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k-symplectic structures and absolutely trianalytic subvarieties in hyperkähler manifolds

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ABSTRACT

Let (M, I, J, K) be a hyperkähler manifold, and $Z \subset (M, I)$ a complex subvariety in (M, I). We say that Z is trianalytic if it is complex analytic with respect to J and K, and absolutely trianalytic if it is trianalytic with respect to any hyperkähler triple of complex structures (M, I, J', K') containing I. For a generic complex structure I on M, all complex subvarieties of (M, I) are absolutely trianalytic. It is known that the normalization Z' of a trianalytic subvariety is smooth; we prove that $b_2(Z') \geqslant b_2(M)$, when M has maximal holonomy (that is, M is IHS).

To study absolutely trianalytic subvarieties further, we define a new geometric structure, called k-symplectic structure; this structure is a generalization of hypersymplectic structure. A k-symplectic structure on a $2d$-dimensional manifold X is a k-dimensional space R of closed 2-forms on X which all have rank $2d$ or d. It is called non-degenerate if the set of all degenerate forms in R is a smooth, non-degenerate quadric hypersurface in R.

We consider absolutely trianalytic tori in a hyperkähler manifold M of maximal holonomy. We prove that any such torus is equipped with a non-degenerate k-symplectic structure, where $k = b_2(M)$. We show that the tangent bundle TX of a k-symplectic manifold is a Clifford module over a Clifford algebra $Cl(k - 1)$. Then an absolutely trianalytic torus in a hyperkähler manifold M with $b_2(M) \geqslant 2r + 1$ is at least 2^{r-1}-dimensional.

1. Introduction

1.1. Absolutely trianalytic subvarieties in hyperkähler manifolds

Let M be a Kähler, compact, holomorphic symplectic manifold. Calabi–Yau theorem [1] implies that M admits a Ricci-flat metric g, unique in each Kähler class. Using Berger's classification of Riemannian holonomies and Bochner vanishing, one shows that the Levi-Civita connection of g preserves a triple of complex structures I, J, K satisfying the quaternionic relation $IJ = -JI = K$ [2]. A Riemannian manifold admitting a triple of complex structures I, J, K satisfying quaternionic relations and Kähler with respect to g is called **hyperkähler**. One can construct a holomorphic symplectic form on any hyperkähler manifold as follows. There are Kähler forms ω_I, ω_J and ω_K associated to the complex structures I, J and K. One can check that the 2-form $\omega_J + \sqrt{-1}\omega_K$ is of Hodge type $(2, 0)$ with respect to I. Since it is also closed it is a holomorphic symplectic form. So we shall treat the terms "hyperkähler" and "holomorphic symplectic" as (essentially) synonyms.

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Given any triple $a, b, c \in \mathbb{R}$, $a^2 + b^2 + c^2 = 1$, the operator $L = aI + bJ + cK$ satisfies $L^2 = -1$ and defines a Kähler structure on (M, g); such a complex structure is called **induced by the hyperkähler structure**. Complex subvarieties of such (M, L) for generic (a, b, c) were studied in [3,4] and [5].

Definition 1.1. Let $Z \subset M$ be a closed subset of a hyperkähler manifold. It is called **trianalytic**, if it is complex analytic with respect to all induced complex structures L.

In the definition above it is enough to require the subvariety Z to be complex analytic with respect only to I and J. Then it will automatically be complex analytic with respect to any induced complex structure. This is clear, because Z is trianalytic if and only if for all smooth points $z \in Z$, the space $T_zZ \subset T_zM$ is preserved by the quaternion algebra $\mathbb{H}$ [4]. However, $\mathbb{H}$ is generated by any two non-collinear elements I, I_1 with $I^2 = I_1^2 = -1$.

Singularities of trianalytic subvarieties always admit a hyperkähler resolution.

Theorem 1.2 (*[6]*)**.** *Let M be a hyperkähler manifold, $Z \subset M$ a trianalytic subvariety, and I an induced complex structure. Consider the normalization*

$$\widetilde{(Z, I)} \xrightarrow{n} (Z, I)$$

of (Z, I). Then $\widetilde{(Z, I)}$ is smooth, and the map $\widetilde{(Z, I)} \longrightarrow M$ is an immersion, inducing a hyperkähler structure on $\widetilde{(Z, I)}$. ∎

Definition 1.3. Let M be a hyperkähler manifold, and S the family of all induced complex structures $L = aI + bJ + cK$, where $a, b, c \in \mathbb{R}$, $a^2 + b^2 + c^2 = 1$. Then S is called **the twistor family** of complex structures.

The following theorem implies that whenever L is a generic element of a twistor family, all subvarieties of (M, L) are trianalytic.

Theorem 1.4 (*[4,5]*)**.** .
Let M be a hyperkähler manifold, S its twistor family. Then there exists a countable subset $S_1 \subset S$, such that for any complex structure $L \in S \setminus S_1$, all compact complex subvarieties of (M, L) are trianalytic. ∎

In [7] (see also [8,9] and [10]), this theorem was used to study subvarieties of generic deformations of a compact holomorphic symplectic manifold M. Recall that the Teichmüller space of M is the quotient $\mathsf{Teich} = \mathsf{Comp}/\mathsf{Diff}^0$ of the (infinite-dimensional) space of all complex structures of hyperkähler type by the group Diff^0 of isotopies [11]. Teich is a complex, non-Hausdorff manifold. If we fix a complex structure I on M, then the connected component of the Teichmüller space containing I can be identified with a connected component of the so-called "marked moduli space" of deformations of (M, I) [11].

Definition 1.5. Let (M, I, J, K) be a compact, holomorphic symplectic, Kähler manifold, and $Z \subset (M, I)$ a complex subvariety, which is trianalytic with respect to any hyperkähler structure compatible with I. Then Z is called **absolutely trianalytic**.

Definition 1.6. For a given complex structure I, consider **the Weil operator** W_I acting on (p, q) forms as $\sqrt{-1}(p - q)$. Let $G_{MT}(M, I)$ be a smallest rational algebraic subgroup of $\mathrm{Aut}(H^*(M, \mathbb{R}))$ containing e^{tW_I}. This group is called **the Mumford–Tate group of** (M, I). A group generated by $G_{MT}(M, I)$ for all complex structures I in a connected component of a deformation space is called **a maximal Mumford–Tate group of** M [12].

It is not hard to check that the Mumford–Tate group $G_{MT}(I)$ of (M, I) is lower semicontinuous as a function of $I \subset \mathsf{Teich}$ in Zariski topology on Teich [12]. This implies that $G_{MT}(I)$ is constant outside of countably many complex subvarieties of positive codimension. We call $I \in \mathsf{Teich}$ **Mumford–Tate generic** if $G_{MT}(I)$ is maximal. If M has maximal holonomy, the maximal Mumford–Tate group is isomorphic to $\mathrm{Spin}(H^2(M, \mathbb{R}), q)$ [8]. Any $I \in \mathsf{Teich}$ outside of a countably many subvarieties of positive codimension is Mumford–Tate generic.

