Victor Domansky, Victoria Kreps, Marina Sandomirskaia, A Survey on Discrete Bidding Games with Asymmetric Information, *Contributions to Game Theory and Management*, 2013, Volume 6, 89–114

A Survey on Discrete Bidding Games with Asymmetric Information *

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Abstract Repeated bidding games were introduced by De Meyer and Saley (2002) to analyze the evolution of the price system at finance markets with asymmetric information. In the paper of De Meyer and Saley arbitrary bids are allowed. It is more realistic to assume that players may assign only discrete bids proportional to a minimal currency unit. This paper represents a survey of author's results on discrete bidding games with asymmetric information.

Keywords: multistage bidding, asymmetric information, price fluctuation, random walk, repeated game, optimal strategy.

1. Introduction

1.1. Modeling financial markets by repeated games

Regular random fluctuations in stock market prices are usually explained by effects from multiple exogenous factors subjected to accidental variations. The work of De Meyer and Saley (2002) proposes a different strategic motivation for these phenomena. The authors assert that the Brownian component in the evolution of prices on the stock market may originate from the asymmetric information of stockbrokers on events determining market prices. "Insiders" are not interested in the immediate revelation of their private information. This forces them to randomize their actions and results in the appearance of an oscillatory component in price evolution.

De Meyer and Saley demonstrate this idea on a model of multistage bidding between two agents for risky assets (shares). The liquidation price of a share depends on a random "state of nature". Before the bidding starts a chance move determines the "state of nature" and therefore the liquidation price of a share once and for all. Player 1 is informed on the "state of nature", but Player 2 is not. Both players know the probability of a chance move. Player 2 knows that Player 1 is an insider.

At each subsequent step $t = 1, 2, ..., n$ both players simultaneously propose their prices for one share. The maximal bid wins and one share is transacted at this price. If the bids are equal, no transaction occurs. Each player aims to maximize the value of his final portfolio (money plus liquidation value of obtained shares).

* This study was partially supported by the Russian Foundation for Basic Research, projects 04-06-80430-a, 07-06-00174-a, 10-06-00348-a and 13-01-00462-a.

In this model the uninformed Player 2 should use informed Player 1's history of moves to update his beliefs about the state of nature. Thus Player 1 must maintain a delicate balance between taking advantage of his private information and concealing it from Player 2.

De Meyer and Saley consider a model where a share's liquidation price takes only two values and players may make arbitrary bids. They reduce this model to a zero-sum repeated game with lack of information on one side, as introduced by Aumann and Maschler (1995), but with continual action sets. De Meyer and Saley show that these n-stage games have the values (i.e. the guaranteed gains of Player 1 are equal to the guaranteed losses of Player 2). They find these values and the optimal strategies of players. As n tends to infinity, the values infinitely grow up with rate $\sqrt{n}$. It is shown that Brownian Motion appears in the asymptotics of transaction prices generated by these strategies.

More exactly, De Meyer and Saley construct continuous time processes $\Pi^n(t)$ with $t \in [0, 1]$ representing these finite random sequences and prove that, as n tends to ∞, the processes Π^n converge in law to the limit process Π expressed by means of Brownian Motion.

In De Meyer (2010) a model of a market with one risky asset and perfectly general trading mechanism was considered. For example, transactions of arbitrary amount of shares at any stage of a game and presence of non-zero bid-ask spread can be implemented by means of this mechanism. The model of De Meyer and Saley is a particular case of this general model. For this general problem the limiting properties as the number of repetitions tends to infinity were investigated. It was shown that when both players use their optimal strategies the price process (expected price of a risky asset giving the history up to a current stage) converges after proper normalization in finite dimensional distributions to a martingale adapted to the natural Brownian filtration with terminal distribution coinciding with prior distribution of the share price. This class of price evolutions was called CMMV (continuous martingales of the maximal variation). The limit of the value and "asymptotically optimal" strategy of informed player were explicitly characterized too. Rather surprisingly, it was found that all the limiting objects do not depend on particular trading mechanism. To obtain these results a breakthrough technique to analyze repeated games with incomplete information was developed. The main idea was to look at the game from the point of view of informed player and to reduce it to some martingale optimization problem, the so-called problem of the maximal variation. This approach then allowed to apply a broad variety of tools from theory of stochastic processes.

The ideas of De Meyer about reduction to the martingale optimization problem were extended by Gensbittel (2010) in his thesis to a general repeated games with incomplete information (not necessary modelling a finance market). Moreover, he considered several improvements of the general trading mechanism of De Meyer: the case of several risky assets and the non-zero sum case (the total amount of money is not conserved).

It is to be mentioned that all the results of De Meyer and Gensbittel are obtained under several assumptions on trading mechanism: invariance with respect to non-risky part of the risky asset (i.e., shift invariance) and invariance with respect to numeraire change (i.e., scale invariance). These assumptions significantly simplify the analysis because they result in very handy linear structure of a game. But both these assumptions do not reflect the properties of real bidding. Indeed, only bids proportional to the minimal currency unit are allowed in real bidding. Therefore, neither shift invariance nor scale invariance really hold.

1.2. Results on discrete bidding games with asymmetric information

De Meyer and Marino (2005), Domansky and Kreps (2005), Domansky (2007) analyze a bidding model analogous to the model of De Meyer and Moussa-Saley (2002), where market makers have to post prices within a discrete grid. it corresponds to prices proportional to a minimal currency unit . The n-stage games $G_n^m(p)$ are considered with two possible values of liquidation price, an integer $m > 0$ with probability p and 0 with probability $1-p$, and with admissible bids being integer numbers.

The works mentioned above show that, unlike the model of De Meyer and Saley, the sequence of values $V_n^m(p)$ of the games $G_n^m(p)$ is bounded from above and converges as n tends to ∞. The authors calculate its limit H^m, that is a continuous, concave, and piecewise linear function with m domains of linearity $[k/m, (k+1)/m]$, $k = 0, \dots, m-1$, and the values at peak points $H^m(k/m) = k(m-k)/2$.

The proof in De Meyer and Marino (2005) differs in essential ways from the proof in Domansky (2007). The last proof is more concise due to exploiting a "reasonable" strategy of Player 2. In fact, this is his optimal strategy for the game with infinite number of steps.

As the sequence $V_n^m(p)$ is bounded from above, it is reasonable to consider the games $G_\infty^m(p)$ with infinite number of steps. The games $G_\infty^m(p)$ are infinitely repeated, non-discounted games with non-averaged payoffs that differs from the classical model of Aumann and Maschler (1995). Unlike the case of $n < \infty$, the existence of a value for the games $G_\infty^m(p)$ has to be proved.

In section 2 following Domansky (2007) we show that the value V_∞^m is equal to H^m and construct explicitly the optimal strategies of players. The fastest optimal strategy of Player 1 provides him the maximal possible expected gain 1/2 per step. For this strategy the posterior probabilities perform a simple random walk over the lattice l/m, $l = 0, \dots, m$, with absorbing extreme points 0 and 1. The absorption of posterior probabilities means revealing of the true value of share by Player 2. For the initial probability k/m, the expected duration of this random walk before absorption is $k(m-k)$. The bidding terminates almost surely in a finite number of steps, and the expected number of steps is also finite. This random time of absorption is a time for disclosure of information. The game terminates naturally when the posterior expectation of liquidation price coincide with its real value.

The set of all optimal strategies of Player 1 for $G_\infty^m(p)$ consists of the described fastest strategy and its slower modifications.

The results of Domansky (2007) cannot be extended to a general transaction mechanism introduced by De Meyer (2010). As mentioned in the last paper, the discretized mechanism does not satisfy axioms of shift- and scale-invariance. Note that in practice a grid of possible bids is not shift- and scale-invariant simultaneously.

Obtaining exact solutions for games $G_n^m(p)$ with finite numbers of steps seems to be a rather hard problem because of combinatorial difficulties as this may be observed at the two simplest case: solutions for one-stage games (Sandomirskaya, Domansky, 2012) and solutions for games with three admissible bids (Kreps, 2009).

In section 3 we describe the set of peak points of value function $V_1^m(p)$ and analyze the structure of bids used in optimal strategies of both players. On the base of this analysis we develop recurrent approach to computing optimal strategies of uninformed player for any probability p. Non-strictly speaking, recursion is on the number of pure strategies used by Player 2 in his optimal mixed strategy. As optimal strategy of insider equalizes the spectrum of optimal strategy of Player 2, we get Player 1' optimal strategies solving the system of difference equations arising from equalizing conditions.

In section 4 we construct the exact solutions for games $G_n^3(p)$ in the explicit form for any number of steps n. The value function $V_n^3(p)$ and the players' optimal strategies are expressed using a second-order recursive sequence.

In section 5 we show that the fastest optimal strategy of Player 1 for the infinitely repeated game $G_\infty^m(p)$ is a ε-optimal strategy of Player 1 for any finitely repeated game $G_n^m(p)$ of length n, where $\varepsilon = O(\cos^n \pi/m)$. This is not so for slower optimal strategies of Player 1 (Sandomirskaya, 2013, unpublished).

In section 6 following Domansky and Kreps (2009) we consider a model where a share liquidation price may take any integer value according to a probability distribution $\mathbf{p}$ over the one-dimensional integer lattice. Any integer bids are admissible. This n-stage model is described by a zero-sum repeated game with countable state and action spaces. The games considered in section 2 can be reduced to particular cases of these games corresponding to probability distributions with two-point supports.

We show that if the liquidation price of a share has a finite expectation, then the values of n-stage games exist. If its variance is finite, then, as n tends to ∞, the sequence of values is bounded from above and converges. The limit H is a continuous, concave, piecewise linear function with a countable number of domains of linearity.

As the sequence of n-stage game values is bounded from above, it is reasonable to consider the games $G_\infty(\mathbf{p})$ with an infinite number of steps. We show that the value $V_\infty(\mathbf{p})$ is equal to $H(\mathbf{p})$.

