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# CONTRIBUTIONS TO GAME THEORY AND MANAGEMENT

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The volume may be recommended for researches and post-graduate students of management, economic and applied mathematics departments.



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**Успехи теории игр и менеджмента. Вып. 4.** Сб. статей четвертой международной конференции по теории игр и менеджменту / Под ред. Л.А. Петросяна, Н.А. Зенкевича. – СПб.: Высшая школа менеджмента СПбГУ, 2011. – 514 с.

Сборник статей содержит работы участников четвертой международной конференции «Теория игр и менеджмент» (28–30 июня 2010 года, Высшая школа менеджмента, Санкт-Петербургский государственный университет, Санкт-Петербург, Россия). Представленные статьи относятся к теории игр и ее приложениям в менеджменте.

Издание представляет интерес для научных работников, аспирантов и студентов старших курсов университетов, специализирующихся по менеджменту, экономике и прикладной математике.

# Contents

# Cooperative Optimality Concepts for Games with Preference Relations

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**Abstract.** In this paper we consider games with preference relations. The cooperative aspect of a game is connected with its coalitions. The main optimality concepts for such games are concepts of equilibrium and acceptance. We introduce a notion of coalition homomorphism for cooperative games with preference relations and study a problem concerning connections between equilibrium points (acceptable outcomes) of games which are in a homomorphic relation. The main results of our work are connected with finding of covariant and contrvariant homomorphisms.

**Keywords:** Nash equilibrium, Equilibrium, Acceptable outcome, Coalition homomorphism

## 1. Introduction

We consider a n-person game with preference relations in the form

$$G = \langle (X_i)_{i\in N}, A, F, (\rho_i)_{i\in N} \rangle \tag{1}$$

where $N = \{1, \ldots, n\}$ is a set of players, $X_i$ is a set of *strategies* of player $i$ $(i \in N)$, $A$ is a set of *outcomes*, realization function $F$ is a mapping of set of *situations* $X = X_1 \times \ldots \times X_n$ in the set of outcomes $A$ and $\rho_i \subseteq A^2$ is a preference relation of player $i$. In general case each $\rho_i$ is an arbitrary reflexive binary relation on $A$. Assertion $a_1 \overset{\rho_i}{\lesssim} a_2$ means that outcome $a_1$ is less preference than $a_2$ for player $i$. Given a preference relation $\rho_i \subseteq A^2$, we denote by $\rho_i^s = \rho_i \cap \rho_i^{-1}$ its symmetric part and $\rho_i^* = \rho_i \backslash \rho_i^s$ its strict part (see Savina, 2010).

The cooperative aspect of a game is connected with its coalitions. In our case we can define for any coalition $T \subseteq N$ its set of strategies $X_T$ in the form

$$X_T = \prod_{i\in T} X_i. \tag{2}$$

We construct a preference relation of coalition $T$ with help of preference relations of players which form the coalition. We denote a preference relation for coalition $T$ by $\rho_T$. The following condition is minimum requirement for preference of coalition $T$:

$$a_1 \overset{\rho_T}{\lesssim} a_2 \;\Rightarrow\; (\forall i \in T)\, a_1 \overset{\rho_i}{\lesssim} a_2. \tag{3}$$

In section 2 we consider some important concordance rules. Let $\mathcal{K}$ be a fix collection of coalitions. In section 3 we introduce the following cooperative optimality concepts: Nash $\mathcal{K}$–equilibrium, $\mathcal{K}$–equilibrium, quite $\mathcal{K}$–acceptance, $\mathcal{K}$–acceptance and connections between these concepts are established in Theorem 1. In next section we consider coalition homomorphisms. The main results of our paper are presented in section 5.

## 2. Concordance rules for preferences of players

To construct a preference relation for coalition $T$ we need to have preference relations of all players its coalition and also certain rule for concordance of preferences of players. Such set of rules is called *concordance rule.* It is known that important concordance rules are the following.

### 2.1. Pareto concordance

**Definition 1.** Outcome $a_2$ is said to (non strict) dominate by Pareto outcome $a_1$ for coalition $T$ if $a_2$ is better (not worse) than $a_1$ for each $i \in T$, i.e.

$$a_1 \overset{\rho_T}{\lesssim} a_2 \;\Leftrightarrow\; (\forall i \in T)\, a_1 \overset{\rho_i}{\lesssim} a_2. \tag{4}$$

In this case symmetric part of preference relation for coalition $T$ is defined by the formula

$$a_1 \overset{\rho_T}{\sim} a_2 \Leftrightarrow (\forall i \in T)\, a_1 \overset{\rho_i}{\sim} a_2 \tag{5}$$

and strict part is defined by the formula

$$a_1 \overset{\rho_T}{<} a_2 \;\Leftrightarrow\; \begin{cases} (\forall i \in T)\; a_1 \overset{\rho_i}{\lesssim} a_2, \\ (\exists j \in T)\;\, a_1 \overset{\rho_j}{<} a_2 \end{cases} \tag{6}$$

Thus, outcome $a_2$ dominate $a_1$ if and only if $a_2$ is better than $a_1$ for all players of coalition $T$ and strictly better at least for one player $j \in T$.