Remark 1.7. Let I be a Mumford–Tate generic complex structure, and η an integer (p, p)-class. Then η is of type (p, p) for any deformation of I.

Absolutely trianalytic subvarieties can be characterized in terms of the Mumford–Tate group, as follows.

Claim 1.8. *Let (M, I, J, K) be a hyperkähler manifold, and $Z \subset (M, I)$ be a complex subvariety. Then Z is absolutely trianalytic if and only if its fundamental class is G-invariant, where G is a maximal Mumford–Tate group of M. In particular, Z is absolutely trianalytic when (M, I) is Mumford–Tate generic.*

Proof. This statement follows from the definitions ([13, Claim 4.4]). ∎

Clearly, the set of absolutely trianalytic subvarieties does not change if one passes from one complex structure in a twistor family S to another complex structure in S. On the other hand, any two complex structures in the same component of Teich can be connected by a sequence of twistor families [14]. This means that the set of absolutely trianalytic subvarieties in (M, I) is determined by the connected component of Teich where I lies.

This can be used to prove the following theorem.

Theorem 1.9. *Let $I_1, I_2 \in$ Teich be points in the same connected component of the Teichmüller space. Then there exists a diffeomorphism $\nu : (M, I_1) \longrightarrow (M, I_2)$ such that any absolutely trianalytic subvariety $Z \subset (M, I_1)$ is mapped to an absolutely trianalytic subvariety $\nu(Z) \subset (M, I_2)$.* ■

Absolutely trianalytic subvarieties were studied in [7], where it was shown that a general deformation of a Hilbert scheme of a K3 surface has no complex (or, equivalently, no absolutely trianalytic) subvarieties. In [15], Kaledin and Verbitsky used the same argument to study absolutely trianalytic subvarieties in generalized Kummer varieties. They "proved" non-existence of such subvarieties, but their argument was faulty (in fact, there exists an absolutely trianalytic subvariety in a generalized Kummer variety). In [9], the error was found, and the argument was repaired to show that any absolutely trianalytic subvariety Z of a generalized Kummer variety is a deformation of a resolution of singularities of a quotient of a torus by a Weyl group action. The idea was to show that Z is a resolution of singularities of a quotient of a flat subtorus in a symmetric power of a torus, and classify the group actions which admit a holomorphic symplectic resolution. Later, Ginzburg and Kaledin have shown that only the Weyl groups A_n, B_n, C_n can occur in quotient maps with quotients which admit holomorphically symplectic resolution of singularities [16].

Non-existence of absolutely trianalytic subvarieties in a Hilbert scheme M of K3 was used in [8] to prove compactness of deformation spaces of certain stable holomorphic bundles on M.

1.2. Hyperkähler manifolds

Before we state our main results, let us introduce the Bogomolov–Beauville–Fujiki form and maximal holonomy manifolds.

Definition 1.10. A compact hyperkähler manifold M is called **simple**, or **maximal holonomy**, or **IHS** (from "Irreducible Holomorphic Symplectic") if $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}$.

Remark 1.11. It follows from Bochner's vanishing and Berger's classification of holonomy groups that a hyperkähler manifold has maximal holonomy $\mathrm{Sp}(n)$ whenever $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$ [2]. This explains the term.

Theorem 1.12 (*Bogomolov's Decomposition; [17]*)**.** *Any compact hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.* ■

Theorem 1.13 (*Fujiki, [18]*)**.** *Let M be a simple hyperkähler manifold of complex dimension $2n$. Then there exists a primitive integral quadratic form q on $H^2(M, \mathbb{Z})$ and a constant c_M such that for any $\eta \in H^2(M, \mathbb{C})$ we have $\int_M \eta^{2n} = c_M q(\eta, \eta)^n$.* ■

Remark 1.14. Theorem 1.13 determines the form q uniquely up to a sign. For $\dim_{\mathbb{R}} M = 4n$, n odd, the sign is also determined. For n even, the sign is determined by the following formula, due to Bogomolov and Beauville.

$$\mu q(\eta, \eta) = (n/2) \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - (1-n) \left(\int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n} \right) \left(\int_X \eta \wedge \Omega^{n} \wedge \overline{\Omega}^{n-1} \right) \tag{1.1}$$

where Ω is the holomorphic symplectic form, and $\mu > 0$ a positive constant.

Definition 1.15. Let M be a hyperkähler manifold of maximal holonomy, and q the form on $H^2(M)$ defined by Remark 1.14 and Theorem 1.13. Then q is called **Bogomolov–Beauville–Fujiki form** (BBF form). Eq. (1.1) (or, even better, a similar equation [19, (1.1)], using the Kähler form instead of the holomorphic symplectic form) can be used to show that q is non-degenerate and has signature $(3, b_2(M) - 3)$.

We will need Fujiki relations in greater generality which we also recall (the proof can be found in [20]). Namely, let M be a simple hyperkähler manifold with $\dim_{\mathbb{C}} M = 2n$. Consider a cohomology class γ which is of Hodge type $(2n-2m, 2n-2m)$ on any small deformation of M. Then there exists a constant $c_\gamma \in \mathbb{R}$ such that for any $\alpha, \beta \in H^2(X, \mathbb{C})$ we have

$$\gamma \cdot \alpha^{2m-1} \cdot \beta = c_\gamma q(\alpha, \alpha)^{m-1} q(\alpha, \beta). \tag{1.2}$$

In this equation on the left hand side we have the intersection product in cohomology of M.

1.3. Main results: k-symplectic structures and absolutely trianalytic subvarieties

In the present paper, we prove two bounds on the Betti numbers of absolutely trianalytic subvarieties, quite restrictive for their geometry. In particular, we prove that the normalization of a proper complex subvariety of a generic deformation of a 10-dimensional O'Grady space must necessarily belong to a new type of hyperkähler manifolds Theorem 1.20.

Our arguments are based on an elementary observation, stated below as Theorem 1.16.

Recall Theorem 1.2 that the normalization of any trianalytic subvariety $Z \hookrightarrow M$ is a smooth hyperkähler manifold $\tilde{Z}$ immersed into M. Therefore, we can replace any trianalytic cycle by an immersed hyperkähler manifold. Note that the complex dimension of any trianalytic subvariety is even.

