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Densities of topological invariants of quasi-periodic subanalytic sets

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Abstract. We establish the existence of the densities of the Betti numbers and torsions of closed quasi-periodic subanalytic sets. In the course of the proof, we introduce the notion of a uniformly subanalytic set, which is useful when estimating the densities of topological invariants.

Keywords: quasi-periodic topology, quasi-periodic function, transversal volume, ergodic theory.

§ 1. Introduction

Definition 1.1. A *quasi-periodic subanalytic set* is the inverse image of a subanalytic set $M \subset \mathbb{R}^n/\mathbb{Z}^n$ under a homomorphism $\mathbb{R}^k \to \mathbb{R}^n/\mathbb{Z}^n$.

Definition 1.2. The *density of a numerical topological invariant* c *of a set* $M \subset \mathbb{R}^k$ is the limit $c(M) = \lim_{r\to\infty} \frac{c(M\cap rB)}{r^k \operatorname{Vol}(B)}$, where $B \subset \mathbb{R}^k$ is the closed ball with unit radius and centre zero.

In Definition 1.2 and what follows, the multiplication of a subset of a vector space by a number means the homothety with centre zero.

Remark 1.3. In Definition 1.2 and what follows, we can take any bounded subanalytic set with non-empty interior instead of the ball.

Theorem 1.4. *The densities of all the Betti numbers and torsions of a closed quasi-periodic subanalytic set exist.*

We obtain a more detailed result (Theorem 4.2), which enables one to calculate the densities of all Betti numbers approximately, and in some cases, even exactly. Before stating this result, we illustrate it by taking an obvious analogue for finite cell complexes. We number the cells of such a complex in some order of non-decreasing dimension. When we paste in a d-dimensional cell to the union of all the lower-numbered cells, either the d-dimensional Betti number increases by 1 (in this case we refer to the d-dimensional cell as a *cell of type* 1), or the $(d-1)$-dimensional Betti number decreases by 1 (in this case we refer to the d-dimensional cell as a *cell of type* 2). Therefore the d-dimensional Betti number of the whole cell complex is equal to the difference between the number of d-dimensional cells of type 1 and the number of $(d+1)$-dimensional cells of type 2.

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In order to apply this observation to the Betti numbers and torsions of quasi-periodic sets, we must generalize it in two directions. First, the coefficient ring may not be a field, and so pasting in a new cell may change the homology of the cell complex in more ways than the two mentioned above, and the cells will have to be classified into more types. Second, the role of cells will be played by the fibres of so-called fibred cells numbered by a continuous function on the base of the fibre bundle rather than by a finite set of numbers. Therefore, instead of the number of cells of a given type, the case of quasi-periodic sets will involve the volume of the set of points of the base whose fibres are cells of that type.

The notions relevant to generalizing in these two directions are introduced in § 2 and § 3, respectively. The main result (Theorem 4.2) is stated in § 4 and reduced to simpler assertions, Theorems 4.6 and 4.7. The proofs of these theorems are given in § 6 and are based on the connection between averaging over the winding and the transversal volume (Theorem 5.2), which is described in § 5 along with properties of uniformly subanalytic sets (Definition 5.6), which are used to obtain technical upper bounds for the density and averaging in quasi-periodic topology. In § 7 we discuss the possibility of, and methods for, the explicit or approximate calculation of the densities of topological invariants using Theorem 4.2; in particular, the density of the Euler characteristic is expressed in a more explicit form. In § 8 we list some properties of the density of the Euler characteristic (Theorems 8.1 and 8.3), analogues of which for the densities of the Betti numbers have not so far been proved.

The main ideas used in this paper consist of applying Definitions 4.4 and 5.6 and Theorem 2.5 to the proofs of Theorems 4.6 and 4.7.

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§ 2. Betti numbers and annihilators of the boundaries of cells

The proofs of the lemmas in this section are omitted because they are standard.

Definition 2.1. The *dimension* of a module M over a finite ring G is the number $\dim_G M = \log_{|G|} |M|$. If G is a finite ring or a field, the *Betti number* $\beta_i(M;G)$ of a topological space M is the number $\dim_G H_i(M;G)$.

In this notation, the equality $\dim_G A + \dim_G C = \dim_G B$ remains valid for a short exact sequence $0 \to A \to B \to C \to 0$, and so do all its consequences, such as the following.

Lemma 2.2 (triangle inequality for exact sequences). *If* $0 \to \cdots \to A_i \to B_i \to C_i \to \cdots \to 0$ *is an exact sequence of vector spaces over a field* G, *or of modules over a finite ring* G, *then*

$$\sum_i |\dim_G A_i - \dim_G B_i| \leqslant \sum_i \dim_G C_i.$$

We make the terminology more precise. For a cell $f\colon D \to M$, where $D \subset \mathbb{R}^d$ is a closed ball and M is a topological space, the *open cell* is defined as the set $f(\operatorname{int} D) \subset M$, and the *boundary* as the set $\partial f = f(\partial D) \subset M$, with the homology cycle $f_*([\partial D]) \in H(\partial f, \mathbb{Z})$.

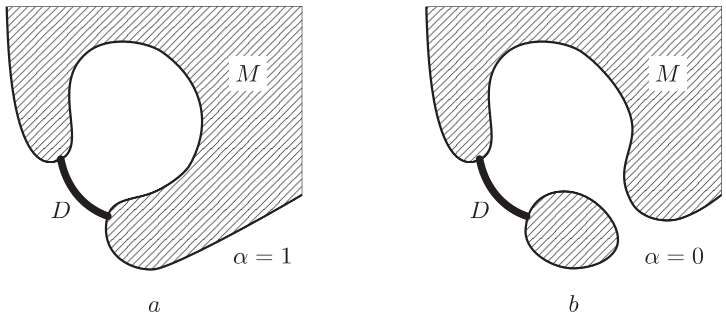

Figure 1

Definition 2.3. The *annihilator* of the boundary of a cell $f\colon D \to M$ in the homology of a set $N \subset M$ containing $f(\partial D)$ with coefficients in a field or a finite ring G is the set of elements $g \in G$ such that $g \cdot f_*([\partial D])$ is the zero element of $H(N; G)$.

The change in the dimension of the homology after pasting in a cell is expressed in terms of the dimension of the annihilator of its boundary.

Lemma 2.4. *Let G be a field or a finite ring and suppose that the dimension of the annihilator of the boundary of a cell $f\colon \partial D^d \to M$ in the homology group $H_{d-1}(M; G)$ is equal to α. Then*

$$\dim_G H_{d-1}\biggl(M \bigcup_f D^d; G\biggr) - \dim_G H_{d-1}(M; G) = -1 + \alpha,$$

$$\dim_G H_d\biggl(M \bigcup_f D^d; G\biggr) - \dim_G H_d(M; G) = \alpha,$$

and

$$\dim_G H_{d'}\biggl(M \bigcup_f D^d; G\biggr) = \dim_G H_{d'}(M; G)$$

in other dimensions d' (see the example in Fig. 1).