### 2.2. Modified Pareto concordance

In this case strict part of preference relation $\rho_T$ is defined by the equivalence

$$a_1 \overset{\rho_T}{<} a_2 \;\Leftrightarrow\; (\forall i \in T)\, a_1 \overset{\rho_i}{<} a_2, \tag{7}$$

and symmetric part is given by

$$a_1 \overset{\rho_T}{\sim} a_2 \;\Leftrightarrow\; (\forall i \in T)\, a_1 \overset{\rho_i}{\sim} a_2. \tag{8}$$

### 2.3. Concordance by majority rule

Outcome $a_2$ is strictly better than outcome $a_1$ for coalition $T$ if and only if $a_2$ is strictly better than $a_1$ for majority of players of coalition $T$, i.e.

$$a_2 \overset{\rho_T}{>} a_1 \Leftrightarrow \left|\left\{i \in T\colon a_2 \overset{\rho_i}{>} a_1\right\}\right| > \left|\frac{T}{2}\right|.$$

For this rule, symmetric part of preference relation $\rho_T$ is given by the equivalence

$$a_1 \overset{\rho_T}{\sim} a_2 \Leftrightarrow \left|\left\{i \in T\colon a_1 \overset{\rho_i}{\sim} a_2\right\}\right| > \left|\frac{T}{2}\right|.$$

### 2.4. Concordance under summation of payoffs

For games with payoff functions in the form $H = \left\langle (X_i)_{i\in N}, (u_i)_{i\in N} \right\rangle$, the following concordance rule of preferences for coalition $T$ is used

$$x^1 \overset{\rho_T}{\lesssim} x^2 \Leftrightarrow \sum_{i\in T} u_i(x^1) \leq \sum_{i\in T} u_i(x^2) \tag{9}$$

and the strict part of $\rho_T$ is given by:

$$x^1 \overset{\rho_T}{<} x^2 \Leftrightarrow \sum_{i\in T} u_i(x^1) < \sum_{i\in T} u_i(x^2).$$

In this case preference relation $\rho_T$ and its strict part are transitive.

**Remark 1.** Let $\{T_1, \ldots, T_m\}$ be partition of set $N$. Then collection of strategies of these coalitions $(x_{T_1}, \ldots, x_{T_m})$ define a single situation $x \in X$ in game $G$. Namely, the situation $x$ is such a situation that its projection on $T_k$ is $x_{T_k}$ $(k = 1, \ldots, m)$. Hence we can define a realization function $F$ by the rule: $F(x_{T_1}, \ldots, x_{T_m}) \overset{df}{=} F(x)$. In particular if $T$ is one fix coalition then the function $F\left(x_T, x_{N\setminus T}\right)$ is defined.

**Remark 2.** Consider a game with payoff functions $H = \left\langle (X_i)_{i\in N}, (u_i)_{i\in N} \right\rangle$ where $u_i\colon \prod_{i\in N} X_i \to \mathbb{R}$ is a payoff function for players $i$. Then we can define the preference relation of player $i$ by the formula

$$x^1 \overset{\rho_i}{\lesssim} x^2 \Leftrightarrow u_i(x^1) \leq u_i(x^2).$$

Let the preference relation of coalition $T$ be Pareto dominance, i.e.

$$x^1 \overset{\rho_T}{\lesssim} x^2 \Leftrightarrow (\forall i \in T)\ u_i(x^1) \leq u_i(x^2).$$

Then considered above concordance rules is becoming well known rules for cooperative games with payoff functions. (see Moulin, 1981).

## 3. Coalitions optimality concepts

In this part we consider games with preference relations of the form (1). For games of this class two types of optimality concepts are introduced and connections between these concepts are established.

Let $\mathcal{K}$ be an arbitrary fixed family of coalitions of players $N$.

### 3.1. Equilibrium concepts

**Definition 2.** A situation $x^0 = (x_i^0)_{i\in N} \in X$ is called *Nash $\mathcal{K}$–equilibrium* (Nash $\mathcal{K}$–equilibrium point) if for any coalition $T \in \mathcal{K}$ and any strategy $x_T \in X_T$ the condition

$$F(x^0 \parallel x_T) \overset{\rho_T}{\lesssim} F(x^0) \tag{10}$$

holds.

**Remark 3.** 1. In the case $\mathcal{K} = \{\{1\}, \ldots, \{n\}\}$, Nash $\mathcal{K}$–equilibrium is Nash equilibrium in the usual sense.

2. In the case $\mathcal{K} = \{N\}$, a situation $x^0$ is Nash $\{N\}$–equilibrium means $F\left(x^0\right)$ is greatest element under preference $\rho_T$.