Theorem 1.16. *Let M be a maximal holonomy hyperkähler manifold and $\tilde{Z} \xrightarrow{\varphi} M$ the normalization of an absolutely trianalytic cycle $Z \subset M$ or a finite covering of such normalization. Consider the polynomial $P_{\tilde{Z}}$ on $H^2(\tilde{Z}, \mathbb{C})$ mapping a cohomology class η to $\int_{\tilde{Z}} \eta^{\dim_{\mathbb{C}} \tilde{Z}}$. Then for any $\alpha \in H^2(M, \mathbb{C})$, one has $P_{\tilde{Z}}(\varphi^*\alpha) = \deg(\varphi) c_Z q(\alpha, \alpha)^{\frac{1}{2}\dim_{\mathbb{C}} Z}$, where c_Z is a constant determined by M and Z.*

Proof. Denote by γ the fundamental class of Z in the cohomology of M. We have $P_{\tilde{Z}}(\varphi^*\alpha) = \deg(\varphi)\gamma \cdot \alpha^{\dim_{\mathbb{C}} Z} = c_Z q(\alpha, \alpha)^{\frac{1}{2}\dim_{\mathbb{C}} Z}$. The last equality follows from (1.2) where we put $\beta = \alpha$ and $c_Z = c_\gamma$. ■

This simple result has very nice consequences.

Corollary 1.17. *Let M be a maximal holonomy hyperkähler manifold, and $\tilde{Z} \xrightarrow{\varphi} M$ the normalization of an absolutely trianalytic cycle $Z \subset M$, or a finite covering of the normalization. Then the induced map $H^2(M, \mathbb{C}) \xrightarrow{\varphi^*} H^2(\tilde{Z}, \mathbb{C})$ is injective.*

Proof. Given $x \in \ker \varphi^* \subset H^2(M, \mathbb{C})$ and any $y \in H^2(M, \mathbb{C})$, one would obtain

$$\begin{aligned}\deg(\varphi) c_Z q(x+y, x+y)^{\frac{1}{2}\dim_{\mathbb{C}} Z} &= P_{\tilde{Z}}(\varphi^*(x+y)) = P_{\tilde{Z}}(\varphi^* y) \\ &= \deg(\varphi) c_Z q(y, y)^{\frac{1}{2}\dim_{\mathbb{C}} Z}.\end{aligned}$$

This gives $q(x, y) = 0$ for all $y \in H^2(M, \mathbb{C})$. However, this implies $x = 0$, because q is non-degenerate. ■

Corollary 1.18. *Let M be a maximal holonomy hyperkähler manifold, $Z \subset M$ an absolutely trianalytic subvariety, and $\tilde{Z} \longrightarrow M$ its normalization. Consider a finite covering $\tilde{Z}_1 \longrightarrow \tilde{Z}$ such that $\tilde{Z}_1$ is a product of a hyperkähler torus T and several maximal holonomy hyperkähler manifolds K_i (such a decomposition always exists by Bogomolov decomposition Theorem 1.12). Then $b_2(T) \geqslant b_2(M)$ and $b_2(K_i) \geqslant b_2(M)$.*

Proof. Any hyperkähler metric on M induces a hyperkähler structure on $\tilde{Z}_1$. Each component in Bogomolov decomposition of $\tilde{Z}_1$ is immersed into M as a hyperkähler subvariety. Hence the images of those components are also absolutely trianalytic subvarieties of M and we can apply Corollary 1.17 to them. ■

In this paper we are also interested in absolutely trianalytic subvarieties $Z \subset M$ such that the normalization $\tilde{Z}$ is a torus. We prove the following theorem.

Theorem 1.19. *Let M be a hyperkähler manifold of maximal holonomy, T a hyperkähler torus, and $T \longrightarrow M$ a hyperkähler immersion with absolutely trianalytic image. Then $b_1(T) \geqslant 2^{\lfloor (b_2(X)-1)/2 \rfloor}$.*

Proof. Corollary 2.15. ■

Together with Corollary 1.18, this allows to prove non-existence of subvarieties of known type in a 10-dimensional O'Grady manifold M (Remark 2.16), which has $b_2(M) = 24$.

Theorem 1.20. *Let $Z \subset M$ be a proper complex subvariety of a general deformation of 10-dimensional O'Grady manifold, $\tilde{Z}$ its normalization, and $\tilde{Z}_1$ its covering equipped with the Bogomolov decomposition obtained as in Corollary 1.18. Then $\tilde{Z}_i = \prod_i K_i$ where K_i are maximal holonomy hyperkähler manifolds with $b_2 \geqslant 24$, that is, previously unknown type.* ■

Remark 1.21. A maximal holonomy hyperkähler manifold K_i with $\dim_{\mathbb{C}} K_i = 2$ satisfies $b_2(K_i) \leqslant 22$ (Enriques–Kodaira classification, see e.g. [2]). When $\dim_{\mathbb{C}} K_i = 4$, one has $b_2(K_i) \leqslant 23$ [21]. Therefore, any absolutely trianalytic subvariety in a 10-dimensional O'Grady manifold (if it exists) satisfies $\dim_{\mathbb{C}} Z \geqslant 6$, and has maximal holonomy.

For a 6-dimensional O'Grady manifold M, one has $b_2(M) = 8$, so we cannot prove analogous result by our methods. However, complex tori are still prohibited.

Theorem 1.22. *Let M be a general deformation of a 6-dimensional O'Grady manifold. Then any holomorphic map from a complex torus to M is trivial.*

Proof. Since an image of such a map is absolutely trianalytic, its normalization is hyperkähler. However, a dominant holomorphic map from a torus T to a manifold Z with trivial canonical bundle is a projection to a quotient torus. This is clear because fibers of such map have trivial canonical bundle by adjunction formula, but a Calabi–Yau submanifold in a torus is also a torus. Then Theorem 1.22 follows from Remark 2.17. ■

2. k-symplectic structures and Clifford representations

The proof of Theorem 1.19 is based on a discovery of a previously unknown geometric structure, called k-symplectic structure. In Section 2.1 we study k-symplectic structures on vector spaces, and in Section 2.2 we apply these linear-algebraic results to algebraic geometry of absolutely trianalytic tori.

k-symplectic structures generalize the hypersymplectic structures known for a long time [22–24], and trisymplectic (3-symplectic) structures defined in [25].

2.1. k-symplectic structures on vector spaces

Consider a complex vector space V of dimension $\dim_{\mathbb{C}} V = 4n$. Let k be a non-negative integer.

Definition 2.1. A k-symplectic structure on V is a subspace $\boldsymbol{\Omega} \subset \Lambda^2 V^*$ of dimension k, such that for some non-zero quadratic form $q \in S^2\boldsymbol{\Omega}^*$ the following condition is satisfied: for any non-zero $\omega \in \boldsymbol{\Omega}$ we have

$$\dim(\ker \omega) = \begin{cases} 2n, & \text{if } q(\omega) = 0; \\ 0, & \text{otherwise.} \end{cases}$$

A k-symplectic structure is called non-degenerate if the quadratic form q is non-degenerate. A vector space with a k-symplectic structure will be called a k-symplectic vector space.

