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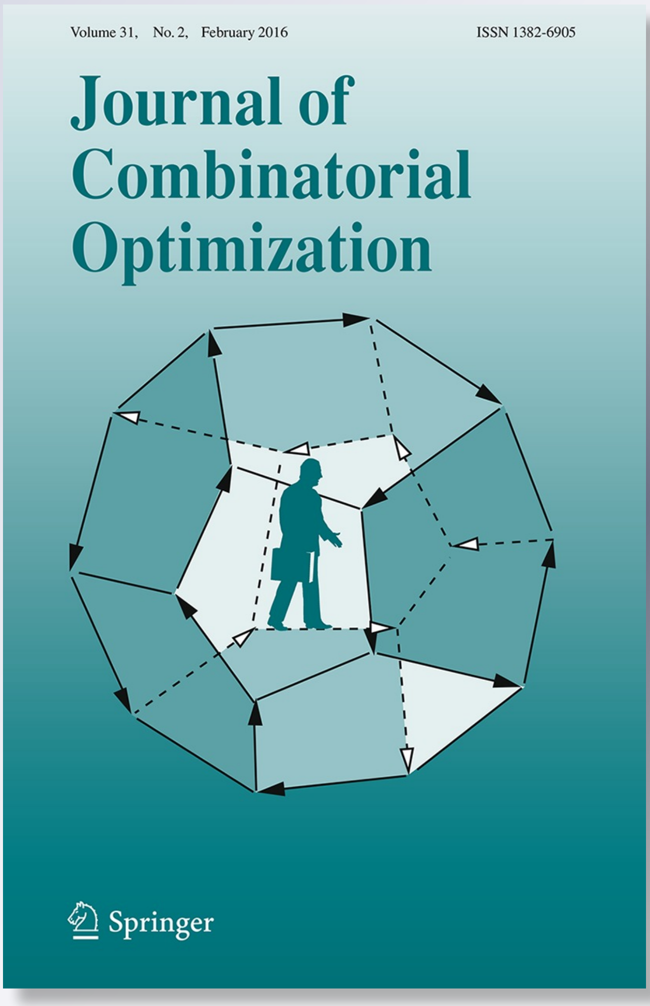

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# Two cases of polynomial-time solvability for the coloring problem

**D. S. Malyshev**

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**Abstract** The complexity of the coloring problem is known for all hereditary classes defined by two connected 5-vertex forbidden induced subgraphs except 13 cases. We update this result by proving polynomial-time solvability of the problem for two of the mentioned 13 classes.

**Keywords** Vertex coloring · Computational complexity · Polynomial-time algorithm

## 1 Introduction

A *coloring* is an arbitrary mapping from the set of vertices of a given graph into a set of colors. A *proper coloring* of a graph is a coloring, where any neighbors are colored by different colors. In other words, a proper coloring of a graph is a partition of its vertices into *independent sets* i.e., sets of pairwise nonadjacent vertices. The *chromatic number* of a graph $G$ (denoted by $\chi(G)$) is the minimal number of colors in proper colorings of $G$ and any corresponding coloring of $G$ is called *optimal*. The COLORING problem for a given graph and a number $k$ is to determine whether its chromatic number is at most $k$.

There are some natural bounds for the chromatic number of a graph. A *clique* of a graph is a set of its pairwise adjacent vertices. The size of a largest clique of a graph $G$ is called the *clique number* of $G$ denoted by $\omega(G)$. Clearly, $\chi(G) \geq \omega(G)$. The gap between the parameters can be arbitrarily large, as there are triangle-free graphs with arbitrarily large chromatic numbers (Mycielski 1955). The situation is similar

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in the computational sense, as the COLORING is NP-complete for triangle-free graphs (Maffray and Preissmann 1996). So, computing the clique number in polynomial time does not yield polynomial-time solvability of the COLORING. But, sometimes finding out the clique number produces an efficient algorithm for the problem. A result of the present paper is based on this idea.

A graph $H$ is called an *induced subgraph* of $G$ if $H$ is obtained from $G$ by deletion of vertices. A *class* is a set of simple unlabeled graphs. A class of graphs is called *hereditary* if it is closed under deletion of vertices. It is well known that any hereditary (and only hereditary) class $\mathcal{X}$ can be defined by a set of its forbidden induced subgraphs $\mathcal{Y}$. We write $\mathcal{X} = Free(\mathcal{Y})$ in this case. If $\mathcal{Y}$ is finite, then the problem whether a given graph $G$ belongs to $\mathcal{X}$ is decided in polynomial time (e.g., by determining in $G$ an induced copy of a graph in $\mathcal{Y}$).

A complete complexity dichotomy for the COLORING is known within the family of hereditary graph classes defined by a single forbidden induced subgraph (Kral' et al. 2001). A study for forbidden pairs was also initialized in the paper. Only partial results are known in the case of two forbidden induced subgraphs (Dabrowski et al. 2012, 2014; Golovach and Paulusma 2013; Kral' et al. 2001; Korpeilainen et al. 2011; Lozin and Malyshev 2014; Schindl 2005). Moreover, a complete classification for pairs is wide open. For example, there are two pairs of 4-vertex forbidden induced structures for which the complexity of the COLORING is still open (Lozin and Malyshev 2014).

The complexity of the COLORING under forbidding two 5-vertex connected induced subgraphs was considered in the paper of Malyshev 2014. It was determined for all pairs of this type except 13 explicitly listed cases. We update the result by proving polynomial-time solvability of the problem for two of the mentioned 13 classes. Namely, we show tractability of the COLORING for $\{claw, bull\}$-free graphs and $\{P_5, sinker\}$-free graphs.

## 2 Notation

We use the standard notation $P_n$, $C_n$, $O_n$ for the simple path, the chordless cycle and the empty graph with $n$ vertices respectively. The graphs $claw$, $bull$, $sinker$ are drawn in the Fig. 1 below.

The *complement graph* of $G$ (denoted by $\overline{G}$) is a graph on the same set of vertices and two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in $G$. The *sum* $G_1 + G_2$ is the disjoint union of $G_1$ and $G_2$ with non-intersected sets of vertices. For a graph $G$ and a set $V' \subseteq V(G)$ the formula $G \setminus V'$ denotes the subgraph of $G$ obtained by deleting all vertices in $V'$.

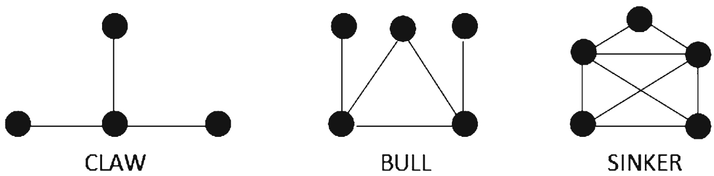


**Fig. 1** The graphs $claw$, $bull$ and $sinker$

We associate the following notation with a given graph $G$: $N(x)$ is the neighborhood of a vertex $x$; $N(V')$ is the neighborhood of $V' \subseteq V(G)$ i.e., the set of vertices outside $V'$ adjacent to at least one vertex in $V'$; $N_{V'}(x) = N(x) \cap V'$; $N_k(V') = \underbrace{N(N(\ldots(N}_{k\ times}(V'))\ldots))$.

