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# ON THE MINIMAL NUMBER OF CRITICAL POINTS OF FUNCTIONS ON $h$-COBORDISMS

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ABSTRACT. Let $(W, M_0, M_1)$ be a non-trivial $h$-cobordism (i.e., the Whitehead torsion of $(W, M_0)$ is non-zero) with $W$ compact, connected and $\dim W \geq 6$. We prove that every smooth function $f : W \to [0,1]$, $f(M_0) = 0, f(M_1) = 1$ has at least 2 critical points. This estimate is sharp: $W$ possesses a function as above with precisely two critical points.

## Introduction

Let $(W, M_0, M_1)$ be an $h$-cobordism, [3]. Here $W$ is always assumed to be smooth, connected and compact and $M_i, i = 0, 1$ is always assumed to be closed. Recall that an $h$-cobordism $(W, M_0, M_1)$ is called trivial if there is a diffeomorphism $(W, M_0, M_1) \cong (M \times [0,1], M_0, M_0)$. We say that a function (not necessarily Morse) $f : W \to [0,1]$ is *regular* if $f^{-1}(M_0) = 0$, $f^{-1}(M_1) = 1$ and both values 0 and 1 are regular values of $f$. It is well known that an $h$-cobordism $(W, M_0, M_1)$ is trivial if and only if $W$ possesses a regular function without critical points. In this note we prove the following theorem.

**Theorem.** *Let $(W, M_0, M_1)$ be a non-trivial $h$-cobordism with* $\dim W \geq 6$. *Then every regular function on $W$ has at least two critical points. Moreover, this estimate is sharp: $W$ possesses a regular function with precisely two critical points.*

We denote by $I$ the closed interval $[0,1]$.

## 1. Preliminaries

Let $f : W \to I$ be a regular Morse function on an $h$-cobordism $(W, M_0, M_1)$. Choose a Riemannian metric on $W$ and consider integral trajectories for the vector field $-\operatorname{grad} f$, the so-called anti-gradient trajectories. We say that an anti-gradient trajectory $a = a(t)$ is *a special trajectory from $p$ to $q$* if $\lim_{t\to-\infty} a(t) = p$ and $\lim_{t\to+\infty} a(t) = q$ where $p$ and $q$ are critical points of $f$ such that the index of $p$ is one more than the index of $q$. We can and shall assume that the number of special trajectories is finite (this is true for generic function and metric).

For every critical point of $f$ we fix orientations of unstable disks (left-hand disks in terminology of [3]). Then every unstable sphere (in a certain level)

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gets an orientation. Moreover, every stable sphere gets a coorientation, i.e., an orientation of its normal bundle in the corresponding level set. Now, for every special trajectory $a$ from $p$ to $q$ we define the number $\varepsilon(a) = \pm 1$ as follows. Take $c \in ]f(q), f(p)[$. Then our trajectory $a$ meets the level $f^{-1}(c)$ in a certain point $x$, which is a point of transversal intersection of the corresponding stable and unstable spheres. We define $\varepsilon(a)$ to be the intersection index at $x$.

## 2. Whitehead torsion

Given a ring $R$, we define a based $R$-module to be a free finite generated left $R$-module $M$ with a fixed $R$-free basis.

Recall the definition of the Whitehead torsion of an $h$-cobordism $(W, M_0, M_1)$. Given a group $\pi$, let $A = A(\pi)$ denote the set of long exact sequences

$$\cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow 0$$

such that each $C_i$ is a based $\mathbb{Z}[\pi]$-module and all but finite number of modules $C_i$ are zero modules. Furthermore, each $\partial_i$ is a $\mathbb{Z}[\pi]$-module homomorphism. Let us call the exact sequence of based $\mathbb{Z}[\pi]$-modules trivial if it has only two non-zero terms and the corresponding isomorphism is given by the identity matrix. The term-wise direct sum operation converts $A$ into an abelian semigroup. Let $R$ be the equivalence relation on $A$ generated by the following operations:

- interchanging of the elements;
- replacement of a basis element by the sum of this element with the multiple of another basis element;
- addition of the trivial exact sequence;
- multiplication of any basis element by the element $\pm g, g \in \pi$.

The above mentioned operation in $A$ induces a group structure in $A/R$. This groups is called the *Whitehead group* of $\pi$ and is denoted by $\mathrm{Wh}(\pi)$, [4]. It turns out to be that $\mathrm{Wh}(\pi)$ is a functor of $\pi$. In particular, every homomorphism $\varphi : \pi \to G$ induces a homomorphism $\mathrm{Wh}(\varphi) : \mathrm{Wh}(\pi) \to \mathrm{Wh}(G)$. Namely, the homomorphism $\varphi$ yields the homomorphism $\mathbb{Z}[\varphi] : \mathbb{Z}[\pi] \to \mathbb{Z}[G]$ of group rings, which turns $\mathbb{Z}[G]$ into the right $\mathbb{Z}[\pi]$-module $\mathbb{Z}[G]_{\mathbb{Z}[\varphi]}$. Now, for every based $\mathbb{Z}[\pi]$-module $C$ we can form the based $\mathbb{Z}[G]$-module $\mathbb{Z}[G]_{\mathbb{Z}[\varphi]} \otimes C$. The sequence $\{\mathbb{Z}[G]_{\mathbb{Z}[\varphi]} \otimes C_n\}$ turns out to be exact because all the $C_n$'s are free, etc.

