AN ANALOGUE OF HILBERT'S THEOREM 90 FOR INFINITE SYMMETRIC GROUPS

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Abstract. Let K be a field and G be a group of its automorphisms. If K is algebraic over the subfield K^G fixed by G then, according to Speiser's generalization of Hilbert's Theorem 90, any smooth (i.e. with open stabilizers) *K-semilinear* representation of the group G is isomorphic to a direct sum of copies of K.

If K is not algebraic over K^G then there exist non-semisimple smooth semilinear representations of G over K, so Hilbert's Theorem 90 does not hold.

Let now G be the group of all permutations of an infinite set Ψ acting naturally on the field $k(\Psi)$ freely generated over a subfield k by the set Ψ. The goal of this note is to present three examples of G-invariant subfields $K \subseteq k(\Psi)$ such that the smooth K-semilinear representations of G of *finite length* admit an explicit description, close to Hilbert's Theorem 90.

Namely, (i) if $K = k(\Psi)$ then any smooth K-semilinear representation of G of finite length is isomorphic to a direct sum of copies of K, (ii) if $K \subset k(\Psi)$ is the subfield of rational homogeneous functions of degree 0 then any smooth K-semilinear representation of G of finite length splits into a direct sum of one-dimensional K-semilinear representations of G, (iii) if $K \subset k(\Psi)$ is the subfield generated over k by $x-y$ for all $x, y \in \Psi$ then there is a unique isomorphism class of indecomposable smooth K-semilinear representations of G of each given finite length.

1. Introduction

Let G be a group of automorphisms of a field K. Then the group G is endowed with the standard topology, whose base is given by the left or right translates of the pointwise stabilizers of finite subsets in K. We are interested in continuous G-actions on discrete sets (i.e., with open stabilizers), called *smooth* in what follows. These G-sets will be K-vector spaces endowed with *semilinear* G-actions.

For an abelian group A and a set S we denote by $A[S]$ the direct sum of copies of A indexed by S. In some cases, $A[S]$ will be endowed with an additional structure, e.g., of a module, a ring, etc.

Denote by $K\langle G\rangle$ the unital associative subring in $\mathrm{End}_{\mathbb{Z}}(K[G])$ generated by the natural left action of K and the diagonal left action of G on $K[G]$. In other words, $K\langle G\rangle$ is the ring of K-valued measures on G with finite support. Then $K\langle G\rangle$ is a central k-algebra, where $k := K^G$ is the fixed field.

More explicitly, the elements of $K\langle G\rangle$ are the finite formal sums $\sum_{i=1}^{N} a_i[g_i]$ for all integer $N \geq 0$, $a_i \in K$, $g_i \in G$. Addition is defined obviously; multiplication is a unique distributive one such that $(a[g])(b[h]) = ab^g[gh]$, where we write a^h for the result of applying of $h \in G$ to $a \in K$.

An additive action of G on an K-vector space V is called *semilinear* if $g(a \cdot v) = a^g \cdot gv$ for any $g \in G$, $v \in V$ and $a \in K$. Then a K-vector space endowed with an additive semilinear G-action is the same as an $K\langle G\rangle$-module.

A *K-semilinear representation* of G is a left $K\langle G\rangle$-module.

Let, as before, K be a field and G be a group of its automorphisms. Then Speiser's generalization of Hilbert's theorem 90, cf. [3, Satz 1], can be interpreted and slightly generalized further as follows.

Proposition 1.1. *The following conditions on the pair (K, G) are equivalent:*

1. *G is precompact (i.e., any open subgroup of G is of finite index),*
2. *K is algebraic over the subfield K^G fixed by G,*
3. *any smooth $\mathbb{Q}$-linear representation of G is semisimple,*
4. *any smooth K-semilinear representation V of G is semisimple,*

(5) *any smooth K-semilinear representation V of G is isomorphic to a direct sum of copies of K, in other words, the natural map $V^G \otimes_{K^G} K \to V$ is an isomorphism.*

Proof. Set $k := K^G$. In the case of finite G the implication (1)⇒(5) is [3, Satz 1], appropriately reformulated. Namely, the natural G-action on K gives rise to a k-algebra homomorphism $K\langle G\rangle \to \mathrm{End}_k(K)$, which is (a) surjective by Jacobson's density theorem and (b) injective by independence of characters. Then (a) any $K\langle G\rangle$-module is isomorphic to a direct sum of copies of K, (b) the field extension $K|k$ is finite, which shows (1)⇒(2).

For arbitrary precompact G, a smooth K-semilinear representation V of G and $v \in V$ the intersection H of all conjugates of the stabilizer of v in G is of finite index. Thus, v is contained in the K^H-semilinear representation V^H of the group G/H. As G/H is finite, $V^H = (V^H)^{G/H} \otimes_{(K^H)^{G/H}} K^H = V^G \otimes_{K^G} K^H$, i.e., v is contained in a subrepresentation isomorphic to a direct sum of copies of K. In particular, any element of K is contained in a finite field extension of k.