## 3 Methodology

In this section we present some algorithmic tools to be used in obtaining the main results. The first natural idea is to drop one or more vertices preserving the chromatic number or changing it in a regular manner. To implement this idea we need the notions of a simplicial vertex and quasi-twins.

A vertex of a graph is called *simplicial* if its neighborhood is a clique. Verifying that a given vertex is simplicial is done in polynomial time by checking its neighborhood for the completeness.

**Lemma 1** *If a vertex $x$ is simplicial in a graph $G$, then* $\chi(G) = \max(|N(x)| + 1, \chi(G \setminus \{x\}))$ *and* $\omega(G) = \max(|N(x)| + 1, \omega(G \setminus \{x\}))$.

*Proof* Let $H = G \setminus \{x\}$. Clearly, $\chi(H) \geq |N(x)|$ and $\chi(G) \geq \max(|N(x)| + 1, \chi(H))$. If $\chi(H) = |N(x)|$, then an optimal coloring of $H$ induces a partial proper coloring of $G$. The vertex $x$ of $G$ can be colored by the $(|N(x)| + 1)$-th color (not participating in the coloring of $H$). Hence, $\chi(G) = |N(x)| + 1$, as there exists a proper coloring of $G$ with $|N(x)| + 1$ colors and $\chi(G) \geq |N(x)| + 1$. Let $\chi(H) \geq |N(x)| + 1$ now. For each optimal coloring of $H$ there is a color $c$ by which every vertex of $N(x)$ is not colored. Therefore, this coloring plus coloring of $x$ by $c$ yield a proper coloring of $G$ with $\chi(H)$ colors. Hence, $\chi(G) = \chi(H)$, as $\chi(G) \geq \chi(H)$. The equation $\omega(G) = \max(|N(x)| + 1, \omega(G \setminus \{x\}))$ is obvious. □

Two vertices of a graph are called *twins* if they have coinciding neighborhoods. Two vertices are called *quasi-twins* if the neighborhood of one of them is included in the neighborhood of the second one. Clearly, checking that given two vertices are quasi-twins is done in linear time. The significance of the quasi-twins notion is certified by the following lemma.

**Lemma 2** *If $G$ is a graph, $x, y \in V(G)$ and $N(x) \subseteq N(y)$, then* $\chi(G) = \chi(G \setminus \{x\})$ *and* $\omega(G) = \omega(G \setminus \{x\})$.

*Proof* Taking an arbitrary optimal coloring of $H = G \setminus \{x\}$ and coloring of $x$ by the $y$'s color produces a proper coloring of $G$ with $\chi(H)$ colors. Hence, $\chi(G) = \chi(H)$. If $x$ belongs to a largest clique $Q$ of $G$, then $Q \setminus \{x\} \cup \{y\}$ is a largest clique of $G \setminus \{x\}$. Thus, $\omega(G) = \omega(G \setminus \{x\})$. □

The second algorithmic idea is to decompose a graph into more simple parts with a connection between the chromatic numbers of the graph and its parts.

A vertex $x$ of a connected graph $G$ is called *cut-vertex* if its deletion disconnects the graph. An $x$-*block* of $G$ is its induced subgraph produced by vertices of a connected component of $G \setminus \{x\}$ and $x$. Verifying that a given vertex is a cut-vertex and computing all its blocks is done in linear time by any depth first search algorithm.

**Lemma 3** *Let $G$ be a connected graph, $x$ be its cut-vertex and $G_1, G_2, \ldots, G_k$ be all $x$-blocks. Then, $\chi(G) = \max(\chi(G_1), \chi(G_2), \ldots, \chi(G_k))$ and $\omega(G) = \max(\omega(G_1), \omega(G_2), \ldots, \omega(G_k))$.*

*Proof* Every graph among $G_1, G_2, \ldots, G_k$ contains $x$. Let $V_i$ be the set of vertices having the same color as $x$ in an optimal coloring of $G_i$. Hence, $\bigcup_{i=1}^{k} V_i$ is an independent set of $G$ and $\chi(G_i \setminus V_i) \leq \chi(G_i) - 1$. Therefore, any connected component of $G \setminus \bigcup_{i=1}^{k} V_i$ can be properly colored by at most $\max\limits_i \chi(G_i) - 1$ colors. As $\bigcup_{i=1}^{k} V_i$ is independent, then $\chi(G) \leq \max\limits_i \chi(G_i)$. Thus, $\chi(G) = \max\limits_i \chi(G_i)$, as the inequality $\chi(G) \geq \max\limits_i \chi(G_i)$ is obvious. The relation $\omega(G) = \max(\omega(G_1), \omega(G_2), \ldots, \omega(G_k))$ is clear. □

Lemmas 1–3 show that the corresponding compression of a graph polynomially reduces the COLORING in a hereditary class to the same problem for its connected graphs without simplicial vertices, quasi-twins and cut-vertices.

The third idea is to extend graphs in a polynomial case preserving tractability.

For a hereditary class $\mathcal{X}$ and a number $k$ the class $[\mathcal{X}]_k$ is a set of graphs, for which one can delete at most $k$ vertices such that the result belongs to $\mathcal{X}$.

**Lemma 4** (see Malyshev 2014) *Let $\mathcal{X}$ be a class with the* COLORING *solvable in polynomial time, the problem whether a graph belongs to $\mathcal{X}$ is polynomial-time solvable and for some fixed number $p$ the inclusion $\mathcal{X} \subseteq Free(\{O_p\})$ $(p \geq 2)$ holds. Then, for any fixed $q$ the* COLORING *is also polynomial in the class $[\mathcal{X}]_q$.*

## 4 Polynomial-time solvability for {claw,bull}-free graphs

**Lemma 5** *Any connected graph in $Free(\{claw, bull\})$ belongs to $Free(\{C_4 + P_1, C_5 + P_1, \ldots\})$.*

*Proof* Let $G \in Free(\{claw, bull\})$ be a connected graph containing $C_n + P_1$ $(n \geq 4)$ as an induced subgraph. Consider in $G$ a shortest induced path $P$ connecting the isolated vertex $v$ of $C_n + P_1$ with the vertices of its $n$-cycle. Let $x \in V(P)$ be the last its vertex counting from $v$. Among vertices of $P$ only $x$ has neighbors in $C_n$ (as $P$ is shortest). If $x$ is adjacent to three consecutive vertices of $C_n$, then $x$, two its nonadjacent neighbors in $C_n$ and its unique neighbor $y \in V(P)$ induce $claw$. Hence, if $(x, z) \in E(G), z \in V(C_n)$, then $C_n$ has a common neighbor of $z$ and $x$ (otherwise, $claw$ is an induced subgraph of $G$) and such a neighbor is unique. But, $x, z$, two neighbors of $z$ in $C_n$ and $y$ induce $bull$. Thus, we have a contradiction with the existence of $G$ above. □

**Lemma 6** *The* COLORING *for $Free(\{claw, bull\})$ is reduced in polynomial time to the* COLORING *for $Free(\{claw, bull, C_4, C_5\})$.*

*Proof* Let $G$ be a connected $\{claw, bull\}$-graph that has induced $C_n$, where $n \in \{4, 5\}$. By Lemma 3, this cycle dominates all vertices of $G$. In other words, each vertex outside $C_n$ has a neighbor in this cycle. We will show that $H = G \setminus V(C_n)$ belongs to $Free(\{O_3\})$. Assume that $H$ contains three pairwise nonadjacent vertices $x, y, z$. For each of them the intersection of its neighborhood with $V(C_n)$ is a set of at least two consecutive vertices. By $S = (i, j, k)$ we denote a tuple with the cardinalities of the intersections sorted by ascending (i.e., $i \leq j \leq k$). All elements of $S$ are at least 2. Without loss of generality we can assume $|N(x) \cap V(C_n)| = i$, $|N(y) \cap V(C_n)| = j$, $|N(z) \cap V(C_n)| = k$.