The optimal strategies are given in an explicit form. For constructing the optimal strategy of Player 1 for the game $G_\infty(\mathbf{p})$ with an arbitrary distribution having an integer expectation, we use the solutions for the games with two-point distributions and the symmetric representation of distributions over one-dimensional integer lattice with fixed integer mean values as convex combinations (probability mixtures) of distributions with two-point supports and with the same mean values.

The insider optimal strategy generates a random walk of posterior expectations over the one-dimensional integer lattice with absorption. The absorption may occur at any stage t if the posterior expectation of share price at this stage coincides with its prior expectation.

For any initial distribution with an integer mean value the expected duration of this random walk is equal to the variance of the liquidation price of a share. The value of infinite game is equal to the expected duration of this random walk multiplied by the constant one-step gain 1/2 of informed Player 1.

In section 7 we consider multistage bidding models where two types of risky assets are traded between two agents that have different information on the liquidation prices of traded assets (Domansky and Kreps, 2013, submitted to RAIRO-Operation Research). These prices are random integer variables that are determined by the initial chance move according to a probability distribution $\mathbf{p}$ over the two-dimensional integer lattice that is known to both players. Player 1 is informed on the prices of both types of shares, but Player 2 is not. The bids may take any integer value.

The model of n-stage bidding is reduced to the zero-sum repeated game with lack of information on one side.

If the expectations of share prices are finite, then the value of such n-stage bidding game does not exceed the sum of values of games modeling the bidding with one-type shares. This means that simultaneous bidding of two types of risky assets is less profitable for the insider than separate bidding of one-type shares. This is explained by the fact that the simultaneous bidding leads to revealing more insider information, because the bids for shares of each type provide information on shares of the other type.

We show that, if liquidation prices of both shares have finite variances, then the sequence of values of n-step games is bounded. This makes it reasonable to consider the bidding of unlimited duration that is reduced to the infinite game.

We begin with constructing solutions for these games with distributions $\mathbf{p}$ having two- and three-point supports (elementary games). Next, using symmetric representations of probability distributions over the two-dimensional plane with given mean values as convex combinations of distributions with supports containing not more than three points and with the same mean values (Domansky, 2013), we build the optimal strategies of Player 1 for bidding games $G_\infty(\mathbf{p})$ with arbitrary distributions $\mathbf{p}$ as convex combinations of his optimal strategies for elementary games.

The optimal strategies of Player 1 generate a random walk of transaction prices. But unlike the case of one-type assets, the symmetry of this random walk is broken at the final stages of the game.

We demonstrate that the value $V_\infty(\mathbf{p})$ is equal to the sum of values of corresponding games with one-type risky asset. Thus, the profit that Player 2 gets under simultaneous n-step bidding in comparison with separate bidding for each type of shares disappears in a game of unbounded duration.

In the bidding models considered in the sections 2-7 players propose only one price for a share at each step, i.e. bid and ask prices coincide. In a more realistic model developed in section 8 both players simultaneously propose their bid and ask prices for one share at each step of bidding. The bid-ask spread s is fixed by rules of bidding. Transaction occurs from a seller to a buyer by a bid price. The simplified model (sections 2-7) corresponds to the case $s = 0$ what is equivalent to $s = 1$ due to the price discreteness.

One-step payoff matrices of corresponding repeated games with incomplete information have more complicated structure than for the case $s = 1$ and solutions of these games are not found.

In section 8, for any integer $s > 1$ and two possible states of nature, by analogy with the zero-spread case of section 2 we construct an upper bound of value function provided by a reasonable strategy of Player 2. We construct a lower bound provided by a strategy of Player 1 that is the best strategy generating a simple random walk of price expectations. The bounds have the same form and coincide for $s = 1$ being equal to the value function of of the game under consideration.

By analogy with zero spread case (see section 6), for any integer $s > 1$ the results are generalized to the case of countable set of possible values for a share price.

As for $s > 1$ the constructed Player 1' strategy is not optimal, we conclude that the insider's optimal strategy does not generate simple random walk of price expectations and leads apparently to non-symmetric price fluctuations.

2. Bidding games with two states of nature

2.1. Bidding games of finite and infinite duration: $G_n^m(p)$ and $G_\infty^m(p)$

In this section we consider the repeated games $G_n^m(p)$ modelling the bidding with two possible random "state of nature", the state space $S = \{L, H\}$. Before bidding starts a chance move determines the "state of nature" L or H and therefore the liquidation value of a share once for all. This value is a positive integer m with probability p at the state H and 0 with probability $1-p$ at the state L. Player 1 is informed about the "state of nature", Player 2 is not. Both players know probability p. Player 2 knows that Player 1 is an insider.

At each subsequent stage $t = 1, \dots, n$ (n may be infinite) of bidding both players simultaneously propose their prices for one share, i_t for Player 1 and j_t for Player 2. Then the pair (i_t, j_t) is announced to both Players before proceeding to the next stage. The maximal bid wins and one share is transacted at this price. Therefore, if $i_t > j_t$, Player 1 gets one share from Player 2 and Player 2 receives the sum of money i_t from Player 1. If $i_t < j_t$, Player 2 gets one share from Player 1 and Player 1 receives the sum l from Player 2. If $i_t = j_t$, then no transaction occurs.

The bids may take arbitrary integer numbers, but the bids $0, 1, 2, \dots, m-1$ are efficient only. Indeed, as the minimal value of a share is 0 and the maximal value is $m > 0$, the bids $k < 0$ and $k > m-1$ are senseless and thus $k = 0, \dots, m-1$. So the action spaces are $I = J = 0, \dots, m-1$.

At state L, i.e. if the liquidation value of the share is equal to zero, the one-step gains of Player 1 are given with the following matrix $A^{L,m}$:

$$\begin{pmatrix} 0 & 1 & 2 & \dots\, m-1 \\ -1 & 0 & 2 & \dots\, m-1 \\ -2 & -2 & 0 & \dots\, m-1 \\ \dots & \dots & \dots & \dots\dots \\ -m+1 & -m+1 & -m+1 & \dots\, 0 \end{pmatrix}$$

At state H, i.e. if the liquidation value of the share is equal to m, then the matrix $A^{H,m}$ of the one-step gains of Player 1 takes the form

$$\begin{pmatrix} 0 & -m+1 & -m+2 & \dots\, -1 \\ m-1 & 0 & -m+2 & \dots\, -1 \\ m-2 & m-2 & 0 & \dots\, -1 \\ \dots & \dots & \dots & \dots\dots \\ 1 & 1 & 1 & \dots\, 0 \end{pmatrix}$$

We consider n-step games $G_n^m(p)$ with total (non-averaged) payoffs

$$K_n^m(p, \sigma, \tau) = \sum_{t=1}^{n} \mathbf{E}_{(\sigma,\tau)}[(1-p)a^{L,m}(i_t^L, j_t) + p \cdot a^{H,m}(i_t^H, j_t)]. \tag{2.1}$$

Note that at step t it is enough for both Players to take into account the sequence $(i_1, \dots, i_{t-1})$ of Player 1's previous actions only. Thus, a strategy σ for Player 1 (insider) is a sequence of moves

$$\sigma = (\sigma_1, \dots, \sigma_t, \dots),$$

where $\sigma_t : S \times I^{t-1} \to \Delta(I)$ is the probability distribution used by Player 1 to select his action at stage t, given the state s and previous observations.

A strategy τ for uninformed Player 2 does not depend on state s and represents a sequence of moves

$$\tau = (\tau_1, \ldots, \tau_t, \ldots),$$

where $\tau_t : I^{t-1} \to \Delta(J)$.

We also consider the infinite games $G^m_\infty(p)$. For certain pairs of strategies (σ, τ), the payoff function $K^m_\infty(p, \sigma, \tau)$, given by the infinite series (2.1), may be indefinite. If we restrict the set of Player 1's admissible strategies to strategies with nonnegative one-step gains

$$\mathbf{E}_{(\sigma_1, j)}[(1-p)a^{L,m}(i^L, j) + p \cdot a^{H,m}(i^H, j)].$$

against any action j of Player 2, then the payoff function of the game $G^m_\infty(p)$ becomes completely definite (may be infinite).

Observe that Player 1 has many strategies, ensuring him a nonnegative one-step gain against any action of Player 2. In fact, any "reasonable" strategy of Player 1 should possess this property.

Remind that, due to the recursive structure of the repeated game, it is sufficient to define the first move for any prior probability p to define the whole strategy of Player 1. Further this move will be played if the current posterior probability becomes equal to p. Thus, if for any prior probability p, the first move is "reasonable", the whole strategy is "reasonable".

The games $G^m_n(p)$ with $n < \infty$, as games with a finite sets of actions, have values $V^m_n(p)$. The values $V^m_n(p)$ are positive and do not decrease, as the number of steps n increases.

2.2. Asymptotics of values $V^m_n(p)$

The next theorem provides an upper bound for the values $V^m_n(p)$.

Theorem 2.1. *The functions V^m_n are bounded from above by a function H^m that is continuous, concave, and piecewise linear with m domains of linearity $[k/m, (k+1)/m]$, $k = 0, \ldots, m-1$. It is completely determined with its values at the peak points k/m, $k = 0, \ldots, m$:*

$$H^m(k/m) = k(m-k)/2.$$

To prove this theorem, we define recursively the set of infinite "reasonable" strategies $\tau^{k,m}, k = 0, \ldots, m-1$ of Player 2, suitable for the games $G^m_n(p)$ with arbitrary n.

Definition 2.1. The first move $\tau^{k,m}_1$ is the action k. The moves $\tau^{k,m}_t$ for $t > 1$ depend on the last observed pair of actions (i_{t-1}, j_{t-1}) only:

$$\tau^{k,m}_t(i_{t-1}, j_{t-1}) = \begin{cases} j_{t-1} - 1, & \text{for } i_{t-1} < j_{t-1}; \\ j_{t-1}, & \text{for } i_{t-1} = j_{t-1}\ ; \\ j_{t-1} + 1, & \text{for } i_{t-1} > j_{t-1}\ . \end{cases}$$

The next theorem provides a lower bound for the values $V^m_n(p)$.

