Bounded limit cycles of polynomial foliations of $\mathbb{C}P^2$

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In this article we prove in a new way that a generic polynomial vector field in $\mathbb{C}^2$ possesses countably many homologically independent limit cycles. The new proof needs no estimates on integrals, provides thinner exceptional set for quadratic vector fields, and provides limit cycles that stay in a bounded domain.

Contents

1   Introduction

Consider a polynomial differential equation in $\mathbb{C}^2$ (with complex time),

$$\dot{x} = P(x,y),$$
$$\dot{y} = Q(x,y), \tag{1}$$

where $\max(\deg P, \deg Q) = n$. The splitting of $\mathbb{C}^2$ into trajectories of this vector field defines a singular analytic foliation of $\mathbb{C}^2$. Denote by $\mathcal{A}_n$ the space of foliations of $\mathbb{C}^2$ defined by vector fields (1) of degree at most n with coprime P and Q. Two vector fields define the same foliation if they are proportional, hence $\mathcal{A}_n$ is a Zariski open subset of the projective space. $\mathcal{A}_n$ is equipped with a natural topology induced from this projective space.

In 1970-s there appeared several results on the properties of generic foliations from $\mathcal{A}_n$. In particular, Yu. Ilyashenko [Il78] proved that a generic foliation (more precisely, each foliation from some full Lebesgue measure subset of $\mathcal{A}_n$) has an infinite number of limit cycles. Later his theorem was improved by E. Rosales-González, L. Ortiz-Bobadilla and A. Shcherbakov [SRO98], namely they replaced a full-measure set by an open dense subset.

Definition. *Limit cycle* on a leaf L of an analytic foliation is an element $[\gamma]$ of the free homotopy group of L such that the holonomy along (any) its representative γ is non-identical.

Definition. A set of limit cycles of a foliation is called *homologically independent*, if for any leaf L all the cycles located on this leaf are linearly independent in $H_1(L)$.

Theorem ([Il78])**.** *For $n \geqslant 2$, there exists a full-measure subset of $\mathcal{A}_n$, such that each foliation from this subset possesses an infinite number of homologically independent limit cycles.*

Theorem ([SRO98])**.** *For $n \geqslant 3$, there exists an open dense subset of $\mathcal{A}_n$, such that each foliation from this subset possesses an infinite number of homologically independent limit cycles.*

The proof of the first theorem in [Il78] is rather technical; the proof of the second one in [SRO98] contains about 10 pages of cumbersome estimates of integrals along the limit cycles. The constructed sequence of representatives γ_j of required limit cycles $[\gamma_j]$ in both theorems converges to the infinite line.

Our results yield another, less technical proof of these theorems, and our limit cycles are detached from the infinite line. Also, our proof works for $n = 2$ for both types of genericity assumptions.

Main Theorem. *For $n \geqslant 2$, there exist*

- *a full-measure subset $\mathcal{A}_n^{LC1} \subset \mathcal{A}_n$,*

- *a complement to a real-analytic subset $\mathcal{A}_n^{LC2} \subset \mathcal{A}_n$,*

such that each $\mathcal{F} \in \mathcal{A}_n^{LC1} \cup \mathcal{A}_n^{LC2}$ possesses an infinite sequence of limit cycles $[\gamma_j]$ such that:

a) the cycles are homologically independent;

    b) the multipliers of the cycles tend to zero;

    c) the cycles are uniformly bounded, i.e., there exists a ball in $\mathbb{C}^2$ that includes all representatives γ_j;

    d) there exists a cross-section such that γ_j intersect it in a dense subset.

The explicit descriptions of the sets $\mathcal{A}_n^{LC1}$ and $\mathcal{A}_n^{LC2}$ are given below, in Sections "Multiplicative density" and "Unsolvable monodromy group", respectively.

The key genericity assumption for $\mathcal{A}_n^{LC1}$ is that the characteristic numbers of two singular points at infinity generate a dense semi-group in $\mathbb{C}/\mathbb{Z}$. The key genericity assumption for $\mathcal{A}_n^{LC2}$ is that the monodromy group at infinity is unsolvable.

Though the exceptional set in the second part is much thinner, we include the first part for two reasons: $\mathcal{A}_n^{LC2}$ does not include $\mathcal{A}_n^{LC1}$, and the first case is technically easier.

Our construction also yields that the infinite number of limit cycles survives in a neighbourhood of $\mathcal{A}_n^{LC1} \cup \mathcal{A}_n^{LC2}$ in the space $\mathcal{B}_{n+1}$ of foliations of $\mathbb{C}P^2$ that are given by a polynomial vector field of degree at most $n+1$ in any affine chart, see Corollary 6.

2 Preliminaries

2.1 Extension to infinity

Let us extend a polynomial foliation $\mathcal{F} \in \mathcal{A}_n$ given by (1) to $\mathbb{C}P^2$. After changing variables, $u = \frac{1}{x}, v = \frac{y}{x}$, and the time change $d\tau = -u^{n-1}dt$, the vector field takes the form

$$\dot{u} = u\widetilde{P}(u,v)$$
$$\dot{v} = v\widetilde{P}(u,v) - \widetilde{Q}(u,v) \tag{2}$$

where $\widetilde{P}(u,v) = P\left(\frac{1}{u}, \frac{v}{u}\right) u^n$ and $\widetilde{Q}(u,v) = Q\left(\frac{1}{u}, \frac{v}{u}\right) u^n$ are two polynomials of degree at most n.

Since $\dot{u}(0,v) \equiv 0$, the infinite line $\{u = 0\}$ is invariant under this vector field. Denote by $h(v)$ the polynomial $\dot{v}(0,v) = v\widetilde{P}(0,v) - \widetilde{Q}(0,v)$. In a generic (*non-dicritical*) case $h(v) \not\equiv 0$; then (2) has isolated singular points $a_j \in \{u = 0\}$ at the roots of h, and $L_\infty := \{u = 0\} \smallsetminus \{a_1, a_2, \ldots\}$ is a leaf of the extension of $\mathcal{F}$ to $\mathbb{C}P^2$.

Denote by $\mathcal{A}'_n$ the set of foliations $\mathcal{F} \in \mathcal{A}_n$ such that h has $n+1$ distinct roots $a_j, j = 1, \ldots, n+1$. In particular, all these foliations are non-dicritical.

For each j, let λ_j be the ratio of the eigenvalues of the linearization of (2) at a_j (the eigenvalue corresponding to L_∞ is in the denominator). One can show that $\sum \lambda_j = 1$, and this is the only relation on λ_j.

For $\mathcal{F} \in \mathcal{A}'_n$, fix a non-singular point $O \in L_\infty$ and a cross-section S at O given by $v = \text{const}$. Let ΩL_∞ be the loop space of (L_∞, O), i. e., the space of all continuous maps $(S^1, pt) \to (L_\infty, O)$. For a loop $\gamma \in \Omega L_\infty$, denote by $\mathbf{M}_\gamma : (S, O) \to (S, O)$ the monodromy map along γ. It is easy to see that $\mathbf{M}_\gamma$ depends only on the class $[\gamma] \in \pi_1(L_\infty, O)$, and the map $\gamma \mapsto \mathbf{M}_\gamma$ reverses the order of multiplication,

$$\mathbf{M}_{\gamma\gamma'} = \mathbf{M}_{\gamma'} \circ \mathbf{M}_\gamma.$$

The set of all possible monodromy maps $\mathbf{M}_\gamma, \gamma \in \Omega L_\infty$, is called the *monodromy pseudogroup* $G = G(\mathcal{F})$. The word "pseudogroup" means that there is no common domain where all elements of G are defined. However we will follow the tradition and write "monodromy group" instead of "monodromy pseudogroup".

