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Ergodic properties of invariant Erdös measure for golden ratio

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Abstract

We study the ergodic properties of the Erdös measure and the invariant Erdös measure in the case of the golden ratio and for all values of the Bernoulli parameter. We provide an effective algorithm for the calculations of the entropy of the invariant Erdös measure. We show that for certain values of the Bernoulli parameter that algorithm gives the Hausdorff dimension of the Erdös measure to the fifteen decimal places.

1 Introduction

Almost seventy years ago Erdös [1] posed the following problem:
What one can say about the distribution function of the random variable :
$\zeta = \zeta_1\rho + \zeta_2\rho^2 + \ldots$, *where* $\zeta_1, \zeta_2, \ldots$ *are independent, identically distributed random variables taking values* 0, 1 *and* $0 < P(\zeta_i = 0) = 1/2$, $(0 < \rho < 1)$. We will call this distribution of the random variable ζ the *Erdös measure* on the real line.

The problem of Erdös has been the subject of the large number of papers.

In [4] the authors gave the definitions of the *Erdös measure* on the unit interval $[0,1]$, on the Fibonacci compactum and the invariant *Erdös measure* on the Fibonacci compactum for the case $\rho = 1/\beta$, where $\beta = (\sqrt{5}+1)/2$ is the golden ratio. In [4] the authors proved that the *Erdös measure* is equivalent the invariant *Erdös*

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measure on the Fibonacci compactum. Vershik posed the problem about the ergodic properties of the invariant *Erdös measure* on the Fibonacci compactum. This problem was solved in [4].

In the previous paper [2] we discovered the connection of the Erdös - Vershik problem with the hidden Markov chains for more general case $0 < P(\zeta_i = 0) = q < 1$, $P(\zeta_i = 1) = p$, $\rho = (\sqrt{5} - 1)/2$.

In [3] we proved that the two-sided shift on the two-sided Fibonacci compactum with the invariant Erdös measure is isomorphic to an integral automorphism over the Bernoulli shift with a countable alphabet. Using of this theorem we obtained an effective algorithm for the calculations of the entropy of the invariant Erdös measure.

Here we give another derivation of that algorithm for the calculations of the entropy of the invariant Erdös measure. The ratio of the entropy of the invariant *Erdös measure* and $\ln(\beta)$ is the Hausdorff dimension of the invariant *Erdös measure* on the Fibonacci compactum with the metric $d(x, y) = \rho^{n(x,y)}$, where $n(x, y)$ is the length of the longest common prefix of the words x and y. This dimension is equal to the Hausdorff dimension of the *Erdös measure* on the real line.

The formula for the Hausdorff dimension of the *Erdös measure* on the real line was obtained by Feng in [7] (theorem 4.29). Our formula coincides with the Feng's formula. Therefore we gave a new derivation of the Feng's formula. The direct calculation of the Hausdorff dimension with the help of Feng's formula is impossible because the series for the Hausdorff dimension converges too slowly for the effective computation.

In our case Lalley [6] obtained another formula for the Hausdorff dimension of the *Erdös measure* on the real line. Using this formula and Monte Carlo method he was able to obtain the estimates of the Hausdorff dimension of the *Erdös measure* with the accuracy (to confidence level .99) to within $\pm.002$ for the various values of p. For those values of p that are in [6] we calculate the Hausdorff dimension of the *Erdös measure* with a larger number of decimal digits. This allows to estimate the accuracy of Lalley's estimates.

In our calculations we use the acceleration of the convergence of the series in the formula for the Hausdorff dimension. This acceleration is analogues to the acceleration from the paper of Alexander - Zagier [5]. Note that the Lalley's calculations are the calculations of the Lyapunov exponent for some sequence of random matrices (see [6]). One can say the same about our calculations but our sequence of random matrices is another sequence.

2 Invariant Erdös measure on Fibonacci compactum.

We shall give the definition of invariant *Erdös measure* on Fibonacci compactum (see [2]). In [2] the problem about the ergodic properties of the invariant *Erdös measure* was reduced to the study of the hidden Markov chain $\{\eta_i = f(\xi_i)\}$, with generating Markov chain $\{\xi_i\}$ with 5 states $1, 2, 3, 4, 5$ and the transition matrix P of form

$$P = \begin{pmatrix} q & 0 & 0 & pq & p^2 \\ q & 0 & qp & 0 & p^2 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The initial distribution l is the stationary distribution, the "gluing" function f equals 0 for the states $1, 2, 3$ and equals 1 for the states $4, 5$.