We now define some generalization of Nash equilibrium.

A strategy $x_T^0 \in X_T$ is called a *refutation of the situation* $x \in X$ *by coalition* $T$ if the condition

$$F(x \parallel x_T^0) \overset{\rho_T}{>} F(x) \tag{11}$$

holds.

**Definition 3.** A situation $x^0 = (x_i^0)_{i \in N} \in X$ is called $\mathcal{K}$*–equilibrium point* if any coalition $T \in \mathcal{K}$ does not have a refutation of this situation, i.e. for any coalition $T \in \mathcal{K}$ and any strategy $x_T \in X_T$ the condition

$$F(x^0 \parallel x_T) \overset{\rho_T}{\not>} F(x^0)$$

holds.

**Remark 4.** 1. In the case $\mathcal{K} = \{\{1\}, \ldots, \{n\}\}$, $\mathcal{K}$–equilibrium is equilibrium in the usual sense.
2. In the case $\mathcal{K} = \{N\}$, $\mathcal{K}$–equilibrium point is Pareto optimal.
3. In the case $\mathcal{K} = 2^N$, $\mathcal{K}$–equilibrium point is called *strong equilibrium* one.

### 3.2. Acceptable outcomes and acceptable situations

A strategy $x_T^0 \in X_T$ is called a *objection* of coalition $T$ *against outcome* $a \in A$ if for any strategy of complementary coalition $x_{N \setminus T} \in X_{N \setminus T}$ the condition

$$F(x_T^0, x_{N \setminus T}) \overset{\rho_T}{>} a \tag{12}$$

holds.

**Definition 4.** An outcome $a \in A$ is called *acceptable* for coalition $T$ if this coalition does not have objections against this outcome.

An outcome $a \in A$ is said to be $\mathcal{K}$*–acceptable* if it is acceptable for all coalitions $T \in \mathcal{K}$, that is

$$(\forall T \in \mathcal{K})(\forall x_T \in X_T)(\exists x_{N \setminus T} \in X_{N \setminus T}) F(x_T, x_{N \setminus T}) \overset{\rho_T}{\not>} a. \tag{13}$$

A strategy $x_T^0 \in X_T$ is called a *objection* of coalition $T$ *against situation* $x^* \in X$ if this strategy is an objection against outcome $F\left(x^*\right)$.

We define also a quite acceptable concept by changing quantifiers: $\forall x_T \; \exists x_{N \setminus T} \to \exists x_{N \setminus T} \; \forall x_T$.

**Definition 5.** An outcome $a$ is called *quite* $\mathcal{K}$*-acceptable* for family of coalitions $\mathcal{K}$ if the condition

$$(\forall T \in \mathcal{K})(\exists x_{N \setminus T} \in X_{N \setminus T})(\forall x_T \in X_T) F(x_{N \setminus T}, x_T) \overset{\rho_T}{\not>} a \tag{14}$$

holds.

A situation $x^0 \in X$ is called *quite* $\mathcal{K}$*-acceptable* if outcome $F\left(x^0\right)$ is quite $\mathcal{K}$-acceptable one.

These optimality concepts are analogous to well known optimality concepts of games with payoff functions (see Moulin, 1981).

Now we consider connections between these optimality concepts.

**Lemma 1.** *Nash $\mathcal{K}$–equilibrium point is also a $\mathcal{K}$–equilibrium point but converse is false.*

*Proof (of lemma).* Let $x^0 = (x_i^0)_{i\in N}$ be Nash $\mathcal{K}$–equilibrium point then for any coalition $T \in \mathcal{K}$ and any strategy $x_T \in X_T$ the condition $F(x^0 \parallel x_T) \overset{\rho_T}{\lesssim} F(x^0)$ holds. Suppose $F(x^0 \parallel x_T) \overset{\rho_T}{>} F(x^0)$. The system of conditions

$$\begin{cases} F(x^0 \parallel x_T) \overset{\rho_T}{\lesssim} F(x^0) \\ F(x^0 \parallel x_T) \overset{\rho_T}{>} F(x^0) \end{cases}$$

is false. Hence, $F(x^0 \parallel x_T) \overset{\rho_T}{\not>} F(x^0)$. □

Thus, Nash $\mathcal{K}$–equilibrium is $\mathcal{K}$–equilibrium. But the converse is false. Indeed, consider

*Example 1.* Consider an antagonistic game $G$ whose realization function $F$ is given by Table 1 and preference relation for player 1 by Diagram 1; preference relation of player 2 is given by inverse order, $\mathcal{K} = \{\{1\},\{2\}\}$.

**Table 1.** Realization function

| $F$ | $t_1$ | $t_2$ |
|---|---|---|
| $s_1$ | $a$ | $b$ |
| $s_2$ | $c$ | $d$ |

Situation $(s_1, t_1)$ is $\mathcal{K}$–equilibrium. Since $F(s_1, t_1) = a$ and $a \| b, a \| c$ (i.e. $a$ and $b$ is incomparable, $a$ and $c$ is incomparable) then $(s_1, t_1)$ is not Nash $\mathcal{K}$–equilibrium.