If G is not precompact then it admits an open subgroup $U \subset G$ of infinite index, while the representations $\mathbb{Q}[G/U]$ and $K[G/U]$ of G have no non-zero vectors fixed by G, unlike their simple quotients $\mathbb{Q}$ and K, respectively. (For a G-set S we consider $K[S]$ as a K-vector space with the diagonal G-action.) This shows implications (3)⇒(1) and (4)⇒(1). The implication (5)⇒(4) is trivial, while (1)⇒(3) is well-known; (2)⇒(1) is evident: K is a union of finite G-invariant extensions of K^G, so G is dense in a profinite group. □

The purpose of this note is to present three examples (Theorems 1.2, 1.3, 1.6) of a field K and a non-precompact group G of its automorphisms such that the smooth *irreducible* K-semilinear representations of G admit an explicit description.

Theorem 1.2. *Let $K = k(\Psi)$ be the field of rational functions over a field k in the variables enumerated by a set Ψ. Let $G = \mathfrak{S}_\Psi$ be the group of all permutations of the set Ψ acting naturally on K. Then any smooth K-semilinear representation of $\mathfrak{S}_\Psi$ of finite length is isomorphic to a direct sum of copies of K.*

Theorem 1.3. *Let $K \subset k(\Psi)$ be the subfield of homogeneous rational functions of degree 0. Let $G = \mathfrak{S}_\Psi$ be the group of all permutations of an infinite set Ψ acting naturally on the fields $k(\Psi)$ and K. Then any smooth K-semilinear representation of $\mathfrak{S}_\Psi$ of finite length is isomorphic to $\bigoplus_{d\in\mathbb{Z}} V_d^{m(d)}$ for a unique function $m : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ with finite support, where $V_d \subseteq k(\Psi)$ is the one-dimensional subspace of homogeneous rational functions of degree d.*

Remark 1.4. Let K be a field and G be a group of automorphisms of K. Let $k \subseteq K^G$ be a subfield. Then any smooth irreducible representation W of G over k can be embedded into a smooth irreducible K-semilinear representation of G. Indeed, W can be embedded into any irreducible quotient of the K-semilinear representation $W \otimes_k K$.

Corollary 1.5. *In notation of Theorem 1.3, any smooth irreducible representation W of $\mathfrak{S}_\Psi$ over a field k can be embedded into the K-semilinear representation $V_d \subset k(\Psi)$ for some integer d.*

This follows from Remark 1.4 and Theorem 1.3. □

Theorem 1.6. *Let k be a field and Ψ be an infinite set. Let $K \subset k(\Psi)$ be the subfield generated over k by the rational functions $x - y$ for all $x, y \in \Psi$. Let $G = \mathfrak{S}_\Psi$ be the group of all permutations of the set Ψ acting naturally on the fields $k(\Psi)$ and K. Then for any integer $N \geq 1$ there exists a unique isomorphism class of smooth K-semilinear indecomposable representations of $\mathfrak{S}_\Psi$ of length N.*

Remark 1.7. We may relax the condition (5) of Proposition 1.1 as follows:

any **irreducible** *smooth K-semilinear representation of G is isomorphic to K.*

If this relaxed condition holds for a pair (K, G) then, according to Remark 1.4, any irreducible smooth k-linear representation W of G can be embedded into K.

However, the converse is not true: in the setup of Theorem 1.3, the group $G = \mathfrak{S}_\Psi$ admits non-trivial irreducible smooth K-semilinear representations V_d for $d \neq 0$, but any irreducible smooth

k-linear representation W of G can be embedded into K if k is of characteristic 0. Namely, it is quite well-known (cf., e.g. [2, Theorem 5.7]), that W can be embedded into $k[\{\text{embeddings of } I \text{ into } \Psi\}]$ for an appropriate finite $I \subset \Psi$. On the other hand, any sufficiently general homogeneous rational function $Q \in k(I)$ of degree 0 gives rise to an embedding $k[\{\text{embeddings of } I \text{ into } \Psi\}] \hookrightarrow K$, $[g] \mapsto gQ$.

2. Open subgroups and permutation modules

For a subset $T \subseteq \Psi$, we denote by $\mathfrak{S}_{\Psi|T}$ the pointwise stabilizer $\mathfrak{S}_{\Psi|T}$ of T in $\mathfrak{S}_\Psi$. Let $\mathfrak{S}_{\Psi,T} := \mathfrak{S}_{\Psi \smallsetminus T} \times \mathfrak{S}_T$ be the group of all permutations of Ψ preserving T (in other words, the setwise stabilizer of T in the group $\mathfrak{S}_\Psi$, or equivalently, the normalizer of $\mathfrak{S}_{\Psi|T}$ in $\mathfrak{S}_\Psi$).

