omega_0^3 l$  as a function of the parameter  $\delta$  for fixed values of the parameter  $\chi$  are shown in Fig. 4. Both of



Fig. 3. Dimensionless phase velocity  $c^*$  as a function of the parameters  $\chi = h_2/h_1$  and  $\delta = l_1/l$  in the form of (a) contour lines and (b) curves for fixed values of  $\chi$ .

the coefficients are always positive and infinitely increase when the size of one of the steps decreases (i.e., for  $\delta \longrightarrow 0$  and  $\delta \longrightarrow 1$ ). The equations for the minimum values of these coefficients can be obtained from Eq. (18), but they are rather cumbersome and we do not present them here. The dispersion parameter  $\beta^*$ monotonically increases with the increase in the parameter  $\chi$ , i.e., when the depth difference decreases. Conversely, the behavior of the nonlinear parameter  $\alpha^*$  is rather complicated. Since the polarity of the solitons in the Korteweg–de Vries equation is determined by the sign of the product of the nonlinear and dispersion coefficients, only positive solitons are possible. It needs to be pointed out that, in the case under study, the coefficients of nonlinearity and dispersion in the KDV equation of the first order for any ratio of the model parameter do not vanish. In addition, the highorder corrections, which no doubt can be found analytically, do not essentially affect the dynamics of the wave process.

#### Topographic Rossby Waves over a Channel with a Flat Sloping Bottom

Let us consider a model of the channel with a width *l* when the depth varies linearly in the transverse direction  $h(y) = \gamma_0 y$  and the Coriolis parameter varies so slightly



**Fig. 4.** Dimensionless coefficients of the KDV equation as a function of the (a) quadratic nonlinearity  $\chi = h_2/h_1$  and (b) dispersion  $\chi = l_1/l$ .



Fig. 5. Normalized eigenfunctions for a linearly sloping bottom.

that it can be assumed to be a constant  $\Omega = \Omega_0 = \text{const.}$  In this case, the ordinary differential equation of the second order, which defines the eigenvalue problem (12), is reduced to the Bessel equation and its solution can be given by

$$F(y) = y J_2 \left( 2 \sqrt{\frac{\Omega_0}{c}} y \right),$$

where  $J_n(z)$  is the Bessel function of the first kind, and the eigenvalue *c* can be found from

$$J_2\left(2\sqrt{\frac{\Omega_0}{c}}l\right) = 0.$$
(19)

Considering a dimensionless case and assuming that l = 1 and  $\Omega_0 = 1$ , we can find numerical values of the roots of Eq. (19) for any mode of the topographic Rossby waves. The phase velocities for the three lowest modes are  $c_0 = 0.1516615$ ,  $c_1 = 0.0564573$ , and  $c_2 = 0.0296251$ . The eigenfunctions, which correspond to these coefficients, are shown in Fig. 5. The coefficients in KDV equation (14) for the lowest mode of the topographic Rossby waves are  $\alpha = 744266$  and  $\beta = -17.7888$  (the coefficient q vanishes identically because the external conditions are independent of the coordinate of the wave propagation x). In this case, the soliton of the KDV equation described by

$$A(t,x) = \frac{a}{\cosh^2\left(\sqrt{\frac{\alpha a}{12\beta}}(t-Vx)\right)}, \quad V = \frac{\alpha a}{3}, \quad (20)$$

should have a negative polarity, since the nonlinear coefficient is positive, whereas the dispersion coefficient is negative. The wave field of the zero mode, which in the first approximation is given by (11), for the soliton amplitude a = -1 is shown in Fig. 6. For a velocity amplitude equal to 0.06 m/s and a channel width 50 km, the size of the eddy along the x axis is 200 km.

The KDV coefficients for the waves of the first mode are  $\alpha = -8261923$  and  $\beta = -113.433$ , and, by virtue of the fact that both of them are negative, the amplitude *a* of the soliton (20) should be positive. The wave field for a = 1 in the first approximation is shown in Fig. 7.