Choose $n + 1$ loops $\gamma_j \in \Omega L_\infty$, $j = 1, 2, \ldots, n + 1$, passing around points a_j, respectively. We suppose that γ_j are simple and intersect only at O. Then the pseudogroup $G(\mathcal{F})$ is generated by monodromy maps $\mathbf{M}_j = \mathbf{M}_{\gamma_j}$. It is easy to see that the multipliers $\mu_j = \mathbf{M}'_j(0)$ are equal to $\exp 2\pi i \lambda_j$.

2.2 Fatou coordinates

The space of germs of analytic parabolic maps $g : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, $z \mapsto z + o(z)$, has a natural filtration by the degree of the leading term of $g(z) - z$. Denote by A_p the set of germs of the form $z \mapsto z + az^{p+1} + o(z^{p+1})$, $a \neq 0$.

In this section we will recall some results on sectorial rectifying charts of parabolic fixed points that will be used in the article. For a more complete exposition, see, e.g., Chapter IV of [IYa07].

We start with describing the formal normal forms for quadratic parabolic germs.

Theorem. *A quadratic parabolic germ $f : z \mapsto z + az^2 + bz^3 + o(z^3)$ is formally conjugated to the time-one flow map f_λ of the vector field $v_\lambda(z) = \frac{z^2}{1 + \lambda z}$, where $\lambda = 1 - \frac{b}{a^2}$. More precisely, there exists a formal series $H(z) = az + \sum_{k=2}^{\infty} h_k z^k$, such that $f_\lambda \circ H = H \circ f$. The series H is uniquely defined modulo a formal composition with a flow map of v_λ.*

Remark. It is easy to see that the map $t_\lambda : z \mapsto -\frac{1}{z} + \lambda \log z$ conjugates f_λ to the map $t \mapsto t + 1$, $t_\lambda(f_\lambda(z)) = t_\lambda(z) + 1$.

We will need the following theorem that describes sectorial rectifying charts for quadratic parabolic germs. Consider the following sectors

$$S_{\alpha,r}^+ = \{\, z \mid |z| < r, |\arg z| < \alpha \,\}, \quad S_{\alpha,r}^- = \{\, z \mid |z| < r, |\arg z - \pi| < \alpha \,\}.$$

Theorem (Sectorial Normalization Theorem). *Let $f : z \mapsto z + az^2 + o(z^2)$ be a quadratic parabolic map, let $H(z) = az + \sum_{k=2}^{\infty} h_k z^k$ be a formal series which conjugates f to its formal normal form f_λ. Then for any $\frac{\pi}{2} < \alpha < \pi$ there exists $r > 0$ and a unique couple of analytic maps $h^\pm : \frac{1}{a} S_{\alpha,r}^\pm \to \mathbb{C}$ with the following properties:*

- *H is an asymptotic series for h^- and h^+: for $N \in \mathbb{N}$, we have $h^\pm(z) = az + \sum_{k=2}^{N} h_k z^k + o(z^N)$ as $z \to 0$ inside $\frac{1}{a} S_{\alpha,r}^\pm$;*

- *$h^\pm$ conjugates f to f_λ: $f_\lambda \circ h^\pm = h^\pm \circ f$.*

Remark. For most parabolic germs f, $h^- \neq h^+$. So, the analytic classification of parabolic germs does not coincide with their formal classification. The analytic classification has functional moduli called *Ecalle–Voronin moduli*, namely the restrictions of $(h^+)^{-1} \circ h^-$ to the sectors $\{\, z \mid |z| < r, \pi - \alpha < \arg z < \alpha \,\}$ and $\{\, z \mid |z| < r, -\alpha < \arg z < \pi + \alpha \,\}$ up to a conjugation by a flow map of v_λ.

Remark. It is easy to check that the image of $h^\pm$ includes a sector of the form $S_{\alpha',r'}^\pm$ for each $\alpha' < \alpha$ and some r', and is included by another sector of the same form. Also, $t_\lambda(S_{\alpha',r'}^-)$ includes a *sector at infinity*:

$$S_{\beta,R}^\infty = \{\, \zeta \mid |\zeta| > R, |\arg \zeta| < \beta \,\} \tag{3}$$

for each $\beta < \alpha'$ and some $R = R(\alpha', r', \lambda, \beta) \gg 1$. Thus the image of $\zeta = t_\lambda \circ h^-$ includes a sector at infinity.

Definition. A *Fatou coordinate* for a parabolic map f in a sector $\frac{1}{a}S^-_{\alpha,r}$ is a coordinate of the form $\zeta = t_\lambda \circ h^-$, where h^- is given by Sectorial Normalization Theorem. A Fatou coordinate ζ conjugates f to the shift $\zeta \mapsto \zeta + 1$ in a domain that includes a sector at infinity (3), and is defined uniquely modulo addition of a complex number.

We shall need the following statement.

Lemma 1. *Let g be a parabolic map of the form $z \mapsto z + az^2 + \dots$. Let ζ be a Fatou chart for g defined in a sector $\frac{1}{a}S^-_{\alpha,r}$. Let S^∞ be the image of a smaller sector $\frac{1}{a}S^-_{\alpha-\varepsilon,r-\varepsilon}$ under ζ. Let $F : \mathbb{C} \to \mathbb{C}$ be an analytic map, $F(0) = 0$, defined in the chart z. Let $\tilde{F} = \zeta \circ F \circ \zeta^{-1}$ be the corresponding map in the chart ζ.*

> *a) If $F(z) = kz + o(z)$, then $\tilde{F}(\zeta) = k^{-1}\zeta + c + o(1)$ as $\zeta \to \infty$ inside S^∞.*

> *b) If $F(z) = z + kz^{p+1} + o(z^{p+1})$, $p \geqslant 1$, then $\tilde{F}(\zeta) = \zeta + (-1)^{p-1}ka^{-p}\frac{1}{\zeta^{p-1}} + o\left(\frac{1}{\zeta^{p-1}}\right)$ in ζ-chart as $\zeta \to \infty$ inside S^∞.*

> *c) If F is a parabolic map, then $\log \tilde{F}'(\zeta) = o(F(\zeta) - \zeta)$ as $\zeta \to \infty$ inside S^∞.*

Proof. Recall that $h(z) - az = O(z^2)$, $z \in \frac{1}{a}S^-_{\alpha,r}$. The Cauchy estimates imply that $h'(z) = a + O(z)$ in $\frac{1}{a}S_{\alpha-\varepsilon,r-\varepsilon}$.