Theorem 2.2. *The following inequalities hold:*

$$L^m_n(p) \le V^m_n(p) \quad \forall p \in [0, 1],$$

where the functions L_n^m are continuous, concave, and piecewise linear on the interval $[0,1]$ with m domains of linearity $[k/m,(k+1)/m]$, $k=0,\dots,m-1$. At the peak points k/m, the values $L_n^m(k/m)$ are given with recursive formulas

$$L_n^m(k/m) = 1/2 + 1/2(L_{n-1}^m((k-1)/m) + L_{n-1}^m((k+1)/m)),$$

with the initial condition $L_0^m(k/m) = 0$, and the boundary conditions $L_n^m(0) = L_n^m(1) = 0$.

To prove this theorem, we define the strategy $\bar{\sigma}^m$ of Player 1 ensuring these lower bounds.

Definition 2.2. For the initial probability k/m, the first move of the strategy $\bar{\sigma}^m$ makes use of two actions $k-1$ and k only. These actions occur with the same total probabilities $q(k-1) = q(k) = 1/2$.

The corresponding conditional posterior probabilities of the state H are

$$p^H(k-1) = (k-1)/m,$$

for the action $k-1$, and

$$p^H(k) = (k+1)/m,$$

for the action k.

Remark 2.1. These lower bounds have the same form as the upper bounds of Theorem 2.1.

Remark 2.2. As all posterior probabilities belong to the set $p = k/m, k = 0,\dots m$, these first moves define the strategy $\bar{\sigma}^m$ for the games $G_n^m(k/m)$ of arbitrary duration.

Corollary 2.1 (Asymptotics of values $V_n^m(p)$). *The following equalities hold:*

$$\lim_{n\to\infty} V_n^m(p) = H^m(p), \quad m = 2, 3, \dots.$$

2.3. Solutions for the games $G_\infty^m(p)$ and random walks

As the values $V_n^m(p)$ are bounded from above on the number of steps n, the consideration of values for the games $G_\infty^m(p)$ with infinite number of steps becomes reasonable.

We restrict the set of Player 1's admissible strategies in these games to the set Σ^+ of strategies employing only the moves ensuring him a nonnegative one-step gain against any action of Player 2. Consequently, the payoff functions $K_\infty^m(p,\sigma,\tau)$ of the games $G_\infty^m(p)$ become definite (may be infinite) at all cases.

We show that the infinite game $G_\infty^m(p)$ has a value and this value is equal to $H^m(p)$.

The existence of values for these games does not follow from common considerations and has to be proved. We prove it by providing the optimal strategies explicitly.

Theorem 2.3. *The game $G_\infty^m(p)$ has a value $V_\infty^m(p)$ equal to $H^m(p)$. Both Players have optimal strategies.*

For $p = k/m, k = 1, \dots, m-1$, the optimal strategy of Player 1 is the strategy $\bar{\sigma}^m$, given by Definition 2.2. For the interior points $p \in (k/m, (k+1)/m)$, the optimal first move of Player 1 is the convex combination of the first moves corresponding to the extreme points of this interval. This optimal first move makes use of three actions $k-1, k$ and $k+1$, using them with total probabilities

$$q(k-1) = 1/2(k+1-mp), q(k) = 1/2, q(k+1) = 1/2(mp-k).$$

Corresponding posterior probabilities are

$$P(H|k-1) = (k-1)/m, \quad P(H|k) = (2k+1-mp)/m, \quad P(H|k+1) = (k+2)/m.$$

For $p \in (k/m, (k+1)/m)$, $k = 0, \dots, m-1$, the optimal strategy $\bar{\tau}^m$ of Player 2 coincides with the strategy $\tau^{k,m}$, given by Definition 2.1. For the peak points k/m, $k = 1, \dots, m-1$, any convex combination of the strategies $\tau^{k-1,m}$ and $\tau^{k,m}$ is optimal.

Corollary 2.2. *For the initial probabilities $p = l/m$, $l = 0, \dots, m$, the random sequence of posterior probabilities, generated with the optimal strategy $\bar{\sigma}^m$ of Player 1, is the elementary symmetric random walk $(\bar{p}_t^m)_{t=1}^{\infty}$, over the points k/m, $k = 0, \dots, m$ with the absorbing extreme points 0 and 1, i.e. the Markov chain with the transition probabilities*

$$P(k/m, (k-1)m) = P(k/m, (k+1)/m) = 1/2, \quad k = 1, \dots, m-1,$$

$$P(0,0) = P(1,1) = 1.$$

For the initial probabilities $p \neq l/m$, $l = 0, \dots, m$, the random sequence of posterior probabilities hits the set $p = k/m$, $k = 0, \dots, m$, with probability 1/2 after each step. Further it continues as the elementary symmetric random walk with the absorbing extreme points 0 and 1.

Further we consider the random process $\{c_t^m\}_{t=1}^{\infty}$, formed by the prices of transactions $c_t^m = \max\{x_t^m, y_t^m\}$ at sequential steps of the infinite game $G_{\infty}^m(p)$. We say that the transaction occurs at step t if $x_t^m \neq y_t^m$.

Theorem 2.4. *a) For each step $t = 1, 2, \dots$, the probability that transaction occurs is 1/2.*
b) For $p_t^m \in [k/m, (k+1)/m]$, under the condition that the transaction occurs at step t, the following random transaction prices occur:

$$c_t^m(p_t^m) = \begin{cases} k & \text{with probability } \textit{k+1-mp}\ ; \\ k+1 & \text{with probability } \textit{mp-k}\ . \end{cases}$$

In particular, for $p_t^m = k/m$, under the condition that the transaction occurs at step t, $c_t^m = p_t^m = k$, and the price process reproduces the random walk of posterior probabilities.
c) Player 1's one-step gain is $1/2$.

3. Solution for one-stage bidding game with incomplete information

In this section we give the solution for the one-stage bidding game $G_1^m(p)$ with arbitrary integer m and with any probability $p \in (0,1)$ of the high share price. The complete description is given in the paper of Sandomirskaia and Domansky (2012).

If the share price is zero (state L), then Player 1 posts the zero bid at the one-stage game $G_1^m(p)$ for any probability p. So the problem is to describe the optimal strategy of Player 1 for the state H and the optimal strategy of Player 2. The latter does not depend on the state of nature.

Thus, solving of the zero-sum game $G_1^m(p)$ with incomplete information is reduced to solving the game with complete information with payoff matrix

$$A^m(i,j) = \begin{cases} (1-p)j + p(m-i), & \text{for } i > j; \\ (1-p)j, & \text{for } i = j; \\ (1-p)j + p(-m+j), & \text{for } i < j, \end{cases}$$

here $i \in I$ is the bid of insider at state H, $j \in J$ is the bid of uninformed player.

We develop recurrent approach to computing optimal strategies of uninformed player for any probability p based on analysis of structure of bids used in optimal strategies of both players. Non-strictly speaking, recursion is on the number of pure strategies used by Player 2 in an optimal mixed strategy.

3.1. Properties of spectra of optimal strategies.

The value $V_1^m(p)$ of the game $G_1^m(p)$ is a continuous concave piecewise linear function over $[0,1]$ with a finite number of linearity intervals. The optimal strategy of the uninformed Player 2 is constant over linearity intervals and is unique in its interiors.

Let $\mathbf{x}^m(p) = (x_0^m(p), \ldots, x_{m-1}^m(p))$ and $\mathbf{y}^m(p) = (y_0^m(p), \ldots, y_{m-1}^m(p))$ be optimal mixed strategies of Players 1 and 2 respectively for an initial probability p.

For probabilities $p \in [0, 1/m]$ and $p \in [(m-1)/m, 1]$ the game $G_1^m(p)$ has solution in pure strategies. Out of these intervals optimal strategies of Player 1 and Player 2 for the game $G_1^m(p)$ are mixed ones.

A change of the set $\operatorname{Spec}\mathbf{y}^m(p)$ (the set of positive components of the optimal strategy $\mathbf{y}^m(p)$) takes place at a point p if and only if p is a peak point of value function $V_1^m(p)$.

Consider the set $P^m = \{p_1, \ldots, p_{m-1}\}$, $0 < p_1 < \ldots < p_{m-1}$:

$$1 - p_1 = \frac{m-1}{m}, \quad 1 - p_2 = \frac{m-2}{m-1}, \quad 1 - p_k = (1 - p_{k-2})\frac{m-k}{m-k+1}.$$

and the set $Q^m = \{q_1, \ldots, q_{m-1}\}$, $1 > q_1 > \cdots > q_{m-1} = p_{m-1}$:

$$1 - q_1 = \frac{1}{m}, \quad 1 - q_2 = \frac{1}{m-1}, \qquad 1 - q_k = \frac{1 - p_{k-2}}{m-k+1}.$$

Proposition 3.1. *If $p \in P^m \cup Q^m$, then p is a peak point of value function $V_1^m(p)$ and*

$$V_1^m(p) = m \cdot p(1-p) \qquad \text{for } p \in P^m \cup Q^m.$$

Corollary 3.1. *The value of one-step bidding game with arbitrary bids being equal to $m \cdot p(1-p)$ (see De Meyer, Saley, 2002) coincides with $V_1^m(p)$ for $p \in P^m \cup Q^m$.*

Corollary 3.2. *As the set $P^m \cup Q^m$ is asymptotically everywhere dense over* $[0,1]$*, it follows that*

$$\lim_{m\to\infty} V_1^m(p)/m = p(1-p).$$

Remark 3.1. For $p < p_{m-1}$ ($p_{m-1} \approx 1/2$), the spectra of optimal strategies of both players expand as p increase until these spectra reach the bid $m-1$. For $p > p_{m-1}$ they narrow down but retaining the bid $m-1$.