Let $n = 4$. Clearly, $i = 2$, otherwise $C_4$ has a common neighbor of $x, y, z$ and $G \notin Free(\{claw\})$. For the same reason, $(j, k) \neq (3, 4)$ and $(j, k) \neq (4, 4)$. Only three cases remain: $(2, 2, 2)$, $(2, 2, 3)$, $(2, 3, 3)$. In the first two cases $x$ and $y$ have no a common neighbor on $C_4$ (otherwise, $G \notin Free(\{claw, bull\})$). Hence, $G$ obligatory contains *bull* or *claw* as an induced subgraph for $S \in \{(2, 2, 2), (2, 2, 3)\}$. In the last case to avoid induced *claw* the set $N(y) \cap N(z) \cap V(C_4)$ must contain two consecutive vertices of the 4-cycle and $N(x) \cap N(y) \cap N(z) \cap V(C_4) = \emptyset$. The vertices $x, y, z$ and two elements of $N(x) \cap V(C_4)$ induce *bull*. Thus, we have a contradiction in all cases.

Let $n = 5$. Clearly, $i \neq 2$ (otherwise, $G \notin Free(\{bull\})$), $k \neq 5$ and $(j, k) \neq (4, 4)$ (otherwise, $x, y, z$ have a common neighbor on $C_5$ and $G$ is not *claw*-free). We have only two remaining cases: $(3, 3, 3)$ and $(3, 3, 4)$. As $G$ is $\{claw, bull\}$-free, then in the 5-cycle $x$ and $y$ must have only one common neighbor. It implies a contradiction for $S = (3, 3, 3)$ (as here $|N(x) \cap N(z) \cap V(C_5)| > 1$ or $|N(y) \cap N(z) \cap V(C_5)| > 1$). Let $S = (3, 3, 4)$. If $N(x) \cap V(C_5) \subseteq N(z) \cap V(C_5)$ or $N(y) \cap V(C_5) \subseteq N(z) \cap V(C_5)$, then $G$ has *claw* as an induced subgraph. If both inclusions do not hold, then $x, y, z$ and two internal vertices of $N(z) \cap V(C_5)$ induce *bull*. We have a contradiction in all cases.

So, $H$ is $O_3$-free. The COLORING is polynomial-time solvable for $O_3$-free graphs, since it is equivalent to finding a maximum matching in the complement graphs, which is well known to be polynomial. Hence, by Lemma 4, the COLORING is solved in polynomial time for $Free(\{claw, bull\}) \setminus Free(\{C_4, C_5\})$. Thus, we have the polynomial-time reduction. □

A *circular-arc graph* is the intersection graph of a set of arcs on a circle. It has one vertex for each arc in the set and an edge between every pair of vertices corresponding to arcs that intersect. A *proper circular-arc graph* is a circular-arc graph that has an intersection model in which no arc properly contains another. A minimal set of forbidden induced subgraphs for the class of proper circular-arc graphs is known and completely described by Lin and Szwarcfiter 2009. It is constituted by $\overline{C_3 + P_1}$, $\overline{C_5 + P_1}$, $\overline{C_6}$, $\overline{C_7 + P_1}$, $\overline{C_8}$, $\overline{C_9 + P_1}$, $\overline{C_{10}}$, ..., by $C_4 + P_1$, $C_5 + P_1$, ... and the graphs drawn in the Fig. 2 below (see also www.graphclasses.org/classes/gc_881.html).

**Theorem 1** *The* COLORING *is polynomial-time solvable for {claw,bull}-free graphs.*

*Proof* Every graph presented in the figure contains $C_4$ or *bull* as an induced fragment. The complements of $C_3 + P_1$ and $C_5$ are *claw* and $C_5$. For every

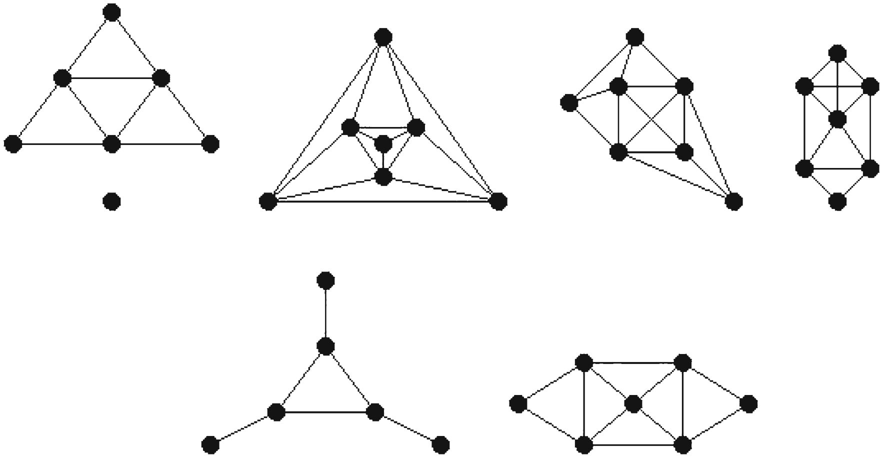

**Fig. 2** Some forbidden induced subgraphs for proper circular-arc graphs

$n \geq 6$ the complement of $C_n$ contains an induced $C_4$, as it is true even for $P_5$. Hence, $Free(\{claw, bull, C_4, C_5, C_6+P_1, C_7+P_1, \ldots\})$ consists of proper circular-arc graphs. The COLORING is solved in polynomial time for proper circular-arc graphs (Bhattacharya et al. 1996). Hence, by Lemmas 5 and 6, it is also true for $Free(\{claw, bull\})$. □

## 5 Polynomial-time solvability for $\{P_5, sinker\}$-free graphs

In the next few lemmas we will suppose that $G$ is a connected $\{P_5, sinker\}$-free graph without simplicial vertices, cut-vertices and quasi-twins and $Q$ is a maximal under inclusion its clique with at least 4 vertices. Every vertex of $N(Q)$ is adjacent to one vertex in $Q$ or to $|Q|-1$ such vertices (as $G$ is $sinker$-free and due to the maximality of $Q$). Moreover, $N_3(Q) = \emptyset$. We will say that $x \in N(Q)$ is *$Q$-simple* if $|N_Q(x)| = 1$. It is *$Q$-complex* otherwise.

