

ON HURWITZ–SEVERI NUMBERS

YURII BURMAN AND BORIS SHAPIRO

ABSTRACT. For a point $p \in \mathbb{CP}^2$ and a triple (g, d, ℓ) of non-negative integers we define a *Hurwitz–Severi number* $\mathfrak{H}_{g,d,\ell}$ as the number of generic irreducible plane curves of genus g and degree $d+\ell$ having an ℓ -fold node at p and at most ordinary nodes as singularities at the other points, such that the projection of the curve from p has a prescribed set of local and remote tangents and lines passing through nodes. In the cases $d+\ell \geq g+2$ and $d+2\ell \geq g+2 > d+\ell$ we express the Hurwitz–Severi numbers via appropriate ordinary Hurwitz numbers. The remaining case $d+2\ell < g+2$ is still widely open.

1. INTRODUCTION AND MAIN RESULTS

In what follows we will always work over the field \mathbb{C} of complex numbers, and by a genus g of a (singular) curve C we mean its geometric genus, i.e. the genus of its normalisation.

Fix a point $p \in \mathbb{CP}^2$ and denote by $\mathcal{W}_{g,d,\ell}$ the set consisting of all reduced irreducible plane curves of degree $d+\ell$, genus g , having an ℓ -fold node at the point p (i.e. ℓ smooth local branches intersecting transversally at p ; $\ell=0$ means that p does not belong to the curve), all the singularities outside p (if any) being ordinary nodes. The set $\mathcal{W}_{g,d,\ell}$ is nonempty if and only if

$$g \leq \binom{d+\ell-1}{2} - \binom{\ell}{2}, \quad (1.1)$$

see [12].

$\mathcal{W}_{g,d,\ell}$ is usually referred to as the (open, generalized) *Severi variety*, the classical case corresponding to $\ell=0$. The study of this variety was initiated by F. Severi [14] back in the 1920s. In a number of celebrated papers (see e.g., [6], [7], [12], [13]) $\mathcal{W}_{g,d,\ell}$ was proved to be irreducible of dimension $3d+2\ell+g-1$. Another well-studied characteristics of the Severi varieties is their degree, see [3].

A Hurwitz–Severi number $\mathfrak{H}_{g,d,\ell}$, which we define below, seems to be an equally natural characteristics of $\mathcal{W}_{g,d,\ell}$ as its degree, but it is apparently more difficult to calculate for general triples (g, d, ℓ) .

The set $\mathcal{W}_{g,d,\ell}$ is acted upon by the group $G \subset \mathrm{PGL}(3, \mathbb{C})$ of projective transformations of \mathbb{CP}^2 preserving p and each line passing through p . Obviously, G is a 3-dimensional Lie group that acts locally freely on $\mathcal{W}_{g,d,\ell}$. (In fact, unions of lines passing through p are the only curves having positive-dimensional stabilizers under this action.) Denote the orbit space of the action by $\widetilde{\mathcal{W}}_{g,d,\ell} := \mathcal{W}_{g,d,\ell}/G$; it is smooth almost everywhere and its dimension equals $3d+2\ell+g-4$.

Let us denote by \mathcal{C} a normalisation of a given plane curve C and by $\kappa : \mathcal{C} \rightarrow C$, the normalisation map. For a curve $C \in \mathcal{W}_{g,d,\ell}$, one defines the associated meromorphic function of degree d

$$\alpha_C := \pi_p \circ \kappa : \mathcal{C} \rightarrow p^\perp \simeq \mathbb{CP}^1$$

2010 *Mathematics Subject Classification*. Primary 14H50, Secondary 14H30.
Key words and phrases. Severi varieties, Hurwitz numbers.

obtained by composing the normalisation map with the standard projection $\pi_p : \mathbb{CP}^2 \setminus p \rightarrow p^\perp$ from the point p to the pencil $p^\perp \simeq \mathbb{CP}^1$ of lines passing through p .

For a generic $C \in \mathcal{W}_{g,d,\ell}$, there are ℓ distinct lines tangent to C at p (*local tangents*), and $2d+2g-2$ distinct lines passing through p and tangent to C elsewhere (*remote tangents*). Additionally, the curve C has

$$\#\text{nodes} = \binom{d+\ell-1}{2} - \binom{\ell}{2} - g = \binom{d-1}{2} + \ell(d-1) - g \geq 0$$

ordinary nodes (outside p), see e.g. [11]. A line passing through p and a remote node will be called *node-detecting*.

For any set X , denote by $X^{(m)}$ its m -th symmetric power, i.e. the quotient of the Cartesian product $X \times X \times \cdots \times X$ of m copies of X by the natural action of the symmetric group S_m permuting the copies. In the case when X is a smooth complex curve, $X^{(m)}$ is naturally interpreted as the set of effective divisors of degree m on X . Below we will denote elements of $X^{(m)}$ as divisors $z_1 + \cdots + z_m$, where $z_1, \dots, z_m \in X$ are not necessarily distinct. The m -th symmetric power $(\mathbb{CP}^1)^{(m)}$ is identified with \mathbb{CP}^m by means of the standard map sending the divisor $[z_1:w_1] + \cdots + [z_m:w_m]$ to $[a_0:\dots:a_m]$, where $\sum_{k=0}^m a_k t^k s^{m-k} := \prod_{k=1}^m (tz_k - sw_k)$.