A few remarks about this definition.

Remark 2.2. The quadratic form q in the definition above is unique up to a non-zero multiplier. We can reformulate the condition from the definition as follows. Consider the Pfaffian variety

$$\mathcal{P} = \{\omega \in \mathbb{P}(\Lambda^2 V^*) \mid \omega^{2n} = 0\}$$

in $\mathbb{P}(\Lambda^2 V^*)$. A subspace $\boldsymbol{\Omega} \subset \Lambda^2 V^*$ defines a k-symplectic structure if and only if $\mathbb{P}\boldsymbol{\Omega} \cap \mathcal{P}$ is a quadric (necessarily of multiplicity n), and all the forms ω lying on this quadric have rank $2n$. The quadratic form q is the one defining this quadric. If it is necessary to mention the form q explicitly, we will denote a k-symplectic structure by $(\boldsymbol{\Omega}, q)$.

Remark 2.3. A 1-symplectic structure is the same thing as a non-zero two-form $\omega \in \Lambda^2 V^*$ which is either non-degenerate or has rank $2n$.

Remark 2.4. Consider a non-degenerate 2-symplectic structure $(\boldsymbol{\Omega}, q)$ on V. The quadric in $\mathbb{P}\boldsymbol{\Omega} = \mathbb{P}^1$ defined by q is non-degenerate, hence it consists of two distinct points. Denote the corresponding two-forms by ω_1 and ω_2. The kernels V_1 and V_2 of these forms have dimension $2n$ by definition of a 2-symplectic structure. Moreover, $V_1 \cap V_2 = 0$, because the generic linear combination of ω_1 and ω_2 must be non-degenerate. Then we have $V = V_1 \oplus V_2$.

Conversely, given a symplectic vector space (W, ω), consider the direct sum $V = W^{\oplus 2}$ and denote by $\pi_i : V \to W$ the projection to the ith summand. Then the subspace $\boldsymbol{\Omega} \subset \Lambda^2 V^*$ spanned by $\pi_1^*\omega$ and $\pi_2^*\omega$ defines a non-degenerate 2-symplectic structure on V.

Remark 2.5. Given a k-symplectic structure $(\boldsymbol{\Omega}, q)$, we can consider any subspace $\boldsymbol{\Omega}' \subset \boldsymbol{\Omega}$. Then $(\boldsymbol{\Omega}', q|_{\boldsymbol{\Omega}'})$ is a k'-symplectic structure for $k' = \dim_{\mathbb{C}} \boldsymbol{\Omega}'$. We will call it a substructure of $(\boldsymbol{\Omega}, q)$. Note that such substructure can be degenerate even if the initial structure was not.

Remark 2.6. Consider a vector space V of complex dimension 4. Then $\boldsymbol{\Omega} = \Lambda^2 V$ gives a 6-symplectic structure on V, since the set of all degenerate two-forms on V is a Plücker quadric, and all non-zero degenerate two-forms have two-dimensional kernel.

There exist examples of non-degenerate k-symplectic structures for every k, as follows from the construction described below.

Recall the definition of Clifford algebras (for the proofs of all statements about Clifford algebras, see [26]). We will always assume that all vector spaces are either real or complex, however the definition below is valid for any field. Consider a vector space E with quadratic form $q \in S^2 E^*$. By definition, the Clifford algebra $\mathcal{C}\ell(E, q)$ is the quotient of the tensor algebra $T^\bullet E$ by the two-sided ideal generated by all tensors of the form $x \otimes x - q(x, x) \cdot 1$, for all $x \in E$. The elements generating the ideal belong to the even part of the tensor algebra, so the Clifford algebra is $\mathbb{Z}/2\mathbb{Z}$-graded, $\mathcal{C}\ell(E, q) = \mathcal{C}\ell^0(E, q) \oplus \mathcal{C}\ell^1(E, q)$. There exists an automorphism $\tau : \mathcal{C}\ell(E, q) \to \mathcal{C}\ell(E, q)$, which acts as the identity on $\mathcal{C}\ell^0(E, q)$ and as multiplication by -1 on $\mathcal{C}\ell^1(E, q)$. There is also an antiautomorphism of transposition which maps $x = x_1 \cdot \cdots \cdot x_k$ to $x^t = x_k \cdot \cdots \cdot x_1$ for any $x_i \in E$. We will use the notation $\bar{x} = \tau(x^t)$ for $x \in \mathcal{C}\ell(E, q)$.

One can check that the space E is embedded into $\mathcal{C}\ell(E, q)$ via the map $E = T^1 E \to \mathcal{C}\ell(E, q)$ induced by the projection $T^\bullet E \to \mathcal{C}\ell(E, q)$. The Clifford algebra has the following universal property. Let $\mathcal{A}$ be any associative algebra with unit and $\alpha : E \to \mathcal{A}$ a map with the property that $\alpha(x)^2 = q(x, x) \cdot 1_{\mathcal{A}}$ for all $x \in E$. Then α can be uniquely extended to a morphism of algebras $\alpha' : \mathcal{C}\ell(E, q) \to \mathcal{A}$ such that $\alpha'|_E = \alpha$.

Recall the definition of the group $\mathrm{Pin}(E, q)$ (see [26]). We will denote by $\mathcal{C}\ell^\times(E, q)$ the multiplicative group of invertible elements. Define

$$\mathrm{Pin}(E, q) = \{x \in \mathcal{C}\ell^\times(E, q) \mid \tau(x)Ex^{-1} = E,\ x\bar{x} = 1\}.$$

One can check that $\mathrm{Pin}(E, q)$ acts on E preserving the quadratic form q and is actually a double covering of the orthogonal group $\mathrm{O}(E, q)$. The group $\mathrm{Spin}(E, q)$ is defined as the subgroup of even elements in $\mathrm{Pin}(E, q)$.

Example 2.7. For each integer $k > 0$ we will construct an example of a k-symplectic structure on some vector space. Start from a real k-dimensional vector space $E_{\mathbb{R}}$ with a negative-definite quadratic form q. Consider the Clifford algebra $\mathcal{C}\ell(E_{\mathbb{R}}, q)$ and the corresponding group $\mathrm{Pin}(E_{\mathbb{R}}, q)$. This group is compact (because q is negative-definite) and it acts on $E_{\mathbb{R}}$ by orthogonal transformations.