#### APPENDIX

When deducing Eq. (14) in the higher order, for  $\varepsilon^2$ , we have from (13)

$$L\frac{\partial \Psi_1}{\partial t} = R_1, \tag{A1}$$

where

$$R_{1} = C_{11} \frac{\partial A}{\partial x_{1}} + C_{12} \frac{\partial^{3} A}{\partial t^{3}} + C_{13} A \frac{\partial A}{\partial t} + C_{14} A,$$



Fig. 6. Wave field of the zero mode defined by the soliton of the KDV equation: (a) stream function contour lines and (b) field of the horizontal velocity of the particles.

$$C_{11} = -F \frac{\partial}{\partial y} \left[ \frac{\Omega}{h} \right], \quad C_{12} = -\frac{F}{c^2 h},$$

$$C_{13} = \frac{F^2}{c^2 h} \left( \frac{\partial^2}{\partial y^2} \left[ \frac{\Omega}{h} \right] - \frac{1}{h} \frac{\partial h}{\partial y} \frac{\partial}{\partial y} \left[ \frac{\Omega}{h} \right] \right),$$

$$C_{14} = \frac{\partial F}{\partial y} \frac{\partial}{\partial x} \left[ \frac{\Omega}{h} \right] - \frac{\partial F}{\partial x} \frac{\partial}{\partial y} \left[ \frac{\Omega}{h} \right].$$

Assuming

$$\frac{\partial A}{\partial x_1} = -\beta \frac{\partial^3 A}{\partial t^3} - \alpha A \frac{\partial A}{\partial t} - qA$$
(A2)

(the coefficients  $\alpha$ ,  $\beta$ , and q will be determined below), we can separate the variables in (A1) assuming



Fig. 7. Wave field of the first mode determined by the soliton of the KDV equation: (a) Stream function contour lines and (b) field of the horizontal velocities of the particles.

$$\Psi_1 = \frac{\partial^2 A}{\partial t^2} T^{(d)}(y, x) + A^2 T^{(n)}(y, x) + \int A dt T^{(x)}(y, x),$$

where  $T^{(d), (n), (x)}(y, x)$  are the dispersion correction, nonlinear correction, and the correction responsible for the weak inhomogeneity along the *x* coordinate to the linear mode F(y, x), respectively, which can be found from the following nonhomogeneous boundary-value problems

$$\begin{cases} LT^{(d)} = -\beta C_{11} + C_{12}, \\ LT^{(n)} = -\alpha C_{11} + C_{13}, \\ LT^{(x)} = -qC_{11} + C_{14}, \end{cases}$$
(A3)

with boundary conditions similar to (10). A necessary condition for nonhomogeneous Eqs. (A3) to be resolvable is the orthogonality of their right-hand sides to the eigenfunctions F of operator L. This results in conditions imposed on the coefficients  $\alpha$ ,  $\beta$ , and q:

$$\alpha = \frac{\int FC_{13} dy}{\int FC_{11} dy}, \quad \beta = \frac{\int FC_{12} dy}{\int FC_{11} dy}, \quad a = \frac{\int FC_{14} dy}{\int FC_{11} dy}.$$
 (A4)

Here, the integration limits are chosen in accordance with the boundary conditions (10): either from 0 to *l* or from  $-\infty$  to  $+\infty$ . Now, substituting (A2) with coefficients (A4) in expansion (8) and holding the terms up to the current order  $\partial A/\partial x_1$  with respect to  $\partial A/\partial x$ inclusively (i.e., up to  $\varepsilon^1$ ), we arrive at the evolution equation (14).