We now define three natural maps

$$\begin{aligned} \mathcal{RT} : \mathcal{W}_{g,d,\ell} &\rightarrow (p^\perp)^{(2d+2g-2)}, \\ \mathcal{LT} : \mathcal{W}_{g,d,\ell} &\rightarrow (p^\perp)^{(\ell)}, \\ \mathcal{ND} : \mathcal{W}_{g,d,\ell} &\rightarrow (p^\perp)^{(\#\text{nodes})}, \end{aligned}$$

where \mathcal{RT} sends $C \in \mathcal{W}_{g,d,\ell}$ to the divisor of its remote tangents, \mathcal{LT} sends $C \in \mathcal{W}_{g,d,\ell}$ to the divisor of its local tangents, and \mathcal{ND} sends $C \in \mathcal{W}_{g,d,\ell}$ to the divisor of its node-detecting lines. Observe also that, for any $C \in \mathcal{W}_{g,d,\ell}$, $\mathcal{RT}(C)$ coincides with the divisor of the critical values of the meromorphic function α_C .

\mathcal{RT} , \mathcal{LT} and \mathcal{ND} are obviously preserved by the action of the group G and, therefore, can be considered as maps defined on $\widetilde{\mathcal{W}}_{g,d,\ell}$.

The triple of maps

$$\text{Br}_{g,d,\ell} := (\mathcal{RT}, \mathcal{LT}, \mathcal{ND}) : \widetilde{\mathcal{W}}_{g,d,\ell} \rightarrow \mathcal{P}_{g,d,\ell}$$

where

$$\mathcal{P}_{g,d,\ell} := (p^\perp)^{(2d+2g-2)} \times (p^\perp)^{(\ell)} \times (p^\perp)^{(\#\text{nodes})},$$

is called the *branching morphism*. An easy calculation shows that for all triples (g, d, ℓ) , one has

$$\begin{aligned} \dim \widetilde{\mathcal{W}}_{g,d,\ell} &= 3d + 2\ell + g - 4 = 2d + 2g - 2 + \ell + \#\text{nodes} - \frac{(d-2)(d+2\ell-3)}{2} \\ &\leq 2d + 2g - 2 + \ell + \#\text{nodes} = \dim \mathcal{P}_{g,d,\ell}; \end{aligned}$$

the equality takes place for the series of triples $(g, 2, \ell)$ with $g \leq \ell$ (cf. (1.1)) and in two additional cases: $(0, 3, 0)$ and $(1, 3, 0)$.

Definition 1.

- A triple (g, d, ℓ) is called *bendable* if $\dim \widetilde{\mathcal{W}}_{g,d,\ell} \geq 2d + 2g - 2 + \ell$. This is equivalent to $d + \ell \geq g + 2$ and means that $\dim \widetilde{\mathcal{W}}_{g,d,\ell}$ is larger than or equal to the sum of the number of branch points of α_C plus the number of local tangents. A triple (g, d, ℓ) is called *strongly bendable* if $\dim \widetilde{\mathcal{W}}_{g,d,\ell} = 2d + 2g - 2 + \ell + \#\text{nodes}$, i.e. $\text{Br}_{g,d,\ell}$ is the map between spaces of equal dimension. (All these cases are listed above.)

- A triple (g, d, ℓ) is called *semi-bendable* if $\dim \widetilde{\mathcal{W}}_{g,d,\ell} < 2d + 2g - 2 + \ell$, but $\dim \widetilde{\mathcal{W}}_{g,d,\ell} \geq 2d + 2g - 2$. This is equivalent to $d + \ell < g + 2 \leq d + 2\ell$ and means that $\dim \widetilde{\mathcal{W}}_{g,d,\ell}$ is larger than or equal to the number of branch points of α_C .
- Otherwise, a triple (g, d, ℓ) is called *unbendable*. It means that $\dim \widetilde{\mathcal{W}}_{g,d,\ell} < 2d + 2g - 2$ or, equivalently, that $d + 2\ell < g + 2$.

Conjecture 1. *For any triple (g, d, ℓ) which is not strongly bendable, $\text{Br}_{g,d,\ell}$ is a birational map of $\widetilde{\mathcal{W}}_{g,d,\ell}$ on its image.*

In view of this conjecture, to get combinatorially meaningful quantities, we need to make the image space smaller so that its dimension would be equal to that of $\widetilde{\mathcal{W}}_{g,d,\ell}$.

Definition 2.

- (1) Given a bendable triple (g, d, ℓ) , let $\bar{a} = a_1 + \cdots + a_{2d+2g-2}$, $\bar{b} = b_1 + \cdots + b_\ell$, and $\bar{x} = x_1 + \cdots + x_m$ be generic divisors on p^\perp of degrees $2d + 2g - 2$, ℓ , and $m := \dim \widetilde{\mathcal{W}}_{g,d,\ell} - (2d + 2g + \ell - 2) = d + \ell - g - 2$, respectively. Then the *Hurwitz–Severi number* $\mathfrak{H}_{g,d,\ell}$ is defined as the number of G -orbits $\mathfrak{D} \in \widetilde{\mathcal{W}}_{g,d,\ell}$ such that $\mathcal{RT}(\mathfrak{D}) = \bar{a}$, $\mathcal{LT}(\mathfrak{D}) = \bar{b}$, and $\mathcal{ND}(\mathfrak{D}) \geq \bar{x}$ (i.e. all the lines x_1, \dots, x_m are node-detecting for any $C \in \mathfrak{D}$, but C may have other node-detecting lines as well).
- (2) Given a semi-bendable triple (g, d, ℓ) , let $\bar{a} = a_1 + \cdots + a_{2d+2g-2}$ and $\bar{b} = b_1 + \cdots + b_m$ be generic divisors on p^\perp of degrees $2d + 2g - 2$ and $m := \dim \widetilde{\mathcal{W}}_{g,d,\ell} - (2d + 2g - 2) = d + 2\ell - g - 2$, respectively. Then the *Hurwitz–Severi number* $\mathfrak{H}_{g,d,\ell}$ is defined as the number of G -orbits $\mathfrak{D} \in \widetilde{\mathcal{W}}_{g,d,\ell}$ such that $\mathcal{RT}(\mathfrak{D}) = \bar{a}$ and $\mathcal{LT}(\mathfrak{D}) \geq \bar{b}$ (i.e. all lines b_1, \dots, b_m are local tangents for any $C \in \mathfrak{D}$, but C may have other local tangents as well).
- (3) Given an unbendable triple (g, d, ℓ) , let $\bar{a} = a_1 + \cdots + a_m$ be a generic divisor on p^\perp of degree $m := \dim \widetilde{\mathcal{W}}_{g,d,\ell}$. Then the *Hurwitz–Severi number* $\mathfrak{H}_{g,d,\ell}$ is defined as the number of orbits $\mathfrak{D} \in \widetilde{\mathcal{W}}_{g,d,\ell}$ such that $\mathcal{RT}(\mathfrak{D}) \geq \bar{a}$ (i.e. all the lines a_1, \dots, a_m are remote tangents for any $C \in \mathfrak{D}$).