Lemma 2.8. *Consider a non-trivial real representation $\rho\colon \mathcal{C}\ell(E_{\mathbb{R}}, q) \to \mathrm{End}(V_{\mathbb{R}})$ in some real vector space $V_{\mathbb{R}}$, $\dim V_{\mathbb{R}} = 4n$ (by this we mean a representation of $\mathcal{C}\ell(E_{\mathbb{R}}, q)$ as an algebra with unit). For any representation $V_{\mathbb{R}}$ of $\mathcal{C}\ell(E_{\mathbb{R}}, q)$ as above, we have a $\mathrm{Spin}(E_{\mathbb{R}}, q)$-equivariant embedding*

$$\alpha_{\mathbb{R}}\colon E_{\mathbb{R}} \hookrightarrow \Lambda^2 V_{\mathbb{R}}^*.$$

All non-zero two-forms in the image of $\alpha_{\mathbb{R}}$ are non-degenerate.

Proof. If we have a representation $V_{\mathbb{R}}$, then the compact group $\mathrm{Pin}(E_{\mathbb{R}}, q)$ acts on $V_{\mathbb{R}}$, and there exists a positive-definite invariant quadratic form $g \in (S^2 V_{\mathbb{R}}^*)^{\mathrm{Pin}(E_{\mathbb{R}}, q)}$.

Note that any element $e \in E_{\mathbb{R}}$ with $e^2 = -1$ lies in $\mathrm{Pin}(E_{\mathbb{R}}, q)$. This follows from direct computation: we have $\tau(e) = -e$, $e\bar{e} = 1$ and for any $x \in E_{\mathbb{R}}$ we have $\tau(e)xe^{-1} = exe = 2q(x, e)e - xe^2 = 2q(x, e)e + x \in E_{\mathbb{R}}$. From this and the invariance of g with respect to $\mathrm{Pin}(E_{\mathbb{R}}, q)$ we conclude that $g(e \cdot u, e \cdot v) = g(u, v)$ for any $u, v \in V_{\mathbb{R}}$.

We can define the two-form $\omega_e \in \Lambda^2 V_{\mathbb{R}}^*$ by the formula $\omega_e(u, v) = g(e \cdot u, v)$ for $u, v \in V_{\mathbb{R}}$. This really defines an element in $\Lambda^2 V_{\mathbb{R}}^*$ since $g(e \cdot u, v) = g(e^2 \cdot u, e \cdot v) = -g(u, e \cdot v) = -g(e \cdot v, u)$.

The quadratic form q was chosen negative-definite, so every element $x \in E_{\mathbb{R}}$ is of the form $x = \xi e$ with $\xi \in \mathbb{R}$ and $e^2 = -1$, so $\omega_x = \xi\omega_e \in \Lambda^2 V_{\mathbb{R}}^*$. We have constructed a map $\alpha_{\mathbb{R}}\colon E_{\mathbb{R}} \hookrightarrow \Lambda^2 V_{\mathbb{R}}^*$, $x \mapsto \omega_x$, which is clearly an embedding.

Note that this embedding is a morphism of $\mathrm{Spin}(E_{\mathbb{R}}, q)$-modules: for any $h \in \mathrm{Spin}(E_{\mathbb{R}}, q)$ we have $\tau(h) = h$, $\alpha_{\mathbb{R}}(h \cdot x) = \alpha_{\mathbb{R}}(hxh^{-1}) = \omega_{hxh^{-1}}$ and $\omega_{hxh^{-1}}(u, v) = g(hxh^{-1} \cdot u, v) = g(xh^{-1} \cdot u, h^{-1} \cdot v) = \omega_x(h^{-1} \cdot u, h^{-1} \cdot v)$, so $\alpha_{\mathbb{R}}(h \cdot x) = h \cdot \omega_x$.

For all non-zero $x \in E_{\mathbb{R}}$ the form ω_x is non-degenerate, because the quadratic form g is chosen positive-definite (and in particular non-degenerate). ∎

Consider the complexifications: $E = E_{\mathbb{R}} \otimes \mathbb{C}$, $\mathcal{C}\ell(E, q) = \mathcal{C}\ell(E_{\mathbb{R}}, q) \otimes \mathbb{C}$ and $V = V_{\mathbb{R}} \otimes \mathbb{C}$ with the corresponding complex representation of $\mathcal{C}\ell(E, q)$. Then we get an embedding of complex vector spaces $\alpha\colon E \hookrightarrow \Lambda^2 V^*$.

So we have an irreducible $\mathrm{Spin}(E, q)$-module E embedded into $\Lambda^2 V^*$. Fix a volume form and identify $\Lambda^{4n} V^*$ with $\mathbb{C}$. Then the wedge product in $\Lambda^\bullet V^*$ induces a $\mathrm{Spin}(E, q)$-invariant polynomial $p \in S^{2n}E^*$, $p(x) = \omega_x^{\wedge 2n}$ on E. This polynomial is non-zero, because the image of α contains non-degenerate two-forms by Lemma 2.8.

The group $\mathrm{Spin}(E, q)$ is a double covering of $\mathrm{SO}(E, q)$, so p is $\mathrm{SO}(E, q)$-invariant and by classical invariant theory this polynomial has to be proportional to the nth power of q.

We can define $\boldsymbol{\Omega}$ to be the image of E in $\Lambda^2 V^*$ under the map α. We have already seen that the set of degenerate two-forms is a quadric. To see that $\boldsymbol{\Omega}$ is a k-symplectic structure it only remains to check that all degenerate two-forms in $\boldsymbol{\Omega}$ have kernels of dimension $2n$. This is proved in the following lemma which we will also need later on.

Lemma 2.9. *Let V be a complex vector space of dimension $4n$. Let $\boldsymbol{\Omega} \subset \Lambda^2 V^*$ be a k-dimensional subspace. Identifying $\Lambda_{\mathbb{C}}^{4n} V^*$ with $\mathbb{C}$, consider a polynomial $p \in S^{2n}\boldsymbol{\Omega}^*$ given by $p(\omega) = \omega^{\wedge 2n}$. Suppose that there exists a non-degenerate quadratic form $q \in S^2\boldsymbol{\Omega}^*$, such that $p = q^n$. Then all degenerate two-forms in $\boldsymbol{\Omega}$ have rank $2n$, in particular $\boldsymbol{\Omega}$ is a k-symplectic structure on V.*

Proof. Let $\omega_1 \in \boldsymbol{\Omega}$ be such that $q(\omega_1, \omega_1) = 0$, and let $\dim(\ker \omega_1) = 2r$. Since q is non-degenerate we can find $\omega_2 \in \boldsymbol{\Omega}$ with $q(\omega_2, \omega_2) \neq 0$ and $q(\omega_1, \omega_2) \neq 0$. Consider the polynomial $\tilde{p}(t) = p(\omega_1 + t\omega_2)$. We have $\tilde{p}(t) = q(\omega_1 + t\omega_2, \omega_1 + t\omega_2)^n$, so $\tilde{p}$ must have zero of order n at $t = 0$. But

$$\tilde{p}(t) = (\omega_1 + t\omega_2)^{\wedge 2n} = t^r \omega_1^{\wedge(2n-r)} \wedge \omega_2^{\wedge r} + t^{r+1}\omega_1^{\wedge(2n-r-1)} \wedge \omega_2^{\wedge(r+1)} + \cdots + t^{2n}\omega_2^{\wedge 2n},$$

and $\omega_1^{\wedge(2n-r)} \wedge \omega_2^{\wedge r} \neq 0$ because ω_2 is non-degenerate. So the order of zero at $t = 0$ is r, hence $r = n$. ∎

We can bound from below the dimension of a vector space carrying a k-symplectic structure. In the proposition below we use the following terminology:

Definition 2.10. A k-symplectic structure $(\boldsymbol{\Omega}, q)$ on a vector space V is called **real** if V is a complexification of a real vector space $V_{\mathbb{R}}$, $\boldsymbol{\Omega}$ is a complexification of a real subspace $\boldsymbol{\Omega}_{\mathbb{R}}$ in $\Lambda^2 V_{\mathbb{R}}^*$, and q is a real quadratic form.