Let us prove b). Note that

$$F(h^{-1}(w)) = h^{-1}(w) + k(h^{-1}(w))^{p+1} + o((h^{-1}(w))^{p+1})$$
$$= h^{-1}(w) + ka^{-p-1}w^{p+1} + o(w^{p+1}),$$

hence

$$(h \circ F \circ h^{-1})(w) = w + \int_{h^{-1}(w)}^{F(h^{-1}(w))} a + O(z)\, dz = w + ka^{-p}w^{p+1} + o(w^{p+1}).$$

Similarly, for $\zeta = t_\lambda(w)$ we have

$$\tilde{F}(\zeta) = \zeta + \int_w^{w+ka^{-p}w^{p+1}+o(w^{p+1})} t'_\lambda(\omega)\, d\omega$$
$$= \zeta + \int_w^{w+ka^{-p}w^{p+1}+o(w^{p+1})} \frac{1}{\omega^2} + \frac{\lambda}{\omega}\, d\omega$$
$$= \zeta + ka^{-p}w^{p-1} + o(w^{p-1})$$
$$= \zeta + (-1)^{p-1}ka^{-p}\frac{1}{\zeta^{p-1}} + o\left(\frac{1}{\zeta^{p-1}}\right)$$

Assertion a) can be proved in the same way. Finally, the last assertion follows from Assertion b) in $\zeta\left(\frac{1}{a}S^-_{\alpha-\frac{\varepsilon}{2},r-\frac{\varepsilon}{2}}\right)$ and Cauchy estimates. $\qquad\square$

2.3 Unsolvability of the monodromy group

In [Shch84], A. Shcherbakov proved that for a generic foliation $\mathcal{F} \in \mathcal{A}_n$, the monodromy group is unsolvable. It turns out that a group of germs $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is unsolvable if and only if it contains a couple of commutators that do not commute with each other. This follows from the next lemma.

Lemma 2. *Let $f(z) = z + az^{p+1} + o(z^{p+1})$ and $g(z) = z + bz^{q+1} + o(z^{q+1})$ be two parabolic germs. Then*

$$[f,g](z) := (f \circ g \circ f^{-1} \circ g^{-1})(z) = z + ab(p-q)z^{p+q+1} + o(z^{p+q+1}).$$

In particular, if $p \neq q$, $a \neq 0$, $b \neq 0$, then $[f,g] \in A_{p+q}$.

Corollary. *If a group G of germs $(\mathbb{C},0) \to (\mathbb{C},0)$ contains two parabolic germs $g_1 \in A_p$, $g_2 \in A_q$ with $p \neq q$, then G is unsolvable.*

Proof. Indeed, none of the commutators $g_3 = [g_1,g_2] \in A_{p+q}$, $g_4 = [g_3,g_2] \in A_{p+2q}$, ... can be the identity map. $\qquad\square$

The main result of this section is the following lemma.

Lemma 3. *There exists an open dense subset $\mathcal{A}_n$ such that for each foliation from this subset*

$$\forall i \neq j \quad [\mathbf{M}_i, \mathbf{M}_j] \in A_1, \quad [\mathbf{M}_i^{-1}, \mathbf{M}_j] \in A_1, \quad [[\mathbf{M}_i, \mathbf{M}_j], [\mathbf{M}_i^{-1}, \mathbf{M}_j]] \in A_3.$$

This lemma is immediately implied by the following two statements.

Lemma 4. *There exists a real analytic subset $\mathcal{E} \subset \mathcal{A}_n$ of positive codimension such that for $\mathcal{F} \notin \mathcal{E}$*

- *all commutators $[\mathbf{M}_i, \mathbf{M}_j]$, $i \neq j$, belong to A_1;*

- *all numbers $\dfrac{S(\mathbf{M}_i)(0)}{\mathbf{M}_i'(0)^2 - 1}$, $i = 1, \ldots, n+1$, are different.*

Here and below $S(f)$ is the Schwarzian derivative of f,

$$S(f)(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

In [Shch84], Shcherbakov proved that for a generic foliation, *at least one* commutator $[\mathbf{M}_i, \mathbf{M}_j]$ belongs to A_1, see Section 6.3 of [Shch06]. But it is easy to extend this result to *all* pairs $i \neq j$ using analytic continuation along loops in $\mathcal{A}_n$ that permute the singular points. The second part is proved in the same article but not explicitly stated, so one needs to go through the proof of Theorem 9 in [Shch06] to verify that the assertion of the corollary after Lemma 5 is the only property of $\mathcal{F}$ used in the proof.

Similar results were achieved in [BLL97, N94].

Lemma. *Consider two hyperbolic germs f, g such that*

- *$f'(0)^2 \neq 1$, $g'(0)^2 \neq 1$;*

- *$[f,g] \in A_1$, i.e., $[f,g]''(0) \neq 0$;*

- *$\dfrac{S(f)(0)}{f'(0)^2 - 1} \neq \dfrac{S(g)(0)}{g'(0)^2 - 1}$.*

Then $[f,g]$ does not commute with $[f^{-1}, g]$; moreover, $[[f,g], [f^{-1}, g]] \in A_3$.

This lemma is motivated by Proposition 7 in [Shch84] (which coincides with the corollary after Lemma 5 in [Shch06]) but provides an explicit pair of commutators that do not commute.

Proof. One can verify that

$$S([f,g])(0) = \left[\left(\frac{S(f)}{(f')^2 - 1} - \frac{S(g)}{(g')^2 - 1}\right)\left(1 - \frac{1}{(f')^2}\right)\left(1 - \frac{1}{(g')^2}\right)\right]_{z=0}$$

thus

$$\frac{S([f^{-1}, g])(0)}{S([f,g])(0)} = -f'(0)^2$$

On the other hand,

$$\frac{[f^{-1}, g]''(0)}{[f,g]''(0)} = -f'(0). \tag{4}$$

Two last equalities prove that

$$\frac{S([f^{-1}, g](0)}{[f^{-1}, g]''(0)} = f'(0)\frac{S([f,g](0)}{[f,g]''(0)}$$

The assertions of the lemma imply that this is not zero, thus

$$\frac{S([f^{-1}, g](0)}{[f^{-1}, g]''(0)} \neq \frac{S([f,g](0)}{[f,g]''(0)}.$$

Finally, expanding $[f,g] \circ [f^{-1}, g] - [f^{-1}, g] \circ [f, g]$ up to the fourth order, one can check that the above inequality is equivalent to

$$([f,g] \circ [f^{-1}, g] - [f^{-1}, g] \circ [f, g])^{(4)}(0) \neq 0,$$

hence $[[f,g], [f^{-1}, g]] \in A_3$. $\qquad\square$

3 Plan of the proof of Main Theorem

We will construct the limit cycles as the lifts of loops in the infinite line. Note that if the monodromy map $\mathbf{M}_{k_1}\mathbf{M}_{k_2}\ldots\mathbf{M}_{k_l} : S \to S$ has a fixed point $p \neq 0$, then the corresponding loop $\gamma = \gamma_{k_l}\gamma_{k_{l-1}}\ldots\gamma_{k_1}$ lifts to a limit cycle c starting from p; the projection of c to the infinite line is γ.

We proceed in two steps. First we construct contracting monodromy maps that satisfy **inclusion**, **contraction** and **covering** assumptions formulated below. This is done in a different way for two types of genericity assumptions, see Section "Construction of contracting maps". Then we use the maps constructed on the first step to obtain limit cycles that satisfy the assertions of Main Theorem. On this step we use no information about the foliation except for existence of maps with prescribed properties, see Section "Construction of limit cycles".

3.1 Step 1: contracting maps

We shall find two topological discs $\Delta^- \subset \Delta^+ \subset S$ in the cross-section, $0 \notin \Delta^+$, an analytic chart ζ in Δ^+ and a tuple of monodromy maps f_j with the following properties. Each f_j is a composition of standard generators $\mathbf{M}_k$ of the monodromy group at infinity. For any splitting of this composition into two parts $f_j = f_j^{(t)} \circ f_j^{(h)}$, we will say that $f_j^{(t)}$ is a *tail* of f_j and $f_j^{(h)}$ is its *head*.

Inclusion: $f_j(\Delta^+) \subset \Delta^+$ for any j.