**Remark 5.** If all preference relations $(\rho_T)_{T\in\mathcal{K}}$ is linear then Nash $\mathcal{K}$–equilibrium and $\mathcal{K}$–equilibrium are equivalent.

**Proposition 1.** *An objection of coalition $T$ against situation $x^*$ is also a refutation of this situation.*

*Proof (of proposition).* Let $x_T^0$ be an objection of coalition $T$ against situation $x^*$. Then according to definition of objection the strategy $x_T^0$ is an objection of coalition $T$ against outcome $F(x^*)$, i.e. for any strategy of complementary coalition $x_{N\setminus T} \in X_{N\setminus T}$ the condition $F(x_T^0, x_{N\setminus T}) \overset{\rho_T}{>} F(x^*)$ holds.

Let us take $x_{N\setminus T} = x^*_{N\setminus T}$ as a strategy of complementary coalition, we have $F(x_T^0, x^*_{N\setminus T}) \overset{\rho_T}{>} F(x^*)$.

Since strategy $x_{N\setminus T}$ is an arbitrary one then we get strategy $x_T^0$ is a refutation of this situation. □

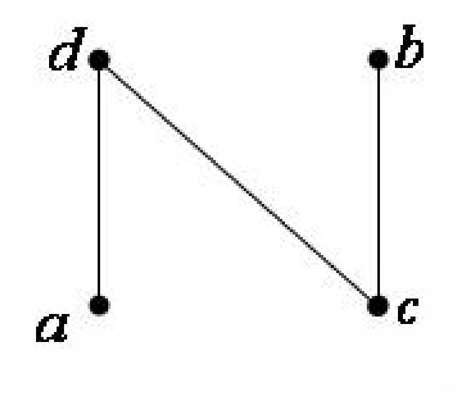


**Fig. 1.** Diagram 1

**Corollary 1.** *Any $\mathcal{K}$–equilibrium point is also $\mathcal{K}$–acceptable.*

We have to prove the more strong assertion.

**Lemma 2.** *Any $\mathcal{K}$–equilibrium point is also quite $\mathcal{K}$–acceptable.*

*Proof (of lemma).* Let $x^0$ be $\mathcal{K}$–equilibrium point. Suppose $x_{N\setminus T} = x^0_{N\setminus T}$ for all coalitions $T \in \mathcal{K}$. Then for any coalition $T \in \mathcal{K}$ we have $F\left(x_{N\setminus T}, x_T\right) = F\left(x^0_{N\setminus T}, x_T\right) = F(x^0 \parallel x_T) \overset{\rho_T}{\not>} F(x^0)$. Hence, $x^0$ is quite $\mathcal{K}$–acceptable. □

**Lemma 3.** *Any quite $\mathcal{K}$–acceptable outcome is $\mathcal{K}$–acceptable.*

The proof of Lemma 3 is obvious.

The main result of the part 3 is the following theorem.

**Theorem 1.** *Consider introduced above coalitions optimality concepts: Nash $\mathcal{K}$–equilibrium, $\mathcal{K}$–equilibrium, quite $\mathcal{K}$–acceptance, $\mathcal{K}$–acceptance. Then each consequent condition is more weak than preceding, i.e.*

*Nash $\mathcal{K}$–equilibrium $\Rightarrow$ $\mathcal{K}$–equilibrium $\Rightarrow$ quite $\mathcal{K}$–acceptance $\Rightarrow$ $\mathcal{K}$–acceptance.*

The proof of Theorem 1 follows from Lemmas 1, 2, 3.

## 4. Coalition homomorphisms for games with preference relations

Let

$$G = \langle (X_i)_{i \in N}, A, F, (\rho_i)_{i \in N} \rangle$$

and

$$\Gamma = \langle (Y_i)_{i \in N}, B, \Phi, (\sigma_i)_{i \in N} \rangle$$

be two games with preference relations of the players $N$.

Any $(n+1)$–system consisting of mappings $f = (\varphi_1, \ldots, \varphi_n, \psi)$ where for any $i = 1, \ldots, n$, $\varphi_i \colon X_i \to Y_i$ and $\psi \colon A \to B$, is called *a homomorphism* from game $G$ into game $\Gamma$ if for any $i = 1, \ldots, n$ and any $a_1, a_2 \in A$ the following two conditions

$$a_1 \overset{\rho_i}{\lesssim} a_2 \;\Rightarrow\; \psi(a_1) \overset{\sigma_i}{\lesssim} \psi(a_2), \tag{15}$$

$$\psi(F(x_1, \ldots, x_n)) = \Phi(\varphi_1(x_1), \varphi_2(x_2), \ldots, \varphi_n(x_n)) \tag{16}$$

are satisfied.