The hidden Markov chain generates the distribution of probabilities μ on the space of its realizations. A useful fact is that the distribution μ is the distribution of the infinite random $0-1$ word $\eta_1\eta_2....\eta_n... = f(\xi_1)f(\xi_2)....f(\xi_n)....$ Its support X is the Fibonacci compactum of the infinite $0-1$ Fibonacci words without the subwords 11. This set is the compact set with the respect to the metric $d(x, y) = \rho^{n(x,y)}$, where $n(x, y)$ is the length of the longest common prefix of words x and y.

Let T be the shift on the space X. The T-invariant measure μ is the invariant *Erdös measure* on the Fibonacci compactum [2].

The shift T on the Fibonacci compactum is an exact endomorphism relative to the measure μ(see [2]). In particular, the invariant *Erdös measure* is ergodic.

In the matrix P we take blocks $P(00), P(01), P(10), P(11)$, corresponding to the partition of the set $\{1, 2, 3, 4, 5\}$ into two subsets, $\{1, 2, 3\} and$ $\{4, 5\}$:

$$P(00) = \begin{pmatrix} q & 0 & 0 \\ q & 0 & qp \\ 0 & 1 & 0 \end{pmatrix}, \ P(01) = \begin{pmatrix} pq & p^2 \\ 0 & p^2 \\ 0 & 0 \end{pmatrix}, \ P(10) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Denote by $l(0)$ the row whose elements are first three elements of the row l, and denote by $l(1)$ the row whose elements are last two elements of the row l, where the row l is the stationary distribution of the Markov chain with the transition matrix P. Also denote by $r(0)$ the column $(1, 1, 1)^\top$ and denote by $r(1)$ the column $(1, 1)^\top$.

Let $n \geq 2$ and $a = a_1...a_n$ be a finite Fibonacci word. We set

$$P(a) = P(a_1a_2)...P(a_{n-1}a_n).$$

Then

$$\mu(\{x : x_1x_2...x_n = a\}) = \mu(a) = l(a_1)P(a)r(a_n), \mu(a_1) = l(a_1)r(a_1).$$

We consider the following coding on the set X_0 words of Fibonacci with initial letter 0: such word x partitioned by overlapping blocks 00 and 010 (last letter of block is the initial letter the following block) and we send the block 00 to 0 and the block 010 to 1 then the word x transfer to infinite 0-1 word $y = y_1y_2....$.

The induced map T' on the set X_0 has the form $T'x = Tx, x = 00..., T'x = T^2x, x = 01...$ Under our coding that map corresponds to the shift S on the space Y.

The conditional measure on the set X_0 transfer to a S - invariant measure on the compact set Y. We denote this measure by μ_0.

Now we derive explicit formula for measure μ_0. Introduce the matrices $t(0) = P(0,0), t(1) = P(0,1)P(1,0)$.

Let $t = t(0) + t(1)$, and $l_t = (q, p^2, p^3 + q)$. The row l_t is the left eigenvector of the matrix t with the eigenvalue 1. The column $r_t = (1,1,1)$ is the right eigenvector of the matrix t with the eigenvalue 1.

It is easy to prove that the measure μ_0 on the compact set Y given by the formula:

$$\mu_0(\{y : y_1 = a_1, y_2 = a_2, ...y_n = a_n\}) = \frac{l_tt(a_1)t(a_2)...t(a_n)r_t}{l_tr_t}.$$

The return function $F(y)$ is equal to 1 for $y_0(y) = 0$ and equal to 2 for $y_0(y) = 1$. Have

$$\int_Y F(y)d\mu_0(y) = \frac{(1+p)(1-pq)}{1-2q^2+3q^3-q^4}.$$

Now we consider the following coding on the set Y of 0-1 words: such word y partitioned by blocks 00, 1, 01 and we send the block 00 to 0 and the block 1 to 1 and the block 01 to 2. Then the word y transfer to infinite word $z = z_1z_2....$ in the alphabet $\{0,1,2\}$.