Contraction: All compositions of the form $f_i^{(t)} \circ f_j^{(h)}$, $f_i^{(h)} \neq \mathrm{id}$, $f_j^{(h)} \neq \mathrm{id}$, contract in $(f_j^{(h)})^{-1} \circ f_i^{(h)}(\Delta^+) \cap \Delta^+$ with respect to the chart ζ. In particular, all f_j contract in Δ^+.

Covering: Images of Δ^- under f_j cover Δ^-.

We will also suppose that the compositions f_j do not contain identical subcompositions, otherwise we remove them. Obviously this does not break any of other requirements on f_j.

3.2 Step 2: limit cycles

Here we use the maps f_j to construct infinitely many homologically independent limit cycles. We will use not a particular construction of f_j, but only the assumptions **inclusion**, **contraction** and **covering**, so the arguments work for both sets A_n^{LC1}, A_n^{LC2}. The main motivation is the following lemma[1].

Lemma 5. *Under assumptions above, for an open $U \subset \Delta^-$ and $\varepsilon > 0$ there exists a word $w = j_1 \ldots j_N$ such that the monodromy map $f_w = f_{j_1} \circ f_{j_2} \circ \ldots \circ f_{j_N}$ satisfies $f_w(\Delta^+) \subset U$ and $|f_w'| < \varepsilon$ in Δ^+.*

Remark. We will use this lemma only for $\varepsilon < 1$. In this case, the map f_w obviously has a fixed point in U. It corresponds to a limit cycle with arbitrarily small multiplier which passes through U.

This lemma enables us to prove assertions b)–d) of Main Theorem. The proof of homological independence is more complicated.

Proof. Take a point $p \in U \subset \Delta^-$. Due to the **covering** assumption, there exists an index j_1 such that $p \in f_{j_1}(\Delta^-)$. Now, take the preimage $f_{j_1}^{-1}(p) \subset \Delta^-$, and repeat the arguments; we obtain a map f_{j_2} such that $p \in f_{j_1}(f_{j_2}(\Delta^-))$.

Repeating the procedure, we get a word $w = j_1 j_2 \ldots j_N$ such that $f_w(\Delta^-) = f_{j_1} \circ f_{j_2} \circ \cdots \circ f_{j_N}(\Delta^-)$ contains p. The diameter of the image $f_w(\Delta^+)$ tends to zero as N tends to infinity, since all maps f_j contract in Δ^+. So, if N is large enough, the fact that $p \in f_w(\Delta^+)$ would imply $f_w(\Delta^+) \subset U$ and $|f_w'| < \varepsilon$ in the whole Δ^+. $\qquad\square$

3.3 A neighborhood in $\mathcal{B}_n$

Note that assumptions **inclusion**, **contraction** and **covering** are robust in the following sense. Considers a foliation $\mathcal{F} \in \mathcal{A}_n$ that possesses a tuple of monodromy maps that satisfy these assumptions. Then there exists a bidisc $D \subset \mathbb{C}^2$ and $\varepsilon > 0$ such that any foliation $\mathcal{F}'$ of D which is ε-close to $\mathcal{F}$ in D possesses monodromy maps that satisfy these assumptions. Since Step 2 relies only on these properties, such foliation $\mathcal{F}'$ satisfies assertions of Main Theorem. In particular, we have the following corollary.

Corollary 6. *Any foliation from some open neighborhood $\mathcal{U}$, $\mathcal{A}_n^{LC1,2} \subset \mathcal{U} \subset \mathcal{B}_{n+1}$, possesses an infinite number of limit cycles satisfying assertions a)–d) of the Main theorem.*

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[1]Some people attribute this statement to Hutchinson [H81], but we failed to find exactly this statement in this article.

4 Construction of contracting maps

4.1 Multiplicative density

We put the following genericity assumptions on the foliation:

- the characteristic numbers of two of the singular points at infinity (say, λ_1 and λ_2) generate a dense subgroup in $\mathbb{C}/\mathbb{Z}$;

- the corresponding monodromy maps $\mathbf{M}_1, \mathbf{M}_2$ do not commute.

Each condition defines a full measure set. For the former condition it is clear, and for the latter one see Lemma 4.

After a holomorphic coordinate change, we may and will assume that the map $\mathbf{M}_1$ is linear. If this map expands, let us replace it with its inverse. Then $\operatorname{Im}\lambda_1 > 0$. Let us pass to the chart $\zeta = \frac{\log z}{2\pi i}$, $\zeta \in \mathbb{C}/\mathbb{Z}$. In this chart, points with large $\operatorname{Im}\zeta$ correspond to points z close to the origin.

Let $\tilde{\mathbf{M}}_1 : \zeta \mapsto \zeta + \lambda_1$ and $\tilde{\mathbf{M}}_2$ be the maps $\mathbf{M}_1$ and $\mathbf{M}_2$ written in the chart ζ. These maps are defined for sufficiently large $\operatorname{Im}\zeta$, and

$$\tilde{\mathbf{M}}_2(\zeta) = \zeta + \lambda_2 + o(1),$$
$$\tilde{\mathbf{M}}_2'(\zeta) = 1 + o(1)$$

as $\operatorname{Im}\zeta \to \infty$. Since $\mathbf{M}_1$ does not commute with $\mathbf{M}_2$, the map $\tilde{\mathbf{M}}_2$ is not linear, hence $\tilde{\mathbf{M}}_2'$ is not identically one. Let Δ^+ be a small closed disc such that either $\tilde{\mathbf{M}}_2$ or its inverse uniformly contracts in Δ^+. Without loss of generality we can and shall assume that it is $\tilde{\mathbf{M}}_2$,

$$\max_{\zeta \in \Delta^+} |\tilde{\mathbf{M}}_2'(\zeta)| < 1. \tag{5}$$

Next, let ζ_0 be the center of Δ^+, put $T = \zeta_0 - \tilde{\mathbf{M}}_2(\zeta_0)$. Note that $\tilde{\mathbf{M}}_2(\Delta^+) + T \Subset \Delta^+$. Choose a much smaller disc $\Delta^- \subset \Delta^+$ with the same center,

$$\operatorname{diam}(\Delta^-) < \operatorname{dist}(\tilde{\mathbf{M}}_2(\partial\Delta^+) + T, \partial\Delta^+). \tag{6}$$

Choose a tuple of vectors $T_j = k_j\lambda_1 + l_j\lambda_2 \in \mathbb{C}/\mathbb{Z}$ such that

$$\Delta^- \subset \bigcup_j (\tilde{\mathbf{M}}_2(\Delta^-) + T_j) \subset \bigcup_j (\tilde{\mathbf{M}}_2(\Delta^+) + T_j) \subset \Delta^+. \tag{7}$$

Due to (6), it is enough to take T_j such that $\tilde{\mathbf{M}}_2(\Delta^-) + T_j$ cover Δ^- and $|T - T_j| < \operatorname{diam}(\Delta^-)$. Due to the density condition, these T_j can be chosen of the form $T_j = k_j\lambda_1 + l_j\lambda_2$.

Now, let us choose f_j so that they approximate the maps $T_j \circ \tilde{\mathbf{M}}_2$ in Δ^+. Note that the map $\tilde{\mathbf{M}}_2^{l_j} \circ \tilde{\mathbf{M}}_1^{k_j}$ approximates the shift $\zeta \mapsto \zeta + T_j$ for large $\operatorname{Im}\zeta$, hence for N large enough, the map $\tilde{\mathbf{M}}_1^{-N} \circ \tilde{\mathbf{M}}_2^{l_j} \circ \tilde{\mathbf{M}}_1^{k_j+N}$ is very close to the shift by T_j in $C^1(\tilde{\mathbf{M}}_2(\Delta^+))$. Therefore, we can take the maps

$$f_j = (\mathbf{M}_1^{-N} \circ \mathbf{M}_2^{l_j} \circ \mathbf{M}_1^{k_j+N}) \circ \mathbf{M}_2.$$

Let $\tilde{f}_j$ be the map f_j written in the chart ζ.

