IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Dispersionless 2D Toda hierarchy, Hurwitz numbers and Riemann theorem

This content has been downloaded from IOPscience. Please scroll down to see the full text.

2016 J. Phys.: Conf. Ser. 670 012036

(http://iopscience.iop.org/1742-6596/670/1/012036)

View the table of contents for this issue, or go to the journal homepage for more

Download details:

IP Address: 128.68.41.10
This content was downloaded on 10/02/2016 at 17:17

Please note that terms and conditions apply.

Journal of Physics: Conference Series **670** (2016) 012036 doi:10.1088/1742-6596/670/1/012036

Dispersionless 2D Toda hierarchy, Hurwitz numbers and Riemann theorem

Sergey M.Natanzon

National Research University Higher School of Economics, 20 Myasnitskaya street, Moscow 101000, Russia
Institute for Theoretical and Experimental Physics, 25 Bolshaya Cheremushkinskaya street, Moscow 117218, Russia
Laboratory of Quantum Topology, Chelyabinsk State University, 129 Brat'ev Kashirinykh street, Chelyabinsk 454001, Russia

E-mail: natanzons@mail.ru

Abstract. We describe all formal symmetric solutions of dispersionless 2D Toda hierarchy. This classification we use for solving of two classical problems: 1) The calculation of conformal mapping of an arbitrary simply connected domain to the standard disk; 2) Calculation of 2-Hurwitz numbers of genus 0.

Explain from the beginning of these 2 problems are reduced to integrable systems.

Calculation of conformal mapping

Let Q be a simply connected domain on the Riemann sphere without two points. According to Riemann theorem, there exists a biholomorphic function into a standard disk

$$w_Q(z) : Q \to \Lambda.$$

The Riemann theorem does not give algorithm for calculation of the function. But this function is very important for applications in gas dynamics, hydrodynamics, the oil industry ets.

After conformal transformation of the Riemann sphere, we may assume that

$$0 \notin Q, \Lambda \ni \infty.$$

Let $\mathcal{H}$ be the set of all such Q with smooth boundary $\gamma = \partial \mathrm{Q}$. The set of moments

$$t_0(Q) = \frac{1}{\pi} \iint\limits_{\overline{\mathbb{C}} \setminus Q} dx\, dy \quad t_k(Q) = -\frac{1}{\pi k} \iint\limits_{Q} z^{-k}\, dx\, dy\,, \quad k \geq 1$$

forms local coordinates $\mathbf{t}(Q) = (t_1(Q), t_2(Q), \ldots,)$ on $\mathcal{H}$ (1992, P.Etingof, A.Varchenko [1]). Moreover, $\mathbf{t}(Q)$ are global coordinates for stars domains Q (1938, P.S.Novikov [2]). Put $\bar{\mathbf{t}}(Q) = (\bar{t}_1(Q), \bar{t}_2(Q)\ , \ldots,)$.

The conformal function $w_Q(z) : Q \to \Lambda$ is single if

$$w_Q(\infty) = \infty, \quad \text{and} \quad w'_Q(\infty) \quad \text{is real positive.}$$

Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.
Published under licence by IOP Publishing Ltd

P.Wiegmann and A.Zabrodin (2000, [3]) proved, that there is a not dependent from Q **symmetric solution of dispersionless 2D Toda hierarchy** $F_{con}(t_0, \mathbf{t}, \bar{\mathbf{t}})$, such that

$$w_Q(z) = z \exp\left(\Big(-\frac{1}{2}\frac{\partial^2}{\partial t_0^2} - \frac{\partial}{\partial t_0}\sum_{k\geq 1}\frac{z^{-k}}{k}\frac{\partial}{\partial t_k}\Big)F_{con}\right)(t_0(Q), \mathbf{t}(Q), \bar{\mathbf{t}}(Q)).$$

Thus, in order to find a conformal mapping for all Q we have to find the universal function $F_{con}(t_0, \mathbf{t}, \bar{\mathbf{t}})$.

Hurwitz numbers

Hurwitz numbers are appeared in the 19th century. They describe a numbers of coverings of surfaces. In recent years they have been actively used in moduli spaces of algebraic curves and mathematical physics. We consider only 2-Hurwitz numbers of genus 0. That is, the number of different rational functions of general position with fixed types divisor of poles and zeros.