The map T_1 on the set Y has the form $T_1y = S^2y, y = 00..., y = 01.., T_1y = Sy, y = 10....$, i.e. $T_1y = S^{C(y)}y$, where $C(y) = 2, y = 00..., y = 01.., , C(y) = 1, y = 1....$ Under our coding that map corresponds the shift S_1 on the space Z of all words $S_1z = z_2z_3...., z = z_1z_2....$ Have

$$\int_Y C(y)d\mu_0(y) = \frac{(1+q)(1-2q^2+3q^3-q^4)}{1-pq}.$$

Introduce the matrices $s(0) = t(0)t(0), s(1) = t(1), s(2) = t(0)t(1)$.

Let $s = s(0) + s(1) + s(2)$, and l_s is a left eigenvector of the matrix s with the eigenvalue 1 and r_s is a right eigenvector of the matrix s with the eigenvalue 1. The measure μ_0 on the set X_0 transfer to a S_1 - invariant measure on the compact set Z. We denote this measure by μ_1. Now we derive explicit formula for measure μ_1. It is easy to prove that the measure μ_1 on the compact set Z given by the formula:

$$\mu_1(\{z : z_1 = a_1, z_2 = a_2, ...z_n = a_n\}) = \frac{l_ss(a_1)s(a_2)...s(a_n)r_s}{l_sr_s}.$$

We delete the third row and the third column of the each matrix $s(j)$. We denote new matrix by $w(j)$:

$$w(0) = \begin{pmatrix} q^2 & 0 \\ q^2 & qp \end{pmatrix}, w(1) = \begin{pmatrix} pq & p^2 \\ 0 & p^2 \end{pmatrix}, w(2) = \begin{pmatrix} pq^2 & p^2q \\ pq^2 & p^2q \end{pmatrix}.$$

Let $w = w(0) + w(1) + w(2)$, and a row l_w is left eigenvector of the matrix w with the eigenvalue 1. The column $r_w = (1,1)$ is right eigenvector of the matrix t with the eigenvalue 1. It is easy to prove that the measure μ_1 on the compact set Z given by the formula:

$$\mu_1(\{z : z_1 = a_1, z_2 = a_2, ...z_n = a_n\}) = \frac{l_ww(a_1)w(a_2)...w(a_n)r_w}{l_wr_w}.$$

Introduce the column $u = (1,1)^\top$ and the row $v = (pq^2, p^2q)$ Remark that $w(2) = uv$. For any 0-1 word $a = a_1a_2....a_n$ define the matrix

$$w(a) = w(a_1)w(a_2)...w(a_n), w(\varnothing) = Id.$$

Also define $p(a)$ as

$$p(a) = vw(a)u.$$

A word $z \in Z$ called the 2-regular word if in this word the letter 2 occur infinitely many times. We consider the set Z_2 of all 2-regular words $z \in Z$ with the initial letter 2. A word $z \in Z_2$ partitioned by the blocks $2A_i(z), i \in N$, where $A_i(z)$ is a 0-1 word. This expansion starts from the first place. It is just the beginning of the first block $2A_1(z)$. The block $2A_2(z)$ starts after the first block , etc.

The induced map S_1' on the set Z_2 has the form $S_1'z = S_1^{F_2(z)}z$, where $F_2(z) = |A_1(z)| + 1$ and $|A_1(z)|$ is the length 0-1 word of $A_1(z)$. Calculate the distribution of random 0-1 word $A_1(z)$ on the set Z_2 with the conditional measure μ_1'.

From definition of the measure μ_1 follow, that

$$\mu_1(\{z \in Z : z(1) = 2\}\mu_1'(\{z \in Z_2 : A_1(z) = a\}) = \mu_1(\{z \in Z : 2A_1(z)2 = 2a2\}) =$$

$$= \frac{l_w uvw(a)uvr_w}{l_w r_w} = p(a)\frac{l_w w(2) r_w}{l_w r_w} =$$
$$= p(a)\mu_1(\{z \in Z : z_1 = 2\}).$$

Hence $p(a) = \mu_1'(\{z \in Z_2 : A_1(z) = a\})$. In particular, we obtain the equality $\sum_a p(a) = 1$. Thus the distribution $p(a)$ is the distribution of the random 0-1 word $A_1(z)$.