For sufficiently large N, these maps satisfy **inclusion** and **covering** assumptions from the plan of the proof. Obviously, each $\tilde{f}_j$ contracts in Δ^+. Now, we have to prove **contraction**

assumption, i.e. that for N large enough all compositions $\tilde{f}_i^{(t)} \circ \tilde{f}_j^{(h)}$ contract. Recall that $\tilde{f}_j^{(h)} \neq \mathrm{id}$, $\tilde{f}_i^{(t)} \neq \tilde{f}_i$.

Since the map $\tilde{\mathbf{M}}_1^{-N} \circ \tilde{\mathbf{M}}_2^{l_j} \circ \tilde{\mathbf{M}}_1^{k_j+N}$ and its heads approximate shifts in $C^1(\Delta^+)$, for N large enough the derivative of $\tilde{f}_j^{(h)}$ is close to the set $\tilde{\mathbf{M}}_2'(\Delta^+)$. On the other hand, the derivative of $\tilde{f}_i^{(t)}$ on the set $\tilde{f}_i^{(h)}(\Delta^+)$ is arbitrarily close to one. Thus we can make the derivative of the composition $\tilde{f}_i^{(t)} \circ \tilde{f}_j^{(h)}$ on the set $\Delta^+ \cap \left((\tilde{f}_j^{(h)})^{-1} \circ \tilde{f}_i^{(h)}(\Delta^+) \right)$ arbitrarily close to $\tilde{\mathbf{M}}_2'(\Delta^+)$. Now (5) yields **contraction** assumption.

4.2 Unsolvable monodromy group

In this case, the construction is similar, but instead of the logarithmic chart we use the Fatou chart for one of the parabolic monodromy maps, and there are more technical difficulties.

4.2.1 Genericity assumptions and preliminary considerations

Let $\mathcal{A}_n^{LC2} \subset \mathcal{A}_n'$ be the set of polynomial foliations such that

- $g_1 := [\mathbf{M}_1, \mathbf{M}_2] \in A_1$, $g_2 := [\mathbf{M}_1^{-1}, \mathbf{M}_2] \in A_1$;

- $g_3 := [g_2, g_1]$ is not the identity map[2];

- $\mu_1 \notin \mathbb{R}$;

- the numbers 1, μ_1, μ_1^{-1}, μ_2^{-1}, $\mu_1^{-1}\mu_2^{-1}$, $\mu_1\mu_2^{-1}$ are all different.

Due to Lemma 3 and the fact that $\sum \lambda_i = 1$ is the only relation on λ_i, the complement $\mathcal{A}_n \setminus \mathcal{A}_n^{LC2}$ is a real analytic subset of $\mathcal{A}_n$.

Consider a foliation $\mathcal{F}$ from this set. Put $g_4 := [g_3, g_2]$. Due to Lemma 2, $g_4 \neq \mathrm{id}$.

Let ζ be a Fatou chart for g_1 in the negative sector.

> **Convention**
>
> In this Section, tilde above means that a map is written in the chart ζ. In particular, $\tilde{g}_1(\zeta) = \zeta + 1$.

Lemma 2 and Item b) of Lemma 1 imply

$$g_1(z) = z + \frac{g_1''(0)}{2}z^2 + o(z^2) \qquad\qquad \tilde{g}_1(\zeta) = \zeta + 1 + o(1)$$

$$g_2(z) = z + \frac{g_2''(0)}{2}z^2 + o(z^2) \qquad\qquad \tilde{g}_2(\zeta) = \zeta + \frac{g_2''(0)}{g_1''(0)} + o(1)$$

$$g_3(z) = z + az^{p+1} + o(z^{p+1}) \qquad\qquad \tilde{g}_3(\zeta) = \zeta - \frac{(-2)^p a}{g_1''(0)^p}\zeta^{1-p} + o(\zeta^{1-p})$$

$$g_4(z) = z + \frac{a(p-1)g_2''(0)}{2}z^{p+2} + o(z^{p+2}) \qquad \tilde{g}_4(\zeta) = \zeta + \frac{(-2)^p a(p-1)g_2''(0)}{g_1''(0)^{p+1}}\zeta^{-p} + o(\zeta^{-p}),$$

---

[2]Lemma 3 implies that for a generic $\mathcal{F}$ we have $g_3 \in A_3$, but our construction works for a slightly broader set of foliations.

where $a \in \mathbb{C} \smallsetminus \{0\}$. Put $\tilde{a} = -\frac{(-2)^p a}{g_1''(0)^p}$, $\tilde{b} = \frac{(-2)^p a (p-1) g_2''(0)}{g_1''(0)^{p+1}}$. Due to (4), $g_2''(0) = -\mu_1 g_1''(0)$, hence

$$\tilde{g}_1(\zeta) = \zeta + 1 + o(1)$$
$$\tilde{g}_2(\zeta) = \zeta - \mu_1 + o(1)$$
$$\tilde{g}_3(\zeta) = \zeta + \frac{\tilde{a}}{\zeta^{p-1}} + o\left(\frac{1}{\zeta^{p-1}}\right)$$
$$\tilde{g}_4(\zeta) = \zeta + \frac{\tilde{b}}{\zeta^{p}} + o\left(\frac{1}{\zeta^{p}}\right).$$

Since $\frac{\tilde{b}}{\tilde{a}} = -\frac{(p-1)g_2''(0)}{g_1''(0)} = (p-1)\mu_1 \notin \mathbb{R}$, each number $T \in \mathbb{C}$ can be represented as

$$T = \xi(T)\tilde{a} + \eta(T)\tilde{b}. \tag{8}$$

We will construct f_j as compositions of the maps g_1, g_2, g_3, g_4.

4.2.2 Construction of f_j

Let Δ^+ be a small disc that we shall choose later. Now we just say that $\tilde{g}_2^{\pm} \in \{\tilde{g}_2, \tilde{g}_2^{-1}\}$ contracts in Δ^+,

$$\max_{\zeta \in \Delta^+} |(\tilde{g}_2^{\pm})'(\zeta)| = q < 1. \tag{9}$$

Since $\tilde{g}_2(\zeta) - \zeta \to -\mu_1$ as $\zeta \to \infty$, we may and will assume that $|\tilde{g}_2^{\pm}(\zeta) - \zeta| < |\mu_1| + 1$ for $\zeta \in \Delta^+$.
As in the previous case, take a disc $\Delta^- \subset \Delta^+$ and a tuple of vectors $T_j \in \mathbb{C}$ such that

$$\Delta^- \subset \bigcup_j (\tilde{g}_2^{\pm}(\Delta^-) + T_j) \subset \bigcup_j (\tilde{g}_2^{\pm}(\Delta^+) + T_j) \subset \Delta^+. \tag{10}$$

It is easy to see that the right inclusion implies $|T_j + \mu_1| < 2$ or $|T_j - \mu_1| < 2$, hence $|T_j| < |\mu_1| + 2$. Put $\xi_j = \xi(T_j)$, $\eta_j = \eta(T_j)$, see (8).
Similarly to the previous case, we will choose the compositions $\tilde{f}_j$ so that they will approximate the maps $\zeta \mapsto \tilde{g}_2^{\pm}(\zeta) + T_j$ in $C^1(\Delta^+)$. It turns out that we can use the compositions $\tilde{f}_j = \tilde{F}_j \circ \tilde{g}_2^{\pm}$, where

$$\tilde{F}_j := \tilde{g}_1^{-N} \circ \tilde{g}_3^{k_j} \circ \tilde{g}_4^{l_j} \circ \tilde{g}_1^{N}, \quad k_j = [N^{p-1}\xi_j], \quad l_j = [N^p \eta_j]. \tag{11}$$

Here N is a large number that we will choose later.
Let us prove that $\tilde{F}_j$ approximate the translations $\zeta \mapsto \zeta + T_j$ in $C^1(\tilde{g}_2^{\pm}(\Delta^+))$.