Remark 3.2. For $m < 5$ there are no other peak points of $V_1^m(p)$ but the point $1/2$.

Denote by $k_1(\mathbf{x}^m(p))$ the maximal element of the set $\operatorname{Spec}\mathbf{x}^m(p)$ of positive components of strategy $\mathbf{x}^m(p)$. At a peak point p we put $k_1(\mathbf{x}^m(p))$ equal to its value to the right adjacent linearity interval.

Analogous notation $k_2(\mathbf{y}^m(p))$ for strategy $\mathbf{y}^m(p)$. The function $k_2(\mathbf{y}^m(p)) = k_2(p)$ is piece-wise constant over $[0,1]$.

Remark 3.3. If $m \geq 5$, then the function $k_2(p)$ has no jump at $p \in P^m \cup Q^m$. But the set $P^m \cup Q^m$ does not cover the set of all peak points p without a jump of $k_2(p)$.

Here we describe an ordering of two subset of peak points such that the function $k_2(p)$ has a jump at these points:

$$S^m = \{s_3, \dots, s_{m-1}\}, \qquad p_2 < s_3 < p_3, ..., p_{m-2} < s_{m-1} < p_{m-1}.$$

At the point s_i the bid i appears at the spectrum of the optimal strategy of Player 2.

$$T^m = \{t_4, \dots, t_{m-1}\} \qquad q_3 > t_4 > q_4, ..., q_{m-2} > t_{m-1} > q_{m-1} = p_{m-1}.$$

At the point t_{m-r}, $r = 2, \dots, m-4$ the bid $m-r$ quits the spectrum of the optimal strategy of Player 2.

Remark 3.4. For $m = 5$ the combination $P^5 \cup Q^5 \cup S^5 \cup T^5$ coincides with the whole set of peak points $V_1^5(p)$.

Definition 3.1. We call a lacuna of a strategy spectrum the set of successive bids that player does not use in this strategy, while using greater and smaller bids with positive probability.

Note that for $m \leq 5$ there are no lacunas in the optimal strategy spectra except of either $\{1\}$ or $\{2\}$. For $m > 5$ the structure of spectra of optimal strategies is more complicated having various lacunas.

Lemma 3.1. *A spectrum of optimal strategies of any player has no lacunas such that the number of its elements is more than 1 and the first element of the spectrum after the lacuna is less than* $m-1$.

3.2. Solutions for games $G_1^m(p)$

Here we restrict ourselves to description solutions of games $G_1^m(p)$ for $p \in (0, p_{m-1})$. The solutions for the interval $(p_{m-1}, 1)$ are analogues and (not strictly speaking) mirror-like with respect to the point p_{m-1}.

We use the following numeration for the linearity intervals of value function $V_1^m(p)$:

$$I_0 = I_{1,0} = [0, p_1], \quad I_1 = I_{1,1} = I_{2,0} = [p_1, p_2] \quad \text{and} \quad I_2 = I_{2,1} = [p_2, s_3];$$

$$I_{k,0} = [s_k, p_k], \quad I_{k,1} = [p_k, s_{k+1}] \quad \text{and} \quad I_k = I_{k,0} \cup I_{k,1}, \quad k = 3, \dots, m-1.$$

The following proposition describes the spectra of optimal strategies over intervals I_k.

Proposition 3.2. *For $p \in I_0$, Player 2 uses the bid 0. Player 1 uses the bid 1. For $p \in I_1 \cup I_2$, Player 1 uses the bids 1 and 2. For $p \in I_1 = I_{2,0}$, Player 2 uses the bids 0 and 1. For $p \in I_2 = I_{2,1}$, Player 2 uses the bids 0 and 2.*
For $p \in I_k$, $k > 2$, Player 1 uses the bids $1, 2, 3, \dots, k$. The maximal bid of Player 2 is k.
For $p \in I_{k,0}$, $k = 3, \dots, m-1$, Player 2 uses the bids $0, 2, 3, \dots, k$, if the number k is odd, and the bids $0, 1, 3, \dots, k$, if k is even.
For $p \in I_{k,1}$, $k = 3, \dots, m-2$, Player 2 uses the bids $0, 1, 3, \dots, k$, if the number k is odd, and the bids $0, 2, 3, \dots, k$, if k is even.

Let $v_{k,i}^H$ and $v_{k,i}^L$ be the gains of Player 1 for the state H and for the state L corresponding to the best reply of Player 1 to the optimal strategy of Player 2 for $p \in I_{k,i}$.

The following theorem provides the recurrent description of value function $V_1^m(p)$ for any linearity domain.

Theorem 3.1. *For $p \in I_{k,i}$,*

$$V_1^m(p) = v_{k,i}^L(1-p) + v_{k,i}^H p,$$

where

$$v_{1,0}^L = 0, \quad v_{1,0}^H = m-1, \quad v_{2,0}^L = \frac{1}{m-1}, \quad v_{2,0}^H = m-2,$$

$$v_{2,1}^L = \frac{2}{m-1}, \quad v_{2,1}^H = \frac{(m-2)^2}{(m-1)},$$

and for $k = 3, \dots, m-2$, $i = 0, 1$, payoffs $v_{k,i}^H$ and $v_{k,i}^L$ are given by the recurrent formulas

$$v_{k,i}^H = \frac{(m-k)^2}{v_{k-1,i+1}^H}, \quad v_{k,i}^L = (v_{k-1,i+1}^L - k)\left(\frac{m-k}{v_{k-1,i+1}^H}\right) + k,$$

Here $i+1$ is calculated modulo 2.

Corollary 3.3. *For any point $p \in (0, 1)$ the inequality*

$$V_1^m(p) \le m \cdot p(1-p) \tag{3.1}$$

holds. According to Proposition 3.1 for any $p \in P^m \cup Q^m$ it turns to be the equality.

Remark 3.5. As the value of one-stage bidding game with arbitrary bids is equal to $m \cdot p(1-p)$, see De Meyer, Saley (2002), the inequality (3.1) implies that this value exceeds the value of one-stage bidding game with discrete bids.

4. Solutions for games $G_n^3(p)$ and recursive sequences

The problem of solution for the n-step games $G_n^m(p)$ still remains open. The case of two admissible bids ($m = 2$) is trivial: the optimal strategy of Player 1 for any a priori probability p is to choose at the first step action 0 in the state L and action 1 in the state H, The both actions of Player 1 are "revealing" and the true price of a share is revealed by Player 2 at the first step.

At the fist step an optimal strategy of Player 2 is to post 1 for $p < 1/2$, and to post 1 for $p > 1/2$. For $p = 1/2$ any of the possible actions or any their probabilistic mixture is optimal. Thus, after the first move the insider's payoff is stabilized, and $V_n^2(p) = V_1^2(p) = \min\{p, 1-p\}$.

In this section we consider the qualitatively more complicated case of three reasonable bids 0, 1 and 2 ($m = 3$). Even the solution for the one-step game $G_1^3(p)$ is nontrivial (see the previous section).

For $m = 3$ the one-step gains of Player 1 are given with the following matrices:

$$A^L = [a_{ij}^L] = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 2 \\ -2 & -2 & 0 \end{bmatrix},$$

$$A^H = [a_{ij}^H] = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}.$$

We construct the exact solutions for games $G_n^3(p)$ in the explicit form for any number of steps n. The value function $V_n^3(p)$ and the optimal players' strategies are expressed using the second-order recursive sequence δ_n, $n = 0, 1, 2, \ldots$, determined by the recurrence relations

$$\delta_{n+1} = 2(\delta_n + \delta_{n-1}), \quad \delta_0 = 0, \quad \delta_1 = 2. \tag{4.1}$$

The theory of recursive sequences can be used to obtain the analytical expression for the sequences δ_n:

$$\delta_n = \frac{(1+\sqrt{3})^n - (1-\sqrt{3})^n}{\sqrt{3}}.$$

We show that the piecewise linear continuous concave value function $V_n^3(p)$ of the game $G_n^3(p)$ has three non-smoothness points on the interval $(0,1)$: $1/3$, $p_n \in (1/3, 2/3)$ and $2/3$, where

$$p_n = (\delta_{n-1} + \delta_n)/(\delta_{n-1} + 2\delta_n).$$

The values of the function $V_n^3(p)$ at these points are also determined using the recursive sequence δ_n.

We demonstrate that

$$V_n^3(p_n) = \max_{0 \le p \le 1} V_n^3(p),$$

i.e. the maximal payoff from private information is obtained by the insider in the case of the largest initial uncertainty of the partner which for the one-step game takes place for the prior probability of high price $p = 1/2$, and for the n-step game with three admissible bids for $p = p_n$.

The insider controls the sequence of posterior probabilities of high stock price, which are calculated with help of his strategies at the preceding steps. We show that the optimal strategy of the insider in the n-step game generates the posterior probability equal to p_{n-1} after the first step, and the posterior probability equal to p_{n-2} after the second step, etc., and finally, before the last step the probability equal to $1/2$.

The optimal first move of Player 2 for n-step game $G_n^3(p)$ is independent of the exact value of p. It depends only on the fact which linearity interval the prior probability p belongs to. The optimal move of Player 2 at the step $t = 2,' ldots, n$ depends only on the interval which the corresponding posterior probability belongs to.

When $n \to \infty$ the sequences of values $V_n^3(1/3)$, $V_n^3(p_n)$ and $V_n^3(2/3)$ converge to 1. Thus, in the limit, the non-smoothness point p_n disappears and the functions $V_n^3(p)$ converge to the value $V_\infty^3(p)$ of the game with unbounded duration $G_\infty^3(p)$ calculated in section 2.

The next theorem gives the exact formulation of the result.