Thus, the Betti numbers of a cell complex are obtained by summing the dimensions of the annihilators of the boundaries of its cells. Moreover, under a small change in a cell complex, not only do these sums change little, but the same is true of all their summands.

Theorem 2.5. *Let G be a field or a finite ring and suppose that a compact cell complex X is represented in the form of a union $Y \sqcup \bigsqcup_{i=1}^{I} \sigma_i$, where the σ_i are open cells and $Y \sqcup \bigsqcup_{i=1}^{j} \sigma_i$ is closed for any j. For each cell σ_j, let $A^j \subset G$ and $A_j \subset G$ be the annihilators of the boundary of σ_j in the homology of the sets $Y \sqcup \bigsqcup_{i=1}^{j-1} \sigma_i$ and $\partial Y \sqcup \bigsqcup_{i=1}^{j-1} \sigma_i$, respectively. Then*

1) $\dim_G H_i(X; G) = \dim_G H_i(Y; G) + \sum\limits_{\substack{j \in \mathbb{N} \\ \dim \sigma_j = i}} \dim_G A^j - \sum\limits_{\substack{j \in \mathbb{N} \\ \dim \sigma_j = i+1}} 1 - \dim_G A^j,$

2) $\sum_j (\dim_G A^j - \dim_G A_j) \leqslant \sum_i \dim_G H_i(Y, \partial Y; G)$.

Proof. Assertion 1) follows from the fact that by Lemma 2.4, the difference

$$\dim_G H_i\biggl(Y \sqcup \bigsqcup_{i=1}^{j} \sigma_i; G\biggr) - \dim_G H_i\biggl(Y \sqcup \bigsqcup_{i=1}^{j-1} \sigma_i; G\biggr)$$

is equal to $\dim_G A^j$ when $\dim \sigma_j = i$, to $\dim_G A^j - 1$ when $\dim \sigma_j = i+1$, and to zero in all other cases (these equalities must be summed over all j).

To prove assertion 2) we consider the function

$$\phi(j) = \sum_i \dim_G H_i\biggl(Y \sqcup \bigsqcup_{i=1}^{j} \sigma_i; G\biggr) - \dim_G H_i\biggl(\partial Y \sqcup \bigsqcup_{i=1}^{j} \sigma_i; G\biggr).$$

This function is non-decreasing, and $\phi(j) - \phi(j-1) = 2(\dim_G A^j - \dim_G A_j)$, and therefore

$$\sum_j (\dim_G A^j - \dim_G A_j) \leqslant \frac{\phi(+\infty) - \phi(-\infty)}{2} \leqslant \frac{|\phi(+\infty)| + |\phi(-\infty)|}{2}.$$

By applying Lemma 2.2 to the exact sequence of the pair $(Y \sqcup \bigsqcup_{i=1}^{j} \sigma_i,\ \partial Y \sqcup \bigsqcup_{i=1}^{j} \sigma_i)$ we obtain that

$$|\phi| \leqslant \sum_i \dim_G H_i\biggl(Y \sqcup \bigsqcup_{i=1}^{j} \sigma_i, \partial Y \sqcup \bigsqcup_{i=1}^{j} \sigma_i; G\biggr) = \sum_i \dim_G H_i(Y, \partial Y; G).$$

From the following lemma we obtain upper and lower bounds for the annihilator of the boundary of a cell f of a cell complex Q in the space $\mathbb{R}^k$ based on the structure of the complex in a neighbourhood of the cell (in the notation of the lemma, the role of a neighbourhood is played by the connected component of the complement of Q_1). The lemma also asserts that this estimate sharpens as the neighbourhood expands (as we pass from Q_1 to Q_2 in the notation of the lemma).

Lemma 2.6. *Suppose that the boundary of a cell $f\colon D \to \mathbb{R}^k$ is contained in a set $Q \subset \mathbb{R}^k$, a locally contractible set $Q_1 \subset \mathbb{R}^k$ is disjoint from $f(D)$, and a set $Q_2 \subset \mathbb{R}^k$ is disjoint from the connected component of $\mathbb{R}^k \setminus Q_1$ containing $f(D)$. Let $\Lambda_1, \dots, \Lambda_5$ be the annihilators of the boundary of f in the homology of the sets $Q \setminus Q_1$, $Q \setminus Q_2$, Q, $Q \cup Q_2$, $Q \cup Q_1$, respectively. Then $\Lambda_1 \subset \dots \subset \Lambda_5$.*

Proof. The proof is obvious if $\mathbb{R}^k \setminus Q_1$ consists of a single connected component. The general case reduces to this since the annihilators $\Lambda_1, \dots, \Lambda_5$ do not change if we add to Q_1 all the connected components of $\mathbb{R}^k \setminus Q_1$ except the component C containing the cell $f(D)$. For $\Lambda_1, \dots, \Lambda_4$ the last fact is obvious, and for Λ_5 it follows from the Mayer–Vietoris theorem applied to the partition of the set $Q \cup Q_1$ into sets U and V such that

$$U \cap V = Q_1, \qquad U \setminus V \subset C, \qquad V \setminus U \subset (\mathbb{R}^k \setminus Q_1) \setminus C.$$

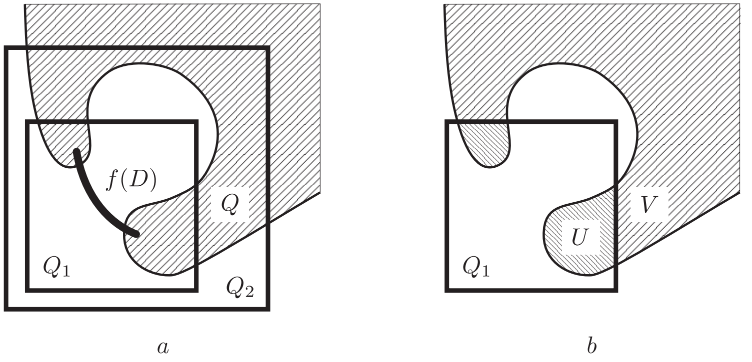

Figure 2

Fig. 2 clarifies the notation used in the statement and proof of the lemma.

Note that the Mayer–Vietoris theorem and the last inclusion in the assertion of the lemma do not always hold without the assumption that Q_1 is locally contractible. All the sets considered below are subanalytic and thus locally contractible.

§ 3. Fibred cell decompositions

Definition 3.1. A *winding of a torus* is a map $j + c\colon \mathbb{R}^k \to \mathbb{R}^n/\mathbb{Z}^n$, where $c \in \mathbb{R}^n/\mathbb{Z}^n$ and $j\colon \mathbb{R}^k \to \mathbb{R}^n/\mathbb{Z}^n$ is an injective homomorphism whose image is dense.

Remark 3.2. Every quasi-periodic subanalytic set can be represented as the inverse image of a subanalytic set under a winding of the torus.