A homomorphism $f$ is said to be *strict homomorphism* if system of the conditions

$$a_1 \overset{\rho_i}{<} a_2 \;\Rightarrow\; \psi(a_1) \overset{\sigma_i}{<} \psi(a_2), \quad (i = 1, \ldots, n) \tag{17}$$

$$a_1 \overset{\rho_i}{\sim} a_2 \;\Rightarrow\; \psi(a_1) \overset{\sigma_i}{\sim} \psi(a_2) \quad (i = 1, \ldots, n) \tag{18}$$

holds instead of condition (15).

A homomorphism $f$ is said to be *regular homomorphism* if the conditions

$$\psi(a_1) \overset{\sigma_i}{<} \psi(a_2) \;\Rightarrow\; a_1 \overset{\rho_i}{<} a_2, \tag{19}$$

$$\psi(a_1) \overset{\sigma_i}{\sim} \psi(a_2) \;\Rightarrow\; \psi(a_1) = \psi(a_2) \tag{20}$$

hold.

A homomorphism $f$ is said to be *homomorphism "onto"*, if each $\varphi_i \, (i = 1, \ldots, n)$ is a mapping "onto".

Now we introduce a concept of coalition homomorphism.

For the first step, we need to fix some rule for concordance of preferences; recall that the preference relation for coalition $T$ denoted by $\rho_T$.

**Definition 6.** A homomorphism $f$ is said to be:

- *a coalition homomorphism* if it preserves preference relations for all coalitions, i.e. for any coalition $T \subseteq N$ the condition

$$a_1 \overset{\rho_T}{\lesssim} a_2 \;\Rightarrow\; \psi(a_1) \overset{\sigma_T}{\lesssim} \psi(a_2) \tag{21}$$

  holds;
- *a strict coalition homomorphism* if for any coalition $T \subseteq N$ the system of the conditions

$$\begin{cases} a_1 \overset{\rho_T}{<} a_2 \Rightarrow \psi(a_1) \overset{\sigma_T}{<} \psi(a_2), \\ a_1 \overset{\rho_T}{\sim} a_2 \Rightarrow \psi(a_1) \overset{\sigma_T}{\sim} \psi(a_2) \end{cases} \tag{22}$$

  is satisfied;

– *a regular coalition homomorphism* if for any coalition $T \subseteq N$ the system of the conditions

$$\begin{cases} \psi(a_1) \overset{\sigma_T}{<} \psi(a_2) \Rightarrow a_1 \overset{\rho_T}{<} a_2, \\ \psi(a_1) \overset{\sigma_T}{\sim} \psi(a_2) \Rightarrow \psi(a_1) = \psi(a_2) \end{cases} \tag{23}$$

is satisfied.

It is easy to see that the following assertion is true.

**Lemma 4.** *For Pareto concordance (and also for modified Pareto concordance), any surjective homomorphism from $G$ into $\Gamma$ is a surjective coalition homomorphism.*

**Lemma 5.** *For Pareto concordance (and also for modified Pareto concordance), any strict homomorphism from $G$ into $\Gamma$ is a strict coalition homomorphism.*

*Proof (of lemma 5).* We consider Pareto concordance for preferences as a concordance rule. Verify the conditions of system (22) for preference relation $\rho_T$. According to defition of Pareto concordance the condition $a_1 \overset{\rho_T}{<} a_2$ is equivalent system

$$\begin{cases} (\forall i \in T)\ a_1 \overset{\rho_i}{\lesssim} a_2, \\ (\exists j \in T)\ a_1 \overset{\rho_j}{<} a_2. \end{cases}$$

Since strict homomorphism is homomorphism then from the first condition of system it follows that $(\forall i \in T)\, \psi(a_1) \overset{\sigma_i}{\lesssim} \psi(a_2)$. Since homomorphism $f$ is strict then $(\exists j \in T)\, \psi(a_1) \overset{\sigma_i}{<} \psi(a_2)$.

From last two conditions we get $\psi(a_1) \overset{\sigma_T}{<} \psi(a_2)$.

Now according to definition of symmetric part of relation $\rho_T$ we have $a_1 \overset{\rho_T}{\sim} a_2 \Leftrightarrow (\forall i \in T)\, a_1 \overset{\rho_i}{\sim} a_2$. Since homomorphism $f$ is strict then we get $(\forall i \in T)\, \psi(a_1) \overset{\sigma_i}{\sim} \psi(a_2)$, i.e. $\psi(a_1) \overset{\sigma_T}{\sim} \psi(a_2)$. □

Now we consider modified Pareto concordance for preferences of players as a concordance rule.