Remark. One can define a branching morphism and a Hurwitz-like number not only for Severi varieties, but for many other natural families of plane algebraic curves. Given a generic curve γ in such a family, take the divisor of all lines passing through a given point $p \in \mathbb{C}\mathbb{P}^2$ which are not in general position with respect to γ . Then one can either define a branching morphism by just mapping γ to this divisor or (as we did above) one can additionally split this divisor into several subdivisors keeping track of different singularities of the intersection of γ with a given line. A Hurwitz-like number will be the number of preimages of a generic subspace of appropriate dimension in the image space under the branching morphism.

Our main results are formulas for the Hurwitz–Severi numbers in the bendable and semi-bendable cases. Consider the set of pairs (\mathcal{C}, α) , where \mathcal{C} is a connected smooth curve of genus g and $\alpha : \mathcal{C} \rightarrow \mathbb{C}\mathbb{P}^1$ is the meromorphic function of degree d with a prescribed set of simple critical values. Such pairs are considered up to an isomorphism: $(\mathcal{C}, \alpha) \sim (\mathcal{C}', \alpha')$ if there exists a holomorphic homeomorphism $\phi : \mathcal{C} \rightarrow \mathcal{C}'$ such that $\alpha = \alpha' \circ \phi$. The number of the pairs is equal to $h_{g,1^d}/d!$, where $h_{g,1^d}$ is the ordinary Hurwitz number of genus g and partition $1^d = (1, \dots, 1)$ of d ; see [8] for the precise definition and an algorithm of computation of the Hurwitz numbers.

Theorem 1. *Let (g, d, ℓ) be a bendable triple. Then the Hurwitz–Severi number $\mathfrak{H}_{g,d,\ell}$ is equal to $\binom{d}{2}^{d+\ell-g-2} d^\ell h_{g,1^d}/d!$.*

Theorem 2. *Let (g, d, ℓ) be a semi-bendable triple. Then the Hurwitz–Severi number $\mathfrak{H}_{g,d,\ell}$ is equal to $d^{d+2\ell-g-2} \binom{2g-d-\ell-1}{g-3} h_{g,1^d}/d!$.*

- Example.*
- (1) Projection of a smooth cubics from a point not lying on it. Here the triple $(g, d, \ell) = (1, 3, 0)$ is bendable. The ordinary Hurwitz number $h_{1,1^3} = 240$ by [8], so the Hurwitz–Severi number $\mathfrak{H}_{1,3,0} = 40$; this result was earlier obtained in [11].
 - (2) Projection of a smooth cubics from a point lying on it. The triple $(g, d, \ell) = (1, 2, 1)$ is bendable. The ordinary Hurwitz number $h_{1,1^2} = 1$ by [8], so the Hurwitz–Severi number $\mathfrak{H}_{1,3,0} = 1$; this can be checked by a direct computation.
 - (3) Projection of a nodal cubics from an outside point corresponds to a bendable triple $(g, d, \ell) = (0, 3, 0)$. Here $h_{0,1^3} = 24$ by [8], implying $\mathfrak{H}_{0,3,0} = 12$. This answer can be checked directly using a computer algebra system.
 - (4) Projection of a nodal cubics from its smooth point corresponds to $(g, d, \ell) = (0, 2, 1)$. The ordinary Hurwitz number is $h_{0,1^2} = 1$, so the Hurwitz–Severi number is $\mathfrak{H}_{0,2,1} = 1$, which is easily checked by hand.
 - (5) Projection of a smooth quartics from its point corresponds to $(g, d, \ell) = (3, 3, 1)$, a semi-bendable triple. The ordinary Hurwitz number computed using the formulas of [8] is $h_{3,1^3} = 19680$, so the Hurwitz–Severi number is $\mathfrak{H}_{g,d,\ell} = 3280$.
 - (6) Projection of a smooth quartics from an outside point corresponds to $(g, d, \ell) = (3, 4, 0)$. This is an unbendable triple not covered by Theorems 1 and 2. This case was investigated by R. Vakil in [15] using different technique.

Theorems 1 and 2 give a complete description of Hurwitz–Severi numbers in the bendable and semi-bendable cases. Unlike them, the unbendable case seems to require completely new ideas. The only result in the unbendable case known to the authors at the time of writing is [15].

Acknowledgements. The authors are grateful to M. Feigin, A. Gorinov, M. Kazarian, S. Lando, S. Lvovski, M. Shapiro, and I. Tyomkin for discussions. The first-named author wishes to thank the Stockholm University for its warm hospitality. The research of the first named author was supported by the Russian Academic Excellence Project ‘5-100’ and the grant No. 15-01-0031 “Hurwitz numbers and graph isomorphism” of the Scientific Fund of the Higher School of Economics. The second named author wants to thank the Department of Mathematics of Higher School of Economics in Moscow for the hospitality and financial support of his visit in April 2014, when this project was initiated. The current project is a continuation of the previous research, see [2] and [11], which was the outcome of the previous visit by the second named author to the HSE in 2013.