A set $M \subset \mathbb{R}^n/\mathbb{Z}^n$ is called a *fibred cell of dimension* d with respect to a winding $\mathbb{R}^k \to \mathbb{R}^n/\mathbb{Z}^n$ if the intersections of M with the images of all the windings parallel to the given one form a continuous family of cells.

Definition 3.3. We represent a winding $j + c\colon \mathbb{R}^k \to \mathbb{R}^n/\mathbb{Z}^n$ as the composite

$$\mathbb{R}^k \xrightarrow{\tilde{j}+c} \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}^n/\mathbb{Z}^n,$$

where the map $\tilde{j}$ is linear and $\pi\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is the natural projection. Let $\pi_j\colon \mathbb{R}^n \to \mathbb{R}^n/\tilde{j}(\mathbb{R}^k)$ be the natural projection and $A \subset \mathbb{R}^n$ a bounded subanalytic set such that the restriction $\pi|_A$ is injective and each fibre of the projection $\pi_j|_A$ is an open d-dimensional cell in $\mathbb{R}^n$ (see Fig. 3). Under these assumptions, the image $\pi(A)$ is called a *fibred subanalytic open d-dimensional cell* in the torus $\mathbb{R}^n/\mathbb{Z}^n$ with respect to the winding $j+c$, the image $\pi_j(A)$ is called the *base* of the fibred cell, the open cell $\pi(\pi_j^{(-1)}(a) \cap A)$ is called the *fibre* corresponding to a point a of the base, and the union of the boundaries of all the fibres is called the *fibrewise boundary*. The *transversal volume form* on the base is defined as the $(n-k)$-form $\mathrm{d}x/(\tilde{j}_*\mathrm{d}y)$ on $\mathbb{R}^n/\tilde{j}(\mathbb{R}^k)$, where $\mathrm{d}x$ and $\mathrm{d}y$ are the standard volume forms on $\mathbb{R}^n/\mathbb{Z}^n$ and $\mathbb{R}^k$, respectively.

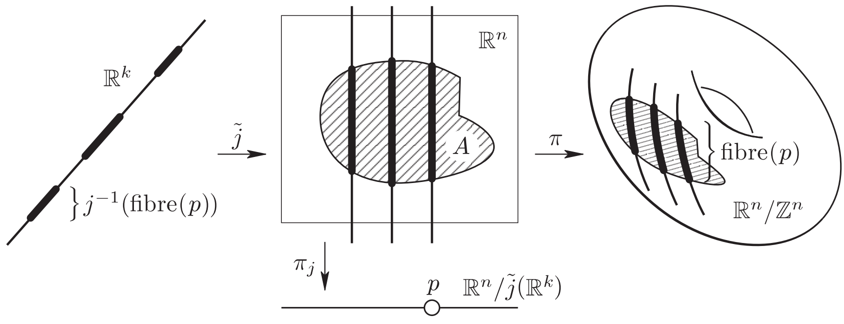

Figure 3

Definition 3.4. A *fibred cell decomposition* $\widetilde{M}$ of a set $M \subset \mathbb{R}^n/\mathbb{Z}^n$ with respect to a given winding $j\colon \mathbb{R}^k \to \mathbb{R}^n/\mathbb{Z}^n$ is a partition of M into finitely many disjoint open fibred cells with respect to this winding such that the fibrewise boundary of each fibred cell is contained in the union of the fibred cells of lower dimension. The *base* $P_{\widetilde{M}}$ of a fibred cell decomposition $\widetilde{M}$ is the disjoint union of the bases of its fibred cells. The union of the bases of the d-dimensional fibred cells is denoted by $P^d_{\widetilde{M}}$. The *transversal volume form* on $P_{\widetilde{M}}$ induced from the bases of the fibred cells is denoted by $\mathrm{d\,Vol}_j$. The fibre of a fibred cell of the decomposition corresponding to a point $p \in P_{\widetilde{M}}$ in the base of this fibred cell is denoted by fibre(p) (Fig. 3).

Lemma 3.5. *Every closed subanalytic set $M \subset \mathbb{R}^n/\mathbb{Z}^n$ admits a fibred cell decomposition with respect to any winding $j\colon \mathbb{R}^k \to \mathbb{R}^n/\mathbb{Z}^n$.*

Proof. Let $\sigma_1, \dots, \sigma_{2^n}$ be the standard cell decomposition of the torus $\mathbb{R}^n/\mathbb{Z}^n$ whose cells $\sigma_i\colon [0,1]^{d_i} \to \mathbb{R}^n/\mathbb{Z}^n$ are numbered in some order of non-increasing dimension, and let $a_{i,q}\colon [0,1]^{d_i} \hookrightarrow [0,1]^{d_q}$, $q < i$, be the attaching maps. We apply the following procedure consecutively to all the cells σ_i.

1) We construct a subanalytic cell decomposition of the set $\sigma_i^{(-1)}(M)$ inscribed in the cell decompositions $a_{i,q}^{(-1)}(\tilde{\sigma}_q)$, where the $\tilde{\sigma}_q$ are the cell decompositions of the sets $\sigma_q^{(-1)}(M)$, $q < i$, constructed in preceding steps (such a decomposition exists, for example, by Theorem 3.2.11 in [1]).

2) We partition the cell decomposition of the set $\sigma_i^{(-1)}(M) \subset [0,1]^{d_i} \subset \mathbb{R}^{d_i}$ constructed in part 1) in such a way that the cells of this subdecomposition are trivially fibred by the natural projection $\mathbb{R}^{d_i} \to \mathbb{R}^{d_i}/L$, where L is a $(k + d_i - n)$-dimensional affine subspace such that $\sigma_i(L \cap [0,1]^{d_i}) \subset j(\mathbb{R}^k)$ (this is possible, for example, by Theorem 9.1.7 in [1]).

Definition 3.6. A *numbering of the fibres* of a fibred cell decomposition $\widetilde{M}$ with respect to a winding $j\colon \mathbb{R}^k \to \mathbb{R}^n/\mathbb{Z}^n$ is a subanalytic function on the base N: $P_{\widetilde{M}} \to \mathbb{R}$ such that the boundary of each fibre is contained in the union of the fibres with smaller numbers, and $N(p_1) \neq N(p_2)$ if the fibres fibre(p_1) and fibre(p_2) are contained in the image of the same winding parallel to $j\colon \mathbb{R}^k \to \mathbb{R}^n/\mathbb{Z}^n$.

Lemma 3.7. *Every fibred cell decomposition admits a numbering.*

Proof. Clearly, it is sufficient to carry out the proof for one fibred cell. In the notation of Definition 3.3, the set $P_0 = \pi_j(\pi^{(-1)}(j(\mathbb{R}^k)))$ is countable. Therefore there exists a linear function l on the space $\mathbb{R}^n/\tilde{j}(\mathbb{R}^k)$ that vanishes at only one point of the set P_0. The restriction of l to the base of the cell can be chosen as a numbering.