**Lemma 6.** *For modified Pareto concordance, any regular homomorphism from $G$ into $\Gamma$ is a regular coalition homomorphism.*

*Proof (of lemma 6).* Verify the condition (23) for strict part of preference relation $\sigma_T$. According to definition of modified Pareto concordance for preferences the condition $\psi(a_1) \overset{\sigma_T}{<} \psi(a_2)$ is equivalent $(\forall i \in T)\, \psi(a_1) \overset{\sigma_i}{<} \psi(a_2)$. Since homomorphism $f$ is regular then we have $(\forall i \in T)\, a_1 \overset{\rho_i}{<} a_2$, i.e. $a_1 \overset{\rho_T}{<} a_2$.

Verify the condition (23) for symmetric part of $\sigma_T$. According to definition of modified Pareto concordance we have

$$\psi(a_1) \overset{\sigma_T}{\sim} \psi(a_2) \Leftrightarrow (\forall i \in T)\, \psi(a_1) \overset{\sigma_i}{\sim} \psi(a_2).$$

Since homomorphism $f$ is regular then from the last condition it follows that $(\forall i \in T)\, \psi(a_1) = \psi(a_2)$, i.e. $\psi(a_1) = \psi(a_2)$. □

## 5. The main results

The main result states a correspondence between sets of $\mathcal{K}$-acceptable outcomes and $\mathcal{K}$-equilibrium situations of games which are in homomorphic relations under indicated types.

A homomorphism $f$ is said to be *covariant* if $f$–image of any optimal solution in game $G$ is an optimal solution in $\varGamma$.

A homomorphism $f$ is said to be *contrvariant* if $f$–preimage of any optimal solution in game $\varGamma$ is an optimal solution in $G$.

**Theorem 2.** *For Nash $\mathcal{K}$–equilibrium, any surjective homomorphism is covariant under Pareto concordance and under modified Pareto concordance also.*

*Proof (of theorem 2).* We consider Pareto concordance for preferences as a concordance rule. Let $x^0$ be Nash $\mathcal{K}$–equilibrium point in game $G$. We have to prove that $\varphi\left(x^0\right)$ is Nash $\mathcal{K}$–equilibrium point in game $\varGamma$.

We fix arbitrary strategy $y_T \in Y_T$. Since $f$ is homomorphism "onto" then according to Lemma 4 we obtain $(\exists x_T^* \in X_T)\, \varphi_T\left(x_T^*\right) = y_T$. For any strategy $x_T$ the condition $F(x_T, x^0_{N\setminus T}) \overset{\rho_T}{\lesssim} F(x^0)$ holds. Hence, for strategy $x_T^*$ the condition $F(x_T^*, x^0_{N\setminus T}) \overset{\rho_T}{\lesssim} F(x^0)$ is satisfied. Since $f$ is homomorphism then $\psi\left(F(x_T^*, x^0_{N\setminus T})\right) \overset{\sigma_T}{\lesssim} \psi\left(F(x^0)\right)$. By condition (16): $\varPhi\left(\varphi_T\left(x_T^*\right), \varphi_{N\setminus T}\left(x^0_{N\setminus T}\right)\right) \overset{\sigma_T}{\lesssim}$ $\varPhi\left(\varphi\left(x^0\right)\right)$, i.e. $\varPhi\left(y_T, \varphi_{N\setminus T}\left(x^0_{N\setminus T}\right)\right) \overset{\sigma_T}{\lesssim} \varPhi\left(\varphi\left(x^0\right)\right)$.

Since strategy $y_T \in Y_T$ is arbitrary one then $\varphi\left(x^0\right)$ is Nash $\mathcal{K}$–equilibrium. □

**Theorem 3.** *For $\mathcal{K}$–equilibrium, any strict surjective homomorphism is contrvariant under Pareto concordance and under modified Pareto concordance also.*

*Proof (of theorem 3).* Consider Pareto concordance for preferences as a concordance rule. Let $y^0$ be $\mathcal{K}$–equilibrium point. We have to prove that situation $x^0$ with $\varphi\left(x^0\right) = y^0$ is $\mathcal{K}$–equilibrium point.

Suppose $x^0 = \left(x_i^0\right)_{i\in N}$ is not $\mathcal{K}$–equilibrium then there exists coalition $T \in \mathcal{K}$ and strategy $x_T^* \in X_T$ such that $F\left(x_T^*, x^0_{N\setminus T}\right) \overset{\rho_T}{>} F\left(x^0\right)$. Since homomorphism $f$ is strict then according to Lemma 5 we get $\psi\left(F\left(x_T^*, x^0_{N\setminus T}\right)\right) \overset{\sigma_T}{>} \psi\left(F\left(x^0\right)\right)$. According to condition (16) we obtain $\varPhi\left(\varphi_T\left(x_T^*\right), \varphi_{N\setminus T}\left(x^0_{N\setminus T}\right)\right) \overset{\sigma_T}{>} \varPhi\left(\varphi\left(x^0\right)\right)$. The last condition means $\varPhi\left(\varphi_T\left(x_T^*\right), y^0_{N\setminus T}\right) \overset{\sigma_T}{>} \varPhi\left(y^0\right)$. Thus, strategy $\varphi_T\left(x_T^*\right)$ is refutation of situation $y^0$ by coalition $T$, which is contradictory with $y^0$ is $\mathcal{K}$–equilibrium point.