For every $h$-cobordism $(W, M_0, M_1)$ with $\pi_1(W) = \pi$ the Whitehead torsion $\tau(W, M_0, M_1) \in \mathrm{Wh}(\pi)$ is defined as follows. Consider a regular Morse function $f : W \to I$, Riemannian metric, etc. as in §1 . Fix a point $x_0 \in W$ and, for every critical point $p$ of $f$, choose a path $u(p)$ from $x_0$ to $p$. Every special trajectory $a$ from $p$ to $q$ gives us a map $a : \mathbb{R} \to W$ which is well defined up to shift of $t \in \mathbb{R}$. We define a path $v = v_a : I \to W$ as follows. Let $\lambda(t) : ]0, 1[ \to \mathbb{R}$ be a function such that

$$\lim \lambda(t)_{t \to 0} = -\infty, \quad \lim \lambda(t)_{t \to 1} = +\infty.$$

We set $v(0) = p$, $v(1) = q$, $v(t) = a(\lambda(t))$. Now, consider the loop $u(p) \circ v \circ (u(q))^{-1}$ (where $\circ$ denotes the product of paths) and define $g(a) \in \pi = \pi_1(W)$ as the based homotopy class of the loop constructed.

Let $p_1, \dots, p_k$ be all the critical points of the index $n$. Define $C_n$ to be the free $\mathbb{Z}[\pi]$-module generated by symbols $[p_1], \dots, [p_k]$. In other words, $C_n$ consists of formal linear combinations

$$\sum_{i=1}^{k} \alpha_i [p_i], \quad \alpha_i \in \mathbb{Z}[\pi].$$

We define the differential $\partial_n : C_n \to C_{n-1}$ to be a $\mathbb{Z}[\pi]$-module homomorphism such that

$$\partial_n [p] = \sum_q \sum_{a \in T(p,q)} \varepsilon(a) g(a) [q]$$

where $q$ runs over all critical points of the index $n-1$ and $T(p,q)$ is the set of special trajectories from $p$ to $q$.

It follows from the Morse theory that $H_*(\{C_n, \partial_n\}) = H_*(\widetilde{W}, \widetilde{M_0})$ where $(\widetilde{W}, \widetilde{M_0})$ is the universal covering of the pair $(W, M_0)$. Since $M_0$ is a deformation retract of $W$, we conclude that $\widetilde{M_0}$ is a deformation retract of $\widetilde{W}$, and therefore the complex $\{C_n, \partial_n\}$ is acyclic, i.e. the sequence

$$\cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow 0$$

is exact. Thus, the above sequence determines a certain element $\tau = \tau(W, M_0) \in \mathrm{Wh}(\pi)$, the so-called *Whitehead torsion* of the $h$-cobordism $(W, M_0, M_1)$.

According to well-known Barden–Mazur–Stallings Theorem, [1, 2, 5], an $h$-cobordsim $(W, M_0, M_1)$ with $\dim W \geq 6$ is trivial if and only if $\tau(W, M_0) = 0$.

**2.1. Lemma.** *Suppose that an $h$-cobordism $(W, M_0, M_1)$ possesses a regular Morse function $f$ such that all the critical points and special trajectories of $f$ are contained in a simply connected domain $U$ of $W$. Then $\tau(W, M_0) = 0$*

*Proof.* Since $\tau(W, M_0)$ does not depend on the choice of the based point $x_0$ and the paths $u(p)$, we can assume that $x_0 \in U$ and every path $u(p)$ belongs to $U$. Then, for every special trajectory $a$, $g(a)$ is the neutral element of $\pi = \pi_1(W)$. Thus,

$$\tau(W, M_0) \in \mathrm{Im}\{\mathrm{Wh}(j) : \mathrm{Wh}\{e\} \to \mathrm{Wh}(\pi)\}$$

where $j : \{e\} \to \pi$ is the inclusion of the trivial subgroup. But it follows from the elementary linear algebra that $\mathrm{Wh}\{e\} = 0$, see e.g. [4]. Thus, $\tau(W, M_0) = 0$.

## 3. Proof of the theorem

Let $f : M \to \mathbb{R}$ be a smooth function (not necessarily Morse) on a Riemannian manifold $M$. Let $U$ be an open ball in $M$ and suppose that $U$ contains precisely one critical point $o$.