Remark 3.8. In many cases there are more convenient methods for obtaining a fibred cell decomposition and a numbering of it than the method described in the proofs of Lemmas 3.5 and 3.7. For example, if $M = \{f \leqslant c\}$, where f is a generic analytic function on the torus, then M can be represented as the union of all Morse cells of the restrictions of f to all the windings parallel to a given one, and these Morse cells turn out to be the fibres of finitely many fibred cells. A numbering of these Morse cells is given by the values of the function f at the corresponding critical points. This construction was used in [2]–[4].

§ 4. The main result

For the rest of the paper we fix a field or finite ring G, a winding $j\colon \mathbb{R}^k \to \mathbb{R}^n/\mathbb{Z}^n$, a cell decomposition $\widetilde{M}$ of a closed subanalytic set $M \subset \mathbb{R}^n/\mathbb{Z}^n$ fibred with respect to j, and a numbering N of $\widetilde{M}$.

Definition 4.1. Let $j_1\colon \mathbb{R}^k \to \mathbb{R}^n/\mathbb{Z}^n$ be a winding parallel to j whose image contains a fibre $\operatorname{fibre}(p)$, $p \in P_{\widetilde{M}}$. We denote by $M_p \subset \mathbb{R}^n/\mathbb{Z}^n$ the union of the fibres $\operatorname{fibre}(p')$ over all the $p' \in P_{\widetilde{M}}$ such that $N(p') < N(p)$. The *type* $T(p) \subset G$ of the fibre $\operatorname{fibre}(p)$ with coefficients in G is the annihilator of the boundary of the cell $j_1^{(-1)}(\operatorname{fibre}(p))$ in the homology group $H(j_1^{(-1)}(M_p); G)$.

Let $\beta_d(j^{(-1)}M; G)$ be the density of the d-dimensional Betti number of the set $j^{(-1)}M$ with coefficients in G, that is, the limit of the ratio

$$\beta_d(j^{(-1)}M \cap rB; G)/\operatorname{Vol}(rB)$$

as $r \to \infty$, where $B \subset \mathbb{R}^k$ is the closed unit ball with centre zero and the Betti number with coefficients in a finite ring is as introduced in Definition 2.1.

Theorem 4.2. *Let G be a field or finite ring, $\widetilde{M}$ a cell decomposition fibred with respect to a winding $j\colon \mathbb{R}^k \to \mathbb{R}^n/\mathbb{Z}^n$, and N a numbering of $\widetilde{M}$. Then the dimension $\varphi(p) = \dim_G T(p)$ is integrable as a function of $p \in P_{\widetilde{M}}$, and the density $\beta_d(j^{(-1)}M; G)$ exists and is equal to $\int_{P^d_{\widetilde{M}}} \varphi \,\mathrm{dVol}_j - \int_{P^{d+1}_{\widetilde{M}}} (1-\varphi)\,\mathrm{dVol}_j$.*

Theorem 1.4 is a special case of Theorem 4.2 since the torsion of the integer homology is expressed in terms of the Betti numbers of the homology with coefficients in $\mathbb{Z}/q^n\mathbb{Z}$, where q is a prime and $n \in \mathbb{N}$.

In the proof of Theorem 4.2 we shall need some auxiliary notions and assertions, as stated below.

Definition 4.3. A *subanalytic* (*bounded subanalytic*) *family of quasi-periodic sets* is a family of sets $M_a = j_a^{(-1)}(M) \subset \mathbb{R}^k$, where $M \subset \mathbb{R}^m \times (\mathbb{R}^n/\mathbb{Z}^n)$ is a subanalytic (bounded subanalytic) set and $j_a(x) = (a, j(x)) \in \mathbb{R}^m \times (\mathbb{R}^n/\mathbb{Z}^n)$ for $a \in \mathbb{R}^m$, $x \in \mathbb{R}^k$.

For closed quasi-periodic sets that form a bounded subanalytic family, the limit in Definition 1.2 converges uniformly. The proof of this fact differs from the proof of Theorem 4.2 only by adding to all the notation the parameter a numbering the sets in this subanalytic family.

The main difficulty in the proof of Theorem 4.2 is that the function φ in the statement of the theorem is not subanalytic and Theorem 5.2 on averaging cannot a priori be applied to it. Therefore we shall bound φ above and below by subanalytic functions which refine the classification of the fibres of a fibred cell decomposition into the types given in Definition 4.1.

For this it is convenient to have a method for associating with a bounded subset $A \subset \mathbb{R}^k$ a point $\operatorname{centre}(A) \in \bar{A}$ in a 'canonical way', in the sense that if a subanalytic set A_t depends subanalytically on a parameter t, then the point $\operatorname{centre}(A_t)$ also depends subanalytically on t. For example, we define $\operatorname{centre}(A)$ as the lexicographically minimal point in $\bar{A}$.

Definition 4.4. In the notation of Definition 4.1, for $p \in P_{\widetilde{M}}$ we denote by $j_p\colon \mathbb{R}^k \to \mathbb{R}^n/\mathbb{Z}^n$ the winding parallel to j such that $j_p(0) = \operatorname{centre}(\operatorname{fibre}(p))$. We denote by $E \subset \mathbb{R}^k$ the complement of the open unit ball with centre zero. For a positive number $R \in \mathbb{R}$, the *lower* $\underline{T}(R,p) \subset G$ and *upper* $\overline{T}(R,p) \subset G$ *types* of the fibre $\operatorname{fibre}(p)$ are the annihilators of the boundary of the cell $j_p^{(-1)}(\operatorname{fibre}(p))$ in the homology of the sets $j_p^{(-1)}(M_p) \setminus \operatorname{int} RE$ and $j_p^{(-1)}(M_p) \cup RE$, respectively. We define the functions $\overline{\varphi}(R,p) = \dim_G \overline{T}(R,p)$ and $\underline{\varphi}(R,p) = \dim_G \underline{T}(R,p)$.

Remark 4.5. By construction, the functions $\underline{\varphi}$ and $\overline{\varphi}$ are subanalytic and take finitely many values, and by Lemma 2.6 the inequalities

$$0 \leqslant \underline{\varphi}(R_1,p) \leqslant \underline{\varphi}(R_2,p) \leqslant \varphi(p) \leqslant \overline{\varphi}(R_2,p) \leqslant \overline{\varphi}(R_1,p) \leqslant 1$$

hold for any $p \in P_{\widetilde{M}}$ and $R_1 < R_2$, where φ is the function defined in the statement of Theorem 4.2. In Fig. 4, a, b we have $\underline{\varphi}(R_1,p) = 0$ and $\overline{\varphi}(R_1,p) = 1$, but here $\underline{\varphi}(R_2,p) = \varphi(p) = \overline{\varphi}(R_2,p) = 1$ in the first case, and $\underline{\varphi}(R_2,p) = \varphi(p) = \overline{\varphi}(R_2,p) = 0$ in the second.