Lemma 7. *For N large enough, each head $\tilde{F}_j^{(h)}$ such that $\left(F_j^{(h)}\right)'(0) = 1$ is close to a translation $\zeta \mapsto \zeta + T$ in $C^1(\tilde{g}_2^{\pm}(\Delta^+))$. Moreover, $\operatorname{Re} T > -|\operatorname{Re} \mu_1| - 2$ and $|\operatorname{Im} T|$ is bounded by a number that does not depend on Δ^+.*
In particular,

    a) $\tilde{g}_1^n$ is the translation by n;

    b) $\tilde{g}_4^n \circ \tilde{g}_1^N$, $|n| \leqslant |k_j|$ is close to the translation by $N + \frac{n\tilde{b}}{N^p}$;

    c) $\tilde{g}_3^n \circ \tilde{g}_4^{l_j} \circ \tilde{g}_1^N$, $|n| \leqslant |k_j|$ is close to the translation by $N + \frac{n\tilde{a}}{N^{p-1}} + \tilde{b}\eta_j$;

    d) $\tilde{g}_1^{-n} \circ \tilde{g}_3^{k_j} \circ \tilde{g}_4^{l_j} \circ \tilde{g}_1^N$, $0 \leqslant n \leqslant N$, is close to the translation by $T_j + N - n$.

Proof. We shall prove this lemma only for $\xi_j > 0$ and $\eta_j > 0$. For other cases, it is enough to replace $\tilde{g}_3$ and (or) $\tilde{g}_4$ by its inverse map.

Let us prove the assertions of the lemma for $\tilde{F}_j^{(h)} = \tilde{g}_4^{l_j} \circ \tilde{g}_1^N$. Recall that $\tilde{g}_4(\zeta) - \zeta = \tilde{b}\zeta^{-p} + o(\zeta^{-p})$, hence

$$\tilde{F}_j^{(h)}(\zeta) = \tilde{g}_4^{l_j}(\zeta + N) = \zeta + N + \frac{\tilde{b}l_j}{(\zeta + N)^p} + o(1) = \zeta + N + \frac{\tilde{b}[N^p \eta_j]}{N^p} + o(1) = \zeta + N + \tilde{b}\eta_j + o(1)$$

as $N \to \infty$, $\zeta \in \tilde{g}_2^{\pm}(\Delta^+)$. Therefore, $\tilde{F}_j^{(h)}$ is C^0-close to the translation by $N + \tilde{b}\eta_j$.

Let us prove that the derivative of $\tilde{F}_j^{(h)}$ is close to one. Due to Item c) of Lemma 1, $\log \tilde{g}_4'(\zeta) = o(\tilde{g}_4(\zeta) - \zeta) = o(\zeta^{-p})$ as $\zeta \to \infty$, hence

$$\log \left(\tilde{F}_j^{(h)} \right)'(\zeta) = \sum_{k=0}^{l_j - 1} \log \tilde{g}_4'(\tilde{g}_4^k(\zeta + N)) \leqslant N^p \eta_j \log \tilde{g}_4'(N + O(1)) = o(1)$$

as $N \to \infty$, $\zeta \in \Delta^+$. Thus

$$\left(\tilde{F}_j^{(h)} \right)'(\zeta) = 1 + o(1).$$

Finally, in this case $\tilde{F}_j^{(h)}$ is C^1-close to the translation by $N + \tilde{b}\eta_j$.

All particular cases listed in the statement of the lemma can be proved in the same way. Also, the estimate $|T_j| < |\mu_1| + 2$ yields a uniform estimate on the imaginary parts of the translation vectors.

Consider a head $\tilde{F}_j^{(h)}$, $\left(F_j^{(h)} \right)'(0) = 1$, not listed explicitly in the statement of the lemma. Since g_1 and g_2 have no heads g with $g'(0) = 1$, $\tilde{F}_j^{(h)}$ differs from a head of type b) or c) by a composition with a head $\tilde{g}$ either of $\tilde{g}_3^{\pm 1}$, or of $\tilde{g}_4^{\pm 1}$ such that $g'(0) = 1$. Since $\tilde{g}$ is applied to points ζ with $\operatorname{Re} \zeta = N + O(1)$, it can be made arbitrarily C^1-close to its "translational part" $\zeta \mapsto \zeta + T$, $T = \lim_{\zeta \to \infty} \tilde{g}(\zeta) - \zeta$. Thus $\tilde{F}_j^{(h)}$ is close to a translation $\zeta \mapsto \zeta + T'$ with bounded $\operatorname{Im} T'$ as well. □

4.2.3 Choice of Δ^+

The construction relies on the following simple observation.

Lemma 8. *For a collection of hyperbolic maps $F_j : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, $F_j'(0) \neq 1$, there exists an arbitrarily thick strip*

$$U = \{ \zeta \mid \operatorname{Re} \zeta > R, n_- < \operatorname{Im} \zeta < n_+ \}, \quad n_+ - n_- > C, \tag{12}$$

such that U does not overlap its images under $\tilde{F}_j$.

Proof. Recall that $\tilde{F}_j(\zeta) = k_j \zeta + b_j + o(1)$, see Lemma 1. For a map F_j with $k_j \in \mathbb{R}$, the affine term $\zeta \mapsto k_j \zeta + b_j$ of $\tilde{F}$ has invariant horizontal line $\operatorname{Im} \zeta = y_j := \frac{\operatorname{Im} b_j}{1 - k_j}$, and for $|\operatorname{Im} \zeta - y_j| > \frac{C}{|k-1|}$ we have $|\operatorname{Im} \zeta - \operatorname{Im}(k\zeta + b)| > C$. Consider a strip U such that $|\operatorname{Im} \zeta - y_j| > \frac{C}{|k_j - 1|}$ whenever $k_j \in \mathbb{R}$. Clearly, for R large enough, all maps $\tilde{F}_j$ will be close enough to their respective affine terms, hence $\tilde{F}_j(U) \cap U = \varnothing$. Finally, we enlarge R so that the assertion is satisfied for the maps $\tilde{F}_j$ with $k_j \notin \mathbb{R}$. □

Let C_1 be the estimate on $|\operatorname{Im} T|$ from Lemma 7; put $C_2 = \max(C_1, |\operatorname{Re}\mu_1| + 2)$. Fix a strip (12), $C = 2C_2 + |\operatorname{Im}\mu_1|$, such that

$$\left(f_i^{(h)}\right)'(0) \neq \left(f_j^{(h)}\right)'(0) \quad \Rightarrow \quad \left(f_i^{(h)}\right)^{-1} \circ \left(f_j^{(h)}\right)(U) \cap U = \varnothing. \tag{13}$$

Recall that $\tilde{g}_2(\zeta) = \zeta - \mu_1 + o(1)$. Hence there exists a small disc $\Delta \subset U$ such that the distance between $\mathbb{C} \setminus U$ and $\Delta \cup \tilde{g}_2(\Delta)$ is greater than C_2, and $|\tilde{g}_2(\zeta) - \zeta + \mu_1| < 1$ for $\zeta \in \Delta$. Shrinking Δ if necessary, we may and will assume that $\forall \zeta \in \Delta$ we have $|\tilde{g}_2'(\zeta)| \neq 1$. If $|\tilde{g}_2'(\zeta)| < 1$ in Δ, then we put $\Delta^+ = \Delta$, $g_2^{\pm} = g_2$, otherwise we put $\Delta^+ = \tilde{g}_2(\Delta)$ and $g_2^{\pm} = g_2^{-1}$. Then $\tilde{g}_2^{\pm}$ contracts in Δ^+, see (9).