In more detail. Fix Young diagrams $\Delta = [\mu_1, \dots, \mu_\ell]$, $\bar{\Delta} = [\bar{\mu}_1, \dots, \bar{\mu}_{\bar{\ell}}]$ of degrees $|\Delta| = |\bar{\Delta}| = d$ and different points $z_1, \dots, z_k \in \mathbb{C} \setminus 0$, where $k = \ell + \bar{\ell} - 2$. Let us consider rational functions $\varphi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of degree d with degrees of its zeros $\Delta = [\mu_1, \dots, \mu_\ell]$, degrees of its poles $\bar{\Delta} = [\bar{\mu}_1, \dots, \bar{\mu}_{\bar{\ell}}]$ and simple critical values $z_1, \dots, z_k$.

Functions φ and $\tilde{\varphi}$ we consider as equivalent if $\tilde{\varphi}(z) = \varphi(hz)$, where h is an automorphism of Riemann sphere. The number $H_0(\Delta, \bar{\Delta})$ of equivalence classes is finite and does not depend on $z_1, \dots, z_k$. For $k > 0$ the number $H_0(\Delta, \bar{\Delta})$ is called a *double Hurwitz number* of genus 0. For $k = 0$ the double Hurwitz number is $\frac{1}{d}$.

Let us look the generating function

$$F(t_0, \mathbf{t}, \bar{\mathbf{t}}) = \frac{t_0^3}{6} + \sum_{|\Delta|=|\bar{\Delta}|}\frac{e^{|\Delta|t_0}\,H_0(\Delta, \bar{\Delta})}{(\ell(\Delta)+\ell(\bar{\Delta})-2)!}\prod_{i=1}^{\ell(\Delta)}\mu_i t_{\mu_i}\prod_{i=1}^{\ell(\bar{\Delta})}\bar{\mu}_i\bar{t}_{\bar{\mu}_i},$$

where $t = \{t_1, t_2, \dots\}$, $\bar{t} = \{\bar{t}_1, \bar{t}_2, \dots\}$, $\Delta = [\mu_1, \dots, \mu_{\ell(\Delta)}]$, $\bar{\Delta} = [\bar{\mu}_1, \dots, \bar{\mu}_{\ell(\bar{\Delta})}]$.

A.Okounkov proved (2000, [4]), that $F(t_0, \mathbf{t}, \bar{\mathbf{t}})$ is a **symmetric solution of dispersionless 2D Toda hierarchy.**

Dispersionless 2D Toda hierarchy

Dispersionless 2D Toda hierarchy was defined by K.Takasaki and T.Takebe (1994, [5]) for application in mathematical physics. In the future, its applications have been found in the topology, complex analysis, and algebraic geometry.

Dispersionless 2D Toda hierarchy are equations on a function $F = F(t_0, \mathbf{t}, \bar{\mathbf{t}})$ from a variable t_0 and two infinite sets of variables , $\mathbf{t} = \{t_1, t_2, \ldots\}$, $\bar{\mathbf{t}} = \{\bar{t}_1, \bar{t}_2, \ldots\}$. To determine the hierarchy is convenient to introduce the auxiliary formal complex variables z, $\bar{z}$, ξ, $\bar{\xi}$.

Let us put

$$\partial_0 = \partial/\partial t_0, \quad \partial_k = \partial/\partial t_k, \quad \bar{\partial}_k = \partial/\partial \bar{t}_k$$

and

$$D(z) = \sum_{k\geq 1}\frac{z^{-k}}{k}\,\partial_k\,, \qquad \bar{D}(\bar{z}) = \sum_{k\geq 1}\frac{\bar{z}^{-k}}{k}\,\bar{\partial}_k\,,$$

The hierarchy is presented by equations

$$(z-\xi)\exp\bigl(D(z)D(\xi)F\bigr) = z\exp\bigl(-\partial_0 D(z)F\bigr) - \xi\exp\bigl(-\partial_0 D(\xi)F\bigr), \tag{1}$$

$$(\bar z-\bar\xi)\exp\bigl(\bar D(\bar z)\bar D(\bar\xi)F\bigr) = \bar z\exp\bigl(-\partial_0 \bar D(\bar z)F\bigr) - \bar\xi\exp\bigl(-\partial_0 \bar D(\bar\xi)F\bigr), \tag{2}$$

$$1-\exp\bigl(-D(z)\bar D(\bar\xi)F\bigr) = \frac{1}{z\bar\xi}\exp\bigl(\partial_0(\partial_0 + D(z) + \bar D(\bar\xi))F\bigr). \tag{3}$$

Differential equations arise, if we expand the right-hand and left-hand sides in the formal Laurent series and equate the coefficients of like powers.