Theorem 4.6. *We have the equality*

$$\lim_{R\to\infty} \int_{P_{\widetilde{M}}} \bigl(\overline{\varphi}(R,\cdot) - \underline{\varphi}(R,\cdot)\bigr)\, \mathrm{dVol}_j = 0.$$

We denote by $\overline{\beta_d}(j^{(-1)}M;G)$ and $\underline{\beta_d}(j^{(-1)}M;G)$, respectively, the upper and lower limits of the ratio $\beta_d(j^{(-1)}M \cap rB;G)/\operatorname{Vol}(rB)$ as $r \to \infty$, where $B \subset \mathbb{R}^k$ is the closed unit ball with centre zero.

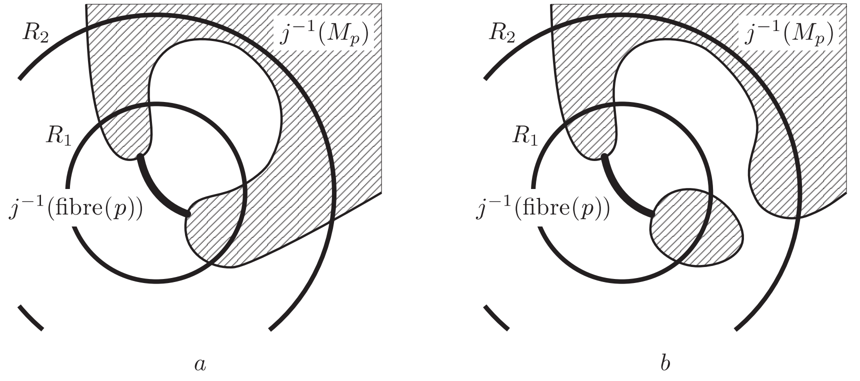

Figure 4

Theorem 4.7. *For any R we have*

$$\overline{\beta_d}(j^{(-1)}M;G) \leqslant \int_{P^d_{\widetilde{M}}} \overline{\varphi}(R,\cdot)\,\mathrm{dVol}_j - \int_{P^{d+1}_{\widetilde{M}}} \bigl(1-\overline{\varphi}(R,\cdot)\bigr)\mathrm{dVol}_j\,,$$

$$\underline{\beta_d}(j^{(-1)}M;G) \geqslant \int_{P^d_{\widetilde{M}}} \underline{\varphi}(R,\cdot)\,\mathrm{dVol}_j - \int_{P^{d+1}_{\widetilde{M}}} \bigl(1-\underline{\varphi}(R,\cdot)\bigr)\,\mathrm{dVol}_j\,.$$

The proof of Theorem 4.2 follows from Theorems 4.6 and 4.7 and Remark 4.5.

It would be useful (see §§ 7, 8) to strengthen Theorem 4.6 as follows.

Conjecture 4.8. *There exists a finite R such that $\overline{\varphi}(R,p)=\underline{\varphi}(R,p)$ for almost all $p\in P_{\widetilde{M}}$.*

§ 5. Averaging and transversal volume

The proofs of Theorems 4.6 and 4.7 are based on the notion of the averaging of a function defined on the base of a fibred cell decomposition.

Definition 5.1. The *averaging* $\mathrm{dens}(F)$ of a subanalytic function $F: P_{\widetilde{M}}\to\mathbb{R}$ on the base of a fibred cell decomposition $\widetilde{M}$ is the limit

$$\lim_{r\to\infty}\frac{F(rB)}{r^k\,\mathrm{Vol}(B)}\,,$$

where $F(rB)$ is the sum of the values $F(p)$ over all the $p\in P_{\widetilde{M}}$ such that $\mathrm{fibre}(p)\subset j(rB)$, and $B\subset\mathbb{R}^k$ is the unit ball with centre zero.

In this notation the main theorem of [5] takes the following form.

Theorem 5.2 [5]. *For any subanalytic function $F\colon P_{\widetilde{M}}\to\mathbb{R}$, the averaging* $\mathrm{dens}(F)$ *exists and is equal to the integral with respect to the transversal volume* $\int_{P_{\widetilde{M}}} F\,\mathrm{dVol}_j$.

In what follows we shall also need a method for estimating from above the contribution of the boundary of the ball over which the averaging is carried out to the density of topological invariants and to the averaging of non-negative functions. This method (Lemma 5.9) is based on the fact that for families of sets that are not too complicated (Definition 5.6, Lemma 5.7), the complexity of their topology (Definition 5.3) is uniformly bounded by the volumes of their neighbourhoods (Definition 5.8).

Definition 5.3. The *complexity* $\mathrm{compl}(M)$ of a set (pair of sets) $M \subset \mathbb{R}^k$ is the sum of its Betti numbers.

Lemma 5.4. *The complexity of a union* $\bigcup_i M_i$ *is at most* ab, *where* a *is the number of non-empty intersections of the form* $\bigcap_\alpha M_{i_\alpha}$ *and* b *is the maximum of the complexities of these intersections. The complexity of a pair* $M \subset N$ *is at most the sum of the complexities of the boundary of* M *in* N *and of the closure of* $N \setminus M$ *in* N.

Proof. The proof follows from the calculation of the Čech homology.

Definition 5.5. A *bounded subanalytic family* of subsets (respectively, of pairs of subsets) in $\mathbb{R}^k$ is the family of fibres of a bounded subanalytic set $M \subset \mathbb{R}^m \times \mathbb{R}^k$ (respectively, of a pair of subanalytic sets $M \subset N \subset \mathbb{R}^m \times \mathbb{R}^k$ such that $N \setminus M$ is bounded) with respect to the natural projection $\mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m$.

The complexity of the (pairs of) sets in a bounded subanalytic family is bounded (see, for example, [1], Theorem 9.1.7).

Definition 5.6. We say that a family of (pairs of) sets $N_\alpha \subset \mathbb{R}^k$, $\alpha \in A$, is *uniformly subanalytic* if the family of (pairs of) sets $\{(N_\alpha - a) \cap [0,1]^k \mid \alpha \in A,\ a \in \mathbb{R}^k\}$ is contained in some bounded subanalytic family of (pairs of) subsets of the space $\mathbb{R}^k$. (We denote by $N_\alpha - a$ the parallel translation of the set N_α by the vector $-a$.)

This definition does not change if we replace the cube $[0,1]^k$ by any other bounded subanalytic subset of $\mathbb{R}^k$ with non-empty interior.

The uniform subanalyticity of the sets we shall consider is proved using the following obvious lemma (the elementary properties of subanalytic sets needed for its proof are given, for example, in [1]).