Finally, since the distance between $\mathbb{C} \setminus U$ and $\Delta \cup \tilde{g}_2(\Delta)$ is greater than C_2, Lemma 7 implies that for N large enough, for each head $\tilde{F}_j^{(h)}$ of $\tilde{F}_j$

$$\left(F_j^{(h)}\right)'(0) = 1 \quad \Rightarrow \quad \tilde{F}_j^{(h)}(\Delta^+) \subset U, \qquad \left|\left(\tilde{F}_j^{(h)}\right)'(\zeta)\right| < \frac{1}{\sqrt{q}}, \qquad \left|\left(\tilde{F}_j^{(t)}\right)'(\zeta)\right| < \frac{1}{\sqrt{q}} \tag{14}$$

for $\zeta \in \tilde{g}_2^{\pm}(\Delta^+)$, where q is given by (9).

4.2.4 Proof of the assumptions

Let us prove that for N large enough, the compositions $\tilde{f}_j$ satisfy the assumptions listed in the plan of the proof. For assumptions **inclusion** and **covering**, this immediately follows from Lemma 7 and the definition of T_j.

Let us prove that (13) and (14) imply the **contraction** property. Consider a composition of the form $\tilde{f}_i^{(t)} \circ \tilde{f}_j^{(h)}$. Recall that $\tilde{f}_i$ and $\tilde{f}_j$ are compositions of the commutators $\tilde{g}_1^{\pm 1}$ and $\tilde{g}_2^{\pm 1}$. Therefore, we can rewrite $\tilde{f}_i^{(h)}$, $\tilde{f}_j^{(h)}$ and the corresponding tails as

$$\tilde{f}_i^{(h)} = \left(\tilde{g}_k^{\pm 1}\right)^{(h)} \circ \tilde{f}_i^{(ph)} \qquad\qquad \tilde{f}_i^{(t)} = \tilde{f}_i^{(pt)} \circ \left(\tilde{g}_k^{\pm 1}\right)^{(t)},$$
$$\tilde{f}_j^{(h)} = \left(\tilde{g}_l^{\pm 1}\right)^{(h)} \circ \tilde{f}_j^{(ph)}, \qquad\qquad \tilde{f}_j^{(t)} = \tilde{f}_j^{(pt)} \circ \left(\tilde{g}_k^{\pm 1}\right)^{(t)},$$

where

- $f_i^{(ph)}, f_i^{(pt)}, f_j^{(ph)}, f_j^{(pt)}$ are compositions of $g_1^{\pm 1}$ and $g_2^{\pm 1}$;

- $\{k, l\} \subset \{1, 2\}$;

- $\left(g_k^{\pm 1}\right)^{(h)}$ and $\left(g_l^{\pm 1}\right)^{(h)}$ may be empty but may not coincide with $g_k^{\pm 1}$ or $g_l^{\pm 1}$.

If the maps $\left(g_k^{\pm 1}\right)^{(h)}$ and $\left(g_l^{\pm 1}\right)^{(h)}$ have different multipliers, then (13) and (14) imply that $(f_j^{(h)})^{-1} \circ f_i^{(h)}(\Delta^+) \cap \Delta^+ = \varnothing$.

Next, suppose that $\left(g_k^{\pm 1}\right)^{(h)}$ and $\left(g_l^{\pm 1}\right)^{(h)}$ have equal multipliers. It is easy to check that our assumption on μ_1, μ_2 implies that in this case $\left(g_k^{\pm 1}\right)^{(t)} \circ \left(g_l^{\pm 1}\right)^{(h)}$ is one of the maps id, $g_k^{\pm 1}$, $g_l^{\pm 1}$. In the first case, we just eliminate the middle part from

$$f_i^{(t)} \circ f_j^{(h)} = \tilde{f}_i^{(pt)} \circ \left[\left(\tilde{g}_k^{\pm 1}\right)^{(t)} \circ \left(\tilde{g}_l^{\pm 1}\right)^{(h)}\right] \circ \tilde{f}_j^{(ph)},$$

and in the two latter cases we can regard the middle part either as a part of $f_i^{(t)}$, or as a part of $f_j^{(h)}$. Hence we can assume that both $f_i^{(t)}$ and $f_j^{(h)}$ are parabolic maps.

Finally, due to (14), for parabolic $f_i^{(t)}$ and $f_j^{(h)}$ we have

$$\left| \left(f_i^{(t)} \circ f_j^{(h)} \right)' (\zeta) \right| < q \times \frac{1}{\sqrt{q}} \times \frac{1}{\sqrt{q}} = 1.$$

Hence, the maps f_j satisfy the **contraction** requirement.

5 Construction of limit cycles

Consider a polynomial foliation $\mathcal{F} \in \mathcal{A}'_n$. Suppose that there exist domains $\Delta^- \subset \Delta^+$, a chart ζ and a tuple of monodromy maps f_j that satisfy the assumptions listed in the plan of the proof. In this section we shall show that such foliation satisfies the assertions of Main Theorem. The proof is based on the following simple observation.

Lemma 9. *Suppose that a collection of limit cycles c_j satisfies the following:*

   a) all cycles c_j are simple, i.e., have no self-intersections;

   b) their multipliers $\mu(c_j)$ satisfy $0 < |\mu(c_j)| < |\mu(c_1) \cdots \mu(c_{j-1})|$;

   c) $c_i \cap c_j = \varnothing$ for $i \neq j$.

   Then these cycles are homologically independent.

Proof. Since all these cycles are simple and do not intersect each other, a possible dependency has the form $\pm[c_{i_1}] \pm [c_{i_2}] \pm \ldots \pm [c_{i_s}] = 0$, $i_1 < i_2 < \ldots < i_s$. However such dependence implies the equality on multipliers, $\mu(c_{i_1})^{\pm 1} \mu(c_{i_2})^{\pm 1} \ldots \mu(c_{i_{s-1}})^{\pm 1} = \mu(c_{i_s})$, which is impossible due to the inequality

$$|\mu(c_{i_s})| < |\mu(c_1) \ldots \mu(c_{i_s - 1})| \leqslant |\mu(c_{i_1}) \ldots \mu(c_{i_{s-1}})| \leqslant |\mu(c_{i_1})^{\pm 1} \mu(c_{i_2})^{\pm 1} \ldots \mu(c_{i_{s-1}})^{\pm 1}|.$$

$\square$

Remark. In earlier papers [Il78, SRO98] the authors used similar arguments, but they estimated $\int_{c_j} x \, dy - y \, dx$ instead of multipliers. This led to much more complicated computations.

As we mentioned above, Lemma 5 enables us to construct cycles with arbitrarily small multipliers, but these cycles may be neither simple, nor disjoint. The following two lemmas fill these gaps.