Lemma 2.1. *For any pair of finite subsets $T_1, T_2 \subset \Psi$ the subgroups $\mathfrak{S}_{\Psi|T_1}$ and $\mathfrak{S}_{\Psi|T_2}$ generate the subgroup $\mathfrak{S}_{\Psi|T_1 \cap T_2}$.*

Proof. Let us show first that $\mathfrak{S}_{\Psi|T_1} \mathfrak{S}_{\Psi|T_2} = \{g \in \mathfrak{S}_{\Psi|T_1 \cap T_2} \mid g(T_2) \cap T_1 = T_1 \cap T_2\} =: \Xi$. The inclusion $\subseteq$ is trivial. On the other hand,

$$\Xi / \mathfrak{S}_{\Psi|T_2} = \{\text{embeddings } T_2 \smallsetminus (T_1 \cap T_2) \hookrightarrow \Psi \smallsetminus T_1\},$$

while the latter is an $\mathfrak{S}_{\Psi|T_1}$-orbit. □

Lemma 2.2. *For any open subgroup U of $\mathfrak{S}_\Psi$ there exists a unique subset $T \subset \Psi$ such that $\mathfrak{S}_{\Psi|T} \subseteq U$ and the following equivalent conditions hold:* (a) *T is minimal;* (b) *$\mathfrak{S}_{\Psi|T}$ is normal in U;* (c) *$\mathfrak{S}_{\Psi|T}$ is of finite index in U. In particular,* (i) *such T is finite,* (ii) *the open subgroups of $\mathfrak{S}_\Psi$ correspond bijectively to the pairs (T, H) consisting of a finite subset $T \subset \Psi$ and a subgroup $H \subseteq \mathrm{Aut}(T)$ under $(T, H) \mapsto \{g \in \mathfrak{S}_{\Psi,T} \mid \text{restriction of } g \text{ to } T \text{ belongs to } H\}$.*

Proof. Any open subgroup U in $\mathfrak{S}_\Psi$ contains the subgroup $\mathfrak{S}_{\Psi|T}$ for a finite subset $T \subset \Psi$. Assume that T is chosen to be minimal. If $\sigma \in U$ then $U \supseteq \sigma \mathfrak{S}_{\Psi|T} \sigma^{-1} = \mathfrak{S}_{\Psi|\sigma(T)}$, and therefore, (i) $\sigma(T)$ is also minimal, (ii) U contains the subgroup generated by $\mathfrak{S}_{\Psi|\sigma(T)}$ and $\mathfrak{S}_{\Psi|T}$. By Lemma 2.1, the subgroup generated by $\mathfrak{S}_{\Psi|\sigma(T)}$ and $\mathfrak{S}_{\Psi|T}$ is $\mathfrak{S}_{\Psi|T \cap \sigma(T)}$, and thus, U contains the subgroup $\mathfrak{S}_{\Psi|T \cap \sigma(T)}$. The minimality of T means that $T = \sigma(T)$, i.e., $U \subseteq \mathfrak{S}_{\Psi,T}$. If $T' \subset \Psi$ is another minimal subset such that $\mathfrak{S}_{\Psi|T'} \subseteq U$ then, by Lemma 2.1, $\mathfrak{S}_{\Psi|T \cap T'} \subseteq U$, so $T = T'$, which proves (b) and (the uniqueness in the case) (a). It follows from (b) that $\mathfrak{S}_{\Psi|T} \subseteq U \subseteq \mathfrak{S}_{\Psi,T}$, so $\mathfrak{S}_{\Psi|T}$ is of finite index in U. As the subgroups $\mathfrak{S}_{\Psi|T}$ and $\mathfrak{S}_{\Psi|T'}$ are not commensurable for $T' \neq T$, we get the uniqueness in the case (c). □

Lemma 2.3. *Let K be a field endowed with an $\mathfrak{S}_\Psi$-action. Let $U \subset \mathfrak{S}_\Psi$ be a proper open subgroup. Then* (i) *index of U in $\mathfrak{S}_\Psi$ is infinite;* (ii) *there are no elements in $\mathfrak{S}_\Psi \smallsetminus U$ acting identically on K^U;* (iii) *there are no irreducible K-semilinear subrepresentations in $K[\mathfrak{S}_\Psi / U]$.*

Example and notation. For an integer $s \geq 0$, we denote by $\binom{\Psi}{s}$ the set of all subsets of Ψ of cardinality s. Let $U \subset \mathfrak{S}_\Psi$ be a maximal proper subgroup, i.e., $U = \mathfrak{S}_{\Psi,I}$ for a finite subset $I \subset \Psi$ (so $\mathfrak{S}_\Psi / U$ can be identified with the set $\binom{\Psi}{\#I}$). Then we are under assumptions of Lemma 2.3, so there are no irreducible K-semilinear subrepresentations in $K[\binom{\Psi}{\#I}]$.