In similar way to calculate joint distribution of random variables $A_1(z)$, $A_2(z)$, ..., $A_m(z)$ on the set Z_2 :

$$\mu_1(\{z \in Z_2 : A_1(z) = a^1, ..., A_m(z) = a^m\}) =$$
$$= l_w uvw(a^1)uvw(a^2)uv...uvw(a^m)uvr_w v =$$
$$= p(a^1)...p(a^m)\mu_1(\{z \in Z : z_1 = 2\}).$$

Thus identically distributed random variables $A_j(z)$ are independent.

Generating function of the distribution $p(a)$ is equal to

$$\varphi(z) = \sum_a p(a) z^{|a|},$$

where $|a|$ is the length of 0-1 word a. Have

$$\varphi(z) = v(Id - z(w(0) + w(1))^{-1}u = \frac{pq}{1-(1-pq)z}.$$

Now we calculate the mean value of the length of 0-1 word :

$$\int_Z |A_1(z)| d\mu_1'(z) = \varphi'(1) = \frac{1-pq}{pq}.$$

Hence

$$\int_{Z_2} F_2(z) d\mu_1'(z) = \frac{1}{pq}.$$

Note that the probability $\mu_1'(\{z \in Z_2 : |A_1(z)| = n\}) = pq(1-pq)^n$.

Let $\alpha = \frac{p}{q}$. Introduce the matrices $\widetilde{w}(0), \widetilde{w}(1)$:

$$\widetilde{w}(0) = \begin{pmatrix} 1 & 0 \\ 1 & \alpha \end{pmatrix}, \widetilde{w}(1) = \begin{pmatrix} \alpha & \alpha^2 \\ 0 & \alpha^2 \end{pmatrix}.$$

Set

$$\widetilde{w}(a) = \widetilde{w}(a_1)...\widetilde{w}(a_n),$$
$$\tilde{u} = (1,1)^{\top}, \tilde{v} = (\alpha, \alpha^2).$$

Have

$$p(a) = \tilde{v}\widetilde{w}(a)\tilde{u}q^{2n+3}, n = |a|.$$

3 The entropy of invariant Erdös measure.

In this section for the definition of the entropy we shall use the binary logarithm. From the Abramov's formula [8] for the entropy of the integral automorphism we obtain that the entropy of the invariant Erdös measure μ equals

$$h(\mu) = \frac{h(\mu_0)}{\int_Y F(y) d\mu_0(y)}.$$

From the Belinskaya's formula [9] we have

$$h(\mu_0) = \frac{h(\mu_1)}{\int_Z C(z) d\mu_1(z)},$$

where $C(z) = 2\ for\ z = 0..., z = 2..., C(z) = 1\ for\ z = 1...$

From the Abramov's formula [8] we have

$$h(\mu_1) = \frac{h(p(\cdot))}{\int_{Z_2} F_2(z) d\mu_1(z)},$$

where

$$h(p(\cdot)) = -\sum_a p(a)\log_2 p(a).$$

We have

$$\int_Y F(y)d\mu_0(y) \int_Z C(z)d\mu_1(z) \int_{Z_2} F_2(z)d\mu_1'(z) =$$
$$= \frac{(1+p)(1-pq)}{1-2q^2+3q^3-q^4} \frac{(1+q)(1-2q^2+3q^3-q^4)}{1-pq} \frac{1}{pq} =$$
$$= 1 + \frac{2}{pq}.$$

Therefore we obtain the following formula for entropy :

$$h(\mu) = \frac{h(p(\cdot))}{1+\frac{2}{pq}}.$$

Recall that $p = q\alpha,\ q = \frac{1}{1+\alpha}$ and

$$p(a) = \tilde{v}\widetilde{w}(a)\tilde{u}q^{2n+3}, n = |a|.$$

Let D_n be the set of all 0-1 words of the length n. Introduce the notation

$$k_n = \sum_{a \in D_n} \log_2(\tilde{v}\widetilde{w}(a)\tilde{u})(\tilde{v}\widetilde{w}(a)\tilde{u}).$$