In particular, the k-symplectic structures from the previous example are real. We will denote by $\mathcal{C}\ell_{r,s}$ the Clifford algebra for a real $(r+s)$-dimensional vector space with quadratic form of signature (r, s), meaning that it has r minuses and s pluses.

Proposition 2.11. 1. *Let $(V, \boldsymbol{\Omega}, q)$ be a k-symplectic vector space. Then V has a structure of a non-trivial $\mathcal{C}\ell^0(\boldsymbol{\Omega}, q)$-module.*

2. *Let $(V, \boldsymbol{\Omega}, q)$ be a vector space with a real k-symplectic structure where the real quadratic form q has signature (r, s). Then the corresponding real vector space $V_{\mathbb{R}}$ has a structure of $\mathcal{C}\ell_{r-1,s}$-module if $r > 0$, and of a $\mathcal{C}\ell_{s-1,r}$-module if $s > 0$, and the module is non-trivial in both cases.*

Proof. Consider a pair of elements $\omega_1, \omega_2 \in \boldsymbol{\Omega}$, such that $q(\omega_1, \omega_1) = -1$ and $q(\omega_1, \omega_2) = 0$. These elements define linear maps $\omega_i: V \to V^*$, and by definition of a k-symplectic structure the map ω_1 is an isomorphism. So we can define the endomorphism $A = \omega_1^{-1}\omega_2 \in \mathrm{End}(V)$. We claim that

$$A^2 = q(\omega_2, \omega_2)Id. \tag{2.1}$$

To prove this we need to find the eigenvalues of A: the operator $A - \lambda Id = \omega_1^{-1}(\omega_2 - \lambda\omega_1)$ is degenerate if and only if the form $\omega_2 - \lambda\omega_1$ has non-trivial kernel. By definition of a k-symplectic structure, this condition is equivalent to $q(\omega_2 - \lambda\omega_1, \omega_2 - \lambda\omega_1) = 0$, hence the eigenvalues of A are $\lambda_\pm = \pm q(\omega_2, \omega_2)^{1/2}$ and the claim follows.

Next fix an element $\omega_1 \in \boldsymbol{\Omega}$ with $q(\omega_1, \omega_1) = -1$ and consider its q-orthogonal complement $W = \{\omega_2 \in \boldsymbol{\Omega} \mid q(\omega_1, \omega_2) = 0\}$. Define a linear map $\alpha: W \to \mathrm{End}(V)$, $\alpha(\omega_2) = \omega_1^{-1}\omega_2$. From Eq. (2.1), we see that $\alpha(\omega_2)^2 - q(\omega_2, \omega_2)Id = 0$. By the universal property of the Clifford algebra, the map α can be extended to $\alpha': \mathcal{C}\ell(W, q|_W) \to \mathrm{End}(V)$.

Observe that $\mathcal{C}\ell(W, q|_W) \simeq \mathcal{C}\ell^0(\boldsymbol{\Omega}, q)$. This isomorphism can be constructed as follows. For any element $\eta \in \mathcal{C}\ell(W, q|_W)$ consider the decomposition $\eta = \eta^0 + \eta^1$ into even and odd parts. Then the map $\eta \mapsto \eta^0 + \omega_1\eta^1 \in \mathcal{C}\ell^0(\boldsymbol{\Omega}, q)$ is the desired isomorphism, which can be checked using the fact that $\omega_1^2 = -1$ and that ω_1 commutes with even elements and anticommutes with odd elements from $\mathcal{C}\ell(W, q|_W)$. This proves the first part of the proposition.

Next consider the case of a real k-symplectic structure with the quadratic form q of signature (r, s). If $r > 0$ we can choose a real element $\omega_1 \in \boldsymbol{\Omega}_\mathbb{R}$ with $q(\omega_1, \omega_1) = -1$. Its real q-orthogonal complement $W_\mathbb{R}$ has a quadratic form of signature $(r - 1, s)$, and we obtain a representation of $\mathcal{C}\ell_{r-1,s}$ in $V_\mathbb{R}$.

If $s > 0$ we can choose a real element $\omega_1 \in \boldsymbol{\Omega}_\mathbb{R}$ with $q(\omega_1, \omega_1) = 1$. In this case for any ω_2 with $q(\omega_1, \omega_2) = 0$ we have an operator $A = \omega_1^{-1}\omega_2$ with $A^2 = -q(\omega_2, \omega_2)Id$. Then we obtain a representation of $\mathcal{C}\ell(W_\mathbb{R}, q')$ in $V_\mathbb{R}$, where $W_\mathbb{R}$ is the q-orthogonal complement to ω_1 as above, and $q' = -q|_{W_\mathbb{R}}$. The form q' has signature $(s - 1, r)$, so the second claim of the proposition follows. ∎

Corollary 2.12. *Let $(V, \boldsymbol{\Omega}, q)$ be a k-symplectic vector space. Then*

$$\dim_\mathbb{C} V = 2^{\lfloor (k-1)/2 \rfloor} m$$

for some positive integer m.