Proof. (i) and (ii) follow from the explicit description of open subgroups in Lemma 2.2.

(iii) Artin's independence of characters theorem (applied to the one-dimensional characters $g : (K^U)^\times \to K^\times$) implies that the morphism $K[\mathfrak{S}_\Psi / U] \to \prod_{(K^U)^\times} K$, given by $\sum_g b_g [g] \mapsto (\sum_g b_g f^g)_{f \in (K^U)^\times}$, is injective. Then, for any non-zero element $\alpha \in K[\mathfrak{S}_\Psi / U]$, there exists an element $Q \in K^U$ such that the morphism $K[\mathfrak{S}_\Psi / U] \to K$, given by $\sum_g b_g [g] \mapsto \sum_g b_g Q^g$, does not vanish on α. Then α generates a subrepresentation V surjecting onto K. If V is irreducible then it is isomorphic to K, so $V^{\mathfrak{S}_\Psi} \neq 0$. In particular, $K[\mathfrak{S}_\Psi / U]^{\mathfrak{S}_\Psi} \neq 0$, which can happen only if index of U in $\mathfrak{S}_\Psi$ is finite. □

Lemma 2.4. *Let K be a field endowed with an $\mathfrak{S}_\Psi$-action. Let $s \geq 0$ be an integer and M be a quotient of the $K\langle\mathfrak{S}_\Psi\rangle$-module $K[\binom{\Psi}{s}]$ by a non-zero submodule M_0. Then there is a finite subset $I \subset \Psi$ such that the $K\langle\mathfrak{S}_{\Psi|I}\rangle$-module M is isomorphic to a quotient of $\bigoplus_{j=0}^{s-1} K[\binom{\Psi\smallsetminus I}{j}]^{\oplus\binom{|I|}{s-j}}$.*

Proof. Let $\alpha = \sum_{S\subseteq J} a_S[S] \in M_0$ be a non-zero element for a finite set $J \subset \Psi$. Fix some $S \subseteq J$ with $a_S \neq 0$. Set $I := J \smallsetminus S$. Then the morphism of $K\langle\mathfrak{S}_{\Psi|I}\rangle$-modules $K\langle\mathfrak{S}_{\Psi|I}\rangle\alpha \oplus \bigoplus_{\varnothing\neq\Lambda\subseteq I} K[\binom{\Psi\smallsetminus I}{s-|\Lambda|}] \to K[\binom{\Psi}{s}]$, given (i) by the inclusion on the first summand and (ii) by $[T] \mapsto [T \cup \Lambda]$ on the summand corresponding to Λ, is surjective. $\square$

Lemma 2.5. *Let K be a field endowed with a smooth faithful $\mathfrak{S}_\Psi$-action. Let S be an infinite set of positive integers. Then the objects $K[\binom{\Psi}{N}]$ for all $N \in S$ form a system of generators of the category of smooth K-semilinear representations of G.*

Proof. Let V be a smooth semilinear representation of $\mathfrak{S}_\Psi$. Then the stabilizer of any vector v is open, i.e., the stabilizer contains the subgroup $\mathfrak{S}_{\Psi|T'}$ for a finite subset $T' \subset \Psi$. Choose a finite subset $T \subset \Psi$ containing T' with $|T| \in S$. The $K^{\mathfrak{S}_{\Psi|T}}$-linear envelope of the (finite) $\mathfrak{S}_T$-orbit of v is a smooth $K^{\mathfrak{S}_{\Psi|T}}$-semilinear representation of $\mathfrak{S}_T$, so it is trivial, i.e., v belongs to the $K^{\mathfrak{S}_{\Psi|T}}$-linear envelope of the $K^{\mathfrak{S}_{\Psi,T}}$-vector subspace fixed by $\mathfrak{S}_{\Psi,T}$. As a consequence, there is a morphism from a finite cartesian power of $K[\mathfrak{S}_\Psi / \mathfrak{S}_{\Psi,T}] \cong K[\binom{\Psi}{|T|}]$ to V, containing v in the image. $\square$

3. Proofs of Theorems 1.2, 1.3 and 1.6

The following result appears also as [2, Lemma 7.1].