2. PROOFS

The symmetric power $\mathcal{C}^{(k)}$ of the curve \mathcal{C} is the set of effective divisors D of degree k on \mathcal{C} . Set $D = x_1 c_1 + \dots + x_m c_m \in \mathcal{C}^{(k)}$ (with $c_1, \dots, c_m \in \mathcal{C}$, $x_1, \dots, x_m \in \mathbb{Z}_{>0}$, and $x_1 + \dots + x_m = k$). Introduce a complex coordinate z_i with $z_i(c_i) = 0$ in an open set $U_i \subset \mathcal{C}$, $c_i \in U_i$, and set $\tilde{D} = p_1 + \dots + p_k \in U$, where U is the image of $U_1 \times \dots \times U_m$ under the standard projection $\mathcal{C}^k \rightarrow \mathcal{C}^{(k)}$. Without loss of generality, it means that $p_1, \dots, p_{k_1} \in U_1$, $p_{k_1+1}, \dots, p_{k_2} \in U_2$, \dots , $p_{k_{m-1}+1}, \dots, p_k \in U_m$ for some $1 \leq k_1 \leq \dots \leq k_{m-1} \leq k$. For every $i = 1, \dots, s$, consider a principal part F_i of a meromorphic function $f_i : U_i \rightarrow \mathbb{C}\mathbb{P}^1$ having at p_i a pole of the degree not

exceeding the multiplicity of p_i in the divisor \tilde{D} and having no other poles in U_i ; let $F = (F_1, \dots, F_s)$. For $c \in \mathcal{C}$, using the above local coordinates, one has

$$F(c) = \frac{a_1 + a_2 z_1(c) + \dots + a_{k_1} z_1(c)^{k_1-1}}{(z_1(c) - z_1(p_1)) \dots (z_1(c) - z_1(p_{k_1}))} + \dots \\ + \frac{a_{k_{m-1}+1} + a_{k_{m-1}+2} z_m(c) + \dots + a_k z_m(c)^{k-k_{m-1}-1}}{(z_m(c) - z_m(p_{k_{m-1}+1})) \dots (z_m(c) - z_m(p_k))}.$$

So, the vectors F of principal parts form a rank k vector bundle on $\mathcal{C}^{(k)}$; the coefficients a_1, \dots, a_k form its trivialisation over the set U . An immediate comparison of the transition maps shows that this bundle is isomorphic to the tangent bundle $T\mathcal{C}^{(k)}$.

Given a 1-form ν holomorphic in U_i , we define a linear functional ν_z on the space of principal parts by the formula

$$\nu_z(F_i) = \text{Res}_z F_i \nu.$$

For a divisor $D = x_1 c_1 + \dots + x_m c_m \in \mathcal{C}^{(k)}$, define $\nu_D := \sum_{i=1}^m \nu_{c_i}$. It follows from the above reasoning that ν_D is a section of the complex cotangent bundle $T^*\mathcal{C}^{(k)}$; cf. the fiber bundle of principal parts introduced in [4].

There exists a natural map $\Phi : \mathcal{O}(D) \rightarrow T_D \mathcal{C}^{(k)}$ sending a meromorphic function $f \in \mathcal{O}(D)$ to the m -tuple $F = (F_1, \dots, F_m)$ of its principal parts at the points c_1, \dots, c_m . By the Riemann–Roch theorem, $F = \Phi(f)$ for some f if and only if $\nu_D(F) = 0$ for every holomorphic 1-form ν on \mathcal{C} . Fixing a basis ν_1, \dots, ν_g of holomorphic 1-forms on \mathcal{C} , we can calculate the dimension $h^0(D) = \dim \mathcal{O}(D)$ as $k - \dim \langle (\nu_1)_D, \dots, (\nu_g)_D \rangle$.

To prove our main results, we need the following technical statement which is apparently well-known to the specialists, but we could not find it explicitly in the literature:

Proposition 3. *Take a pair (\mathcal{C}, α) , where \mathcal{C} is a smooth curve of genus g and $\alpha : \mathcal{C} \rightarrow \mathbb{C}\mathbb{P}^1$ is a meromorphic function of degree d , and suppose that (\mathcal{C}, α) is generic among such pairs. Set $D_\alpha := z_1 + \dots + z_d$, where all z_i are pairwise distinct. If $m \geq g + 2 \geq d$ and z_{d+1}, \dots, z_m are generic pairwise distinct points, then the divisor $z_1 + \dots + z_m$ is non-special.*

Generic divisors are never special, but $z_1 + \dots + z_m$ may be non-generic because $z_1 + \dots + z_m \geq D_\alpha$ for some α of degree d .

Proof. Let ν_1, \dots, ν_g be a basis of holomorphic 1-forms on \mathcal{C} . Since $z_1 + \dots + z_d = D_\alpha$, one has $h^0(z_1 + \dots + z_d) \geq 2$. If there exists $\psi \in \mathcal{O}(D_\alpha)$ not proportional to α , then there exists their non-constant linear combination with no pole at z_d . Since a generic d -gonal curve \mathcal{C} is not $(d-1)$ -gonal [5], this is impossible, and therefore $h^0(z_1 + \dots + z_d) = 2$.