The subsequent expansion and using equations (5)–(8) for  $\varepsilon^3$  yield

$$L\frac{\partial \Psi_2}{\partial t} = R_2, \tag{A5}$$

where

$$\begin{split} R_{2} &= C_{21} \frac{\partial A}{\partial x_{2}} + C_{22} \frac{\partial^{5} A}{\partial t^{5}} + C_{23} A^{2} \frac{\partial A}{\partial t} + C_{24} A \frac{\partial^{3} A}{\partial t^{3}} \\ &+ C_{25} \frac{\partial A}{\partial t} \frac{\partial^{2} A}{\partial t^{2}} + C_{26} \frac{\partial^{2} A}{\partial t^{2}} + C_{27} A^{2} + C_{28} \int A dt + C_{29} \frac{\partial A}{\partial t} \int A dt, \\ C_{21} &= -F \frac{\partial}{\partial y} \Big[ \frac{\Omega}{h} \Big], \\ C_{22} &= -\beta T^{(d)} \frac{\partial}{\partial y} \Big[ \frac{\Omega}{h} \Big] + 2\beta \frac{F}{ch} - \frac{T^{(d)}}{c^{2} h}, \\ C_{23} &= -\alpha T^{(n)} \frac{\partial}{\partial y} \Big[ \frac{\Omega}{h} \Big] + \frac{\alpha}{2} \frac{\partial}{\partial y} \Big[ \Big( \frac{1}{h} \frac{\partial F}{\partial y} \Big)^{2} \Big] \\ &- \alpha F \Big[ \frac{\partial^{2}}{\partial y^{2}} \Big( \frac{1}{h^{2}} \frac{\partial F}{\partial y} \Big) + \frac{1}{h^{2}} \frac{\partial h}{\partial y} \frac{\partial}{\partial y} \Big( \frac{1}{h} \frac{\partial F}{\partial y} \Big) \\ &+ \frac{1}{h^{2}} \frac{\partial F}{\partial y} \frac{\partial}{\partial y} \Big( \frac{1}{h} \frac{\partial h}{\partial y} \Big) \Big] - \frac{1}{ch^{4}} \Big( h^{2} \Big[ 2 \frac{\partial F}{\partial y} \frac{\partial^{2} T^{(n)}}{\partial y^{2}} \\ &- 2 \frac{\partial^{3} F}{\partial y^{3}} T^{(n)} + \frac{\partial^{2} F}{\partial y^{2}} \frac{\partial T^{(n)}}{\partial y} - F \frac{\partial^{3} T^{(n)}}{\partial y^{3}} \Big] \\ &+ 3h \frac{\partial h}{\partial y} \Big[ F \frac{\partial^{2} T^{(n)}}{\partial y^{2}} - \frac{\partial F}{\partial y} \frac{\partial T^{(n)}}{\partial y} + 2 \frac{\partial^{2} F}{\partial y} T^{(n)} \Big] \\ &+ \Big[ h \frac{\partial^{2} h}{\partial y^{2}} - 3 \Big( \frac{\partial h}{\partial y} \Big)^{2} \Big] \Big[ F \frac{\partial T^{(n)}}{\partial y} + 2 \frac{\partial F}{\partial y} T^{(n)} \Big] \Big), \end{split}$$