**Lemma 7** *If there are no $Q$-complex vertices, then $\chi(G) = \chi(G \setminus Q)$.*

*Proof* As $G$ does not contain simplicial vertices, then each element of $Q$ has an adjacent $Q$-simple vertex. Assume, that a vertex $x' \in N(Q)$ with the neighbor $x \in Q$ has a neighbor $x'' \notin N(x)$. If for some $x^* \in Q$ we have $(x^*, x'') \in E(G)$ (i.e., $x'' \in N(Q)$), then any vertex $y' \in N(Q) \setminus (N(x) \cup N(x^*))$ with the neighbor $y \in Q$ is adjacent to $x'$ and $x''$ simultaneously. Why? If $(x', y') \notin E(G)$, $(x'', y') \notin E(G)$, then $x'', x', x, y, y'$ induce the 5-path. If $(x', y') \in E(G)$, $(x'', y') \notin E(G)$ (or vice versa), $y', x', x'', y$ and a vertex in $Q \setminus \{x, x^*, y\}$ induce $P_5$.

Let $x'' \in N_2(Q)$ now. For each vertex $y' \in N(Q) \setminus N(x)$ with the neighbor $y \in Q$ we have $(x', y') \in E(G)$, $(x'', y') \in E(G)$. Indeed, if $(x', y') \notin E(G)$, $(x'', y') \notin E(G)$, then $x'', x', x, y, y'$ induce the 5-path. If $(x', y') \notin E(G)$, $(x'', y') \in E(G)$, then $y', x'', x', x$ and a vertex in $Q \setminus \{x, y\}$ induce the path. If $(x', y') \in E(G)$, $(x'', y') \notin E(G)$, then $x'', x', y', y$ and a vertex in $Q \setminus \{x, y\}$ induce $P_5$.

The set $N(Q)$ must contain two adjacent vertices with different neighbors in $Q$ (otherwise, by the previous paragraph, $N_2(Q) = \emptyset$ and each element of $Q$ is a cut-vertex). Let $Q'$ be a maximal (under inclusion) clique of $N(Q)$ containing at least two such adjacent vertices. Each vertex of $Q$ has an adjacent $Q$-simple vertex belonging to $Q'$. Indeed, if for some vertex $x \in Q$ we have $N(x) \cap Q' = \emptyset$, then for any $y' \in Q'$ and $z' \in Q'$ with $N_Q(y') \neq N_Q(z')$ we have $N(x) \setminus Q \subseteq N(y')$ and $N(x) \setminus Q \subseteq N(z')$ by the first paragraph. The clique $Q'$ is not maximal in this case. Each vertex in $Q$ has only one neighbor in $Q'$, otherwise, $G$ has an induced $sinker$ or $Q'$ has a non-$Q$-simple vertex $\langle 1 \rangle$.

Assume, there is $x' \in N(Q) \setminus Q'$. It has a neighbor $x \in Q$ and there is a vertex $x'' \in Q'$ being a neighbor of $x$. As $x''$ is the unique neighbor of $x$ in $Q'$, then $x'$ is adjacent to all vertices of $Q' \setminus \{x''\}$ (see the first paragraph and $\langle 1 \rangle$). Due to the maximality of $Q'$, $(x', x'') \notin E(G)$. As $x'$ and $x''$ are not quasi-twins, then there exists a vertex $x'''$ with $(x''', x') \in E(G)$ and $(x''', x'') \notin E(G)$. If $x''' \in N(x)$, then it must be adjacent to all elements of $Q' \setminus \{x''\}$ (by analogy with the corresponding property of $x'$ above). If $x''' \in N_2(Q)$, then $x'''$ is adjacent to all vertices in $Q' \setminus \{x''\}$ (by the second paragraph and $\langle 1 \rangle$). If $x''' \in N(Q) \setminus N(x)$, then, by the first paragraph and $\langle 1 \rangle$, $x'''$ is adjacent to at least $|Q'| - 2$ vertices of $Q' \setminus \{x''\}$. In all cases $x', x'', x'''$ and any two elements of $Q' \cap N(x''')$ induce $sinker$. So, there are no elements in $N(Q) \setminus Q'$ i.e., $Q' = N(Q)$.

As $|N(Q)| = |Q|$ and $N(Q)$ is a clique, then, by $\langle 1 \rangle$, any optimal coloring of $G \setminus N(Q)$ can be extended to a proper coloring of $G$ with the same number of colors. Hence, $\chi(G) = \chi(G \setminus Q)$. □

**Lemma 8** *Two $Q$-complex vertices $x$ and $y$ are adjacent if and only if $N_Q(x) \neq N_Q(y)$. A $Q$-simple vertex $u$ and a $Q$-complex vertex $v$ can not have a common neighbor in $Q$.*

*Proof* If $x, y$ are adjacent and $N_Q(x) = N_Q(y)$, then $x, y$, two vertices of $N_Q(x)$ and the remaining vertex of $Q$ induce $sinker$. If $x, y$ are not adjacent and $N_Q(x) \neq N_Q(y)$, then $x, y$, the vertex in $N_Q(x) \setminus N_Q(y)$ and any two vertices in $N_Q(x) \cap N_Q(y)$ induce $sinker$.

Assume that there is a common neighbor $t \in Q$ of $u$ and $v$. The vertices $u$ and $v$ can not be adjacent, otherwise $t$, two other elements of $N_Q(v)$, $u$ and $v$ induce $sinker$. There exists a neighbor $w'$ of the vertex $w \in Q \setminus N_Q(v)$, which is not adjacent to $v$ (otherwise, $v$ and $w$ are quasi-twins). The vertex $w'$ is $Q$-simple (as $N_Q(v) \neq N_Q(w')$ and $v, w'$ are not adjacent). The vertices $u$ and $w'$ can not be adjacent, as $v$, any vertex in $N_Q(v) \setminus \{t\}, w, w', u$ induce $P_5$. There is a vertex $v'$, which is adjacent to $v$ and nonadjacent to $w$ (otherwise, $v$ and $w$ are quasi-twins). Clearly, $v'$ can not be $Q$-simple or $Q$-complex (as $(v, v') \in E(G)$ and $w \notin N_Q(v')$). Hence, $v' \in N_2(Q)$. If $(v', w') \notin E(G)$, then $v', v, t, w, w'$ induce the 5-path. If $(v', w') \in E(G)$, $(v', u) \notin E(G)$, then $v', w', w, t, u$ induce the path. If $(v', u) \in E(G)$, $(v', w') \in E(G)$, then $w', v', u, t$ and a vertex in $N_Q(v) \setminus \{t\}$ induce $P_5$. □

**Lemma 9** *There are no adjacent $Q$-complex vertices.*

*Proof* Assume, $x$ and $y$ are adjacent $Q$-complex vertices. Let $u \in Q$ be the non-neighbor of $x$ and $v \in Q$ be the non-neighbor of $y$. There exist vertices $u', v'$ outside

$Q$, such that $(v, v')$ and $(u, u')$ are edges of $G$ and $(v', y) \notin E(G)$, $(u', x) \notin E(G)$. Indeed, the opposite would imply that $v, y$ or $u, x$ are quasi-twins. Both vertices $v'$ and $u'$ are $Q$-simple (by Lemma 8). But, $v'$ and $x$ have a common neighbor in $Q$. A contradiction (by Lemma 8). □

Let $Q$ really have $Q$-complex vertices. By Lemmas 8 and 9, withdrawing a vertex from $Q$ yields a clique $Q^*$ with independent $N(Q^*)$. The set $N(Q^*)$ consists of all $Q$-complex vertices and one vertex of $Q$. Hence, $|N(Q^*)| > 1$. Any element of $N(Q^*)$ is adjacent to all vertices of $Q^*$. Denote $G \setminus Q^*$ by $G^*$.