The coefficient for $z^{-i}\xi^{-j}$ in (1) for positive i,j gives, for example an equation

$$\partial_i\partial_j F = \sum_{m=1}^{\infty}\sum_{\substack{s_1+\dots+s_m=\\ i+j;\ s_k>1}} \widetilde{P}_{ij}(s_1,\dots,s_m)\partial_1\partial_{s_1-1}F\dots\partial_1\partial_{s_m-1}F.$$

Formal solutions

We consider *formal solutions* i.e. solutions in form of formal Taylor series

$$F(t_0,\mathbf{t},\bar{\mathbf{t}}) =$$

$$\sum_{\substack{\mu_1\ge\mu_2\ge\dots\ge\mu_\ell>0\\ \bar\mu_1\ge\bar\mu_2\ge\dots\ge\bar\mu_{\bar\ell}>0}} F(\mu_1,\mu_2,\dots,\mu_\ell|\bar\mu_1,\bar\mu_2,\dots,\bar\mu_{\bar\ell}|t_0)t_{\mu_1}t_{\mu_2}\dots t_{\mu_k}\bar t_{\bar\mu_1}\bar t_{\bar\mu_2}\dots\bar t_{\mu_{\bar\ell}}.$$

The indexes $\mu_1\ge\mu_2\ge\dots\ge\mu_\ell>0$ and $\bar\mu_1\ge\bar\mu_2\ge\dots\ge\bar\mu_{\bar\ell}>0$ form Young diagrammes Δ and $\bar\Delta$.

Denote by $t_\Delta = t_{\mu_1}t_{\mu_2}\dots t_{\mu_k}$, and $t_{\bar\Delta} = \bar t_{\bar\mu_1}\bar t_{\bar\mu_2}\dots\bar t_{\mu_{\bar\ell}}$. Then

$$F(t_0,\mathbf{t},\bar{\mathbf{t}}) = F(\emptyset|\emptyset|t_0) + \sum_{\Delta} F(\Delta|\emptyset|t_0)t_\Delta + \sum_{\bar\Delta} F(\emptyset|\bar\Delta|t_0)\bar t_{\bar\Delta}$$

$$+\sum_{\Delta,\bar\Delta} F(\Delta|\bar\Delta|t_0)\, t_\Delta\bar t_{\bar\Delta}.$$

Main Theorem

We say that a function $F(t_0,\mathbf{t},\bar{\mathbf{t}})$ is *symmetric* if

$$\partial_k F(t_0,0,0) = \bar\partial_k F(t_0,0,0) = 0 \quad \text{for} \quad k>0.$$

Theorem 1 (2015, S.Natanzon, A.Zabrodin [6]). Any formal symmetric solution of 2D dispersionless Toda hierarchy F is defined by any formal function $\Phi(t_0)$ and it is equal

$$F = \Phi(t_0) + \sum_{i>0}^{\infty} i f^i t_i \bar t_i +$$

$$\sum_{|\Delta|=|\bar\Delta|}\ \sum_{\substack{s_1+\dots+s_m=|\Delta|\\ r_1+\dots+r_m=\ell(\Delta)+\ell(\bar\Delta)-2>0}} N_{(\Delta|\bar\Delta)}\begin{pmatrix} s_1\dots s_m\\ r_1\dots r_m\end{pmatrix}\partial_0^{r_1}(f^{s_1})\dots\partial_0^{r_m}(f^{s_m})\, t_\Delta\bar t_{\bar\Delta}, \tag{4}$$

where $f(t_0) = \exp(\Phi'')(t_0)$ and the sum is given by all Young diagrams Δ, $\bar\Delta$ and all positive integer indexes.