Lemma 3.1. *Let $K = k(\Psi)$ for a field k. Then any finite-dimensional smooth K-semilinear representation V of $\mathfrak{S}_\Psi$ is isomorphic to a direct sum of copies of K.*

Proof. Let $b \subset V$ be a K-basis, pointwise fixed by an open subgroup of $\mathfrak{S}_\Psi$, so $b \subset V_I := V^{\mathfrak{S}_{\Psi|I}}$ for a finite subset $I \subset \Psi$. It is easy to see, cf. e.g. [1, Lemma 2.3] with $\rho \equiv 1$, that the multiplication maps $V_I \otimes_{K_I} K = (V_I \otimes_{K_I} K_J) \otimes_{K_J} K \to V_J \otimes_{K_J} K \to V$ are injective for any subset $J \subseteq \Psi$ containing I, where $K_J := K^{\mathfrak{S}_{\Psi|J}}$. The composition is an isomorphism, so $V_I \otimes_{K_I} K_J \to V_J$ is an isomorphism as well. In particular, $f_\sigma = id_V$ if $\sigma \in \mathfrak{S}_{\Psi|I}$, where $(f_\sigma \in \mathrm{GL}_K(V))_\sigma$ is the 1-cocycle of the $\mathfrak{S}_\Psi$-action in the basis b. Clearly, (i) f_σ depends only on the class $\sigma|_I$ of σ in $\mathfrak{S}_\Psi / \mathfrak{S}_{\Psi|I} = \{$emdeddings of I into $\Psi\}$, (ii) $f_\sigma \in \mathrm{GL}_{K_{I\cup\sigma(I)}}(V_{I\cup\sigma(I)})$.

Assume that $I, \sigma(I), \tau\sigma(I)$ are disjoint, X, Y, Z are the standard collections of the elementary symmetric functions in $I, \tau(I), \tau\sigma(I)$, respectively. Then the cocycle condition $f_{\tau\sigma} = f_\tau f_\sigma^\tau$ (where $f_\sigma^\tau \in \mathrm{GL}_{K_{\tau(I)\cup\tau\sigma(I)}}(V_{\tau(I)\cup\tau\sigma(I)})$) becomes $\Phi(X,Z) = \Phi(X,Y)\Phi(Y,Z)$ and $\Phi(Y,X) = \Phi(X,Y)^{-1}$, where $f_{\tau\sigma} = \Phi(X,Z)$, etc. If k is infinite then there is a k-point Y_0, where $\Phi(X,Y)$ and $\Phi(Y,Z)$ are regular. If k is finite then there is a finite field extension $k'|k$ and a k'-point Y_0, where $\Phi(X,Y)$ and $\Phi(Y,Z)$ are regular. Specializing Y to such Y_0, we get $\Phi(X,Z) = \Phi(X,Y_0)\Phi(Y_0,Z) = \Phi(X,Y_0)\Phi(Z,Y_0)^{-1}$. Then $\Phi(X,Y_0)$ transforms b to a basis fixed by all $\sigma \in \mathfrak{S}_\Psi$ such that $\sigma(I)$ does not meet I, i.e. fixed by entire $\mathfrak{S}_\Psi$. This gives an embedding of V into a (finite) direct sum of copies of $K \otimes_k k'$, which is itself a (finite) direct sum of copies of K, and finally, so is V as well. $\square$

The following lemma asserts that, in a sense, restriction to an open subgroup cannot trivialize the irreducible subquotients of a semilinear representation with a non-trivial irreducible subquotient.

Lemma 3.2. *Let Ψ be a set, $\Psi' \subseteq \Psi$ be a subset of the same cardinality as Ψ, $K := k(\Psi)$ and $K' = k(\Psi')$. Set $I := \Psi \smallsetminus \Psi'$. Then any smooth simple $K\langle\mathfrak{S}_\Psi\rangle$-module M admits a simple $K'\langle\mathfrak{S}_{\Psi|I}\rangle$-submodule M' with $\dim_K M = \dim_{K'} M'$.*

Proof. For any $\mathfrak{S}_\Psi$-set M set $M' := \varinjlim_J M^{\mathfrak{S}_{\Psi|J}} \subseteq M^{\mathfrak{S}_{\Psi|\Psi'}}$, where J runs over finite subsets of Ψ'. [This does not lead to confusion in the cases $M = \Psi$ and $M = K$, since $\Psi' = \varinjlim_J J = \varinjlim_J \Psi^{\mathfrak{S}_{\Psi|J}}$ and $K' = k(\Psi') = \varinjlim_J k(J) = \varinjlim_J k(\Psi)^{\mathfrak{S}_{\Psi|J}}$.] Clearly, the group $\mathfrak{S}_{\Psi|I}$ acts on M'. We note that restriction to Ψ' identifies the groups $\mathfrak{S}_{\Psi|I}$ and the automorphism group $\mathfrak{S}_{\Psi'}$ of Ψ', while $\mathfrak{S}_{\Psi'}$ is identified with $\mathfrak{S}_{\Psi,\Psi'} / \mathfrak{S}_{\Psi|\Psi'}$.