We now prove by induction that if $d + s \leq g + 1$ and the points z_{d+1}, \dots, z_{d+s} are in general position, then

$$\dim \langle (\nu_1)_{z_1 + \dots + z_{d+s}}, \dots, (\nu_g)_{z_1 + \dots + z_{d+s}} \rangle = d + s - 1.$$

Assume that starting with some s the statement fails. It means that for $z_{d+1}, \dots, z_{d+s-1}$ in general position,

$$\dim \langle (\nu_1)_{z_1 + \dots + z_{d+s-1}}, \dots, (\nu_g)_{z_1 + \dots + z_{d+s-1}} \rangle = d + s - 2,$$

but there exists a non-empty open set $\Omega \subset \mathcal{C}$ such that if $z_{d+s} \in \Omega$, then

$$\dim \langle (\nu_1)_{z_1 + \dots + z_{d+s}}, \dots, (\nu_g)_{z_1 + \dots + z_{d+s}} \rangle = d + s - 2$$

as well. In other words, vector $\vec{\nu}_{z_{d+s}} := ((\nu_1)_{z_{d+s}}, \dots, (\nu_g)_{z_{d+s}})$ is a linear combination of $\vec{\nu}_{z_i}$, $i = 1, \dots, d + s - 1$.

For an arbitrary positive integer q , consider q points $z_{d+s}, \dots, z_{d+s+q-1} \in \Omega$. Then for $j = d + s, \dots, d + s + q - 1$, every $\vec{\nu}_{z_j}$ is a linear combination of $\vec{\nu}_{z_i}$, $i = 1, \dots, d + s - 1$, and therefore

$$\dim\langle(\nu_1)_{z_1+\dots+z_{d+s+q-1}}, \dots, (\nu_g)_{z_1+\dots+z_{d+s+q-1}}\rangle = d + s - 2$$

for any q . Hence the divisor $z_{d+s} + \dots + z_{d+s+q-1}$ is special for any collection $z_{d+s}, \dots, z_{d+s+q-1} \in \Omega$, and therefore the set of special divisors of any degree $q > g$ on \mathcal{C} contains an open subset $\Omega^{(q)} \subset \mathcal{C}^{(q)}$. The latter claim is false since the set of special divisors of any sufficiently large degree is nowhere dense. \square

Proof of Theorem 1. Take $p = [0:1:0]$ and suppose without loss of generality that a curve $C \in \mathcal{W}_{g,d,\ell}$ does not contain the point $[1:0:0]$. Then the normalisation map $\kappa : \mathcal{C} \rightarrow C$ is given by

$$\kappa(z) = [\beta(z):1:\alpha(z)],$$

where $\alpha, \beta : \mathcal{C} \rightarrow \mathbb{CP}^1$ are meromorphic functions of degrees d and $d + \ell$, respectively, such that $D_\beta \geq D_\alpha$, i.e. $D_\beta - D_\alpha$ is an effective divisor on \mathcal{C} .

Take a generic divisor $\bar{a} = a_1 + \dots + a_{2d+2g-2}$ on \mathbb{CP}^1 . As it was noted above, there exist $h_{g,1d}/d!$ pairs (\mathcal{C}, α) such that \bar{a} is the divisor of the critical values of α . For any β , one can take $C := \kappa(\mathcal{C})$, with the map $\kappa : \mathcal{C} \rightarrow \mathbb{CP}^2$ as above. Then $\mathcal{RT}(C) = \bar{a}$ regardless of the choice of β . Set $D_\alpha := z_1 + \dots + z_d$ and notice that in general position the points $z_1, \dots, z_d \in \mathcal{C}$ are pairwise distinct (i.e. α has only simple poles).

Now take a generic divisor $\bar{b} = b_1 + \dots + b_\ell$ on \mathbb{CP}^1 and choose points $z_{d+1}, \dots, z_{d+\ell} \in \mathcal{C}$ such that $\bar{b} = \alpha(z_{d+1}) + \dots + \alpha(z_{d+\ell})$. Since the degree of α is d , there are d^ℓ ways to do this; the points $z_{d+1}, \dots, z_{d+\ell}$ are pairwise distinct if \bar{b} is in general position. This guarantees the equality $\mathcal{LT}(C) = \bar{b}$ for the curve $C = \kappa(\mathcal{C})$, provided that $D_\beta = z_1 + \dots + z_{d+\ell}$.

Assume now that $\bar{x} = x_1 + \dots + x_{d+\ell-g-2}$ is a generic divisor on \mathbb{CP}^1 . For each i , take a pair of points $u_i \neq v_i \in \mathcal{C}$ such that $\alpha(u_i) = \alpha(v_i) = x_i$; there are $\binom{d}{2}^{d+\ell-g-2}$ ways to do this. For $i = 1, \dots, d + \ell - g - 2$, define the functionals $\rho_i : \mathcal{O}(z_1 + \dots + z_{d+\ell}) \rightarrow \mathbb{C}$ by

$$\rho_i(\beta) := \beta(u_i) - \beta(v_i).$$

Apparently, $\rho_i(\beta) = 0$ if and only if the line x_i is node-detecting for the curve $C = \kappa(\mathcal{C})$.

Lemma 4. *For a generic choice of α and $z_{d+1}, \dots, z_{d+\ell}$, the functionals ρ_i , $i = 1, \dots, d + \ell - g - 2$ are linearly independent.*

Proof. By the Riemann–Roch theorem, $h^0(z_1 + \dots + z_{d+\ell}) \geq d + \ell - g + 1$. Thus, for any $k \leq d + \ell - g - 2$, there exists a function $\beta \in \mathcal{O}(z_1 + \dots + z_{d+\ell})$ such that $\rho_1(\beta) = \dots = \rho_{k-1}(\beta) = 0$. If for generic α and $z_{d+1}, \dots, z_{d+\ell}$, the functional ρ_k is a linear combination of ρ_i , $1 \leq i \leq k - 1$, then there exists an open subset $\Omega \subset \mathcal{C}$ with the following property. If $\alpha(u) = \alpha(v) = x \in \Omega$, then $\beta(u) = \beta(v)$, implying that $\beta(z)$ is a function of $\alpha(z)$ for $z \in \alpha^{-1}(\Omega)$. Therefore, for any $z \in \alpha^{-1}(\Omega)$, the line $\alpha(z) \in p^\perp$ intersects the curve C at exactly one point. Since $\alpha^{-1}(\Omega)$ is open, there exists $z_* \in \alpha^{-1}(\Omega)$ such that the intersection is transversal. But if $d > 1$, this is impossible. \square

Now by Proposition 3, one has generically

$$h^0(z_1 + \dots + z_{d+\ell}) = d + \ell - g + 1$$

implying that the solutions of the equations $\rho_1(\beta) = \cdots = \rho_{d+\ell-g-2}(\beta) = 0$ form a 3-dimensional space. The group G acts transitively upon this space, and Theorem 1 follows. \square

The proof of Theorem 2 is based on the following statement.