Proof. By the previous proposition, V is a non-trivial $\mathcal{C}\ell^0(\boldsymbol{\Omega}, q)$-module, so we have a map $\alpha': \mathcal{C}\ell^0(\boldsymbol{\Omega}, q) \to \mathrm{End}(V)$. The map α' is non-zero, so its image is isomorphic to a quotient of $\mathcal{C}\ell^0(\boldsymbol{\Omega}, q)$ by a proper two-sided ideal. But $\mathcal{C}\ell^0(\boldsymbol{\Omega}, q)$ is either the matrix algebra $\mathrm{Mat}(2^{(k-1)/2}, \mathbb{C})$ if k is odd or the sum of two copies of the matrix algebra $\mathrm{Mat}(2^{k/2-1}, \mathbb{C})$ if k is even. In any case, V is a direct sum of m copies of the standard representation of the matrix algebra for some m. Hence the desired equality for $\dim V$. ∎

Remark 2.13. Both the construction from the previous example and the proof of the previous proposition involve Clifford algebras. Note, however, that these two constructions are not inverse to each other. Not every k-symplectic structure arises from a real representation of a Clifford algebra associated with a k-dimensional vector space. One example when this is not the case was already mentioned above: this is a 6-symplectic structure on a 4-dimensional vector space V. In this case, the proof of the previous proposition gives us a representation $\alpha': \mathcal{C}\ell(W, q) \to \mathrm{End}(V)$ with 5-dimensional vector space W and $\mathcal{C}\ell(W, q)$ isomorphic to $\mathrm{Mat}(4, \mathbb{C}) \oplus \mathrm{Mat}(4, \mathbb{C})$. The map α' here is not injective. Note that by dimension reasons V cannot be a representation of a Clifford algebra associated to a 6-dimensional vector space. If we consider V as a complexification of a real 4-dimensional vector space, then the corresponding quadratic form on $\Lambda^2 V_\mathbb{R}^*$ will be of signature (3, 3), and by the previous proposition we have a $\mathcal{C}\ell_{2,3}$-module structure on $V_\mathbb{R}$. The algebra $\mathcal{C}\ell_{2,3}$ is isomorphic to $\mathrm{Mat}(4, \mathbb{R}) \oplus \mathrm{Mat}(4, \mathbb{R})$.

2.2. Applications to hyperkähler geometry

Next we will use our observations about k-symplectic structures to investigate submanifolds in a very general irreducible holomorphic symplectic (IHS) manifold (see Definition 1.10).

Consider an IHS manifold X of dimension $2n$ with symplectic form σ. We will call X very general if it represents a point in the moduli space which lies in the complement to a countable union of proper analytic subvarieties. One can prove that a very general X cannot contain submanifolds of odd dimension. Moreover, if there exists a submanifold of dimension $2m$ inside such X, and $\gamma \in H^{4(n-m)}(X, \mathbb{Z})$ is its fundamental class, then γ stays of Hodge type $(2n - 2m, 2n - 2m)$ on any small deformation of X (such a subvariety is called **absolutely trianalytic**, see Definition 1.5). The proof of these well-known facts can be found for example in [20], section 26.3.

Proposition 2.14. *Let X be an IHS manifold, $k = b_2(X)$ its second Betti number, and T a compact complex torus of dimension 2m immersed into X. Assume that T is absolutely trianalytic (this would follow if X is sufficiently general). Then $H_1(T, \mathbb{C})$ carries a non-degenerate real k-symplectic structure. The corresponding quadratic form has signature $(k - 3, 3)$.*

Proof. Denote by $j: T \to X$ the immersion. Let $V = H_1(T, \mathbb{C})$, then $\Lambda^2 V^* = H^2(T, \mathbb{C})$ and we have the restriction map $j^*: H^2(X, \mathbb{C}) \to H^2(T, \mathbb{C})$. Fujiki relations (1.2) imply that j^* is injective: for any $\alpha, \beta \in H^2(X, \mathbb{C})$ we have

$$(j^*\alpha)^{2m-1} \cdot j^*\beta = \gamma \cdot \alpha^{2m-1} \cdot \beta = c_\gamma q(\alpha, \alpha)^{m-1} q(\alpha, \beta),$$

and if $j^*\beta = 0$, then we must have $q(\alpha, \beta) = 0$ for any α with $q(\alpha, \alpha) \neq 0$. Taking α to be a Kähler form we see that $\beta = 0$.

We define $\boldsymbol{\Omega} = \mathrm{Im}(j^*) \subset \Lambda^2 V^*$. Then by Fujiki relations (1.2) we have $(j^*\alpha)^{2m} = c_\gamma q(\alpha, \alpha)^m$ and by Lemma 2.9 $\boldsymbol{\Omega}$ is a k-symplectic structure on V. This k-symplectic structure is real, because the Beauville–Bogomolov form is real (the real structure on V is the standard real structure on homology). The signature of Beauville–Bogomolov form is known to be $(b_2(X) - 3, 3)$. ∎

Corollary 2.15. *If a torus T is immersed into a very general IHS manifold X, then*

$$\dim_{\mathbb{C}} T \geqslant 2^{\lfloor (b_2(X)-1)/2 \rfloor - 1}.$$

Proof. This follows directly from Proposition 2.14 and Corollary 2.12 (note that the complex dimension of a torus is twice its first Betti number). ∎

Remark 2.16. In particular, this shows that a very general deformation of a 10-dimensional IHS manifold M constructed by O'Grady cannot contain a complex torus, because it is known that in this case $b_2(M) = 24$, and this would imply $\dim_{\mathbb{C}} T \geqslant 2^{10}$, which is impossible.

Remark 2.17. For the 6-dimensional IHS manifold of O'Grady's, we have $b_2 = 8$ and our estimate gives $\dim_{\mathbb{C}} T \geqslant 4$. But in this case we can use the fact that the real homology $H_1(T, \mathbb{R})$ must carry a real 8-symplectic structure with quadratic form of signature $(5, 3)$. In this case $H_1(T, \mathbb{R})$ has to be a $\mathcal{C}\ell_{4,3}$-module (and a $\mathcal{C}\ell_{2,5}$-module). The algebras $\mathcal{C}\ell_{4,3}$ and $\mathcal{C}\ell_{2,5}$ are isomorphic to $\mathrm{Mat}(8, \mathbb{C})$ as real algebras. The minimal non-trivial real representation of $\mathrm{Mat}(8, \mathbb{C})$ is $\mathbb{C}^8 \simeq \mathbb{R}^{16}$. This means that $H_1(T, \mathbb{R})$ has to be at least 16-dimensional which gives a contradiction.

3. Further remarks and open questions

3.1. *k-symplectic structures on manifolds*

The notion of k-symplectic structure generalizes that of a trisymplectic structure, which was originally defined in [25]. Jardim and Verbitsky define a trisymplectic structure on a complex manifold M to be a triple $\langle \Omega_1, \Omega_2, \Omega_3 \rangle$ of symplectic forms satisfying the following condition. Let $\boldsymbol{\Omega}$ be the vector space generated by $\langle \Omega_1, \Omega_2, \Omega_3 \rangle$, and $Q \subset \boldsymbol{\Omega}$ the set of degenerate forms. Then Q is a non-degenerate quadric hypersurface, all $v \in Q$ have constant rank, and all non-zero $v \in Q$ have rank $\frac{1}{2} \dim M$.