Lemma 5.7. 1) *If two families of sets* $M_\alpha \subset \mathbb{R}^k$, $\alpha \in A$, *and* $M_\beta \subset \mathbb{R}^k$, $\beta \in B$, *are uniformly subanalytic, then so are the families* $\{M_\alpha \mid \alpha \in A \cup B\}$, $\{M_\alpha \cup M_\beta \mid \alpha \in A,\ \beta \in B\}$, $\{M_\alpha \cap M_\beta \mid \alpha \in A,\ \beta \in B\}$, $\{M_\alpha \setminus M_\beta \mid \alpha \in A,\ \beta \in B\}$, *and the families of closures, interiors, boundaries, connected components and* ε*-neighbourhoods of the sets* M_α, *and the family of pairs of sets* $\{(M_\alpha, M_\alpha \cap M_\beta) \mid \alpha \in A,\ \beta \in B\}$.

2) *A bounded subanalytic family of quasi-periodic sets* (*Definition* 4.3) *is uniformly subanalytic. In particular, a quasi-periodic subanalytic set is uniformly subanalytic.*

3) *The family of all balls in* $\mathbb{R}^k$ *is uniformly subanalytic.*

The following property of uniformly subanalytic families will be important for us.

Definition 5.8. The *size* $\operatorname{size}(M)$ of a set $M \subset \mathbb{R}^k$ (respectively, of a pair $M \subset N \subset \mathbb{R}^k$) is the number of elementary cubes of the form $([0,1]^k + a) \subset \mathbb{R}^k$, $a \in \mathbb{Z}^k$, that intersect the set M (respectively, the closure of the difference $N \setminus M$).

Lemma 5.9. *For any uniformly subanalytic family of (pairs of) sets $M_\alpha \subset \mathbb{R}^k$, there exists a $C \in \mathbb{R}$ such that $\operatorname{compl}(M_\alpha) < C \operatorname{size}(M_\alpha)$ for any α.*

Proof. The proof consists of applying Lemma 5.4 to the partition of the (pairs of) sets M_α by elementary cubes of the form $([0,1]^k + a) \subset \mathbb{R}^k$, $a \in \mathbb{Z}^k$.

§ 6. Proof of Theorems 4.6 and 4.7

Proof of Theorem 4.7. The proof of the first part of the theorem consists of applying the following lemma (one must divide the inequality stated in it by r^k, let r tend to infinity, and apply Theorem 5.2 to the right-hand side). The second part is proved in similar fashion.

Lemma 6.1. *There exists a $C \in \mathbb{R}$ such that*

$$\dim_G H_i\bigl(j^{(-1)}(M) \cap rB; G\bigr) < CRr^{k-1} + \sum_{\substack{p \in P^i_{\widetilde{M}} \\ \operatorname{fibre}(p) \subset j(rB)}} \overline{\varphi}(R,p) - \sum_{\substack{p \in P^{i+1}_{\widetilde{M}} \\ \operatorname{fibre}(p) \subset j(rB)}} \bigl(1 - \overline{\varphi}(R,p)\bigr)$$

for any $r > C$ and $R < r - C$, where $B \subset \mathbb{R}^k$ is the unit ball with centre zero.

Proof. The proof is based on the application of Theorem 2.5, 1) in the case when $X = j^{(-1)}(M) \cap rB$ and Y is the neighbourhood of the intersection of X with the boundary of rB defined below. In Fig. 5, a, b, the sets X and Y are depicted in the case when $j^{(-1)}(M) = \mathbb{R}^k$, and the 'deformed graph paper' depicts the cell decomposition of $j^{(-1)}(M)$ induced by the fibred cell decomposition $\widetilde{M}$.

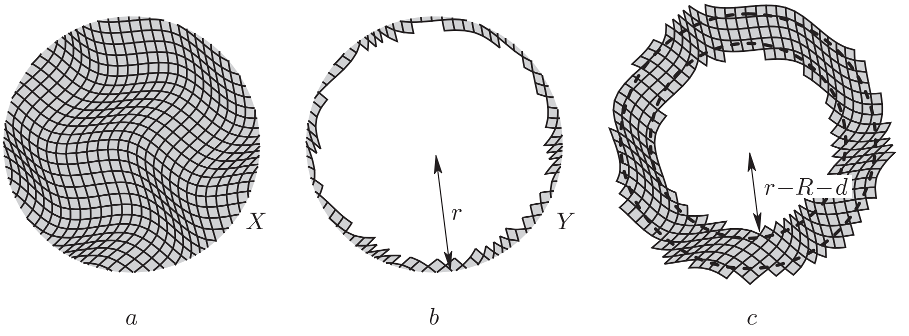

Figure 5

We denote by $E \subset \mathbb{R}^k$ the complement of the open unit ball with centre zero and by $M'(r)$ the difference between the union of all the open cells of the form $j^{(-1)}(\operatorname{fibre}(p))$, $p \in P_{\widetilde{M}}$, that intersect rE, and the interior of rE. We denote by d the supremum of the diameters of the cells $j^{(-1)}(\operatorname{fibre}(p))$, $p \in P_{\widetilde{M}}$, and by α

the number of values of the parameter $p \in P_{\widetilde{M}}$ for which the cell $j^{(-1)}(\text{fibre}(p))$ intersects $(r-R-d)E \setminus rE$ (such cells are depicted in Fig. 5, c).

By Theorem 2.5, 1), the difference

$$\dim_G H_i(j^{(-1)}(M) \cap Br; G) - \dim_G H_i(\overline{M'}(r); G)$$

is at most α greater than the sum $\sum \overline{\varphi}(R,p) - \sum\bigl(1-\overline{\varphi}(R,p)\bigr)$ in the statement of the lemma. Indeed, in the hypothesis of Theorem 2.5, 1), we set $X = j^{(-1)}(M) \cap rB$, $Y = \overline{M'}(r)$ and $\sigma_q = j^{(-1)}(\text{fibre}(p_q))$, where p_q runs, in the order of an increasing numbering N, over all the values of the parameter $p \in P_{\widetilde{M}}$ for which the open cell fibre(p_q) is contained in $j(X \setminus Y)$. Then, in the notation of that theorem, $\dim_G A^q \leqslant 1$ for all p_q and, moreover, by Lemma 2.6 we have $\dim_G A^q \leqslant \overline{\varphi}(R, p_q)$ for all p_q such that the cell $j^{(-1)}(\text{fibre}(p_q))$ does not intersect $(r-R-d)E$. The number $r-R-d$ is chosen in such a way as to make the ball of radius R centred at a point of the cell $j^{(-1)}(\text{fibre}(p_q))$ disjoint from Y and to make it possible to choose Q_1, Q_2 and Q in the statement of Lemma 2.6 to be the boundary of this ball, Y and $j^{(-1)}(M_{p_q})$, respectively (see the notation M_{p_q} in Definition 4.1). Thus, the assertion of Theorem 2.5, 1) implies the estimate

$$\begin{aligned}
&\dim_G H_i(j^{(-1)}(M) \cap Br; G) \\
&\quad= \dim_G H_i(\overline{M'}(r); G) + \sum_{q\in\mathbb{N},\, \dim\sigma_q = i} \dim_G A^q - \sum_{q\in\mathbb{N},\, \dim\sigma_q = i+1} (1 - \dim_G A^q) \\
&\quad\leqslant \dim_G H_i(\overline{M'}(r); G) + \sum_{p_q \in P^i_{\widetilde{M}}} \overline{\varphi}(R, p_q) - \sum_{p_q \in P^{i+1}_{\widetilde{M}}} (1 - \overline{\varphi}(R, p_q)) + \alpha.
\end{aligned}$$

In turn, $\dim_G H_i(\overline{M'}(r); G)$ is estimated from above as Cr^{k-1} by Lemma 5.9 since the family $\overline{M'}(r)$, $r \in \mathbb{R}$, is uniformly subanalytic by Lemma 5.7. For the same reason, α is estimated from above as CRr^{k-1}.