Proposition 5. *Let (g, d, ℓ) be a semi-bendable triple. Then for a generic pair (\mathcal{C}, α) , where \mathcal{C} is a smooth curve and $\alpha : \mathcal{C} \rightarrow \mathbb{C}\mathbb{P}^1$ is a degree d meromorphic function, and for a generic divisor D_0 of degree $d + 2\ell - g - 2$ on \mathcal{C} , the set $\mathcal{D} := \{D \in \mathcal{C}^{(g+2-d-\ell)} \mid h^0(D_\alpha + D_0 + D) \geq 3\}$ is finite and contains $\binom{2g-d-\ell-1}{g-3}$ elements. Additionally, for any $D \in \mathcal{D}$, one has $h^0(D_\alpha + D_0 + D) = 3$.*

Proof. Choose $D \in \mathcal{D}$ and $\beta \in \mathcal{O}(D_\alpha + D_0 + D)$ and define a plane curve $C_{D,\beta} := \kappa(\mathcal{C})$, where $\kappa(z) := [\beta(z):1:\alpha(z)]$. The group G acts on $\mathcal{O}(D_\alpha + D_0 + D)$ by $\beta \mapsto a\beta + b\alpha + c$, where $a, b, c \in \mathbb{C}, a \neq 0$. Thus G acts on the set of all $C_{D,\beta}$ with a fixed D .

Prove first that \mathcal{D} is finite and that $h^0(D_\alpha + D_0 + D) = 3$ for any $D \in \mathcal{D}$. Consider the orbit space $\widetilde{\mathcal{D}} := \{(D, \beta) \mid D \in \mathcal{D}, \beta \in \mathcal{O}(D_\alpha + D_0 + D)\}/G$ and let $\delta := \dim \widetilde{\mathcal{D}}$ be its dimension. The pair (\mathcal{C}, α) is determined, up to a finite choice, by the divisor of the critical values of α , which has degree $2d + 2g - 2$; so the set of all such pairs has dimension $2d + 2g - 2$. The dimension of the set of all divisors D_0 is equal to $\deg D_0 = d + 2\ell - g - 2$. The choice of $(D, \beta) \in \widetilde{\mathcal{D}}$ determines a curve $C_{D,\beta}$ up to the action of G , that is, it determines a point in $\widetilde{\mathcal{W}}_{g,d,\ell}$. On the other hand, for a given curve $C_{D,\beta} \in \mathcal{W}_{g,d,\ell}$, one can uniquely restore the divisor D on the normalisation \mathcal{C} of $C_{D,\beta}$ noticing that its points are the poles of β or, equivalently, the points of \mathcal{C} sent by the normalisation map to the base point $p \in C_{D,\beta}$. So, different choices of $D \in \mathcal{D}$ and different orbits of the G -action on $\mathcal{O}(D_\alpha + D_0 + D)$ for a fixed D correspond to different points of $\widetilde{\mathcal{W}}_{g,d,\ell}$. This implies the inequality

$$\dim \widetilde{\mathcal{W}}_{g,d,\ell} \geq (2d + 2g - 2) + (d + 2\ell - g - 2) + \delta.$$

Since $\dim \widetilde{\mathcal{W}}_{g,d,\ell} = 3d + 2\ell + g - 4$ (see e.g., [6]), one gets that $\delta = 0$. Thus, \mathcal{D} consists of a finite number of points, and for any such point D , the number of G -orbits in $\mathcal{O}(D_\alpha + D_0 + D)$ is finite, which means that $h^0(D_\alpha + D_0 + D) = 3$.

Count now the points $D \in \mathcal{D}$. Set $D_\alpha := z_1 + \cdots + z_d$, $D_0 := z_{d+1} + \cdots + z_{2d+2\ell-g-2}$, $D := z_{2d+2\ell-g-1} + \cdots + z_{d+\ell}$ and denote

$$D' := D_\alpha + D_0 + D - z_d = z_1 + \cdots + z_{d-1} + z_{d+1} + \cdots + z_{d+\ell}.$$

As was shown above, $h^0(D_\alpha + D_0 + D) \geq 3$ if and only if

$$\dim \langle (\nu_1)_{D_\alpha + D_0 + D}, \dots, (\nu_g)_{D_\alpha + D_0 + D} \rangle \leq d + \ell - 2$$

or, equivalently, $\dim \langle \vec{\nu}_{z_1}, \dots, \vec{\nu}_{z_{d+\ell}} \rangle \leq d + \ell - 2$. \mathcal{C} is a generic d -gonal curve, therefore it is not $(d-1)$ -gonal implying that $h^0(D_\alpha) = 2$. Thus the vector $\vec{\nu}_{z_d}$ is a linear combination of $\vec{\nu}_{z_1}, \dots, \vec{\nu}_{z_{d-1}}$ (see the proof of Proposition 3 for notation), which means that the last condition is equivalent to

$$\dim \langle \vec{\nu}_{z_1}, \dots, \vec{\nu}_{z_{d-1}}, \vec{\nu}_{z_{d+1}}, \dots, \vec{\nu}_{z_{d+\ell}} \rangle \leq d + \ell - 2,$$

i.e. $\dim \langle (\nu_1)_{D'}, \dots, (\nu_g)_{D'} \rangle \leq d + \ell - 2$.