**Lemma 10** *If* $|N(Q^*)| \geq 4$, *then any two elements of* $N(Q^*)$ *dominate all vertices of* $N_2(Q^*)$.

*Proof* Let $N(Q^*) = \{x_1, x_2, \ldots, x_k\}$ and $k > 3$. Suppose that $x_1$ and $x_2$ do not dominate the whole set $N_2(Q^*)$. Hence, $N(Q^*)$ contains a vertex (say, $x_3$) having a neighbor that is not adjacent to $x_1$ and $x_2$ simultaneously. Denote the set $N(x_i) \setminus (N(x_1) \cup N(x_2))$ by $N'(x_i)$ $(i \in \{3, 4\})$, $N'(x_3) \neq \emptyset$. The sets $N(x_1) \setminus N(x_2)$ and $N(x_2) \setminus N(x_1)$ are not empty (as $G$ does not have quasi-twins). Any vertex in $N(x_1) \setminus N(x_2)$ is adjacent to any vertex in $N(x_2) \setminus N(x_1)$ (otherwise, some vertex in $N(x_1) \setminus N(x_2)$, $x_1$, an element of $Q^*$, $x_2$, some vertex in $N(x_2) \setminus N(x_1)$ induce the 5-path). Any vertex of $N(x_1) \otimes N(x_2)$ can not be adjacent to a vertex in $N'(x_3) \cup N'(x_4)$ (if a vertex $v \in N(x_1) \otimes N(x_2)$ is adjacent to a vertex $u \in N'(x_3) \cup N'(x_4)$, then $v, u, x_1, x_2$ and a vertex of $Q^*$ induce $P_5$) $\langle 1 \rangle$. Any vertex $y \in N(x_1) \otimes N(x_2)$ must be adjacent to $x_3$, otherwise, $y$, its neighbor in $\{x_1, x_2\}$, an element of $Q^*$, $x_3$ and a vertex (that obligatory nonadjacent to $y$ by $\langle 1 \rangle$) in $N'(x_3)$ induce $P_5$ $\langle 2 \rangle$. By analogy, if $N'(x_4) \neq \emptyset$, then any vertex in $N(x_1) \otimes N(x_2)$ is adjacent to $x_4$ $\langle 3 \rangle$.

Let $z$ be an element of $N(x_1) \cap N(x_2)$ that is not adjacent to $x_3$. It must be adjacent to all vertices of $N(x_1) \otimes N(x_2)$, otherwise, $x_3$, a non-neighbor of $z$ in $N(x_1) \otimes N(x_2)$ and $x_1, x_2, z$ induce the 5-path (take into account $\langle 2 \rangle$). The vertex $x_4$ has at least one neighbor in each of the sets $N(x_1) \setminus N(x_2)$ and $N(x_2) \setminus N(x_1)$ (say, $a \in N(x_1) \cap N(x_4) \setminus N(x_2)$ and $b \in N(x_2) \cap N(x_4) \setminus N(x_1)$). It is trivial, when $N'(x_4) \neq \emptyset$ (by $\langle 3 \rangle$). If $N'(x_4)$ is empty (i.e., $N(x_4) \subseteq N(x_1) \cup N(x_2)$), then it follows from the facts that $(x_1, x_4)$ and $(x_2, x_4)$ are not pairs of quasi-twins. Hence, $x_4$ and $z$ can not be adjacent, since $a, b, z, x_3, x_4$ induce *sinker* otherwise (keep in mind $\langle 2 \rangle$). Therefore, any neighbor of $x_4$ in $N(x_1) \cap N(x_2)$ is adjacent to $x_3$. This and $\langle 2 \rangle$ imply $N(x_4) \cap (N(x_1) \cup N(x_2)) \subseteq N(x_3) \cap (N(x_1) \cup N(x_2))$ $\langle 4 \rangle$.

As $x_3$ and $x_4$ are not quasi-twins, then there exist $x' \in N(x_4) \setminus N(x_3)$ and $x'' \in N(x_3) \setminus N(x_4)$. Moreover, $N(x_4) \setminus N(x_3) = N'(x_4) \setminus N'(x_3)$ (by $\langle 4 \rangle$) and $N(x_3) \setminus N(x_4) = (N'(x_3) \cup N(x_1) \cap N(x_2) \cap N(x_3)) \setminus N(x_4)$ (by $\langle 2 \rangle$, $\langle 3 \rangle$ and $\langle 4 \rangle$). If $x'' \in N'(x_3)$, then $x'', x_3$, a vertex in $Q^*$, $x_4, x'$ (when $(x'', x') \notin E(G^*)$) or $x', x'', x_3$, a vertex in $Q^*, x_1$ (when $(x'', x') \in E(G^*)$) induce $P_5$. Let $x'' \in N(x_1) \cap N(x_2) \cap N(x_3)$. By $\langle 2 \rangle$, to avoid an induced *sinker* there is a vertex $x^* \in N(x_1) \otimes N(x_2)$ with $(x'', x^*) \notin E(G)$. We have $(x^*, x_4) \in E(G)$, $(x^*, x') \notin E(G)$ (by $\langle 3 \rangle$ and $\langle 1 \rangle$). If $(x'', x') \notin E(G)$, then $x'', x_2$, a vertex in $Q^*$, $x_4, x'$ induce the 5-path. If $(x'', x') \in E(G)$, then $x^*, x_4, x', x'', x_2$ induce the 5-path. We obtain a contradiction. □

**Lemma 11** *If* $|N(Q^*)| \geq 4$, *then there exists an optimal coloring of* $G^*$ *with monochromatic* $N(Q^*)$.

*Proof* If there is an optimal coloring of $G^*$ with two vertices in $N(Q^*)$ having the same color $c$, then, by Lemma 10, all vertices of $N(Q^*)$ can be recolored by $c$ keeping the optimality and the properness. Hence, we may assume that elements of $N(Q^*)$ have pairwise different colors. For each of such colors there is a vertex in $N_2(Q^*)$ having this color (otherwise, repainting by one color also works). So, there are $|N(Q^*)|$ vertices in $N_2(Q^*)$ with pairwise different colors. We will show that any two such vertices $x, y$ must be adjacent. There are $x', y' \in N(Q^*)$, such that $x, x'$ are monochromatic and $y, y'$ have the same color. This observation and Lemma 10 imply that $(x, y') \in E(G^*)$, $(y, x') \in E(G^*)$. But, $x, x', y, y'$ and a vertex in $Q^*$ induce the 5-path. We have a contradiction. Hence, the vertices form a clique $Q'$ with $|N(Q^*)| \geq 4$ vertices. Every vertex in $N(Q^*)$ has a neighbor in $Q'$, as this vertex and any other vertex in $N(Q^*)$ will not dominate $Q'$ otherwise (no one element $z \in N(Q^*)$ can dominate $Q'$, as $Q'$ has a vertex with the same color as $z$ has). By the previous lemma, $N(Q^*)$ does not contain two $Q'$-simple vertices. Hence, there are at least two $Q'$-complex vertices in $N(Q^*)$ and they can not dominate $Q'$ (by Lemma 8). We have a contradiction (by Lemma 10) with our assumption. □