Proof of Theorem 4.6. This consists of applying the following lemma (one must divide the inequality stated in it by r^k, let r tend to infinity, and apply Theorem 5.2 to the left-hand side).

Lemma 6.2. *There exists a $C \in \mathbb{R}$ such that for $R > m\sqrt{k}$ the inequality*

$$\sum_{\substack{p \in P_{\widetilde{M}} \\ \text{fibre}(p) \subset j(rB)}} \bigl(\overline{\varphi}(R,p) - \underline{\varphi}(R,p)\bigr) < \frac{Cr^k}{m}$$

holds for any $r \in \mathbb{R}$ and $m \in \mathbb{N}$, where $B \subset \mathbb{R}^k$ is the unit ball with centre zero.

We define the *grid* $c \subset \mathbb{R}^k$ as the set of all points at least one of whose coordinates is an integer and denote by $E \subset \mathbb{R}^k$ the complement of the open unit ball with centre zero.

Proof of Lemma 6.2. This is based on the application of Theorem 2.5, 2) in the case when $X = j^{(-1)}(M) \cup rE \cup mc$ and Y is the neighbourhood of the intersection of X with $rE \cup mc$ defined below. In Fig. 6, a, b, the sets X and Y are depicted

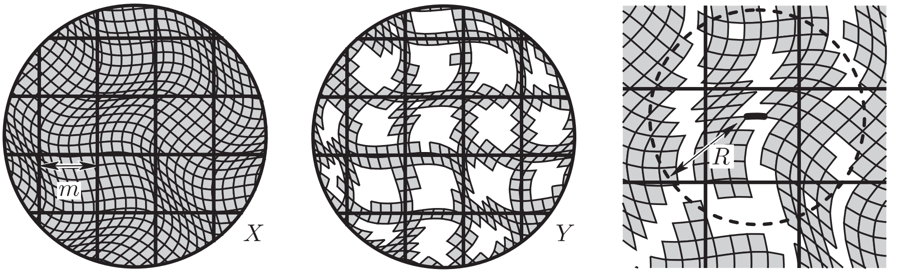

Figure 6

in the case when $j^{(-1)}(M)=\mathbb{R}^k$, and the 'deformed graph paper' depicts the cell decomposition of $j^{(-1)}(M)$ induced by the fibred cell decomposition $\widetilde{M}$.

We denote by $M''(r,m)$ the union of the set $rE\cup mc$ and all the open cells of the form $j^{(-1)}(\mathrm{fibre}(p))$, $p\in P_{\widetilde{M}}$, that intersect $rE\cup mc$. We break the sum $\sum(\overline{\varphi}(R,p)-\underline{\varphi}(R,p))$ in the statement of the lemma into two parts:

1) the summands for which the cell $j^{(-1)}(\mathrm{fibre}(p))$ is contained in the closure $\overline{M''}(r,m)$,

2) all the other summands.

The sum 2) is estimated from above as

$$\sum_i \dim_G H_i\bigl(\overline{M''}(r,m),\overline{M''}(r,m)\setminus M''(r,m);G\bigr)$$

using Theorem 2.5, 2). Indeed, if in this theorem we set $X=j^{(-1)}(M)\cup\overline{M''}(r,m)$, $Y=\overline{M''}(r,m)$ and $\sigma_q=j^{(-1)}(\mathrm{fibre}(p_q))$, where p_q runs over all the values of the parameter p corresponding to the summands of the sum 2) in the order of an increasing numbering N, then, in the notation of that theorem, $\dim_G A^q\geqslant\overline{\varphi}(R,p_q)\geqslant\underline{\varphi}(R,p_q)\geqslant\dim_G A_q$ by Lemma 2.6. The condition $R>m\sqrt{k}$ guarantees that the cube of the grid mc containing the cell $j^{(-1)}(\mathrm{fibre}(p_q))$ does not intersect the sphere S of radius R centred at a point of the cell $j^{(-1)}(\mathrm{fibre}(p_q))$ (see Fig. 6, c), and we can apply Lemma 2.6 to this cell setting Q_1, Q_2 and Q to be equal to Y, S and $j^{(-1)}(M_{p_q})$, respectively (see the notation M_{p_q} in Definition 4.1). Thus, the assertion of Theorem 2.5, 2) gives the estimate

$$\begin{aligned}\sum_q(\overline{\varphi}(R,p_q)-\underline{\varphi}(R,p_q))&\leqslant\sum_q(\dim_G A^q-\dim_G A_q)\\&\leqslant\sum_i\dim_G H_i\bigl(\overline{M''}(r,m),\overline{M''}(r,m)\setminus M''(r,m);G\bigr).\end{aligned}$$

In turn, the sum

$$\begin{aligned}&\sum_i\dim_G H_i\bigl(\overline{M''}(r,m),\overline{M''}(r,m)\setminus M''(r,m);G\bigr)\\&\qquad=\sum_i\dim_G H_i\bigl(\overline{M''}(r,m)\setminus\operatorname{int} rE,\overline{M''}(r,m)\setminus M''(r,m);G\bigr)\end{aligned}$$

is estimated from above as $\frac{Cr^k}{m}$ by Lemma 5.9 since the family of pairs

$$\bigl(\overline{M''}(r,m)\setminus \operatorname{int} rE,\ \overline{M''}(r,m)\setminus M''(r,m)\bigr), \qquad r\in\mathbb{R},\ m\in\mathbb{N},$$

is uniformly subanalytic by Lemma 5.7. For the same reason, the number of summands of type 1) in the sum is estimated from above as $\frac{Cr^k}{m}$.

§ 7. Questions of computability

Theorem 4.2 represents the densities of topological invariants of quasi-periodic sets in the form of linear combinations of certain integrals. We now state some assertions that help to evaluate these integrals.

In the notation of Theorem 4.2,

$$\beta_d(j^{(-1)}M;G) = a_d + a_{d+1} - b_{d+1},$$

where

$$a_d = \int_{P^d_{\widetilde{M}}} \varphi \,\mathrm{dVol}_j, \quad b_d = \int_{P^d_{\widetilde{M}}} 1 \,\mathrm{dVol}_j, \qquad 0\leqslant a_d\leqslant b_d.$$

The numbers b_d can be calculated explicitly. Their definition implies the following lemma.