Any bijection $\iota : \Psi \xrightarrow{\sim} \Psi'$ induces a topological group isomorphism $\iota_{\mathfrak{S}} : \mathfrak{S}_\Psi \xrightarrow{\sim} \mathfrak{S}_{\Psi'}$, $g \mapsto [i \mapsto \iota g(\iota^{-1}(i))]$. For a smooth $\mathfrak{S}_\Psi$-set M the bijection ι induces a bijection $\iota_M : M \xrightarrow{\sim} M'$, $m \mapsto \sigma_m m$ for any $\sigma \in \mathfrak{S}_\Psi$ with $\sigma_m|_J = \iota|_J$ if $m \in M^{\mathfrak{S}_{\Psi|J}}$ for a finite $J \subset \Psi$. This bijection is compatible with $\mathfrak{S}_\Psi$- and $\mathfrak{S}_{\Psi'}$-actions, i.e., the following diagram commutes

$$\begin{array}{ccc} \mathfrak{S}_\Psi \times M & \xrightarrow{\times} & M \\ \downarrow \iota_{\mathfrak{S}} \times \iota_M & & \downarrow \iota_M \\ \mathfrak{S}_{\Psi|I} \times M' & \xrightarrow{\times} & M' \end{array}$$

Clearly, ι induces a ring isomorphism $\iota_{K\langle\mathfrak{S}_\Psi\rangle} : K\langle\mathfrak{S}_\Psi\rangle \xrightarrow{\sim} K'\langle\mathfrak{S}_{\Psi'}\rangle$. Now, if M is a smooth $K\langle\mathfrak{S}_\Psi\rangle$-module then ι_M is compatible with $K\langle\mathfrak{S}_\Psi\rangle$- and $K\langle\mathfrak{S}_{\Psi'}\rangle$-module structures, i.e., the following diagram commutes

$$\begin{array}{ccc} K\langle\mathfrak{S}_\Psi\rangle \times M & \xrightarrow{\times} & M \\ \downarrow \iota_{K\langle\mathfrak{S}_\Psi\rangle} \times \iota_M & & \downarrow \iota_M \\ K'\langle\mathfrak{S}_{\Psi|I}\rangle \times M' & \xrightarrow{\times} & M' \end{array}$$

In particular, $\dim_K M = \dim_{K'} M'$. Moreover, if M is a simple $K\langle\mathfrak{S}_\Psi\rangle$-module then M' is a simple $K'\langle\mathfrak{S}_{\Psi'}\rangle$-module as well. □

Remark 3.3. Let Ψ be an infinite set and EndlMeng be the following site: the underlying category is opposite to the category of finite sets and their embeddings, any morphism is covering. It may be noticed that Lemma 3.2 is based on the existence of an equivalence between the category of smooth $\mathfrak{S}_\Psi$-sets and the category of sheaves of sets on EndlMeng (sending a sheaf $\mathcal{F}$ to the $\mathfrak{S}_\Psi$-set $\varinjlim_{J\subset\Psi} \mathcal{F}(J)$).

Proof of Theorem 1.2. By Lemma 2.5, any smooth simple $K\langle\mathfrak{S}_\Psi\rangle$-module is isomorphic to a quotient of $K[\binom{\Psi}{s}]$ for some s.

Let us show by induction on s that for any field K endowed with a smooth faithful $\mathfrak{S}_\Psi$-action any simple subquotient of the $K\langle\mathfrak{S}_\Psi\rangle$-module $K[\binom{\Psi}{s}]$ is isomorphic to K, the case $s = 0$ being trivial.

By Lemma 2.3, there are no simple $K\langle\mathfrak{S}_\Psi\rangle$-submodules in $K[\binom{\Psi}{s}]$ if $s \geq 1$, and therefore, any simple subquotient M of $K[\binom{\Psi}{s}]$ is contained in a quotient by some non-zero $K\langle\mathfrak{S}_\Psi\rangle$-submodule. By Lemma 2.4, there is a finite subset $I \subset \Psi$ such that the $K\langle\mathfrak{S}_{\Psi|I}\rangle$-module M is isomorphic to a subquotient of $\bigoplus_{j=0}^{s-1} K[\binom{\Psi\smallsetminus I}{j}]^{\oplus\binom{\#I}{s-j}}$. By the induction assumption, any simple subquotient of the $K\langle\mathfrak{S}_{\Psi|I}\rangle$-module M is isomorphic to K. In particular, in notation of Lemma 3.2 with $\Psi' := \Psi \smallsetminus I$, any simple subquotient of the $K\langle\mathfrak{S}_{\Psi|I}\rangle$-module $K \otimes_{K'} M' \subseteq M$ is isomorphic to K, and therefore, there is a surjection of $K\langle\mathfrak{S}_{\Psi|I}\rangle$-modules $\pi : K \otimes_{K'} M' \to K$ identifying M' with a $K'\langle\mathfrak{S}_{\Psi|I}\rangle$-submodule of K.