Vol. 45

2005

No. 5

OCEANOLOGY

 $C_{24} = \frac{\beta}{ch^3} \left( \Omega \left[ \frac{\partial^2 h}{\partial v^2} F^2 - \frac{\partial h}{\partial v} F \frac{\partial F}{\partial v} - \frac{3}{h} \left( \frac{\partial h}{\partial v} \right)^2 F^2 \right]$ +  $2ch\frac{\partial h}{\partial y}T^{(n)}$  +  $\frac{\partial \Omega}{\partial y}\left[hF\frac{\partial F}{\partial y} + 3\frac{\partial h}{\partial y}F^2 - 2ch^2T^{(n)}\right]$  $-h\frac{\partial^2\Omega}{\partial v^2}F^2 + \frac{2\alpha F}{ch} - \alpha T^{(d)}\frac{\partial}{\partial v}\left[\frac{\Omega}{h}\right]$ +  $\frac{1}{c^2 h^4} \left[ FT^{(d)} \right] \Omega \left\{ 3 \left( \frac{\partial h}{\partial y} \right)^2 - h \frac{\partial^2 h}{\partial y^2} \right\}$  $-3h\frac{\partial h}{\partial y}\frac{\partial \Omega}{\partial y} + h^2\frac{\partial^2 \Omega}{\partial y^2} - 2h^3 T^{(n)} \Big),$  $C_{25} = -\frac{\beta}{ch^3} \left( \Omega \left[ -\frac{\partial^2 h}{\partial v^2} F^2 - \frac{\partial h}{\partial v} F \frac{\partial F}{\partial v} + \frac{3}{h} \left( \frac{\partial h}{\partial v} \right)^2 F^2 \right]$  $+\frac{\partial\Omega}{\partial\nu}\left[hF\frac{\partial F}{\partial\nu}-3\frac{\partial h}{\partial\nu}F^{2}\right]+h\frac{\partial^{2}\Omega}{\partial\nu^{2}}F^{2}\right]$  $+\frac{6\alpha F}{ch}+3\alpha T^{(d)}\frac{\partial}{\partial y}\left[\frac{\Omega}{h}\right]-\frac{1}{c^{2}h^{2}}\left(FT^{(d)}\right)\left[\Omega\left\{-3\left(\frac{\partial h}{\partial y}\right)^{2}\right\}\right]$  $+h\frac{\partial^{2}h}{\partial y^{2}} + 3h\frac{\partial h}{\partial y}\frac{\partial \Omega}{\partial y} - h^{2}\frac{\partial^{2}\Omega}{\partial y^{2}} + 6h^{3}T^{(n)} \Big|,$  $C_{26} = -\left(\beta T^{(x)} + q T^{(d)}\right) \frac{\partial}{\partial v} \left[\frac{\Omega}{h}\right] + \frac{2qF}{hc} - \frac{T^{(x)}}{hc^2}$  $-\frac{1}{h^{2}a}\left(\Omega c\left[\frac{\partial h}{\partial x}\frac{\partial T^{(d)}}{\partial y}-\frac{\partial h}{\partial y}\frac{\partial T^{(d)}}{\partial x}\right]-2h\frac{\partial F}{\partial x}\right)$  $+ch\left[\frac{\partial\Omega}{\partial y}\frac{\partial T^{(d)}}{\partial x}-\frac{\partial\Omega}{\partial x}\frac{\partial T^{(d)}}{\partial y}\right]+\frac{\partial ch}{\partial x}F\right),$  $C_{27} = -\frac{q}{h^4} \left[ h^2 \left[ F \frac{\partial^3 F}{\partial y^3} - \frac{\partial F}{\partial y} \frac{\partial^2 F}{\partial y^2} - 2T^{(n)} \left\{ \Omega \frac{\partial h}{\partial y} - 2 \frac{\partial \Omega}{\partial y} h \right\} \right]$  $+h\frac{\partial h}{\partial y}\left[\left(\frac{\partial F}{\partial y}\right)^2-3F\frac{\partial^2 F}{\partial y^2}\right]$  $+ \left[ 3 \left( \frac{\partial h}{\partial y} \right)^2 - h \frac{\partial^2 h}{\partial y^2} \right] F \frac{\partial F}{\partial y} - \frac{\alpha}{2} T^{(x)} \frac{\partial}{\partial y} \left[ \frac{\Omega}{h} \right]$  $-\frac{1}{ch^4}\left(-ch^2\frac{\partial F}{\partial y}\frac{\partial^3 F}{\partial y^2\partial x}-c\Omega h^2\frac{\partial h}{\partial y}\frac{\partial T^{(n)}}{\partial x}\right)$ +  $ch\frac{\partial h}{\partial v}\frac{\partial F}{\partial v}\frac{\partial^2 F}{\partial v\partial x} - h^2\frac{\partial^3 F}{\partial v^3}T^{(x)} - 3\left(\frac{\partial h}{\partial v}\right)^2\frac{\partial F}{\partial v}T^{(x)}$ 