**3.1. Lemma.** *There exists a regular Morse function $g$ which is $C^\infty$-close to $f$ in the Whitney topology and such that every special $g$-trajectory is contained in $U$ whenever its ends are contained in $U$.*

*Proof.* Let $D(r) = \{m \in M \mid d(m, o) < r)$ where $d$ is the distance function on $M$. We can and shall assume that the injectivity radius at $o$ is at least one and that $U = D(1)$. Then there are positive constants $C$ and $E$ such that, for every function $g$ which is $C^\infty$-close to $f$, the following estimates hold in $D(1) \setminus D(1/2)$:

$$|\operatorname{grad} g| \geq E, \quad |L_{\operatorname{grad} g}\, d(m, o)| \leq C.$$

Choose a function $g$ close to $f$. Let $p$ and $q$ be two critical points of $g$ which belong to $U$. Suppose that there is a special trajectory $a(t)$ from $p$ to $q$ which meets the boundary of $D(3/4)$. We claim that in this case

$$g(p) - g(q) \geq \frac{E^2}{4C}.$$

Indeed, since $L_{\operatorname{grad} g}\, d(m, o)| \leq C$, we conclude that

$$a\left[t - \frac{1}{4C}, t + \frac{1}{4C}\right]$$

does not meet $D(1/2)$ whenever $a(t) \notin D(3/4)$ . So, if $a(t_0) \notin D(3/4)$ then

$$g(p) - g(q) \geq \int_{t_0 - \frac{1}{4C}}^{t_0 + \frac{1}{4C}} dg(a(t)) = \int_{t_0 - \frac{1}{4C}}^{t_0 + \frac{1}{4C}} |\operatorname{grad} g|^2 dt \geq \frac{E^2}{4C}.$$

Now we can finish the proof as follows. Since $f$ has only one critical point, there exists $g$ close to $f$ and such that $g(p) - g(q)$ is small enough for all critical points $p$ and $q$ of $g$. This is a contradiction. □

**3.2. Corollary.** *If an $h$-cobordism $(W, M_0, M_1)$ possesses a regular function $f$ with one critical point $p$, then $\tau(W, M_0) = 0$. In particular, if $\dim W \geq 6$ then the $h$-cobordism is trivial.*

*Proof.* Because of Lemma 3.1, we can perturb the function $f$ in a small neighborhood of the critical point and get a function $f_1$ such that all its critical points and special trajectories belong to a disk neighborhood of $p$. Now the result follows from Lemma 2.1. □

**3.3. Proposition.** *Every $h$-cobordism $(W, M_0, M_1)$ , $\dim W \geq 6$ possesses a regular function with at most 2 critical points.*

*Proof.* Consider a regular Morse function $f : W \to I$. Asserting as in [1, Lemme 1] and [3, §4], we can modify $f$ and to get a regular Morse function which has at most two critical levels $a, b$, $a < b$ and index of each of critical points is equal to 2 or 3. Because of this, every critical level is path connected. Now, following [6, Th. 2.7 and Prop.2.9], we can contract the critical points in each of the levels and get a regular function with at most 2 critical points. □

Clearly, Corollary 3.2 and Proposition 3.3 together imply the Theorem.

**3.4. Remarks.** 1. Asserting as in 3.2, one can show that, for every regular function $f$ on a non-trivial $h$-cobordism, the number of critical levels of $f$ is at least 2 provided that all the critical points of $f$ are isolated.

2. Every $h$-cobordism $(W, M_0, M_1)$ possesses a regular function with 1 critical level. Namely, choose collars of the boundary components and define $f$ to be constant on complements of collars and depending on the "vertical" coordinate only for collars. In greater detail, consider a smooth function

$$\varphi : I \to I, \quad \varphi(t) = \begin{cases} t & \text{if } 0 \le t \le \varepsilon/4 \text{ or } 1 - \varepsilon/4 \le t \le 1, \\ 1/2 & \text{if } \varepsilon/2 \le t \le 1 - \varepsilon/2 \end{cases}$$

for $\varepsilon > 0$ small enough. Choose collars $M_0 \times [0, \varepsilon]$ and $M_1 \times [1 - \varepsilon, 1]$ and define $f : W \to I$ by setting

$$f(x) = \begin{cases} \varphi(t) & \text{if } x = (m, t) \in M_0 \times [0, \varepsilon], \\ \varphi(t) & \text{if } x = (m, t) \in M_1 \times [1, 1 - \varepsilon], \\ 1/2 & \text{else.} \end{cases}$$

3. Every trivial $h$-cobordism $(M \times I, M, M)$ possesses a regular function with 1 critical point. Indeed, consider a function $\varphi : M \to I$ such that $\varphi^{-1}(1)$ is a point $m_0$ (and therefore $m_0$ is a critical point of $\varphi$) and define

$$f : M \times I \to I, \quad f(m, t) = (t - 1/2)(1 - \varphi(m)) + \varphi(m)(t - 1/2)^3.$$

It is easy to see that $f$ has just one critical point $(m_0, 1/2)$.

4. Notice that, for every $h$-cobordism $(W, M_0, M_1)$, the relative Lusternik–Schnirelmann category $\operatorname{cat}(W, M_0) = 0$, while every regular function on any non-trivial $h$-cobordism $(W, M_0, M_1)$ has at least two critical points.

5. It is easy to see that, because of the collar theorem, the regularity condition for $f$ in the Theorem can be weaken as follows: $f(M_0) = 0$ and $f(M_1) = 1$.

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