This notion was studied in [25] when M is a complex manifold, and $\Omega_1, \Omega_2, \Omega_3$ holomorphic symplectic forms. In this case M admits an action of $\mathrm{Mat}(2, \mathbb{C})$ in its tangent space, preserving the space $\boldsymbol{\Omega}$, and a torsion-free holomorphic connection preserving the $\boldsymbol{\Omega}$ and $\mathrm{Mat}(2, \mathbb{C})$-action. Trisymplectic structures arise naturally in connection with the mathematical instantons on $\mathbb{C}P^3$; indeed, the moduli space of framed mathematical instantons on $\mathbb{C}P^3$ is trisymplectic.

Trisymplectic structure is a special case of the 3-web, explored in [27], where it was studied using the geometric approach coming back to Chern's doctoral thesis [28]. A 3-web is a triple of involutive sub-bundles $S_1, S_2, S_3 \subset TM$ such that $TM = S_i \oplus S_j$ for any $i \neq j$. Chern has constructed a canonical connection on a manifold equipped with a 3-web; for 3-webs arising from trisymplectic structures, this connection turns out to be torsion-free.

Trisymplectic manifold can be characterized in terms of holonomy of this connection. These are manifolds equipped with a torsion free connection ∇ such that its holonomy group $\mathrm{Hol}(\nabla)$ is contained in a group $G = SL(2, \mathbb{C}) \cdot Sp(2n, \mathbb{C})$ acting on $4n$-dimensional complex space $\mathbb{C}^{4n} = \mathbb{C}^2 \otimes \mathbb{C}^{2n}$, with $SL(2, \mathbb{C})$ acting tautologically on the first tensor factor, and $Sp(2n, \mathbb{C})$ on the second tensor factor.

Trisymplectic structures occur naturally in hyperkähler geometry, for the following reason. Let M be a hyperkähler manifold, and $\mathrm{Tw}(M)$ its twistor space, that is, a total space of its twistor family Definition 1.3. Twistor space is a complex manifold, but it is non-algebraic and non-Kähler when M is compact.

In [29] (see also [30] and [6]) the twistor space was used to define (possibly singular) hyperkähler varieties. It turns out that a component $\mathrm{Sec}(M)$ of the moduli of rational curves on $\mathrm{Tw}(M)$ is equipped with a real structure, in such a way that a connected component $\mathrm{Sec}_{\mathbb{R}}(M)$ of its set of real points is identified with M. Then one can describe the hyperkähler structure on M as a certain geometric structure on $\mathrm{Sec}_{\mathbb{R}}(M)$.

In [25] it was shown that $\mathrm{Sec}(M)$ is equipped with a trisymplectic structure, and its restriction to $M = \mathrm{Sec}_{\mathbb{R}}(M)$ gives the triple of symplectic forms $\omega_I, \omega_J, \omega_K$. One can understand this result by identifying $\mathrm{Sec}(M)$ and a complexification of M. Then the trisymplectic structure on $\mathrm{Sec}(M)$ is obtained as a complexification of the triple $\omega_I, \omega_J, \omega_K$.

A related notion of even Clifford structures on Riemannian manifolds was introduced by A. Moroianu and U. Semmelmann in [31]. An **even Clifford structure** on a Riemannian manifold M is a sub-bundle $A \subset \mathrm{End}(TM)$ which is closed under multiplication and fiberwise isomorphic to an even Clifford algebra acting on the tangent bundle TM orthogonally. An even Clifford structure is called **parallel** if A is preserved by the Levi-Civita connection. If A is trivial, this Clifford structure is called **flat**. Moroianu and Semmelmann classified all manifolds admitting a parallel Clifford structure.

If we are given a k-symplectic structure, then it induces an action of an even Clifford algebra, as one can see from Proposition 2.11. However, the metric induced by symplectic forms and the Clifford algebra action is not necessarily positive definite. If one removes the positive definiteness condition, then any k-symplectic structure induces an even, flat Clifford structure, in the sense of Moroianu–Semmelmann.

For $k = 3$, the corresponding flat Clifford structure is also preserved by the Levi-Civita connection, hence parallel [25]. It is not clear whether the same is true for $k > 3$ (see Question 3.2).

In [32], A. Moroianu and M. Pilca obtain topological restrictions on manifolds admitting Clifford structures. They prove that a manifold with non-vanishing Euler characteristic cannot admit a Clifford structure with rank bigger than 16.

3.2. Open questions and possible directions of research

The notion of a k-symplectic manifold is certainly an intriguing one, and still very little understood.

Let $\boldsymbol{\Omega}$ be a k-symplectic structure on M. A general l-dimensional subspace $\boldsymbol{\Omega}' \subset \boldsymbol{\Omega}$ obviously gives an l-symplectic structure on M. This means, in particular, that any k-symplectic manifold is equipped with a $3(k-3)$-dimensional family R of 3-symplectic structures. As shown in [25], a 3-symplectic manifold is equipped with a canonical torsion-free connection preserving the 3-symplectic structure (this connection is called **the Chern connection**; not to be confused with the Chern connection defined on holomorphic Hermitian bundles). Given a 3-symplectic structure $r \in R$, denote the corresponding Chern connection by ∇_r.

Question 3.1. Are all ∇_r equal for all $r \in R$? If so, what is the holonomy of the Chern connection associated with a k-symplectic structure this way?

Question 3.2. Let M be a complex manifold equipped with a real structure ι, and $\boldsymbol{\Omega}$ an ι-invariant complex k-symplectic structure. The real part $\boldsymbol{\Omega}_{\mathbb{R}}$ restricted to the real part M^ι gives a k-dimensional space of symplectic forms of constant rank. Degenerate forms $\omega \in \boldsymbol{\Omega}$ correspond to the real points in the quadric Q. However, if the set of real points of Q is empty, all forms in $\boldsymbol{\Omega}_{\mathbb{R}} \subset \Lambda^2_{\mathbb{R}}(M^\iota)$ are non-degenerate. When $k = 3$, $\boldsymbol{\Omega}$ defines a hyperkähler structure on M^ι, and any hyperkähler structure can be defined this way. What happens if we start with k-symplectic structure? Presumably, we obtain a generalization of hyperkähler structures, with the quaternion algebra replaced by a bigger-dimensional Clifford algebra. Such a manifold would admit a very special family of hyperkähler structures, parametrized by a Grassmannian of 3-dimensional subspaces in a k-dimensional space.

In [25], a geometric reduction construction was established for trisymplectic structures. Trisymplectic quotient was defined in such a way that a trisymplectic quotient of a trisymplectic manifold M equipped with an action of d-dimensional compact Lie group G preserving the trisymplectic structure is a $(\dim M - 4d)$-dimensional complex manifold equipped with a trisymplectic structure. This construction can be considered as a complexification of the hyperkähler reduction of [29].

Question 3.3. Is there a geometric reduction construction for k-symplectic structures?

3-symplectic geometrical reductions are known in complex [25] and real versions [33].

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