Lemma 10. *Let $D \subset \mathbb{R}^n$ be a closed disc, $g_1, g_2 : D \to D$ be two injective continuous maps such that $g_1(D) \cap g_2(D) = \varnothing$, $\Sigma \subset D$ be a finite subset. Then for m large enough there exists a periodic orbit*

$$p_0, p_1, \ldots, p_m = p_0, p_{i+1} \in \{ g_1(p_i), g_2(p_i) \} \tag{15}$$

that never meets Σ.

Lemma 11. *Given*

   • *an open subset $U \Subset \Delta^-$;*

   • *two maps $g_1 = f_{i_1} \circ \ldots \circ f_{i_s}$, $g_2 = f_{j_1} \circ \ldots \circ f_{j_r}$, $g_i : \Delta^+ \to U$ with disjoint images;*

   • *a finite set $\Sigma \subset S$;*

- *a positive number ε,*

there exists a finite set $\Sigma' \subset \Delta^+$ such that the following holds. Suppose that a periodic orbit (15) never visits Σ'. Since p_0 is a fixed point of some monodromy map, it corresponds to a cycle c. Let c' be its simple subcycle. Then c' visits U, never visits Σ, and the modulus of its multiplier is less than ε.

Let us deduce Main Theorem from these two lemmas.

Proof of Main Theorem. Fix a sequence of points x_k in the interior of Δ^- dense in Δ^-. Let U_k be the intersection of Δ^- with the $(1/k)$-neighborhood of x_k. Now we construct the sequence c_j by induction. Suppose that simple homologically independent cycles $c_1, c_2, \ldots, c_{k-1}$ are already constructed and have multipliers $\mu(c_j)$, $|\mu(c_j)| < 1$. Put $\Sigma = \bigcup_j c_j \cap S$, $\varepsilon = |\mu(c_1)\mu(c_2)\cdots\mu(c_{k-1})|$.

Take two disjoint domains $V_1, V_2 \subset U_k$. Due to Lemma 5, there exist two contracting compositions $g_1 : \Delta^+ \to V_1$ and $g_2 : \Delta^+ \to V_2$. According to the previous two lemmas, there exists a simple cycle c_k with multiplier less than ε that intersects U_k but does not visit Σ. Note that this cycle, as well as all previous ones, projects to the curve of the form $\gamma_{l_1} \gamma_{l_2} \ldots \gamma_{l_r}$ on the infinite line, and $\gamma_i \cap \gamma_j = \{O\}$ for $i \neq j$. Thus if $c_i \cap c_k \neq \varnothing$ for some $i < k$, then $c_k \cap c_i \cap S \neq \varnothing$, hence $c_k \cap \Sigma \neq \varnothing$ which contradicts the choice of c_k. Due to Lemma 9, the cycles $c_1, \ldots, c_k$ are homologically independent. □

Now let us prove the lemmas formulated above.

Proof of Lemma 10. Fix a large number m. Due to Brouwer Theorem, for each word $w = w_1 \ldots w_m$, $w_i \in \{1, 2\}$, the corresponding map

$$g_w = g_{w_1} \circ \ldots \circ g_{w_m} : D \to D$$

has a fixed point. Our goal is to find a word w such that the corresponding periodic orbit will never visit Σ. Since $g_1(D) \cap g_2(D) = \varnothing$, the images of 2^m maps g_w, $|w| = m$, are pairwise disjoint. Hence, given a point $p \in \Sigma$, there is at most one word w such that $g_w(p) = p$. Their cyclic shifts are the only words w such that the corresponding periodic orbit visits Σ, thus there are at most $|\Sigma| \cdot m$ of them. Clearly, for m large enough we have $|\Sigma| \cdot m < 2^m$, hence there exists a periodic orbit of length m that never visits Σ. □

Proof of Lemma 11. Consider a composition $g_w = g_{w_1} \circ \ldots \circ g_{w_s} = \mathbf{M}_{i_1} \circ \ldots \circ \mathbf{M}_{i_k}$, the corresponding periodic orbit (15) and the corresponding limit cycle c. If (15) does not visit the finite set $\Sigma_1 = \bigcup_{g_i^{(h)}} \left(g_i^{(h)}\right)^{-1}(\Sigma)$, then $c \cap \Sigma = \varnothing$.

Subcycles of c correspond to representations $g_w = g^{(t)} \circ g^{(m)} \circ g^{(h)}$ with non-empty $g^{(m)}$ such that $g^{(h)}(p_0)$ is a fixed point of $g^{(m)}$.

Let us prove that sufficiently long compositions $g^{(m)}$ correspond to cycles c' that satisfy the assertions of the lemma, and fixed points of "short" compositions can be avoided by avoiding a finite set Σ_2.

If $g^{(m)}$ is not a subcomposition of one of f_j, then it can be represented in the form

$$g^{(m)} = f_{s_1}^{(h)} \circ f_{s_2} \circ \ldots \circ f_{s_{k-1}} \circ f_{s_k}^{(t)}. \tag{16}$$

Recall that $f_{s_k}^{(t)} \circ f_{s_1}^{(h)}$ contracts in the chart ζ due to **contraction** property, thus $(f_{s_1}^{(h)})^{-1} \circ g^{(m)} \circ f_{s_1}^{(h)}$ contracts, and we have

$$\mu(c') \leqslant \left(\max_{\zeta \in \Delta^+} f_i'(\zeta)\right)^{k-2}. \tag{17}$$

Let $\mathrm{len}(\cdot)$ be the length of a composition of $\mathbf{M}_j$. For sufficiently large L, $\mathrm{len}(g^{(m)}) \geqslant L$ implies that the subcycle c' corresponding to $g^{(m)}$ satisfies the assertions of the lemma. Indeed, the multiplier of c' can be made arbitrarily small due to (17); for any $L > \max(\mathrm{len}\, g_1, \mathrm{len}\, g_2)$, the corresponding c' visits U because it contains a point of the form $g_{w_{i+1}} \circ \ldots \circ g_{w_s}(p_0) \in U$.

Now, it is sufficient to avoid fixed points of "short" compositions $g^{(m)}$, $\mathrm{len}(g^{(m)}) < L$. Let us prove that none of the compositions $g^{(m)}$ are identical in $g^{(h)}(\Delta^+)$. Suppose the contrary. We have eliminated all identical subcompositions from f_j, so $g^{(m)}$ cannot be a subcomposition of some f_j. Thus $g^{(m)}$ has the form (16), and (17) yields that $g^{(m)}$ contracts. Hence $g^{(m)} \neq \mathrm{id}$.

Therefore, each composition $g^{(m)}$ has only finitely many fixed points in $g_{w_j}^{(h)}(\Delta^+)$, where $g_{w_j}^{(h)}(\Delta^+)$ is defined by $g^{(h)} = g_{w_j}^{(h)} \circ g_{w_{j+1}} \circ \ldots \circ g_{w_s}$. In order to guarantee that a subcycle c' corresponds to a long composition $g^{(m)}$, $\mathrm{len}\, g^{(m)} > L$, it is sufficient to require that the periodic orbit (15) avoids the finite set

$$\Sigma_2 = \{ (g_{w_j}^{(h)})^{-1} \mathrm{Fix}\, g^{(m)} \mid \mathrm{len}(g^{(m)}) < L \}.$$

The required exceptional set is $\Sigma' = \Sigma_1 \cup \Sigma_2$. $\qquad\qquad\square$

6 Acknowledgements

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