**Lemma 12** *If $N(Q^*) = \{x_1, x_2, x_3\}$, then $\chi(G^*) \leq 3$ or there is an optimal coloring of $G^*$ in which elements of $N(Q^*)$ have at most two different colors.*

*Proof* One may assume that $\chi(G^*) > 3$ and all vertices of $N(Q^*)$ have different colors in any optimal coloring of $G^*$. Let us fix some optimal coloring, $x_i$ has a color $c_i$ and $c^*$ be a color in the coloring different from $c_1, c_2, c_3$. A vertex $x \in N_2(Q^*)$ will be called an *$i$-vertex* if $|N(x) \cap N(Q^*)| = i$. A 2-vertex $x$ will be called a *$(2, x_i)$-vertex* if $N(x) \cap N(Q^*) = N(Q^*) \setminus \{x_i\}$. Any 1-vertex and any 2-vertex do not have neighbors in $N_3(Q^*)$ (as $G$ is $P_5$-free) $\langle 1 \rangle$. For the same reason, all 1-vertices are adjacent to the same element of $N(Q^*)$ (say, $x_1$), any 1-vertex is adjacent to each $(2, x_1)$-vertex, a $(2, x_i)$-vertex and a $(2, x_j)$-vertex ($i \neq j$) must be adjacent $\langle 2 \rangle$.

Let us show that for every $x_i$ the set of $(2, x_i)$-vertices is nonempty and independent $\langle 3 \rangle$. The set coincides with $N(x_j) \cap N(x_k) \setminus (N(x_1) \cap N(x_2) \cap N(x_3))$ ($\{x_j, x_k\} = N(Q^*) \setminus \{x_i\}$). Clearly, $N(x_j) \setminus N(x_k)$ and $N(x_k) \setminus N(x_j)$ are nonempty (otherwise, $x_j$ and $x_k$ are quasi-twins of $G$). If $x_i$ has an adjacent 1-vertex (hence, $i = 1$), then $x_1$ is adjacent to all vertices of $N(x_2) \otimes N(x_3)$ (otherwise, $N(x_2) \otimes N(x_3)$ has an 1-vertex and this contradicts $\langle 2 \rangle$). This fact implies that there exist $(2, x_1)$-vertices, as $x_1, x_2$ are quasi-twins otherwise. Let $x_i$ does not have adjacent 1-vertices. Either $N(x_j) \setminus N(x_k) \subseteq N(x_i)$ or $N(x_k) \setminus N(x_j) \subseteq N(x_i)$ is true, as $x_j$ and $x_k$ have adjacent 1-vertices otherwise (a contradiction with $\langle 2 \rangle$). There are $(2, x_i)$-vertices, $(N(x_j) \setminus N(x_k)) \cap N(x_i) \neq \emptyset$ and $(N(x_k) \setminus N(x_j)) \cap N(x_i) \neq \emptyset$ (otherwise, $(x_i, x_j)$ or $(x_i, x_k)$ is a pair of quasi-twins of $G$). Hence, there are neighbors $x'_j \in N(x_j) \setminus N(x_k)$, $x'_k \in N(x_k) \setminus N(x_j)$ of $x_i$. If $x'_i$ is a $(2, x_i)$-vertex, then $(x'_j, x'_i) \in E(G^*)$, $(x'_k, x'_i) \in E(G^*)$, otherwise, $x_i, x'_k, x_k, x'_i, x_j$ or $x_i, x'_j, x_j, x'_i, x_k$ induce the 5-path. Any two $(2, x_i)$-vertices can not be adjacent, as they and $x'_j, x_j, x_k$ induce *sinker* otherwise.

An 1-vertex $v$ can not have an adjacent $(2, x_i)$-vertex $v'$ for $i \in \{2, 3\}$ $\langle 4 \rangle$. Indeed, $v, v', x_j$ ($j \neq 1, j \neq i$), a vertex in $Q^*$, $x_i$ induce $P_5$ otherwise.

A 3-vertex $x$ can not be adjacent to a $(2, x_i)$-vertex and to a $(2, x_j)$-vertex simultaneously for $i \neq j$, otherwise $x$, these two vertices, $x_k$ ($k \neq i$ and $k \neq j$) and $x_j$

induce *sinker* (see also ⟨2⟩) ⟨5⟩. The vertex $x$ can not have a neighbor in $N_3(Q^*)$, as $G^*$ contains an induced 5-path otherwise (by ⟨1⟩ and ⟨2⟩, the neighbor, $x, x_i$ for some $i$, a $(2, x_j)$-vertex and a $(2, x_i)$-vertex induce $P_5$) ⟨6⟩. If $x$ is adjacent to an 1-vertex $v$, then $v$ can not have adjacent 1-vertices ⟨7⟩. If there is such a neighbor $v'$, then $v', v, x, x_2$ and a vertex in $Q^*$ induce $P_5$, when $(v', x) \notin E(G^*)$). If $(v', x) \in E(G^*)$, then for any $(2, x_1)$-vertex $u$ the subgraph *sinker* is induced by $x_1, x, v, v', u$ (when $(u, x) \notin E(G^*)$) or by $u, x, v, v', x_3$ (when $(u, x) \in E(G^*)$) (see also ⟨2⟩).

Denote by $\hat{V}_i$ the set of 3-vertices colored by $c^*$ and having a neighbor $(2, x_i)$-vertex. By ⟨5⟩, $\hat{V}_1 \cap \hat{V}_2$, $\hat{V}_2 \cap \hat{V}_3$, $\hat{V}_1 \cap \hat{V}_3$ are empty. The set $\hat{V}_0$ is the set of 3-vertices colored by $c^*$ that do not have a neighbor 2-vertex. Clearly, $\hat{V}_i \cap \hat{V}_0 = \emptyset$ for any $i$. The set of 3-vertices colored by $c^*$ coincides with $\hat{V}_0 \cup \hat{V}_1 \cup \hat{V}_2 \cup \hat{V}_3$. The set of 1-vertices having a neighbor 3-vertex will be denoted by $\hat{V}$. It is independent and any vertex of $\hat{V}$ does not adjacent to an 1-vertex outside $\hat{V}$ (by ⟨7⟩). By ⟨1⟩ and ⟨6⟩, $\{x_1, x_2, x_3\}$ dominates all other vertices of $G^*$. Hence, only $x_1$ and some $(2, x_1)$-vertices were colored by $c_1$. For the same reason, only $x_2$ (resp. $x_3$), some 1-vertices and some $(2, x_2)$-vertices (resp. $(2, x_3)$-vertices) were colored by $c_2$ (resp. $c_3$). We recolor $\hat{V}, x_2, x_3$ by $c^*$, all $(2, x_1)$-vertices by $c_1$, all $(2, x_2)$-vertices, $\hat{V}_1$, $\hat{V}_3$ and $\hat{V}_0$ by $c_2$, all $(2, x_3)$-vertices and $\hat{V}_2$ by $c_3$. By the comments above and ⟨4⟩, the result is a proper coloring and, hence, an optimal coloring. □