Hence, $x^0$ is $\mathcal{K}$–equilibrium point. □

**Theorem 4.** *For $\mathcal{K}$–acceptance, any strict surjective homomorphism is contrvariant under Pareto concordance and under modified Pareto concordance also.*

*Proof (of theorem 4).* Consider Pareto concordance for preferences as a concordance rule. Let outcome $b$ with $\psi(a) = b$ be $\mathcal{K}$–acceptable one in game $\varGamma$. Assume that outcome $a$ is not acceptable for all coalitions $T \in \mathcal{K}$, i.e. there exists such strategy $x_T^0 \in X_T$ that for any strategy $x_{N\setminus T} \in X_{N\setminus T}$ the condition

$$F(x_T^0, x_{N\setminus T}) \overset{\rho_T}{>} a \tag{24}$$

holds.

Let $y_{N\setminus T} = (y_j)_{j\in N\setminus T}$ be arbitrary strategy of complementary coalition $N \setminus T$ in game $\Gamma$. Since $f$ is homomorphism "onto" then according to Lemma 4 we have $\left(\exists x^*_{N\setminus T} \in X_{N\setminus T}\right) \varphi_{N\setminus T}\left(x^*_{N\setminus T}\right) = y_{N\setminus T}$. By (24) the condition $F(x_T^0, x^*_{N\setminus T}) \overset{\rho_T}{>} a$ holds. According to Lemma 5 we get $\psi\left(F(x_T^0, x^*_{N\setminus T})\right) \overset{\sigma_T}{>} \psi(a)$. By (16) we have $\psi\left(F(x_T^0, x^*_{N\setminus T})\right) = \Phi\left(\varphi_T\left(x_T^0\right), \varphi_{N\setminus T}\left(x^*_{N\setminus T}\right)\right)$. Thus, the condition $\Phi\left(\varphi_T\left(x_T^0\right), y_{N\setminus T}\right) \overset{\sigma_T}{>} \psi(a)$ is satisfied. Hence, strategy $\varphi_T\left(x_T^0\right)$ is objection of coalition $T$ against outcome $b$ which is contadictory with $b$ is $\mathcal{K}$–acceptable outcome.

Hence, outcome $a$ is $\mathcal{K}$–acceptable. □

**Theorem 5.** *For $\mathcal{K}$–equilibrium, any regular surjective homomorphism is covariant under modified Pareto concordance.*

*Proof (of theorem 5).* Let $x^0$ be $\mathcal{K}$–equilibrium. We have to prove that situation $\varphi\left(x^0\right)$ is $\mathcal{K}$–equilibrium.

Suppose $\varphi\left(x^0\right)$ is not $\mathcal{K}$–equilibrium, i.e.

$$(\exists T \in \mathcal{K})\ (\exists y_T \in Y_T)\ \Phi\left(\varphi\left(x^0\right) \| y_T\right) \overset{\sigma_T}{>} \Phi\left(\varphi\left(x^0\right)\right) \tag{25}$$

Since homomorphism $f$ is surjective then according to Lemma 4 we have $(\exists x_T^* \in X_T)\, \varphi_T\left(x_T^*\right) = y_T$. Hence, the condition $\Phi\left(\varphi_T\left(x_T^*\right), \varphi_{N\setminus T}\left(x^0_{N\setminus T}\right)\right) \overset{\sigma_T}{>} \Phi\left(\varphi\left(x^0\right)\right)$ holds. By (16) we get $\Phi\left(\varphi_T\left(x_T^*\right), \varphi_{N\setminus T}\left(x^0_{N\setminus T}\right)\right) = \psi\left(F\left(x_T^*, x^0_{N\setminus T}\right)\right)$. Thus, $\psi\left(F\left(x^0 \| x_T^*\right)\right) \overset{\sigma_T}{>} \psi\left(F\left(x^0\right)\right)$. Because homomorphism $f$ is regular then according to Lemma 6 we obtain $F\left(x^0 \| x_T^*\right) \overset{\rho_T}{>} F\left(x^0\right)$. Thus, strategy $x_T^*$ is refutation of situation $x^0$ by coalition $T$, which is contradictory with $x^0$ is $\mathcal{K}$–equilibrium.

Hence, $\varphi\left(x^0\right)$ is $\mathcal{K}$–equilibrium in game $\Gamma$. □

## Appendix

Consider the example conserning of concordance rules.

Let $G$ be a game of three players with set of outcomes $A = \{a, b, c, d, e\}$. Preference relations for each player are given by Diagrams 2,3,4.