$$\begin{split} &-3ch\frac{\partial h}{\partial y}\frac{\partial F}{\partial x}\frac{\partial^{2} F}{\partial y^{2}}-ch^{3}\frac{\partial \Omega}{\partial x}\frac{\partial T^{(n)}}{\partial y}+2ch\frac{\partial h}{\partial x}\frac{\partial F}{\partial y}\frac{\partial^{2} F}{\partial y^{2}}\\ &-ch\frac{\partial^{2}h}{\partial y^{2}}\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}+ch\frac{\partial^{2}h}{\partial y\partial x}\left(\frac{\partial F}{\partial y}\right)^{2}\\ &-h\frac{\partial h}{\partial y}\frac{\partial F}{\partial y}\frac{\partial T^{(x)}}{\partial y}+h\frac{\partial^{2}h}{\partial y^{2}}\frac{\partial F}{\partial y}T^{(x)}+c\Omega t^{2}\frac{\partial h}{\partial x}\frac{\partial T^{(n)}}{\partial y}\\ &-3c\frac{\partial h}{\partial y}\frac{\partial h}{\partial x}\left(\frac{\partial F}{\partial y}\right)^{2}+h^{2}\frac{\partial F}{\partial y}\frac{\partial^{2} T^{(x)}}{\partial y^{2}}+3h\frac{\partial h}{\partial y}\frac{\partial^{2} F}{\partial y^{2}}T^{(x)}\\ &+ch^{3}\frac{\partial \Omega}{\partial y}\frac{\partial T^{(n)}}{\partial x}+3c\left(\frac{\partial h}{\partial y}\right)^{2}\frac{\partial F}{\partial y}\frac{\partial F}{\partial x}+ch^{2}\frac{\partial F}{\partial x}\frac{\partial^{3} F}{\partial y^{3}}\right),\\ C_{28}&=-qT^{(x)}\frac{\partial}{\partial y}\left[\frac{\Omega}{h}\right]+\frac{1}{h^{2}}\left(h\left[\frac{\partial \Omega}{\partial x}\frac{\partial T^{(x)}}{\partial y}-\frac{\partial \Omega}{\partial y}\frac{\partial T^{(x)}}{\partial x}\right]\right),\\ C_{29}&=\frac{F}{c^{2}h^{4}}\left(\Omega h\frac{\partial h}{\partial y}\frac{\partial T^{(x)}}{\partial y}-h^{2}\frac{\partial \Omega}{\partial y}+3c\left(\frac{\partial h}{\partial y}\right)^{2}\frac{\partial T^{(x)}}{\partial y}\right),\\ C_{29}&=\frac{F}{c^{2}h^{4}}\left(\Omega h\frac{\partial h}{\partial y}\frac{\partial T^{(x)}}{\partial y}-h^{2}\frac{\partial \Omega}{\partial y}+3c\left(\frac{\partial h}{\partial y}\right)^{2}\frac{\partial T^{(x)}}{\partial y}\right).\\ \end{array}$$

Assuming

$$\frac{\partial A}{\partial x_2} = -s_2 \frac{\partial^5 A}{\partial t^5} - s_3 A^2 \frac{\partial A}{\partial t} - s_4 A \frac{\partial^3 A}{\partial t^3} - s_5 \frac{\partial A}{\partial t} \frac{\partial^2 A}{\partial t^2} - s_6 \frac{\partial^2 A}{\partial t^2} - s_7 A^2 - s_8 \int A dt - s_9 \frac{\partial A}{\partial t} \int A dt$$
(A6)