Let $Q \in K^\times$ be a non-zero element of $\pi(M')$. As $K' = k(\Psi \smallsetminus I)$, so $K = K'(I)$, we can consider Q as a rational function in variables in I over K'. If k is infinite then, specializing the elements of I to elements of k so that Q has neither zero nor pole at chosen collection, we get a non-zero morphism of $K'\langle\mathfrak{S}_{\Psi|I}\rangle$-modules $\pi(M') \to K'$, so $M' \cong K'$, and thus, $M \cong K$.

If k is finite then there is a finite field extension $k'|k$ such that $Q(I)$ has neither zero nor pole at some collection of elements of k'. Specializing the elements of I to such collection, we get a non-zero morphism of $K' \otimes_k k'\langle\mathfrak{S}_{\Psi|I}\rangle$-modules $\pi(M') \otimes_k k' \to K' \otimes_k k'$. As the $K'\langle\mathfrak{S}_{\Psi|I}\rangle$-modules $\pi(M') \otimes_k k'$ and $K' \otimes_k k'$ are isomorphic to (finite) direct sums of copies, respectively, of M' and of K', we get again $M' \cong K'$ and $M \cong K$.

Therefore, any smooth K-semilinear representation V of $\mathfrak{S}_\Psi$ of finite length is finite-dimensional. Finally, by Lemma 3.1, V is isomorphic to a direct sum of copies of K. □

Corollary 3.4. *Let k be a field and Ψ be an infinite set. Let $\mathfrak{S}_\Psi$ be the group of all permutations of the set Ψ acting naturally on the field $k(\Psi)$. Let $K \subset k(\Psi)$ be an $\mathfrak{S}_\Psi$-invariant subfield over k. Then any smooth K-semilinear irreducible representation of $\mathfrak{S}_\Psi$ can be embedded into $k(\Psi)$.*

Proof. For any smooth simple $K\langle\mathfrak{S}_\Psi\rangle$-module V the $k(\Psi)\langle\mathfrak{S}_\Psi\rangle$-module $V \otimes_K k(\Psi)$ admits a simple quotient isomorphic, by Theorem 1.2, to $k(\Psi)$. This means that V can be embedded into $k(\Psi)$. $\square$

Proof of Theorem 1.3. For any smooth simple $K\langle\mathfrak{S}_\Psi\rangle$-module V the $k(\Psi)\langle\mathfrak{S}_\Psi\rangle$-module $V \otimes_K k(\Psi)$ admits a simple quotient isomorphic, by Theorem 1.2, to $k(\Psi)$. This means that V can be embedded into $k(\Psi)$.

Let us show that any simple $K\langle\mathfrak{S}_\Psi\rangle$-submodule $V \subset k(\Psi)$ coincides with V_d for some $d \in \mathbb{Z}$. Let $P/Q \in V$ be a non-zero element for some polynomials $P, Q \in k[\Psi]$. Then there is a non-zero morphism $V \to V_{\deg P - \deg Q}$ sending P/Q to $P_{\deg P}/Q_{\deg Q}$, where $P_{\deg P}$ and $Q_{\deg Q}$ denote the homogeneous components of maximal degrees of P and Q, respectively. As V is simple, this morphism should be bijective. Then P/Q is homogeneous, since otherwise V would be infinite-dimensional over K, and therefore, $V = V_{\deg P - \deg Q}$.

Thus, any smooth $K\langle\mathfrak{S}_\Psi\rangle$-module V of finite length is a finite-dimensional K-vector space. Set $N := \dim_K V$. By Lemma 3.1, the $\mathfrak{S}_\Psi$-action on V in a fixed basis is given by the 1-cocycle $f_\sigma = \Phi(I)\Phi(\sigma I)^{-1}$ for some finite $I \subset \Psi$ and some $\Phi(X) \in \mathrm{GL}_N k(I)$. As $f_\sigma \in \mathrm{GL}_N K$, one has $\Phi(\lambda I)\Phi(\lambda\sigma I)^{-1} = \Phi(I)\Phi(\sigma I)^{-1}$ for any $\lambda \in \overline{k}$ and any $\sigma \in \mathfrak{S}_\Psi$, and therefore, $\Phi(I)^{-1}\Phi(\lambda I) \in (\mathrm{GL}_N k(I))^{\mathfrak{S}_\Psi} = \mathrm{GL}_N k$. Then $\lambda \mapsto \Phi(I)^{-1}\Phi(\lambda I)$ gives rise to a homomorphism of algebraic k-groups $\mathbb{G}_{m,k} \to \mathrm{GL}_{N,k}$. Changing the basis, we may assume that $\Phi(I)^{-1}\Phi(\lambda I)$ is diagonal with powers of λ on the diagonal. This means that the colums of $\Phi(I)$ are homogeneous of the same degree, i.e., V is isomorphic to a direct sum of several V_d's for some integer d. The spaces $V_d \subseteq k(\Psi)$ are pairwise non-isomorphic one-dimensional K-semilinear representations of $\mathfrak{S}_\Psi$, since $V_d = V_1^{\otimes_K^d}$. $\square$