Theorem 4.1. *The piecewise linear continuous value function $V_n^3(p)$ of the game $G_n^3(p)$ on the interval $[0,1]$ has three non-smoothness points: $1/3$, p_n, $2/3$. The function $V_n^3(p)$ is determined by its values at the ends of the interval $V_n^3(0) = V_n^3(1) = 0$ and the peak points:*

$$V_n^3(p) = \begin{cases} 1 - 2/3\delta_n & for\ p = 1/3, \\ 1 - 1/(2\delta_n + \delta_{n-1}) & for\ p = p_n, \\ 1 - 1/3\delta_{n-1} & for\ p = 2/3, \end{cases}$$

and

$$V_n^3(p) = \begin{cases} (3 - 2/\delta_n)p, & if\ p \in [0, 1/3], \\ (1 - 1/\delta_n)(1-p) + p, & if\ p \in [1/3, p_n], \\ (1 + 1/\delta_{n-1})(1-p) + (1 - 1/\delta_{n-1})p, & if\ p \in [p_n, 2/3], \\ (3 - 1/\delta_{n-1})(1-p), & if\ p \in [2/3, 1]. \end{cases}$$

Both players have the optimal strategies σ^{*n} è τ^{*n}, which on the four corresponding linearity intervals of the function V_2^3, enumerated by the Roman figures I, II, III, IV, have the following structure:

I. The interval $p \in [0, 1/3]$. The first move of the strategy $\sigma^{*n}(p, I)$ is

$$\sigma_1^{*n}(L, p, I) = (1 - 2p\delta_{n-1}/(1-p)\delta_n, 2p\delta_{n-1}/(1-p)\delta_n, 0), \quad \sigma_1^{*n}(H, p, I) = (0, 1, 0).$$

The first move $\tau_1^{*n}(I)$ of the strategy $\tau^{*n}(I)$ is $(1,0,0)$.

The continuation $\tau^{*n}(\cdot|i, I)$ of the strategy $\tau^{*n}(I)$ after observation of the bid i is determined by the relations

$$\tau^{*n}(\cdot|i, I) = \begin{cases} \tau^{*(n-1)}(I), & \text{if } i = 0, \\ (\tau^{*(n-1)}(II)\delta_{n-1} + \tau^{*(n-1)}(III)\delta_{n-2})/(\delta_{n-1} + \delta_{n-2}), & \text{if } i = 1. \end{cases}$$

II. The interval $p \in [1/3, p_n]$. The first move of the strategy $\sigma^{*n}(p, II)$ is

$$\sigma_1^n(L, p, II) = (1 - \delta_{n-1}/\delta_n, \delta_{n-1}/\delta_n, 0), \quad \sigma_1^n(H, p, II) = (0, (1-p)/2p, (3p-1)/2p).$$

The first move $\tau_1^{*n}(II)$ of the strategy $\tau^{*n}(II)$ is $(1/\delta_n, 1 - 1/\delta_n, 0)$.

The continuation $\tau^{*n}(\cdot|i,II)$ of the strategy $\tau^{*n}(II)$ after observation of the bid i is determined by the relations

$$\tau^{*n}(\cdot|i,II)=\begin{cases}\tau^{*(n-1)}(I), & \text{if } i=0,\\ (\tau^{*(n-1)}(II)\delta_{n-1}+\tau^{*(n-1)}(III)\delta_{n-2})/(\delta_{n-1}+\delta_{n-2}), & \text{if } i=1,\\ \tau^{*(n-1)}(IV), & \text{if } i=2.\end{cases}$$

III. The interval $p\in[p_n,2/3]$. The first move of the strategy $\sigma^{*n}(p,III)$ is

$$\sigma_1^n(L,p,III)=((2-3p)/(1-p),(2p-1)/(1-p),0),$$

$$\sigma_1^n(H,p,III)=(0,(2p-1)\delta_n/2p\delta_{n-1},1-(2p-1)\delta_n/2p\delta_{n-1}).$$

The first move $\tau_1^{*n}(III)$ of the strategy $\tau^{*n}(III)$ is $(0,1-1/\delta_{n-1},1/\delta_{n-1})$.

The continuation $\tau^{*n}(\cdot|i,III)$ of the strategy $\tau^{*n}(III)$ after observation of the bid i is determined by the relations

$$\tau^{*n}(\cdot|i,III)=\begin{cases}\tau^{*(n-1)}(I), & \text{if } i=0,\\ \tau^{*(n-1)}(II), & \text{if } i=1,\\ \tau^{*(n-1)}(IV), & \text{if } i=2.\end{cases}$$

IV. The interval $p\in[2/3,1]$. The first move of the strategy $\sigma^{*n}(p,IV)$ is

$$\sigma_1^n(L,p,IV)=(0,1,0),\quad \sigma_1^n(p,H)=(0,(1-p)\delta_n/2p\delta_{n-1},1-(1-p)\delta_n/2p\delta_{n-1}).$$

The first move $\tau_1^{*n}(IV)$ of the strategy $\tau^{*n}(IV)$ is $(0,0,1)$.

The continuation $\tau^{*n}(\cdot|i,IV)$ of the strategy $\tau^{*n}(IV)$ after observation of the bid i is determined by the relations

$$\tau^{*n}(\cdot|i,IV)=\begin{cases}\tau^{*(n-1)}(II), & \text{if } i=1,\\ \tau^{*(n-1)}(IV), & \text{if } i=2.\end{cases}$$

5. Analysis of lower bounds for values $V_n^m(p)$ of games $G_n^m(p)$.

In section 2 we constructed the Player 1' fastest optimal strategy $\bar{\sigma}^m$ for the bidding game $G_\infty^m(p)$ of unlimited duration (see Definition 2.2.). The strategy σ^m provides Player 1 the maximal possible expected gain 1/2 per step. For this strategy the posterior probabilities perform a simple random walk over the grid l/m, $l=0,\dots,m$, with absorbing extreme points 0 and 1. At the random time Θ^m of absorption of posterior probabilities revealing the true share value by Player 2 occurs. For the initial probability k/m, the expected duration $\beta_\infty^m(k)=\mathbf{E}_k[\Theta^m]$ of this random walk before absorption is $k(m-k)$, where $\mathbf{E}_k$ is the expectation for the random walk starting at the point k/m.

For the n-stage game $G_n^m(p)$ the strategy $\bar{\sigma}^m$ ensures the Player 1' gain that does not exceed $L_n^m(p)$. The Player 1' guaranteed gain is equal to $L_n^m(p)$ if he uses the strategy $\bar{\sigma}^m$.

The continuous, concave, and piecewise linear lower bound $L_n^m(p)$ for value $V_n^m(p)$ at its peak points k/m is given with recursive formulas (section 2, Theorem 2.2).

In this section we obtain an explicit formula for $L_n^m(p)$, i.e. for the guaranteed gain of Player 1 in the n-stage game if he applies his optimal strategy σ^m for the game $G_\infty^m(p)$ of unlimited duration. Let $W_n^m(\sigma,\tau|p)$ be the payoff function of the game $G_n^m(p)$.

Theorem 5.1. *If Player 1 exploits the strategy σ^m in the game $G_n^m(k/m)$, then his guaranteed gain $L_n^m(k/m) = \inf_\tau W_n^m(\sigma^m, \tau|k/m)$ is given with the formula*

$$L_n^m(k/m) = \frac{(m-k)k}{2} - \varepsilon_n^m(k), \tag{5.1}$$

where

$$\varepsilon_n^m(k) = \frac{1}{2m}\sum_{l=1}^{[m/2]} \cos^n\frac{\pi(2l-1)}{m}\sin\frac{\pi k(2l-1)}{m}\mathrm{ctg}\frac{\pi(2l-1)}{2m}\left(1+\mathrm{ctg}^2\frac{\pi(2l-1)}{2m}\right), \tag{5.2}$$

with $[\alpha]$ being the integer part of α.

Sketch of the proof. Let $\beta_n^m(k) = \mathbf{E}_k[\Theta^m \wedge n]$ denote the average number of steps of the simple random walk of posterior probabilities starting at the point k/m in the n-stage game $G_n^m(p)$. Then the expected insider's profit is given by $W_n^m(k) = \frac{1}{2}\beta_n^m(k)$.

The recursive equations for $\beta_n^m(k)$ hold $\beta_{n+1}^m(k) = \frac{1}{2}\beta_n^m(k+1) + \frac{1}{2}\beta_n^m(k-1) + 1$ with the boundary conditions $\beta_n^m(0) = \beta_n^m(m) = 0$ and with the initial condition $\beta_0^m(k) = 0$.

The values $\beta_\infty^m(k)$ satisfy the equations $\beta_\infty^m(k) = \frac{1}{2}\beta_\infty^m(k+1) + \frac{1}{2}\beta_\infty^m(k-1) + 1$ with the boundary conditions $\beta_\infty^m(0) = \beta_\infty^m(m) = 0$.

Thus the differences $\varepsilon_n^m(k) = \frac{1}{2}\left(\beta_\infty^m(k) - \beta_n^m(k)\right)$ satisfy the homogeneous recursive equations

$$\varepsilon_{n+1}^m(k) = \frac{1}{2}\varepsilon_n^m(k+1) + \frac{1}{2}\varepsilon_n^m(k-1)$$

with the boundary conditions $\varepsilon_n^m(0) = \varepsilon_n^m(m) = 0$ and with the initial condition $\varepsilon_0^m(k) = \beta_\infty^m(k)/2$.

Solving these equations we obtain the representation (5.2) for $\varepsilon_n^m(k)$.

Corollary 5.1. *The strategy σ^m is a ε_n^m-optimal strategy of Player 1 for the finitely repeated game $G_n^m(p)$ of length n, where $\varepsilon_n^m = O(\cos^n \pi/m)$, i.e. the "error term" $\varepsilon_n^m(k)$ decreases exponentially.*

This is not so for slower optimal strategies of Player 1.

The case m=3.

For $m = 3$ the above result means

$$\varepsilon_n^3(k) = \frac{1}{2^n}, \quad k = 1, 2.$$

As the exact solutions for the n-stage games $G_n^3(p)$ are known (see section 4), we may refine the values of the "error term" estimating the difference between the value $V_n^3(p)$ and the lower bound $L_n^3(p)$, not only $(V_\infty^3(p) - L_n^3(p))$.