In Theorem 1 and in Theorem 2 we are correcting misprints in [6] in indexes of sums. The correct formula given also in S.Natanzon, A.Zabrodin arxiv.org/1302.7288v4 .

Calculation of $N_{(\Delta|\bar\Delta)}\begin{pmatrix} s_1\dots s_m\\ r_1\dots r_m\end{pmatrix}$

We find also recursion algorithm for calculation of $N_{(\Delta|\bar\Delta)}\begin{pmatrix} s_1\dots s_m\\ r_1\dots r_m\end{pmatrix}$. It consists of several steps.

1. **Denote** by $P_{ij}(r_1,\dots,r_m)$ is the number of sequences of positive integers $(i_1,\dots,i_m)$, $(j_1,\dots,j_m)$ such that $i_1+\dots+i_m=i$, $j_1+\dots+j_m=j$ and $r_k=i_k+j_k$.

Put $T_{ij}(p_1,\dots,p_m)=\sum\limits_{\substack{k>0,\,n_i>0\\ n_1+\dots+n_k=m}}\frac{(-1)^{m+1}}{k\,n_1!\dots n_k!}P_{ij}\left(\sum\limits_{i=1}^{n_1}p_i,\ \sum\limits_{i=n_1+1}^{n_1+n_2}p_i,\ \dots,\ \sum\limits_{i=n_1+\dots+n_{k-1}+1}^{m}p_i\right)$.

2. **Define**

$$T_{i_1i_2}\begin{pmatrix} s_1\dots s_m\\ \ell_1\dots \ell_m\end{pmatrix}=\begin{cases} T_{i_1i_2}(s_1,\dots,s_m), & \text{if } \ell_1=\dots=\ell_m=1\\ 0 & \text{otherwise,}\end{cases}$$

and

$$T_{i_1\dots i_k}\begin{pmatrix} s_1\dots s_m\\ \ell_1\dots \ell_m\end{pmatrix}=\sum_{1\leqslant i\leqslant j\leqslant m}\frac{\ell!}{(\ell_i-1)!\dots(\ell_j-1)!}\times$$

$$T_{i_1\dots i_{k-1}}\begin{pmatrix} s_1\dots s_{i-1}\, s\, s_{j+1}\dots s_m\\ \ell_1\dots \ell_{i-1}\,\ell\,\ell_{j+1}\dots \ell_m\end{pmatrix}T_{s,i_k}(s_i,s_{i+1},\dots,s_j),$$

where

$$s=s_i+s_{i+1}+\dots+s_j-i_k>0,\quad \ell=(\ell_i-1)+\dots+(\ell_j-1)>0.$$

3. **Consider** $\tilde N_{\left(\begin{smallmatrix} i_1\dots i_k\\ \bar i_1\dots \bar i_{\bar k}\end{smallmatrix}\right)}\begin{pmatrix} s_1\dots s_m\\ r_1\dots r_m\end{pmatrix}=$

$$\frac{i_1\dots i_k\,\bar i_1\dots \bar i_{\bar k}}{s_1\dots s_m}\sum T_{i_1\dots i_k}\begin{pmatrix} s_1\ \dots\ s_m\\ r_1-n_1+1\dots r_m-n_m+1\end{pmatrix},$$

where the summation is carried over all representation of the set $\{\bar i_1,...,\bar i_{\bar k}\}$ as a union of non-intersecting non-empty subsequences $\{b^j_1,...,b^j_{n_j}\}\subset\{\bar i_1,...,\bar i_{\bar k}\}$ such that $b^j_1+\dots+b^j_{n_j}=s_j$ and $j=1,\dots,m$.

Put

$$N_{(\Delta|\bar\Delta)}\begin{pmatrix} s_1\dots s_m\\ r_1\dots r_m\end{pmatrix}=\frac{1}{\sigma(\Delta)\sigma(\bar\Delta)}\tilde N_{\left(\begin{smallmatrix} \mu_1\dots\mu_k\\ \bar\mu_1\dots\bar\mu_{\bar k}\end{smallmatrix}\right)}\begin{pmatrix} s_1\dots s_m\\ r_1\dots r_m\end{pmatrix},\qquad(5)$$

where Δ, $\bar\Delta$ are Young diagrams with rows $[\bar\mu_1,\dots,\bar\mu_\ell]$ and $[\mu_1,\dots,\mu_\ell]$ and $\sigma(\Delta)$, $\sigma(\bar\Delta)$ are orders of the automorphism groups of rows for the Young diagrams Δ, $\bar\Delta$.

