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Nonreflected Vertical Propagation of Acoustic Waves in a Strongly Inhomogeneous Atmosphere

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Abstract—Using the linear theory of waves in a compressible atmosphere located in a gravitational field, we found a family of sound speed profiles for which the wave field can be represented by a traveling wave with no reflection. The vertical flux of wave energy on these nonreflected profiles is retained, which proves that the energy transfer may occur over long distances.

Keywords: waves, atmosphere, energy, sound speed, profile.

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INTRODUCTION

Acoustic gravity waves (AGWs) have remained a key subject of investigation for geophysicists and astrophysicists [1–7]. In atmospheric geophysics, this fact is explained primarily by the importance of AGWs in the energy transfer from the troposphere to the upper atmosphere. Special emphasis is placed on the problem of propagation of AGWs excited in the lower atmosphere by explosions, earthquakes, etc. [8–9]. In particular, reaching the ionosphere, the AGWs affect the quality factor of the magnetospheric resonator for electromagnetic waves [10].

In solar astrophysics, the study of AGWs is substantiated by the fact that the heating of chromospheres and coronas of the Sun and other stars is still an open problem. Historically, the acoustic noise generated by subphotospheric convection was recognized as the main candidate for carrying energy in the upper layers of the solar atmosphere [11]. However, calculations indicated that these waves, especially long waves (with a period of around 300 s), are subjected to a strong reflection from the inhomogeneity in the temperature profile [5–7]. However, only long waves can reach upper chromospheric heights without managing to dissipate in lower layers on impact ruptures [6]. In connection with this, recent preference is given to other mechanisms that can supply energy to the corona [5].

The first model that initiated a study of AGWs was the so-called isothermal atmosphere: a flat-layered medium in a constant gravitational field directed perpendicular to these layers, the unperturbed temperature of which is the same everywhere. A rather comprehensive linear theory of AGWs in this atmosphere has been developed [12, 13]. Using a number of transformations, the original linearized system of gas-dynamic equations is reduced to a wave equation with constant coefficients, which made it possible to obtain a dispersion relation describing two branches of waves: acoustic and internal gravity. Acoustic waves in this atmosphere propagate as in a homogeneous medium without reflection on any height, although their amplitude certainly depends on height. Unlike acoustic waves, gravitational waves can only propagate at an angle to the horizon.

The simplest nonisothermal model for AGWs is the polytropic atmosphere where temperature changes with height linearly. In [14, 15], the authors found that, although all the basic unperturbed physical parameters (temperature, pressure, and density) vary monotonically, any layer of this atmosphere is a refractive waveguide for oblique harmonic AGWs with certain parameters. In turn, for each wave there is a layer where this wave is captured. This is caused by the fact that, at one and the same depth, relatively short ($k > 1/H$) and relatively long ($k < 1/H$) waves deviated due to refraction in opposite (relative to the vertical axis) directions (here k is the horizontal wave number and $H(z)$ is the height of equivalent homogeneous atmosphere at level z). As a result, one and the same wave at different heights can be reflected in opposite directions as a short wave in some cases and as a long wave in other cases. In [16] it was shown that these features are found in AGWs in a medium with an exponential temperature change. Direct numerical calculations of AGWs in atmospheres with more complex temperature profiles also testify that there are energetic losses due to wave reflection [17].

Waves in an inhomogeneous medium are known generally to be reflected (see, for example, [18]); therefore, it is commonly accepted that the AGWs in a nonisothermal atmosphere cannot propagate to higher altitudes, which is confirmed by the results of analytical and numerical calculations [14–17]. Thus, all studies conducted so far show that nonisothermal atmospheres are characterized to some extent by either reflection or the refraction of AGWs. This factor certainly affects the energetic balance of the medium. In our opinion, this makes it important to find the conditions under which the wave reflection is minimal or absent completely.

Recently it has been shown that, in a strongly inhomogeneous medium under specific profiles of inhomogeneity, wave propagation is possible over long distances without reflection and examples of the dynamics of long surface waves in a shallow ocean [19–22] and internal gravity waves propagating into the oceanic depth [23–25] are presented. The approach used here is based on Lie algebra and transform conversions and was developed in mathematics for systems of differential equations of rather general form [26–31], although it has never been treated in terms of the nonreflected propagation of waves. In this paper, this approach is used to investigate the vertical propagation of AGWs in a strongly inhomogeneous atmosphere. The main equations of the model are briefly described in Section 2. Possible profiles of sound speed ensuring that the propagation of AGWs is nonreflected are described in Section 3. The wave field and energy flow in a medium with nonreflecting characteristics are discussed in Section 4. The Conclusions summarize the main results.