In section 4 the value functions $V_n^3(p)$ are expressed by means of a second-order recursive sequence. They converge to the value $V_\infty^3(p)$ of the game with unbounded duration $G_\infty^3(p)$. Using the theory of recurrent sequences it is easy to estimate function $V_n^3(p)$ at the peak points $p = 1/3$ and $p = 2/3$,

$$V_n^3(1/3) \approx 1 - \frac{2}{\sqrt{3}(1+\sqrt{3})^n}, \qquad V_n^3(2/3) \approx 1 - \frac{1+\sqrt{3}}{\sqrt{3}(1+\sqrt{3})^n},$$

and to get the refined values

$$\bar{\varepsilon}^3_n(1) = (V^3_n(1/3) - L^3_n(1/3)) \approx \frac{1}{2^n} - \frac{2}{\sqrt{3}(1+\sqrt{3})^n},$$

$$\bar{\varepsilon}^3_n(2) = (V^3_n(2/3) - L^3_n(2/3)) \approx \frac{1}{2^n} - \frac{1+\sqrt{3}}{\sqrt{3}(1+\sqrt{3})^n}.$$

So for sufficiently large n the optimal strategy of the insider for the bidding game of infinite duration is a rather good approximation of his optimal strategy for the n-stage game.

6. Bidding games $G_n(\mathbf{p})$ and $G_\infty(\mathbf{p})$ with countable state space

In this section we consider the model where any integer non-negative bids are admissible and the liquidation price of a share $C_{\mathbf{p}}$ may take any nonnegative integer values $k = 0, 1, 2, \ldots$ according to a probability distribution $\mathbf{p} = (p_0, p_1, p_2, \ldots)$.

At stage 0 a chance move determines the liquidation value of a share for the whole period of bidding n according to the probability distribution $\mathbf{p} = (p_0, p_1, p_2, \ldots)$ over the one-dimensional integer lattice, $S = \mathbb{Z}_+$. Structure of information and trading mechanism are the same as in section 2 for the case of two possible states of nature.

This n-stage model is described by a zero-sum repeated game $G_n(\mathbf{p})$ with incomplete information of Player 2 and with countable state space $S = \mathbb{Z}_+$ and with countable action spaces $I = \mathbb{Z}_+$ and $J = \mathbb{Z}_+$. One-step gains of Player 1 are given with the matrices $A^s = [a^s(i,j)]_{i \in I, j \in J}$, $s \in S$,

$$a^s(i,j) = \begin{cases} j - s, & \text{for } i < j \ ; \\ 0, & \text{for } i = j \ ; \\ -i + s, & \text{for } i > j \ . \end{cases}$$

At the end of the game Player 2 pays to Player 1 the sum

$$\sum_{t=1}^{n} a^s(i_t, j_t).$$

This description is common knowledge to both Players. The games $G^m_n(p)$ considered in section 2 represent particular cases of these games corresponding to probability distributions with two-point supports, $p_0 = 1 - p$ and $p_m = p$.

Theorem 6.1. *If the random variable $C_{\mathbf{p}}$, determining the liquidation price of a share has a finite mathematical expectation $\mathbf{E}[C_{\mathbf{p}}]$, then the values $V_n(\mathbf{p})$ of n-stage games $G_n(\mathbf{p})$ exist The values $V_n(\mathbf{p})$ are positive and do not decrease, as the number of steps n increases.*

The theorem follows from the fact that for this case the payoff of game $G_n(\mathbf{p})$ can be approximated by payoffs of games $G_n(\mathbf{p}_k)$ with probability distributions $\mathbf{p}_k$ having finite support.

6.1. Upper bound for values $V_n(\mathbf{p})$

If the variance $\mathbf{D}[C_\mathbf{p}]$ is infinite, then, as n tends to ∞, the sequence $V_n(\mathbf{p})$ diverges.

The next theorem demonstrates that on the contrary, if the variance $\mathbf{D}[C_\mathbf{p}]$ is finite, then, as n tends to ∞, the sequence of values $V_n(\mathbf{p})$ of the games $G_n(\mathbf{p})$ is bounded from above.

Theorem 6.2. *For $\mathbf{p}$ such that $\mathbf{D}[C_\mathbf{p}] < \infty$, the values $V_n(\mathbf{p})$ are bounded from above by a continuous, concave, and piecewise linear function $H(\mathbf{p})$. Its domains of linearity are*

$$L(k) = \{\mathbf{p} : \mathbf{E}[\mathbf{p}] \in [k, k+1]\}, \quad k = 0, 1, \dots.$$

Its domains of non-smoothness are

$$\Theta(k) = \{\mathbf{p} : \mathbf{E}[\mathbf{p}] = k\}.$$

The equality holds

$$H(\mathbf{p}) = (\mathbf{D}[\mathbf{p}] - \alpha(\mathbf{p})(1 - \alpha(\mathbf{p})))/2, \tag{6.1}$$

where $\alpha(\mathbf{p}) = \mathbf{E}[\mathbf{p}] - ent[\mathbf{E}[\mathbf{p}]]$ and $ent[x]$, $x \in R^1$ is the integer part of x.

The result is provided by a "reasonable" strategy of Player 2. The strategy is analogous to his optimal strategy for two-states game $G_n^m(p)$ (see section 2): at the first move Player 2 posts $ent[\mathbf{E}[\mathbf{p}]]$ and then his moves depend on the last observed pair of actions only.

6.2. Solutions for games $G_\infty(\mathbf{p})$ with arbitrary $\mathbf{p}$

As the sequence $V_n(\mathbf{p})$ is bounded from above, it is reasonable to consider the games $G_\infty(\mathbf{p})$ with infinite number of steps. We show that the value $V_\infty(\mathbf{p})$ is equal to $H(\mathbf{p})$. We get solutions for these games in the explicit form.

The optimal strategy of Player 2 is his "reasonable" strategy mentioned above. We construct the optimal strategy of Player 1 for the game $G_\infty(\mathbf{p})$ with an arbitrary distribution having an integer expectation on the base of the solutions for the games with two-point distributions obtained in section 2. The result is due to the symmetric representation of distributions over the one-dimensional integer lattice with fixed integer mean values as convex combinations (probability mixture) of distributions with two-point supports and with the same mean values (see, e.g. Obloy, 2004).

Symmetric representation of distributions over the one-dimensional integer lattice. Let $\mathbf{p}$ be a probability distribution over the set of integers $\mathbb{Z}^1$ with mean value equal to an integer r. Then

$$\mathbf{p} = p_r \cdot \delta^r + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{k+l}{\sum_{t=1}^{\infty} t \cdot p_{r+t}} p_{r-l} p_{r+k} \cdot \mathbf{p}_{r+k,r-l}^r, \tag{6.2}$$

where $\mathbf{p}_{r+k,r-l}^r$ is the probability distribution with the two-point support $r-l, r+k$ and with mean value equal to r.

We treat coefficients

$$\mathbf{P}_\mathbf{p}(\mathbf{p}_{r+k,r-l}^r) = \frac{k+l}{\sum_{t=1}^{\infty} t \cdot p_{r+t}} p_{r-l} p_{r+k}$$

of decomposition (6.2) as probabilities of corresponding distributions with two-point supports $(r+k)$, $(r-l)$ in this probability mixture.

Given one point z (equal to $r+k$ or to $r-l$) in the support of two-point distribution, the conditional probability of complementary point ($r-l$ or $r+k$) may be calculated

$$\mathbf{P}_{\mathbf{p}}(r+k|r-l) = \frac{k \cdot p_{r+k}}{\sum_{t=1}^{\infty} t \cdot p_{r+t}}, \quad \mathbf{P}_{\mathbf{p}}(r-l|r+k) = \frac{l \cdot p_{r-l}}{\sum_{t=1}^{\infty} t \cdot p_{r+t}}. \tag{6.3}$$

Player 1' optimal strategy σ^*. We construct Player 1' optimal strategy σ^* for the game $G_\infty(\mathbf{p})$ making use of the obtained decomposition for the initial distribution $\mathbf{p}$ with mean value equal to an integer r (the prior expectation of share price is r).
a) If the state chosen by chance move is r, then Player 1 stops the game (Player 1' informational advantage disappears).
b) If chance move chooses $z = r+k$ (or $z = r-l$), where k, l are integer positive numbers, then Player 1 chooses a point $z_2 = r-l$ (or $z_2 = r+k$) by means of lottery with probabilities (6.3) and plays his optimal strategy for the state z in the two-point game $G(\mathbf{p}^r_{r+k,r-l})$ (see section 2).

The described optimal strategy of Player 1 generates a symmetric random walk of posterior mathematical expectations of liquidation price with absorption. The absorption may occur at any stage if the posterior expectation of share price at this stage coincides with its prior expectation. If the liquidation price chosen by the chance move coincides with its prior expectation, then the absorption occurs at the first stage. Note that it is impossible for two-point support distributions.

The expected duration of this random walk is equal to the initial variance of liquidation price. The guaranteed total gain of Player 1 (the value of the game) is equal to this expected duration multiplied with the fixed gain per step.

7. Repeated games with asymmetric information modeling financial markets with two risky assets

In this section we consider multistage bidding models where two types of risky assets are traded. Two players with opposite interests have money and two types of shares. The liquidation prices of both share types may take any integer values x and y. At stage 0 a chance move determines the "state of nature" s and therefore the liquidation prices of shares (s^1, s^2) for the whole period of bidding n according to the probability distribution $\mathbf{p}$ over the two-dimensional integer lattice known to both Players. Player 1 is informed about the result of chance move z, Player 2 is not. Player 2 knows that Player 1 is an insider.

At each step of bidding both players simultaneously make their integer bids, i.e. they post their prices for each type of shares. The player who posts the larger price for a share of a given type buys one share of this type from his opponent at this price. Any integer bids are admissible. Players aim to maximize the values of their final portfolios, calculated as money plus obtained shares evaluated by their liquidation prices.