(the coefficients  $s_k$ , k = 2...9 will be determined below), we can separate the variables in (A5)

$$\begin{aligned} \frac{\partial \Psi_2}{\partial t} &= \frac{\partial^5 A}{\partial t^5} T_2(y, x) + A^2 \frac{\partial A}{\partial t} T_3(y, x) \\ &+ A \frac{\partial^3 A}{\partial t^3} T_4(y, x) + \frac{\partial A}{\partial t} \frac{\partial^2 A}{\partial t^2} T_5(y, x) + \frac{\partial^2 A}{\partial t^2} T_6(y, x) \\ &+ A^2 T_7(y, x) + \int A dt T_8(y, x) + \frac{\partial A}{\partial t} \int A dt T_9(y, x). \end{aligned}$$

The corrections of the second order  $T_k$ , k = 2...9 to the linear mode F(y, x) can be found from the following nonhomogeneous boundary problems

$$LT_{k} = -s_{k}C_{21} + C_{2k}, \quad k = 2...9,$$
(A7)

with the boundary conditions similar to (10). The conditions of solvability of boundary problems (A7) give us an equation for the coefficients  $s_k$ , k = 2...9

$$s_k = \frac{\int FC_{2k} dy}{\int FC_{21} dy}.$$

The substitution of  $\partial A/\partial x_1$  from (A2) and  $\partial A/\partial x_2$  from (A6) in the expansion for  $\partial A/\partial x$  (Eq. (8)) and the use of the terms up to the current order *i* with respect to  $\varepsilon$  inclusively (i.e., up to  $\varepsilon^2$ ) result in the nonlinear evolution equation (15).

#### ACKNOWLEDGMENTS

This study was supported by the Russian Foundation for Basic Research, project no. 03-05-64975; INTAS, project no. 03-51-4286; and the Scientific School of Corresponding Member of the Russian Academy of Sciences B.V. Levin, no. NSh-2104.2003.5.

#### REFERENCES

- A. L. Berestov, "Solitary Rossby Waves," Izv. Akad. Nauk, Fiz. Atmos. Okeana 15 (6), 648–654 (1979).
- A. L. Berestov and A. S. Monin, "Solitary Rossby Waves," Usp. Mekh. 3, 3–34 (1980).
- V. M. Volosov, "Nonlinear Topographic Rossby Waves," Okeanologiya 16 (3), 389–396 (1976).
- 4. K. V. Konyaev and K. D. Sabinin, *Waves Inside the Ocean* (Gidrometeoizdat, St. Petersburg, 1992) [in Russian].
- 5. P. Le Blond and L. Mysak, *Waves in the Ocean:* (Elsevier, Amsterdam, 1978; Mir, Moscow, 1981).
- Ch. Li and N. R. Sigbatullin, "Refined Theory of Long Waves on the Water Surface," Prikl. Mat. Mekh. 61 (2), 184–189 (1997).
- 7. A. Newell, *Solitons in Mathematics and Physics* (IO NFMI **320**, 1998) [in Russian].
- A. B. Odulo and E. N. Pelinovskii, "On Nonlinear Topographic Rossby Waves," Okeanologiya 18 (1), 16–19 (1978).
- E. N. Pelinovskii, V. E. Fridman, and Yu. K. Engel'brekht, Nonlinear Evolution Equations (Valgus, Tallinn, 1984) [in Russian].
- 10. J. P. Boyd, "Equatorial Solitary Waves," J. Phys. Oceanagr. **10**, 1699–1717 (1980).
- A. S. Fokas and Q. M. Liu, "Asymptotic Integrability of Water Waves," Phys. Rev. Lett. 77, 2347–2351 (1996).
- T. R. Marchant and N. F. Smyth, "Soliton Interaction for the Extended Korteweg–De Vries Equation," J. Appl. Math. 56, 157–176 (1996).
- S. Pierini, A. M. Fincham, D. Renouard, *et al.*, "Laboratory Modeling of Topographic Rossby Normal Modes," Dyn. Atmos. Oceans 35, 205–225 (2002).
- 14. L. G. Redekopp, "On the Theory of Solitary Rossby Waves," J. Fluid Mech. **82** (4), 725–745 (1977).
- 15. R. Smith, "The Ray Paths of Topographic Rossby Waves," Deep-Sea Res. 18, 477–483 (1971).