Using Diagrams 2 – 4 we can define preference relations in the following form:

$$\begin{aligned}
\rho_1 &: a < b, b \sim c, c \sim d, b < e \\
\rho_2 &: a \sim b, b \sim c, c < d, e < d \\
\rho_3 &: a < c, b \sim c, c < d, b \sim e, d \sim e.
\end{aligned}$$

Then according to Pareto concordance (see 2.1) for coalition $T = \{1, 2\}$ we have $\rho_T\colon a \lesssim b, b \lesssim c, c \lesssim d$ where strict part consists of two conditions $a \overset{\rho_T}{<} b, c \overset{\rho_T}{<} d$ and symmetric part is $b \overset{\rho_T}{\sim} c$.

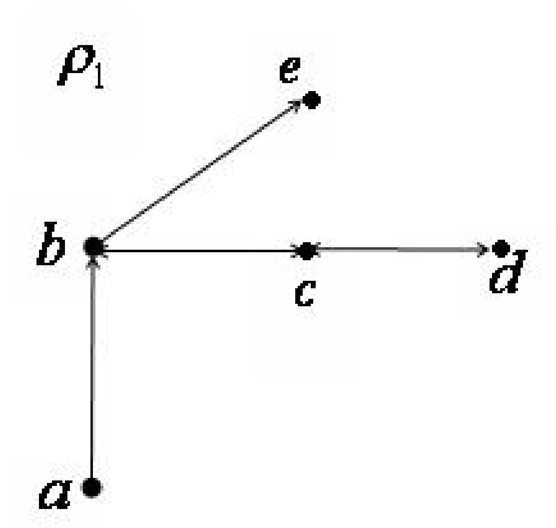


**Fig. 2.** Diagram 2

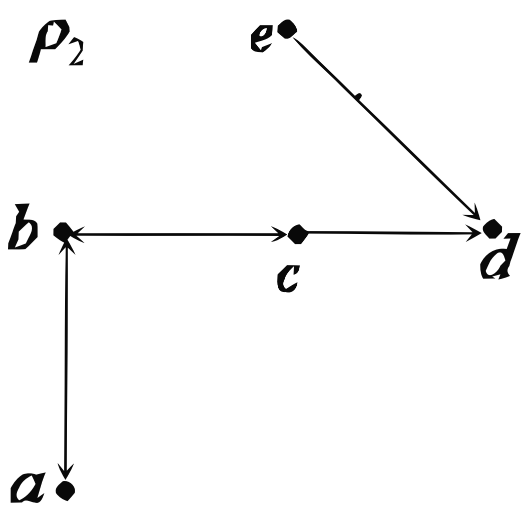


**Fig. 3.** Diagram 3

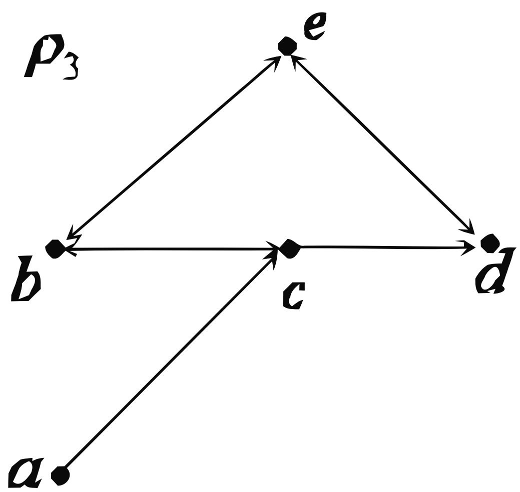


**Fig. 4.** Diagram 4

For $T = \{1, 3\}$ a preference relation $\rho_T$ is defined by $b \lesssim c, c \lesssim d, b \lesssim e$ where strict part is $c \overset{\rho_T}{<} d, b \overset{\rho_T}{<} e$ and symmetric part is $b \overset{\rho_T}{\sim} c$.

For $T = \{2, 3\}$ relation $\rho_T$ is $b \lesssim c, c \lesssim d, e \lesssim d$ where $c \overset{\rho_T}{<} d, e \overset{\rho_T}{<} d, b \overset{\rho_T}{\sim} c$.

For $T = \{1, 2, 3\}$ relation $\rho_T$ is $b \lesssim c, c \lesssim d$ where $c \overset{\rho_T}{<} d, b \overset{\rho_T}{\sim} c$.

According to modified Pareto concordance (see 2.2) for coalition $T = \{1, 2\}$ strict part $\rho_T$ is empty set and symmetric part consists of one condition $b \overset{\rho_T}{\sim} c$.

For $T = \{2, 3\}$ strict part of preference relation $\rho_T$ is defined by $c \overset{\rho_T}{<} d$ and symmetric part is $b \overset{\rho_T}{\sim} c$.

Preference relation $\rho_T$ for coalition $T = \{1, 2, 3\}$ in the game with majority rule (see 2.3): $a \overset{\rho_T}{\lesssim} b, b \overset{\rho_T}{\sim} c, c \overset{\rho_T}{<} d, b \overset{\rho_T}{\lesssim} e, e \overset{\rho_T}{\lesssim} d$.

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