The described model of n-stage bidding is reduced to the zero-sum repeated game $G_n(\mathbf{p})$ with lack of information on one side and with two-dimensional one-step actions with components corresponding to bids for each type of assets. The countable state space is $S = \mathbb{Z}^2$ and the countable action spaces are $I = \mathbb{Z}^2$ and $J = \mathbb{Z}^2$. The one-step gain $a(s, i, j)$ of Player 1 corresponding to the state $s = (s^1, s^2)$ and the actions $i = (i^1, i^2)$ and $j = (j^1, j^2)$ is given with the sum

$\sum_{e=1}^{2} a^e(s^e, i^e, j^e)$, where

$$a^e(s^e, i^e, j^e) = \begin{cases} j^e - s^e, & \text{for } i^e < j^e; \\ 0, & \text{for } i^e = j^e; \\ -i^e + s^e, & \text{for } i^e > j^e. \end{cases}$$

At the end of the game Player 2 pays to Player 1 the sum

$$\sum_{t=1}^{n} a(s, i_t, j_t),$$

where s is the result of a chance move. This description is a common knowledge of both Players.

It is easy to show that if the expectations of share prices are finite, then the value of such n-stage bidding game does not exceed the sum of values of games modeling the bidding with one-type shares. This means that simultaneous bidding of two types of risky assets is less profitable for the insider than separate bidding of one-type shares. This is explained by the fact that the simultaneous bidding leads to revealing more insider information, because the bids for shares of each type provide information on shares of the other type.

We show that, if liquidation prices of both shares have finite variances, then the value $V_n(\mathbf{p})$ of n-stage bidding games does not exceed the function $H(\mathbf{p})$ which is the smallest piecewise linear function equal to the one half of the sum of share price variances for distributions with integer expectations of both share prices.

This makes it reasonable to consider the bidding of unlimited duration that is reduced to the infinite game $G_\infty(\mathbf{p})$. We give the solutions for these games with arbitrary probability distributions over the two-dimensional integer lattice with finite component variances.

Both players have optimal strategies. The optimal strategy for Player 2 is a direct combination of his optimal strategies for the games with one-type of risky asset (see section 6).

We begin with constructing Player 1' optimal strategies for games $G_\infty(\mathbf{p})$ with distributions $\mathbf{p}$ having two- and three-point supports – elementary games. Next, using symmetric representations of probability distributions over the two-dimensional plane with given mean values as convex combinations of distributions with supports containing not more than three points and with the same mean values (Domansky, 2013), we build the optimal strategies of Player 1 for bidding games $G_\infty(\mathbf{p})$ with arbitrary distributions $\mathbf{p}$ as convex combinations of his optimal strategies for elementary games.

The optimal strategy of Player 1 generates a random walk of transaction prices. But unlike the case of one-type assets, the symmetry of this random walk is broken at the final stages of the game.

We show that this game terminates naturally when the posterior expectations of both liquidation prices come close enough to their real values. We demonstrate that the value $V_\infty(\mathbf{p})$ coincides with $H(\mathbf{p})$. So it is equal to the sum of values of corresponding games with one-type risky asset. Thus, the profit that Player 2 gets under simultaneous n-step bidding in comparison with separate bidding for each type of shares disappears in a game of unbounded duration.

7.1. Solutions for games $G_\infty(\mathbf{p})$ with $\mathbf{p}$ having two-point supports

For games $G_\infty(\mathbf{p})$ with the support of distribution $\mathbf{p}$ containing two states, we show that the value $V_\infty(\mathbf{p})$ is equal to $H(\mathbf{p})$.

To construct optimal strategies σ^* of Player 1 for games $G_\infty(\mathbf{p})$ with two states we use the results for games with one-type assets and with two states. But the fastest optimal strategy of Player 1 described in section 2 is not sufficient for this purpose. We use Player 1' slower optimal strategies.

Without loss of generality we assume that one of support points is $(0, 0)$. Thus there are two states $0 = (0, 0)$ and $z = (x, y)$, where x and y are integers and $x > 0$. The distribution $\mathbf{p}$ can be depicted with a scalar parameter $p \in [0, 1]$ being the probability of state z. For definiteness set $y > 0$.

The strategy σ^* of Player 1 generates an asymmetric random walk of posterior probabilities by adjacent points of the irregular lattice

$$Lat(x, y) = \{k/x, k = 0, \ldots, x\} \cup \{l/y, l = 0, \ldots, y\}$$

formed with those probabilities where at least one of the price expectations has an integer value. The probabilities of jumps provide martingale characteristics of posterior probabilities and with absorption at extreme points 0 and 1.

7.2. Solutions for games $G_\infty(\mathbf{p})$ with $\mathbf{p}$ having three-point supports

We construct optimal strategies σ^* of Player 1 that ensure $H(\mathbf{p})$ for games $G_\infty(\mathbf{p})$ with three states $z_1, z_2, z_3 \in \mathbf{Z}^2$.

Denote $\triangle(z_1, z_2, z_3)$ the triangle spanned across the support points of distribution. A distribution $\mathbf{p}$ with the support z_1, z_2, z_3 is uniquely determined with a vector $w = (u, v) \in \triangle(z_1, z_2, z_3)$ of expectations of coordinates (the barycenter of distribution $\mathbf{p}$). Denote it $\mathbf{p}^w_{z_1,z_2,z_3}$.

For $\mathbf{p}^w_{z_1,z_2,z_3}$ the first step of optimal strategy σ^* may efficiently use the actions $(u-1, v-1)$, $(u, v-1)$, $(u-1, v)$ and (u, v). With the help of these actions Player 1 can perform moves such that the modulus of difference between posterior expectations of each coordinate and its initial expectation is not more than one.

There are several types of optimal first moves of Player 1, in particular, the first moves σ_1^{NE-SW} (north-east – south-west), σ_1^{NW-SE}, and their probabilistic mixtures. Denote $e = (1, 1), \bar{e} = (1, -1)$. The first move σ_1^{NE-SW} exploits only two actions $w - e$ and w with posterior expectations $w - b \cdot e$ and $w + a \cdot e$. The first move σ_1^{NW-SE} makes use of actions $(u - 1, v)$ and $(u, v - 1)$ with posterior expectations $w - b\bar{e}$ and $w + a\bar{e}$.

The martingale of posterior expectations generated by the optimal strategy of Player 1 for the game $G_\infty(\mathbf{p}^w_{z_1,z_2,z_3})$ represents a symmetric random walk over points of integer lattice lying within the triangle $\triangle(z_1, z_2, z_3)$.

The symmetry is broken at the moment that the walk hits the triangle boundary. From this moment, the game turns into one of games with distributions having two-point supports.

7.3. Solutions for games $G_\infty(\mathbf{p})$ with arbitrary $\mathbf{p}$

We construct Player 1's optimal strategy for the game $G_\infty(\mathbf{p})$ with an arbitrary distribution $\mathbf{p}$ having an integer expectation vector (k, l), as a convex combination (a probability mixture) of his optimal strategies for games with distributions having not more than three-point supports and the same expectation vector (k, l).

To realize the idea we use symmetric representations of probability distributions over the two-dimensional plane with given mean values as convex combinations of elementary distributions – distributions with supports containing not more than three points and with the same mean values (Domansky, 2013).

This decomposition is a generalization of the analogous decomposition of one-dimensional distributions into a convex combination of distributions with no more than two-point supports and with the same expectation that was used in section 6 for constructing solutions for bidding games with a one-type risky asset.

The coefficient at an elementary distribution may be regarded as its probability in this probability mixture. Given one point z in the support of elementary distribution, the conditional probability of any elementary distribution having z in its support may be calculated. Then we obtain the conditional probability $\mathbf{P_p}(2|z)$ of elementary two-point support distributions and the conditional probability $\mathbf{P_p}(3|z)$ of elementary three-point support distributions. For constructing the optimal Player 1' strategy we use also conditional probabilities $\mathbf{P_p}(z_2|z,2)$ of a complementary point z_2 for the two-point support (z, z_2) and conditional probabilities $\mathbf{P_p}(z_2, z_3|z, 3)$ of complementary points z_2, z_3 for the three-point support (z, z_2, z_3).

The optimal strategy of Player 1 is given by the following algorithm:

1. If the state $z = (x, y)$ chosen by chance move coincides with the price expectation vector, $(x, y) = (k, l)$, then Player 1 stops the game. In this case he cannot receive any profit from his informational advantage.

2. If not, $z = (x, y) \neq (k, l)$, then Player 1' optimal strategy is constructed with help of a two-stage lottery.

a) To choose between two-point and three-point distributions Player 1 realizes the Bernoulli trial with probabilities $\mathbf{P_p}(2|z)$ and $\mathbf{P_p}(3|z)$.

b) If two-point distributions are chosen, then Player 1 plays his optimal strategy in a game with two-point support (z, z_2) choosing a complementary point z_2 by means of the lottery with conditional probabilities $\mathbf{P_p}(z_2|z, 2)$.

If three-point distributions are chosen, then Player 1 plays his optimal strategy in a game with three-point support (z, z_2, z_3) choosing two complementary points by means of the lottery with conditional probabilities $\mathbf{P_p}(z_2, z_3|z, 3)$.

8. Bidding models with non-zero bid-ask spread

We generalize bidding models with one-type risky assets investigated in the previous sections where players proposed only one price for a share at each step, i.e. bid and ask prices coincide. Here we drop this restriction. We assume that at each step of bidding both players simultaneously propose their bid and ask prices for one share. The bid-ask spread s is fixed by rules of bidding. Transaction occurs from seller to buyer by bid price. The simplified model (sections 2-7) corresponds to the case $s = 0$ what is equivalent to $s = 1$ due to the price discreteness.

The model is reduced to a repeated game with incomplete information. Depending on bid-ask spread s one-step payoff matrices for these games have more complicated structure to compare with the case $s = 1$.