Then the entropy $h(\mu)$ of the invariant Erdös measure is equal to

$$H = \log_2(1+\alpha) - \frac{1}{1+\frac{2(1+\alpha)^2}{\alpha}} \sum_{n=0}^{\infty} k_n \left(\frac{1}{1+\alpha}\right)^{2n+3}.$$

If $q = 1/2$, then

$$H = 1 - \frac{1}{9}\sum_{n=0}^{\infty} k_n \frac{1}{2^{2n+3}}.$$

Note that the formula for the Hausdorff dimension of the invariant Erdös measure $H/\log_2(\beta)$ coincides with the Alexander - Zagier formula for the Hausdorff dimension of the Hausdorff dimension of the Erdös measure on the real line [5]. The Alexander - Zagier formula was obtained with the help of the combinatorics of the Euclidean tree. It is possible that our formula corresponds to the combinatorics of $\alpha-$ Euclidean tree.

The main difficulty for the calculation of the entropy H is the slow convergence of the corresponding series. The series for H converges too slowly for the effective computation. Following the approach of Alexander - Zagier [5], we use some rearrangement of the series for H.

Introduce

$$\mu_n = k_n - [3]_\alpha k_{n-1},\ k_{-1} = 0,$$

where $[3]_\alpha = 1 + \alpha + \alpha^2$. Then

$$(1-[3]_\alpha x)\left(\sum_{n=0}^{\infty} k_n x^n\right) = \sum_{n=0}^{\infty} \mu_n x^n$$

Consider

$$\lambda_n = 2\lambda_{n-1} - \lambda_{n-2} + \mu_n - [3]_\alpha \mu_{n-1},\ \lambda_0 = \mu_0, \lambda_1 = 2\lambda_0 + \mu_1 - [3]_\alpha \mu_0.$$

It is clear that

$$\sum_{n=0}^{\infty} k_n x^n = \frac{1}{1-[3]_\alpha x}\sum_{n=0}^{\infty} \mu_n x^n$$

$$(1-x)^2 \sum_{n=0}^{\infty} \lambda_n x^n = (1-[3]_\alpha x)\sum_{n=0}^{\infty} \mu_n x^n$$

$$\sum_{n=0}^{\infty} k_n x^n = \frac{(1-x)^2}{(1-[3]_\alpha x)^2}\sum_{n=0}^{\infty} \lambda_n x^n$$

Use this relation and set $x = (\frac{1}{1+\alpha})^2$, obtain

$$H = \log_2(1+\alpha) - \frac{\alpha(2+\alpha)}{(1+2\alpha)} \sum_{n=0}^{\infty} \lambda_n \left(\frac{1}{1+\alpha}\right)^{2n+3}.$$

This series converges more rapidly than the initial series.

The following relation holds: entropy H under the substitution α by $\frac{1}{\alpha}$ is the same. Of course, the series for $\alpha > 1$ converges more rapidly. Below we use this relation for the calculation of the Hausdorff dimension.

4 Results of calculations.

In the following table we give the values of the Hausdorff dimension $H_{dim} = H/\log_2\beta$ of the invariant *Erdös measure* on the Fibonacci compactum with the metric $d(x,y) = \rho^{n(x,y)}$, where $n(x,y)$ is the length of the longest common prefix of the words x and y.

In the table the second column gives the values of the Hausdorff dimension of the *Erdös measure* for different probabilities p. In the third column it is shown how many terms of the series are chosen in the formula for the Hausdorff dimension of the *Erdös* measure. In the fourth column shows the results of Lalley calculations.

p	H_{dim}	n	Lalley
0.05	0.392167680782199076	15	0.3877 ± 0.03
0.05	0.392167680782199076	14	
0.1	0.6101383374950678578	20	0.6085 ± 0.008
0.1	0.6101383374950678578	19	
0.2	$0.84990339802715197\underline{6}$	23	0.8499 ± 0.004
0.2	$0.84990339802715197\underline{2}$	22	
0.3	$0.9513889802259\underline{8}70$	24	0.9501 ± 0.002
0.3	$0.9513889802259\underline{8}69$	23	
0.4	$0.987545683253293\underline{8}$	25	0.9868 ± 0.001
0.4	$0.987545683253293\underline{1}$	24	
0.5	$0.99571312668555552\underline{6}$	24	0.9954 ± 0.0008
0.5	$0.995713126685555556\underline{0}$	23	

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