For any k denote by $S_k : \mathcal{C}^k \rightarrow \mathcal{C}^{(k)}$ the natural projection; for any vector $X \in \mathcal{C}^k$ denote by $\iota_X : \mathcal{C}^k \rightarrow \mathcal{C}^{m+k}$ the natural embedding (coordinates of X are written before the coordinates of the argument). Take any point $Z = (z_1, \dots, z_{d-1}, z_{d+1}, \dots, z_{2d+2\ell-g-2})$ such that $S_{2d+2\ell-g-3}(Z) = D_0 + D_1 - z_d$ and consider the vector bundle $E = \iota_Z^* S_{d+\ell-1}^* T^* \mathcal{C}^{(d+\ell-1)}$ of rank $d + \ell - 1$ on $\mathcal{C}^{g+2-d-\ell}$. (In other words, Z is an arbitrary ordering of $z_1, \dots, z_{d-1}, z_{d+1}, \dots, z_{2d+2\ell-g-2}$.)

The Riemann–Roch theorem implies that $D \in \mathcal{D}$ if and only if for any $W \in S_{g+2-d-\ell}^{-1}(D_1)$, one has

$$\dim \langle \iota_Z^* S_{d+\ell-1}^*(\nu_1)_{D'}(W), \dots, \iota_Z^* S_{d+\ell}^*(\nu_g)_{D'} \rangle \leq d + \ell - 2.$$

Since we have shown that variety $S_{g+2-d-\ell}^{-1}(\mathcal{D})$ is 0-dimensional, its number of points is given by the Porteous formula [9]:

$$\#S_{g+2-d-\ell}^{-1}(\mathcal{D})Y = \det \begin{pmatrix} c_1(E) & c_2(E) & \dots & c_{g+1-d-\ell}(E) & c_{g+3-d-\ell}(E) \\ 1 & c_1(E) & \dots & c_{g-d-\ell}(E) & c_{g+1-d-\ell}(E) \\ 0 & 1 & \dots & c_{g-1-d-\ell}(E) & c_{g-d-\ell}(E) \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & c_1(E) \end{pmatrix}, \quad (2.1)$$

where $Y \in H^{2(g+2-d-\ell)}(\mathcal{C}^{g+2-d-\ell})$ is the generator of the top-dimensional cohomology, i.e. it is the Poincaré dual of a point.

Set $m := g+2-d-\ell$ for brevity, and denote by $\mathcal{N}_{k,m}$ the lower right $(k \times k)$ -minor of (2.1). In particular, $\#S_{g+2-d-\ell}^{-1}(\mathcal{D})Y = \mathcal{N}_{m,m}$. Developing the determinant by its first column, one obtains $\mathcal{N}_{m,m} = \sum_{k=1}^m (-1)^{k+1} c_k(E) \mathcal{N}_{m-k,m}$, implying

$$\sum_{k=0}^m \mathcal{N}_{k,m} = \left(\sum_{k=0}^m (-1)^k c_k(E) \right)^{-1}. \quad (2.2)$$

Denote by $x \in H^2(\mathcal{C}^{(k)})$ the class dual to the fundamental homological class of the diagonal $\{kz \mid z \in \mathcal{C}\} \subset \mathcal{C}^{(k)}$. It follows from the general formula of [10] that $c_k(T^* \mathcal{C}^{(m)}) = \binom{2g-2+k-m}{k} x^k$. One has $S_m^* x = x_1 + \dots + x_m := X$, where $x_i \in H^2(\mathcal{C}^m)$ is the class dual to the fundamental class of the i -th copy of \mathcal{C} in the product $\mathcal{C} \times \dots \times \mathcal{C} = \mathcal{C}^k$. Additionally, one has $\iota_Z^* x_i = x_{i-m} \in H^2(\mathcal{C}^k)$ if $i > m$ and $\iota_Z^* x_i = 0$ if $i \leq m$. Therefore $c_k(E) = \binom{g-3+m+k}{k} X^k$, implying that

$$\sum_{k=0}^m (-1)^k \binom{g-3+m+k}{k} X^k = (1+X)^{-(g-3+m)}.$$

Thus, (2.2) implies that $\sum_{k=0}^m \mathcal{N}_{k,m} = (1+X)^{-(g-3+m)}$, giving $\mathcal{N}_{m,m} = \binom{g-3+m}{g-3} X^m$. Since $Y = X^m/m!$, then $\#S_m^{-1}(\mathcal{D}) = m! \binom{g-3+m}{g-3}$. By dimensional reasons, in generic situation all the elements \mathcal{D} are sums of exactly m distinct points, meaning that

$$\#\mathcal{D} = \frac{1}{m!} \#S_m^{-1}(\mathcal{D}) = \binom{g-3+m}{g-3} = \binom{2g-1-d-\ell}{g-3}.$$

□

Proof of Theorem 2. Similarly to the proof of Theorem 1, for a generic divisor $\bar{a} = a_1 + \dots + a_{2d+2g-2}$, there are $h_{g,1^d}/d!$ ways to choose a curve \mathcal{C} of genus g and a degree d meromorphic function $\alpha : \mathcal{C} \rightarrow \mathbb{CP}^1$ such that \bar{a} is its divisor of critical values.

Let D_α be the pole divisor of α ; choose $d+2\ell-g-2$ points $z_{d+1}, \dots, z_{2d+2\ell-g-2} \in \mathcal{C}$ such that $\bar{b} = \alpha(z_{d+1}) + \dots + \alpha(z_{2d+2\ell-g-2})$. For generic \bar{b} , there are $d^{d+2\ell-g-2}$ ways to do that.