As for the zero bid-ask spread models we start with the case of two possible states of nature (two possible values for a share price). We generalize the results of section 2 for multistage games: we construct the upper and lower bounds for the values of n-stage games as $n \to \infty$. The bounds coincide for $s = 1$.

We generalize the developed in section 3 recursive approach to solutions of one-stage bidding games (see Sandomirskaya, 2012). The spectrum structure of optimal strategies becomes more complicated as lacunas longer than in the case $s = 1$ appear. The idea of equalizing insider's spectrum and obtaining recurrent relations on weights in the Player 2's optimal strategy remain applicable, however the difficulties concerned with explicit weight representation increase enormously.

Here we go to the case of two-point state of nature and generalize the results of sections 2 for bidding games with bid-ask spread. After this we make necessary comment on how to extend results for two-point state of nature to the case of countable one.

8.1. The model of bidding with two possible values for a share price

As for the zero bid-ask case we start with bidding games with two states of nature: the state m (integer positive) with probability p and the state 0 with probability $1 - p$. In this model any integer bids are admissible. For the sake of simplicity we assume that $m \bmod s = 0$. A chance move and an information structure of its outcome are the same as for models with zero bid-ask spread.

At each subsequent stage $t = 1, \ldots, n$ of bidding both players simultaneously propose their integer bid prices and integer ask prices for one share. The bid-ask spread s is fixed by rules of bidding. It is the same for both players. Denote i_t a bid price for Player 1 at stage t and j_t a bid price for Player 2 at stage t. Then $i_t + s$ and $j_t + s$ are ask prices for Player 1 and for Player 2 at stage t.

At stage t transaction of one share occurs if and only if an ask price of one player does not exceed a bid price of his opponent, i.e. either $i_t + s \leq j_t$, or $j_t + s \leq i_t$. If so, then a player-buyer gets one share from his opponent-seller according to his (buyer) bid price.: if $i_t + s \leq j_t$, then at stage t Player 2 buys one share from Player 1 for the price j_t; if $j_t + s \leq i_t$, then at stage t Player 1 buys one share from Player 2 for the price i_t. Thus, at stage t there is no transactions if and only if $|i_t - j_t| < s$.

This n-stage model with the bid-ask spread equal to s is described by a zero-sum repeated game $G_n^{m,s}(p)$ with incomplete information of Player 2 and with countable state and action spaces. The corresponding games $G_n^{m,s}(p)$ are given by the two matrices of one-step payoffs.

$$a^{L,m,s}(i,j) = \begin{cases} -i, & \text{if } i \geq j + s, \\ 0, & \text{if } |i - j| < s, \\ j, & \text{if } j \geq x + s, \end{cases}$$

$$a^{H,m,s}(i,j) = \begin{cases} m - i, & \text{if } i \geq j + s, \\ 0, & \text{if } |i - j| < s, \\ -m + j, & \text{if } j \geq i + s. \end{cases}$$

For $s = 0$, zero elements of the matrices appear at the principal diagonal only. For $s > 1$, zero elements fill a "band of s-range" along the principal diagonal. For $s > 1$ the more complicated structure of payoff matrices makes an analysis of games $G_n^{m,s}(p)$ more difficult.

8.2. Upper and lower bounds for the game value $V_n^{m,s}(p)$

Following the guideline of section 2 we get upper and lower bounds for for value function $V_n^{m,s}(p)$ provided by a "reasonable" strategy of Player 2 and a "reasonable" strategy of Player 1.

Upper bound for $V_n^{m,s}(p)$.

Theorem 8.1. *For any number of steps n functions $V_n^{m,s}$ are bounded from above by a function $H^{m,s}$ that is continuous, concave, and piecewise linear with m/s linearity domains $[sk/m, s(k+1)/m]$, $k = 0, 1, \ldots, m/s-1$. The function $H^{m,s}$ is completely determined with the values at its peak points $p_k = sk/m$, $k = 0, 1, \ldots, m/s$:*

$$H^{m,s}(p_k) = \frac{m^2}{2s} p_k(1 - p_k). \tag{8.1}$$

To prove the theorem we construct the following "reasonable" strategy $\tau^{m,s}$ of Player 2 that is an analogue of his optimal strategy in the game of infinite duration with $s = 1$ (see section 2).

For the initial probability $p \in \left[\frac{sk}{m}, \frac{s(k+1)}{m}\right)$ the first move of Player 2 strategy $\tau^{m,s}$ is to propose the bid price sk. Then at step t, $t = 2, 3, \ldots$, Player 2 shifts his bid price by s upwards or downwards depending on the insider's bid at the previous step:

$$\tau_t^{m,s}(i_{t-1}, j_{t-1}) = \begin{cases} j_{t-1} - s, & \text{if } i_{t-1} \le j_{t-1} - s; \\ j_{t-1}, & \text{if } |i_{t-1} - j_{t-1}| < s \\ j_{t-1} + s, & \text{if } i_{t-1} \ge j_{t-1} + s; \end{cases}$$

As the values $V_n^{m,s}$ are bounded from above as $n \to \infty$, the consideration of games with infinite number of steps becomes reasonable.

Lower bound for $V_\infty(p)$.

Theorem 8.2. *The function $V_\infty^{m,s}$ is bounded from below by a function $L^{m,s}$ that is continuous, concave, and piecewise linear with m/s linearity domains $[sk/m, s(k+1)/m]$, $k = 0, 1, \ldots, m/s-1$. The function $L^{m,s}$ has the following values at the peak points $p_k = sk/m$, $k = 0, 1, \ldots, m/s$:*

$$L^{m,s}(p_k) = V_1(s) \frac{m^2}{s^2} p_k(1 - p_k). \tag{8.2}$$

Value $V_1(s)$ is a guaranteed insider's gain per step, explicit formula will be given a few below.

Remark 8.1. The obtained upper and lower bounds have the same form.

Sketch of the proof for Theorem 8.2. As for the case $s = 1$ the Player 1' optimal strategy in the game of infinite duration generates the simple random walk (SRW) on the lattice of posterior probabilities of share prices, for the case $s > 1$ it is natural to investigate the class Σ^{SRW} of strategies with SRW-property on the lattice $\left\{\frac{sk}{m} \mid k = 0, ..m/s\right\}$ corresponding to the case $s > 1$.

Below we construct the best strategy in the class Σ^{SRW} and show that this strategy provides the result of Theorem 8.2.

To determine this strategy we use the following notation,

$$g(d) = \frac{1}{s} + \frac{1}{s-1} + \ldots + \frac{1}{s-d},$$

$$d^* = \max\{d \mid g(d) \le 1\},$$

$$\varepsilon^* = 1 - g(d^*).$$

For probability $p_k = sk/m$ the first move of insider's strategy $\sigma^{k,m,s}$ is to mix bid prices $\{sk - 2s\}$ and $\{sk, sk+1, \dots, sk+d^*, sk+d^*+1\}$ in accordance with total probabilities

$$\sigma_1^{k,m,s}(sk-2s|H) = \tfrac{1}{2},$$
$$\sigma_1^{k,m,s}(sk+d|H) = \tfrac{1}{2(s-d)}, \quad d = 0, 1, \dots, d^*,$$
$$\sigma_1^{k,m,s}(sk+d^*+1|H) = \tfrac{1}{2}\varepsilon^*.$$

Conditional probabilities of these bids are calculated so that corresponding posterior probabilities of high share price will be the following

$$p(i = sk-2s) = s(k-1)/m = p_{k-1},$$
$$p(i = sk+d) = s(k+1)/m = p_{k+1}, \quad d = 0, 1, \dots, d^*, d^*+1.$$

At the next step insider must apply the same strategy, but for the posterior probability calculated at the previous step.

This strategy generates the simple random walk over the lattice sk/m with absorption at extreme points, insider's profit per step being equal to $V(1)$ given by

$$V_1(s) = \frac{1}{2}\left(d^* + 1 + \varepsilon^*(s - d^* - 1)\right). \tag{8.3}$$

It is the best strategy in the class Σ^{SRW}.

Remark 8.2. For the minimal nontrivial case $s = 2$ the constructed "reasonable" strategy of insider is not his optimal strategy for the game of infinite duration.

Therefore, we conclude that the insider's optimal strategy does not generate simple random walk of price expectations and leads apparently to non-symmetric price fluctuations.

Relationships between upper and lower bounds. For the case $s = 1$, the obtained upper and lower bounds coincide and give the value function of bidding game of unlimited duration at its peak points $p_k = k/m$:

$$H^{m,1}(p_k) = L^{m,1}(p_k) = \frac{m^2}{2} p_k(1-p_k) = V_\infty^{m,1}(p_k).$$

For the case of minimal nontrivial bid-ask spread $s = 2$ the following equality holds at the points $p_k = 2k/m$,

$$L^{m,2}(p_k) = 3/4 H^{m,2}(p_k).$$

As $s \to \infty$ the ratio between L and H decreases and in the limit yields

$$L^{m,s}(p_k) \approx 0,63 \cdot H^{m,s}(p_k).$$

As shown above, bid-ask spread plays a role of regulator for transaction activity on stock market. As bid-ask spread increases transactions occur less frequently and expected insider's profit falls at least by s times to compare with the model without spread.

Generalization of the model with non-zero bid-ask spread to the case of countable set of possible values for a share price The results above are generalized to the case of countable set of possible values for a share price. We analyze the model where this price can take values on the lattice sk, $k \in \mathbb{Z}$ by analogy with section 6. The principal idea is to represent distributions on the integer lattice with given first moment as convex combinations (probability mixtures) of two-point distributions with the same first moments. It is shown that upper and lower bounds obtained above preserve their form with replacement of the term $sk(m - sk)$ by the variance $\mathbf{D}(\mathbf{p})$ for distributions with mean values $\mathbf{E}(\mathbf{p}) = sk$, $k \in \mathbb{Z}$. We construct the insider's strategies for these games as probability mixtures of strategies for two-point games implementing a preliminary additional lottery for the choice of two-point distribution.

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