Lemma 7.1. *Let $M_d\subset\mathbb{R}^n/\mathbb{Z}^n$ be a piecewise-smooth set of codimension k such that almost every point of the set M_d is contained in a d-dimensional fibred cell and for almost every $p\in P^d_{\widetilde{M}}$ the fibre* fibre(p) *contains exactly one point of the set M_d (here 'almost' means 'except for a $(n-k-1)$-dimensional set'). Then*

$$b_d = \int_{M_d} \mathrm{d}x/(j_*\mathrm{d}y),$$

where dx *and* dy *are the standard volume forms on $\mathbb{R}^n$ and $\mathbb{R}^k$.*

Theoretically, the numbers a_d can be calculated effectively with any preassigned accuracy using the following lemma since the functions $\overline{\varphi}$ and $\underline{\varphi}$ occurring in it are effectively determined by the fibred cell decomposition $\widetilde{M}$ (see Definition 4.4).

Lemma 7.2. *For every fibred cell decomposition $\widetilde{M}$ there exists a C such that for any $R\in\mathbb{R}$ we have*

1) $\displaystyle\int_{P^d_{\widetilde{M}}} \underline{\varphi}(R,\cdot)\,\mathrm{dVol}_j \leqslant a_d \leqslant \int_{P^d_{\widetilde{M}}} \overline{\varphi}(R,\cdot)\,\mathrm{dVol}_j,$

2) $\displaystyle\int_{P^d_{\widetilde{M}}} \overline{\varphi}(R,\cdot)\,\mathrm{dVol}_j - \int_{P^d_{\widetilde{M}}} \underline{\varphi}(R,\cdot)\,\mathrm{dVol}_j < \frac{C}{R}.$

Proof. Part 1) follows from Remark 4.5 while part 2) is the more detailed formulation of Theorem 4.6 in which this theorem is actually proved in § 6. In particular, in this proof the constant C is effectively determined by $\widetilde{M}$.

If Conjecture 4.8 is true, then for large R an explicit formula for the number a_d follows from part 1) of Lemma 7.2. The simplest case in which this conjecture is true is described in the following obvious lemma.

Lemma 7.3. *If for all d-dimensional fibres* $\operatorname{fibre}(p)\colon D^d \to \mathbb{R}^n/\mathbb{Z}^n$, $p \in P^d_{\widetilde{M}}$, *of a fibred cell decomposition* $\widetilde{M}$, *the boundary of the fibre*

$$(\operatorname{fibre}(p))_*[\partial D^d] \in H_{d-1}\bigl((\operatorname{fibre}(p))(\partial D^d); G\bigr)$$

is equal to zero, then $a_d = b_d$.

The hypothesis of Lemma 7.3 holds if, for example, in the fibred cell decomposition $\widetilde{M}$ there are no $(d-1)$-dimensional fibred cells. In particular, $a_0 = b_0$. We also note that the numbers a_d cancel out in the alternating sum of the densities of the Betti numbers.

Lemma 7.4. *The density of the Euler characteristic of a quasi-periodic set* $j^{(-1)}M$ *is equal to* $\sum_{d=0}^{n}(-1)^d b_d$.

§ 8. Density of the Euler characteristic

Lemma 7.4 and well-known properties of the volume of sets that form a subanalytic family imply the following.

Theorem 8.1. *The density of the Euler characteristic of closed quasi-periodic sets* M_a *that form a one-parameter subanalytic family* (*Definition* 4.3) *is a subanalytic function of the parameter* a *everywhere except for a discrete set of points.*

Remark 8.2. It would be useful to investigate the nature of such a dependence for the density of the Betti numbers since the arguments of this paper yield only the continuity of the densities of the Betti numbers everywhere except for a countable set of points (the author also knows a proof of the fact that the densities of the Betti numbers are piecewise Lipschitz).

Lemma 7.4 can easily be strengthened. We say that the density of a topological invariant c of a set $Q \subset \mathbb{R}^k$ converges rapidly to its limit c_0 if

$$\varlimsup_{r\to\infty}\left|\frac{c(Q\cap rB)}{r^{k-1}\operatorname{Vol}(B)} - rc_0\right| < \infty.$$

Theorem 8.3. *The density of the Euler characteristic of a subanalytic quasi-periodic set converges rapidly.*

Remark 8.4. In this connection, it would be interesting to investigate the rate of convergence of the densities of the Betti numbers.

Remark 8.5. The questions suggested in Remarks 8.2 and 8.4 are special cases of the following more important and difficult question. Is the dependence of the densities of the Betti numbers of subanalytic quasi-periodic sets on the initial data analytic? (For the density of the Euler characteristic, the analyticity follows from Lemma 7.4.) In particular, if Conjecture 4.8 is true, it would be interesting to investigate the dependence of the number R in its statement on the initial data. As a first step towards formulating plausible conjectures, it would be useful to calculate the functions $\underline{\varphi}$ and $\overline{\varphi}$ approximately for concrete examples (the realization of such a calculation presents technical difficulties even in the simplest cases).

Remark 8.6. The proof given below can be applied not only to a closed, but also to an arbitrary, subanalytic set $M \subset \mathbb{R}^n/\mathbb{Z}^n$. To do so, one must represent M as the union of the fibred cells that are a part of a fibred cell decomposition of the closure of M. The other results in this paper can likewise be easily extended to arbitrary subanalytic sets, but the technical methods needed for such a generalization are different from those considered above and will be described in a future paper.

Proof of Theorem 8.3. We prove the theorem for the additive Euler characteristic, that is, the Euler characteristic of the homology with compact support. (Recall that for a topological space that can be represented as the union of several cells $\sigma_1, \dots, \sigma_q$ of a cell complex, the additive Euler characteristic is equal to $\sum_i (-1)^{\dim \sigma_i}$.) It is sufficient to prove that for some C,

$$\biggl| \chi\bigl(j^{(-1)}(M) \cap rB\bigr) - \sum_{\substack{p \in P_{\widetilde{M}} \\ \operatorname{fibre}(p) \subset j(rB)}} (-1)^{\dim \operatorname{fibre}(p)} \biggr| < Cr^{k-1} \qquad (*)$$

for all r, divide this inequality by r^{k-1}, let r tend to infinity, and apply to the sum on the left-hand side the strengthened version of Theorem 5.2 in which convergence is replaced by rapid convergence.

By the additivity of the Euler characteristic, the difference on the left-hand side of inequality $(*)$ is equal to the Euler characteristic of the set $M'(r)$ (recall that $M'(r)$ is the difference between the union of all the open cells of the form $j^{(-1)}(\operatorname{fibre}(p))$, $p \in P_{\widetilde{M}}$, that intersect rE, and the interior of rE, where $E \subset \mathbb{R}^k$ is the complement of the open unit ball with centre zero). Since the family of sets $M'(r)$ is uniformly subanalytic, there exists a $C \in \mathbb{R}$ such that the Euler characteristics of these sets are less than Cr^{k-1}.

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