Similar to the proof of Proposition 5, denote $D_0 := z_{d+1} + \dots + z_{2d+2\ell-g-2}$ for short, and denote by $\mathcal{D} \subset \mathcal{C}^{(g+2-d-\ell)}$ the set of effective divisors $D := z_{2d+2\ell-g-1} + \dots + z_{d+\ell}$ of degree $g+2-d-\ell$ such that $h^0(D_\alpha + D_0 + D) \geq 3$. By Proposition 5, the set \mathcal{D} is finite, and for any $D \in \mathcal{D}$, the space $\mathcal{O}(D_\alpha + D_0 + D)$ has dimension

3 and contains exactly one orbit of the group G . Therefore the number of points $C \in \widetilde{W}_{g,d,\ell}$ with $\mathcal{RT}(C) = a$ and $\mathcal{LT}(C) \geq b$ is equal to

$$d^{d+2\ell-g-2} h_{g,1^d}/d! \# \mathcal{D} = d^{d+2\ell-g-2} \binom{2g-d-\ell-1}{g-3} h_{g,1^d}/d!.$$

□

3. FINAL REMARKS

1. The definition of Hurwitz–Severi numbers given above can be easily extended from the class of Severi varieties $W_{g,d,\ell}$ to a somewhat broader class $W_{g,d,\ell,\mu}$ which appeared earlier in several papers of J. Harris and Z. Ran. Namely, one can additionally require that curves under consideration have a given set μ of tangency multiplicities to a given line passing through the point p . One might expect Theorems 1 and 2 to have straightforward analogs in this more general setup.

2. The problem of calculation of Hurwitz–Severi numbers for the simplest unbendable case $g = (d-1)(d-2)/2, \ell = 0$, i.e. when a smooth plane curve of degree d is projected from a point not lying on it, bears a strong resemblance with the problem of calculation of Zeuten numbers, see [16]. Namely, in a special case, Zeuten’s problem asks how many smooth plane curves of degree d are tangent to a given set of $\frac{d(d+3)}{2}$ lines in general position. The Hurwitz–Severi number for the case $g = (d-1)(d-2)/2$ and $\ell = 0$ counts the number of G -orbits of smooth curves of degree d which are tangent to a given set of $\frac{d(d+3)}{2} - 3$ generic lines passing through a given point p . To the best of our knowledge, both problems are unsolved at present and apparently are quite difficult.

3. One possible approach to the calculation of Hurwitz numbers in the unstable case (such as $((d-1)(d-2)/2, d, 0)$) might be the use of tropical algebraic geometry. For example, in [1] the authors studied tropical analogs of Zeuten numbers and were able to recover some of the classical Zeuten numbers through their tropical analogs.

4. It would be interesting to study possible relation of the above Hurwitz–Severi numbers to appropriate Gromov–Witten invariants of plane curves.

REFERENCES

- [1] B. Bertrand, E. Brugalle, G. Mikhalkin, Genus 0 characteristic numbers of the tropical projective plane, *Compositio Mathematica*, 150(1) (2014), 46–104.
- [2] Yu. Burman, S. Lвовский. On projections of smooth and nodal plane curves, *Moscow Math Journal*, 15(1) (2015), 31–48.
- [3] L. Caporaso, J. Harris, Counting plane curves of any genus, *Invent. Math.* 131(2) (1998), 345–392.
- [4] T. Ekedahl, S. Lando, M. Shapiro and A. Vainshtein, Hurwitz numbers and intersections on moduli spaces of curves, *Invent. Math.* 146 (2001), 297–327.
- [5] G. Farkas, Brill–Noether loci and the gonality stratification of M_g , *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2001(539):185–200.
- [6] J. Harris, On the Severi problem, *Invent. Math.* 84(3) (1986), 445–461.
- [7] J. Harris, I. Morrison, *Moduli of curves*, Springer, Graduate Texts in Mathematics, 1998.
- [8] M. E. Kazarian, S. K. Lando, An algebro-geometric proof of Witten’s conjecture. *J. Amer. Math. Soc.* 20(4) (2007), 1079–1089.
- [9] G. Kempf, D. Laksov, The determinantal formula of Schubert calculus, *Acta Mathematica*, 132(1) (1974), 153–162.
- [10] T. Ohmoto, Generating functions for orbifold Chern classes I: symmetric products, *Math. Proc. Cambridge Philos. Soc.* 144(2) (2008), 423–438.
- [11] J. Ongaro, B. Shapiro, A note on planarity stratification of Hurwitz spaces, *Canadian Mathematical Bulletin* 58(3) (2015), 596–609.

- [12] Z. Ran, Families of plane curves and their limits: Enriques' conjecture and beyond, *Ann. Math.* 130(1) (1989), 121–157.
- [13] Z. Ran, Enumerative geometry of singular plane curves, *Inv. Math.*, 97 (1989), 447–465.
- [14] F. Severi, *Vorlesungen über algebraische Geometrie*, Teubner-Verlag, 1921.
- [15] R. Vakil, Twelve points on the projective line, branched covers, and rational elliptic fibrations, *Math. Ann.* 320 (2001), 33–54.
- [16] H. G. Zeuten, *Almindelige Egenskaber ved Systemer af plane Kurver*, Kongelige Danske Videnskabernes Selskabs Skrifter — Naturvidenskabelig och Matematisk, 10 (1873), 287–393.

NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS (NRU-HSE), RUSSIA, AND
INDEPENDENT UNIVERSITY OF MOSCOW, B. VLASSIEVSKII PER., 11, MOSCOW, 119002, RUSSIA
E-mail address: `burman@ccme.ru`

DEPARTMENT OF MATHEMATICS, STOCKHOLM UNIVERSITY, S-10691, STOCKHOLM, SWEDEN
E-mail address: `shapiro@math.su.se`