**Lemma 13** *If* $\omega(G) \geq 4$ *and* $Q$ *is not largest, then* $\chi(G) = \max(\chi(G^*), \omega(G))$.

*Proof* Clearly, $\chi(G) \leq \max(\chi(G^*), |Q^*|) + 1$ (by coloring $N(Q^*)$ by one color and the remaining two disconnected parts by at most $\max(\chi(G^*), |Q^*|)$ other colors). As $Q$ is not largest, then $|Q^*| \leq \omega(G) - 2$. As $\omega(G) \leq \chi(G)$ and $\chi(G) \leq \max(\chi(G^*), \omega(G)-2)+1$, then $\chi(G^*) \geq \omega(G)-1$. As $\chi(G) \geq \max(\chi(G^*), \omega(G))$, then the lemma is obvious if $\chi(G^*) = \omega(G) - 1$ (here $\max(\chi(G^*), \omega(G)) = \max(\chi(G^*), |Q^*|) + 1 = \omega(G)$). So, one may assume that $\chi(G^*) \geq \omega(G) \geq 4$. Hence, by Lemmas 11 and 12, there is an optimal coloring of $G^*$ with at most two colors arising in $N(Q^*)$. Hence, an optimal coloring of $G^*$ can be extended to an optimal coloring of $G$ by coloring $Q^*$ by at most $\chi(G^*) - 2$ colors from $G^* \setminus N(Q^*)$. Thus, $\chi(G) = \chi(G^*)$. □

A connected $\{P_5, sinker\}$-free graph will be called *incompressible* if:

- it does not contain simplicial vertices, cut-vertices and quasi-twins
- any its maximal clique with at least 4 vertices has a complex vertex
- any its maximal clique with at least 4 vertices is largest

For any largest clique $Q$, $|Q| \geq 4$ of an incompressible graph $G$ there is a vertex $x$, such that $N(Q \setminus \{x\})$ is independent and any its vertex is adjacent to all vertices of $Q \setminus \{x\}$ (by Lemmas 8 and 9). The clique $Q \setminus \{x\}$ will be called *pre-largest*.

**Lemma 14** *The* COLORING *for* $\{P_5, sinker\}$*-free graphs is reduced in polynomial time to the same problem for incompressible graphs. A list of pre-largest cliques of an incompressible graph is obtained in polynomial time.*

*Proof* The *independent set problem* is to find a largest independent set in a given graph. For a graph it is equivalent to finding a largest clique for its complement.

Note that $\overline{sinker} = P_3 + P_1 + P_1$. If $Free\{G_1, G_2, \ldots\}$ is a hereditary class for which the independent set problem is polynomial-time solvable, then it is so for $Free\{G_1 + P_1, G_2, \ldots\}$ (Malyshev 2012, Theorem 1). As $P_3$-free graphs are the disjoint unions of cliques, then the independent set problem is polynomial-time solvable for $Free(\{\overline{P_5}, \overline{sinker}\})$. So, finding a largest clique for $\{P_5, sinker\}$-free graphs is also polynomial.

The COLORING for $\{P_5, sinker\}$-free graphs can be polynomially reduced to connected graphs in the class without simplicial vertices, cut-vertices and quasi-twins (by Lemmas 1–3). Let $H$ be a connected $\{P_5, sinker\}$-free graph without those vertices. It contains a maximal clique having at least 4 vertices and only simple vertices if and only if there are vertices $a, b, c, d$, such that $(a, b), (b, c), (c, d)$ are edges of $H$, $(a, c) \notin E(H), (b, d) \notin E(H)$ and the set $(N(b) \cap N(c)) \setminus (N(a) \cup N(d))$ has two adjacent vertices. To make sure of this fact see the second part of Lemma 8. There is a trivial polynomial algorithm to check whether the four vertices exist or not. If $Q_1$ and $Q_2$ are maximal cliques of $H$ containing at least two common vertices and $\max(|Q_1|, |Q_2|) \geq 4$, then $|Q_1| = |Q_2|$. Let $|Q_1| \geq |Q_2|$. Clearly, $Q_2 \setminus Q_1 \neq \emptyset$ (as $Q_2$ is maximal) and any element of the set is a $Q_1$-complex vertex (as $H$ is $sinker$-free and $Q_1$ is maximal). Hence, by Lemma 9, $Q_2 \setminus Q_1$ contains only one element. Thus, $Q_2$ consists of $|Q_1| - 1$ vertices of $Q_1$ and one vertex outside $Q_1$. In other words, $|Q_1| = |Q_2|$. Hence, the graph $H$ has a maximal clique with at least 4 vertices that is not largest if and only if it has two adjacent vertices $x$ and $y$, such that the clique number of the subgraph induced by $\{x\} \cup \{y\} \cup (N(x) \cap N(y))$ is at least 4 and at most $w(H) - 1$. It can be verified in polynomial time.Thus, by Lemmas 7 and 13, we have a polynomial reduction to incompressible graphs.

Let $Q_1^*$ and $Q_2^*$ be pre-largest cliques of an incompressible graph $G$. The set $Q_1^* \cap Q_2^*$ is empty (otherwise, neither $N(Q_1^*)$ nor $N(Q_2^*)$ is independent). Hence, any vertex of $G$ belongs to at most one pre-largest clique. This implies that constructing a list of pre-largest cliques of $G$ can be solved in polynomial time. □

**Lemma 15** *For any pre-largest cliques $Q_1^*, Q_2^*$ of an incompressible graph $G$ we have $Q_1^* \cap Q_2^* = N(Q_1^*) \cap Q_2^* = N(Q_2^*) \cap Q_1^* = \emptyset$. If $N(Q_1^*) \subseteq N(Q_2^*)$ or $N(Q_2^*) \subseteq N(Q_1^*)$, then $N(Q_1^*) = N(Q_2^*)$. If $N(Q_1^*) \neq N(Q_2^*)$, then each vertex of $N(Q_1^*) \setminus N(Q_2^*)$ is adjacent to each vertex of $N(Q_2^*) \setminus N(Q_1^*)$.*

*Proof* The emptiness of $Q_1^* \cap Q_2^*$ was proved in the previous lemma. There are no an edge $(x, y)$ between any two vertices $x \in Q_1^*, y \in Q_2^*$. The opposite implies that $x \in N(Q_2^*), y \in N(Q_1^*)$, that is $Q_2^* \subseteq N(x), Q_1^* \subseteq N(y)$ (hence, $Q_1^* \subseteq N(Q_2^*)$, $Q_2^* \subseteq N(Q_1^*)$) and $N(Q_1^*)$, $N(Q_2^*)$ are not independent.