Proof of Theorem 1.3. By Corollary 3.4, any smooth simple $K\langle\mathfrak{S}_\Psi\rangle$-module can be embedded into $k(\Psi)$. Let us show that any simple $K\langle\mathfrak{S}_\Psi\rangle$-submodule $V \subset k(\Psi)$ coincides with K.

Fix some $x \in \Psi$. One has $k(\Psi) = K[x] \oplus \bigoplus_R \varinjlim_{0 \le j < m \deg R} V_R^{(j,m)}$, where R runs over the $\mathfrak{S}_\Psi$-orbits of non-constant irreducible monic polynomials in $K[x]$ and $V_R^{(j,m)}$ is the K-linear envelope of $P(x)/Q^m$ for all $Q \in R$ and $P \in K[x]$ with $\deg P \le j$. Clearly, these decomposition and filtrations are independent of x. It suffices to show that the only simple $K\langle\mathfrak{S}_\Psi\rangle$-submodule $K[x]$ is K and there are no simple $K\langle\mathfrak{S}_\Psi\rangle$-submodules in $V_R^{(j,m)}$ for any R, m and j.

Suppose first that $V \subset K[x]$. Let $Q \in V$ be a (non-zero) monic polynomial in x of minimal degree. Then V contains $Q - \sigma Q$ for any $\sigma \in \mathfrak{S}_\Psi$. If $\sigma Q \neq Q$ for some $\sigma \in \mathfrak{S}_\Psi$ then $Q - \sigma Q \neq 0$ and $\deg(Q - \sigma Q) < \deg Q$, contradicting our assumption, so $\sigma Q = Q$ for any $\sigma \in \mathfrak{S}_\Psi$, i.e., $Q \in k$.

Suppose now that $V \subset V_R^{(j,m)}$. One has isomorphisms

$$x^j \cdot : V_R^{(0,m)} \xrightarrow{\sim} V_R^{(j,m)}/V_R^{(j-1,m)}$$

for all $0 < j < m \deg R$, so it suffices to check that $V_R^{(0,m)}$ admits no simple $K\langle\mathfrak{S}_\Psi\rangle$-submodules. Fix some $Q \in R$. Then the morphism $K[\mathfrak{S}_\Psi/\mathrm{Stab}_Q] \to V_R^{(0,m)}$, $[g] \mapsto (gQ)^{-m}$, is an isomorphism. By [2, Lemma 6.2], there are no simple submodules in $K[\mathfrak{S}_\Psi/\mathrm{Stab}_Q]$.

Thus, any smooth $K\langle\mathfrak{S}_\Psi\rangle$-module V of finite length is a finite-dimensional K-vector space. Set $N := \dim_K V$. By Lemma 3.1, the $\mathfrak{S}_\Psi$-action on V in a fixed basis is given by the 1-cocycle $f_\sigma = \Phi(I)\Phi(\sigma I)^{-1}$ for some finite $I \subset \Psi$ and some $\Phi(X) \in \mathrm{GL}_N k(I)$. As $f_\sigma \in \mathrm{GL}_N K$, one has $\Phi(T_\lambda I)\Phi(T_\lambda \sigma I)^{-1} = \Phi(I)\Phi(\sigma I)^{-1}$ for any $\lambda \in \overline{k}$ and any $\sigma \in \mathfrak{S}_\Psi$, where $T_\lambda x = x + \lambda$ for any $x \in \Psi \subset k(\Psi)$, and therefore, $\Phi(I)^{-1}\Phi(T_\lambda I) \in (\mathrm{GL}_N k(I))^{\mathfrak{S}_\Psi} = \mathrm{GL}_N k$. Then $\lambda \mapsto \Phi(I)^{-1}\Phi(T_\lambda I)$ gives rise to a homomorphism of algebraic k-groups $\mathbb{G}_{a,k} \to \mathrm{GL}_{N,k}$. Changing the basis, we may assume that $\Phi(I)^{-1}\Phi(T_\lambda I)$ is block-diagonal with unipotent blocks corresponding to indecomposable direct summands of V. For any integer $N \geq 1$ the unique isomorphism class of smooth K-semilinear indecomposable representations of $\mathfrak{S}_\Psi$ of length N is presented by $\bigoplus_{j=0}^{N-1} x^j K \subset k(\Psi)$ for any $x \in \Psi$. □

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