Example 1. If $\Delta=[\mu_1,\dots,\mu_l]$, $\bar\Delta=[\bar\mu_1,\dots,\bar\mu_{\bar l}]$ and $d=|\Delta|=|\bar\Delta|$, then

$$N_{(\Delta|\bar\Delta)}\begin{pmatrix} d\\ \ell(\Delta)+\ell(\bar\Delta)-2\end{pmatrix}=\frac{\rho(\Delta)\rho(\bar\Delta)}{d\sigma(\Delta)\sigma(\bar\Delta)},$$

where $\rho(\Delta)=\mu_1\cdots\mu_l$. For other cases $N_{(\Delta|\bar\Delta)}\begin{pmatrix} s\\ r\end{pmatrix}=0$.

Journal of Physics: Conference Series **670** (2016) 012036 doi:10.1088/1742-6596/670/1/012036

Example 2.

$$N_{\binom{i_1 i_2}{\bar{i}_1 \bar{i}_2}}\binom{\bar{i}_1\bar{i}_2}{11} = N_{\binom{i_1 i_2}{\bar{i}_1 \bar{i}_2}}\binom{\bar{i}_2\bar{i}_1}{11} = -\frac{i_1 i_2}{2\sigma([i_1,i_2])\sigma([\bar{i}_1,\bar{i}_2])}\min\{i_1,i_2,\bar{i}_1,\bar{i}_2\}).$$

For other cases $N_{\binom{i_1 i_2}{\bar{i}_1 \bar{i}_2}}\binom{s_1 s_2}{r_1 r_2} = 0.$

Application to Hurwitz numbers

Okounkov's theorem says that

$$F(t_0,\mathbf{t},\bar{\mathbf{t}}) = \frac{t_0^3}{6} + \sum_{|\Delta|=|\bar{\Delta}|}\frac{e^{|\Delta|t_0}\,H_0(\Delta,\bar{\Delta})}{(\ell(\Delta)+\ell(\bar{\Delta})-2)!}\prod_{i=1}^{\ell(\Delta)}\mu_i t_{\mu_i}\prod_{i=1}^{\ell(\bar{\Delta})}\bar{\mu}_i\bar{t}_{\bar{\mu}_i}$$

is a symmetric formal solutions of dispersionless 2D Toda hierarchy with $\Phi(t_0) = \frac{t_0^3}{6}$ and therefore $f(t_0) = \exp(t_0)$. Thus, Theorem 1 gives

Theorem 2 (2015, S.Natanzon, A.Zabrodin [6]). Double Hurwitz numbers of genus 0 for $\ell(\Delta)+\ell(\bar{\Delta}) > 2$ are

$$H_0(\Delta|\bar{\Delta}) = \frac{(\ell(\Delta)+\ell(\bar{\Delta})-2)!}{\rho(\Delta)\,\rho(\bar{\Delta})}\sum s_1^{r_1}\dots s_m^{r_m}\,N_{(\Delta|\bar{\Delta})}\binom{s_1\dots s_m}{r_1\dots r_m},$$

where the sum is carried over all matrixes $\binom{s_1\dots s_m}{r_1\dots r_m}$ such that $s_1+\ldots+s_m = |\Delta|$ and $r_1+\ldots+r_m = \ell(\Delta)+\ell(\bar{\Delta})-2$.

In particulary examples 1 and 2 give

Example 3.

The double Hurwitz numbers of polynomials for $\ell(\Delta) > 1$ are

$$H_0(\Delta|[n]) = \frac{(\ell(\Delta)-1)!}{\sigma(\Delta)}\,n^{\ell(\Delta)-2}.$$

Example 4.