If $N(Q_1^*) \subset N(Q_2^*)$ or $N(Q_2^*) \subset N(Q_1^*)$, then any vertex in $N(Q_1^*) \otimes N(Q_2^*)$ has a neighbor outside $N(Q_1^*) \cup Q_1^* \cup N(Q_2^*) \cup Q_2^*$ (as $N(Q_1^*)$ or $N(Q_2^*)$ has quasi-twins otherwise). Therefore, $G$ contains induced $P_5$. Hence, $N(Q_1^*) = N(Q_2^*)$.

Let $N(Q_1^*) \neq N(Q_2^*)$. Then, $N(Q_1^*) \setminus N(Q_2^*)$ and $N(Q_2^*) \setminus N(Q_1^*)$ are not empty (by the previous paragraph). If $N(Q_1^*) \cap N(Q_2^*)$ is not empty, then the claim is obvious (to avoid induced $P_5$). Let $N(Q_1^*) \cap N(Q_2^*) = \emptyset$. A shortest path between a vertex of $Q_1^*$ and a vertex of $Q_2^*$ exists (as $G$ is connected) and it has at most three edges (as $G$ is $P_5$-free). Hence, a vertex $x \in N(Q_1^*)$ is adjacent to a vertex $y \in N(Q_2^*)$. Assume that there are $x' \in N(Q_1^*)$ and $y' \in N(Q_2^*)$, such that $(x', y') \notin E(G)$. If $(x, y') \in E(G)$, then $x' \neq x$. The vertex $x'$, a vertex in $Q_1^*$, $x$, $y'$ and an element of $Q_2^*$ induce $P_5$. Let $(x, y') \notin E(G)$. Then, $y'$, a vertex in $Q_2^*$, $y$, $x$ and an element of $Q_1^*$ induce the 5-path. So, our assumption was false. □

Two different pre-largest cliques $Q_1^*$, $Q_2^*$ of an incompressible graph will be called *near* if $N(Q_1^*) \cap N(Q_2^*) \neq \emptyset$ and $N(Q_1^*) \neq N(Q_2^*)$. If an incompressible graph $G$ has near pre-largest cliques, then $\chi(G) \geq \omega(G) + 1$ (by Lemma 15). Moreover, if $\omega(G) \geq 15$, then $\chi(G) = \omega(G) + 1$ by the following lemma.

**Lemma 16** *For any $\{P_5, sinker\}$-free graph $H$ we have $\chi(H) \leq \max(15, \omega(H)) + 1$.*

*Proof* We will prove the bound using the mathematical induction on the number of vertices in graphs. Its base case (for the one-vertex graph) is clear. Suppose that the inequality holds for all $\{P_5, sinker\}$-free with less than $n$ vertices and $H$ is such a graph having $n$ vertices. By Lemmas 1-3, one can consider that $H$ is a connected graph without simplicial vertices, cut-vertices and quasi-twins. A. Guarfas showed that any $P_5$-free graph $H'$ has the chromatic number at most $4^{\omega(H')-1}$ (Gyarfas 1987, Theorem 2.4). Hence, the bound is trivial, when $\omega(H) \leq 3$. We assume that $\omega(H) \geq 4$.

Let $Q$ be any largest clique of $G$. If there are no $Q$-complex vertices, then $\chi(H) = \chi(H \setminus Q)$ (by Lemma 7). By the induction hypothesis, $\chi(H \setminus Q) \leq \max(15, \omega(H \setminus Q)) + 1 \leq \max(15, \omega(H)) + 1$. Hence, $\chi(H) \leq \max(15, \omega(H)) + 1$. Let there are $Q$-complex vertices. By analogy with the proof of Lemma 13, for some $x \in Q$ we have $\chi(H) \leq \max(\chi(H \setminus Q^*), |Q^*|) + 1 = \max(\chi(H \setminus Q^*), w(H) - 1) + 1$, where $Q^* = Q \setminus \{x\}$. One can assume that $\chi(H \setminus Q^*) \geq 16$ and $\chi(H \setminus Q^*) \geq \omega(H) + 1$ (otherwise, the target inequality is obvious). Taking into account the induction hypothesis for $H \setminus Q^*$ and the inequality $\omega(H \setminus Q^*) \leq \omega(H)$ we conclude $\chi(H \setminus Q^*) = \omega(H) + 1$. By Lemmas 11 and 12, an optimal coloring of $H \setminus Q^*$ is extendable to a proper coloring of $H$ with $\omega(H) + 1$ colors by coloring $Q^*$ by $w(H) - 1$ colors from $H \setminus (N(Q^*) \cup Q^*)$. □

**Lemma 17** *Let $G$ be an incompressible graph with $\omega(G) \geq 18$. If $G$ has two near pre-largest cliques, then $\chi(G) = \omega(G) + 1$. Otherwise, $\chi(G) = \omega(G)$.*

*Proof* The lemma is clear if $G$ really has near pre-largest cliques. Let $Q_1^*, \ldots, Q_k^*$ be all pre-largest its cliques and there are no two near pre-largest cliques. Any two pre-largest cliques have coinciding neighborhoods or their neighborhoods do not intersect (by Lemma 15). Let $\{N(Q_1^*), \ldots, N(Q_k^*)\} = \{N(Q_{i_1}^*), \ldots, N(Q_{i_s}^*)\}$. An element of $N(Q_{i_1}^*)$, an element of $N(Q_{i_2}^*)$,…, an element of $N(Q_{i_s}^*)$ constitute the $s$-clique (by Lemma 15). If $s > 3$, then some maximal under inclusion clique of $G$ containing the $s$-clique is largest (by the definition of an incompressible graph). Deleting some its vertex produces a pre-largest clique that must belong to $\{Q_1^*, \ldots, Q_k^*\}$. We have a contradiction. Hence, $s \leq 3$. For the same reason, $H = G \setminus \bigcup_{i=1}^{k}(Q_i^* \cup N(Q_i^*))$ can not have a clique with at least 4 vertices. We have $\chi(H) \leq 16$ (Guarfas 1987, Theorem 2.4). As $N(Q_{i_1}^*), \ldots, N(Q_{i_s}^*)$ are independent, then each of them can be properly colored with its own color. Thus, to extend it to a proper coloring of $G$ we need $\omega(G) - s$ additional colors (as every $Q_i^*$ is colorable by $\omega(G) - 1$ colors depending on a color of $N(Q_i^*)$ and $\chi(H) \leq 16 \leq \chi(G) + 1 - s$). Hence, $\chi(G) = \omega(G)$. □

The following theorem is the main claim of the section.

**Theorem 2** *The* COLORING *is polynomially solvable for* $\{P_5, sinker\}$*-free graphs.*

*Proof* By Lemma 14, we may consider only incompressible graphs with known lists of pre-largest cliques. For each fixed $k$ the set of $P_5$-free graphs without the $k$-clique consists of a polynomial case for the problem (Hoang et al. 2010). Hence, we may consider incompressible graphs with known lists of pre-largest cliques and the clique number at least 18 (as the fact that the clique number of a graph is at least 18 is decided in polynomial time). Let $G$ be a graph of this type. If $G$ has two near pre-largest cliques (this fact is verified in polynomial time), then $\chi(G) = \omega(G) + 1$ (by Lemma 17). Otherwise, $\chi(G) = \omega(G)$ (by Lemma 17). □

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