The double Hurwitz numbers of simplest Laurent polynomials are

$$H_0([i_1,i_2]|[\bar{i}_1,\bar{i}_2]) = 2\frac{d-\min\{i_1,i_2,\bar{i}_1,\bar{i}_2\}}{(1+\delta_{i_1 i_2})(1+\delta_{\bar{i}_1\bar{i}_2})}.$$

These nice formulas were at first found by integration on compactification of moduli spaces of algebraic curves (2002, S.Lando [7]; 2005, I.Goulden, D.Jackson, Vakil [8]; 2008, S.Shadrin, M.Shapiro, A.Vainshtein [9]). Theorem 2 enables us to find a lot of other formulas of this type.

Application to Riemann theorem

Wiegmann-Zabrodin function F_{con} is a symmetric solutions to dispersionless 2D Toda hierarchy, $\Phi(t_0) = F_{con}(t_0, 0, 0) = \frac{1}{2}t_0^2 \ln t_0 - \frac{3}{4}t_0^2$ and therefore $f(x) = t_0$. Thus, theorem 1 gives

Theorem 3 (2005, S.Natanzon [10]). The conformal map $w_Q : Q \to \Lambda$ is

$$w_Q(z) = z \exp\left(\Big(-\frac{1}{2}\frac{\partial^2}{\partial t_0^2} - \frac{\partial}{\partial t_0}\sum_{k\geq 1}\frac{z^{-k}}{k}\frac{\partial}{\partial t_k}\Big)F_{con}\right)(t_0(Q), \mathbf{t}(Q), \bar{\mathbf{t}}(Q)),$$

where

$$F_{con}(t_0, \mathbf{t}, \bar{\mathbf{t}}) = \frac{1}{2}t_0^2 \ln t_0 - \frac{3}{4}t_0^2 + \sum_{i>0}^{\infty} i t_0^i t_i \bar{t}_i +$$

$$\sum_{|\Delta|=|\bar{\Delta}|} \sum_{\substack{s_1+\ldots+s_m=|\Delta| \\ r_1+\ldots+r_m=\ell(\Delta)+\ell(\bar{\Delta})-2>0}} N_{(\Delta|\bar{\Delta})} \binom{s_1 \ldots s_m}{r_1 \ldots r_m} \partial_0^{r_1}(t_0^{s_1}) \ldots \partial_0^{r_m}(t_0^{s_m})\, t_\Delta \bar{t}_{\bar{\Delta}}.$$

Acknowledgments

The author was partially supported in part by RFBR grant 13-01-00755 and partially supported by Laboratory of Quantum Topology of Chelyabinsk State University (Russian Federation government grant 14.Z50.31.0020).

References

[1] Varchenko A and Etingof P 1992 Why the boundary of a round drop becomes a curve of order four *University Lecture Series, AMS Povidence RI* **3** 72.

[2] Novikov P.S 1938 On uniqueness of the solution of the inverse problem for a potential *Soviet Math.Dok.* **18:3** 165-168.

[3] Wiegmann P and Zabrodin A 2000 Conformal maps and integrable hierarchies *Commun. Math. Phys.* **213** 523-538.

[4] Okounkov A 2000 Toda equations for Hurwitz numbers *Math. Res. Lett.* **7** 447-453.

[5] Takasaki K and Takebe T 1991 SDiff(2) Toda equation - hierachy, tau function and symmetries *Lett. Math. Phys.* **23** 205-214.

[6] Natanzon S and Zabrodin A 2015 Simmetric solutions to dispersionless 2D Toda hierachy, Hurwitz numbers and conformal dynamics *Int. Math. Res. Notices* **2015** 2082-2110.

[7] Lando S 2002 Ramifield coverings of the two-dimensional sphere and the intersection theory in spaces of meromorphic functions on algebraic curves *Russ. Math. Surv.* **57:3** 463-533.

[8] Gulden I, Jeckson D and Vakil R 2005 Towards the geometry of double Hurwitz numbers *Adv. Math.* **198** 43-92.

[9] Shadrin S, Shapiro M and Wainshtein A 2008 Chamber behevior of double Hurwitz numbers in genus 0 *Adv. Math.* **217** 79-96.

[10] Natanzon S 2005 Towards an effectivisation of the Riemann theorm *Ann. Global Anal. Geom.* **28** 233-255.