2. MAIN EQUATION FOR ACOUSTIC–GRAVITY WAVES IN AN INHOMOGENEOUS COMPRESSIBLE ATMOSPHERE

To analyze the conditions for the propagation of AGWs in a flat-layered atmosphere in a constant gravitational field, we use the classical system of equations of gas dynamics for adiabatic perturbations propagating vertically. In the linear approximation, the waves propagating in an inhomogeneous atmosphere are described by the equation for $\chi(z, t) = \partial V/\partial z$ (V is the vertical velocity of fluid particles) [13]:

$$\frac{\partial^2\chi}{\partial t^2} = c^2(z)\frac{\partial^2\chi}{\partial z^2} + \left[\frac{dc^2(z)}{dz} + \gamma g\right]\frac{\partial\chi}{\partial z} \qquad (1)$$

with coefficients depending on a single variable: the sound speed $c(z) = (\gamma p_0/\rho_0)^{1/2}$. Here, the equilibrium values of atmospheric pressure and density are determined by the vertical distribution of temperature $T_0(z)$ in the atmosphere:

$$p_0(z) = p(0)\exp\left[-\int_0^z \frac{dz'}{H(z')}\right],$$
$$\rho_0(z) = \rho(0)\frac{T(0)}{T(z)}\exp\left[-\int_0^z \frac{dz'}{H(z')}\right], \qquad (2)$$

where $p(0)$, $\rho(0)$, and $T(0)$ are the pressure, density, and temperature, respectively, at a fixed level ($z = 0$); $H(z) = c^2(z)/\gamma g$ is the height of the equivalent homogeneous atmosphere.

Equation (1) has been investigated earlier in a number of papers mentioned in the Introduction with the conclusion that AGWs in a nonisothermal atmosphere cannot propagate to large altitudes.

As will be shown below, there are certain profiles of temperature (sound speed) at which the nonreflected propagation of AGWs at long distances is possible.

3. NONREFLECTED PROFILES OF SOUND SPEED

We formulate the problem of whether there is a transformation of variables so that Eq. (1) with variable coefficients is reduced to a wave equation with constant coefficients. If such transformations exist, we can find nonreflected waves in an inhomogeneous medium because hyperbolic equations with constant coefficients always involve such waves. The idea of conversion follows from an analysis of wave processes in smoothly inhomogeneous media, where the amplitude and phase of a traveling wave are variable but the wave retains its shape in space. However, here we consider the case when the sound speed does not change slowly.

We seek the solution of Eq. (1) in the form

$$\chi(z,t) = A(z)W(Z,t), \quad Z = Z(z), \qquad (3)$$

where all the functions are to be determined. Inserting (3) into (1), we obtain the so-called Klein–Gordon equation with variable coefficients

$$A(z)\left[\frac{\partial^2 W}{\partial t^2} - c^2(z)\left(\frac{dZ}{dz}\right)^2\frac{\partial^2 W}{\partial Z^2}\right]$$
$$-\left[\frac{d}{dz}\left(c^2 A\frac{dZ}{dz}\right) + \left(c^2\frac{dA}{dz} + \gamma g A\right)\frac{dZ}{dz}\right]\frac{\partial W}{\partial Z} \qquad (4)$$
$$-\frac{d}{dz}\left(c^2\frac{dA}{dz} + \gamma g A\right)W = 0.$$

To provide this equation with constant coefficients, we must impose several conditions. One of them is evidently in the d'Alembertiane (in the first square brackets) and allows function $Z(z)$to be found:

$$Z(z) = \int \frac{dz}{c(z)}. \tag{5}$$

Naturally, the sign of the integral in (5) can be arbitrary; actually, it corresponds to the wave direction, as will be seen later. The physical meaning of function $Z(z)$ is obvious: it is the time of wave propagation in an inhomogeneous medium.

The second condition is that the content of the second square brackets in (4) vanishes to ensure that the wave field grows at both ends:

$$A(z) \sim \frac{1}{\sqrt{c(z)}} \exp\left[-\int \frac{dz}{2H(z)}\right]. \tag{6}$$

Thus, the wave-field amplitude was defined. If the problem were considered in the Wentzel–Kramers–Brillouin (WKB) approximation for a smoothly changing medium, the expression for amplitude (6) would be the same, giving additional arguments to justify the solutions obtained in the form of nonreflected waves.

Using (5) and (6), Eq. (4) is reduced to a simpler form

$$\frac{\partial^2 W}{\partial t^2} - \frac{\partial^2 W}{\partial Z^2} = Q(z)W, \tag{7}$$

where

$$Q = \frac{1}{A}\frac{d}{dz}\left[c^2(z)\frac{dA}{dz} + \gamma g A(z)\right]. \tag{8}$$

We require that $Q = \text{const}$, which, in view of (6), leads to the second-order ordinary differential equation to find the nonreflected sound speed profiles

$$\frac{1}{4}\frac{d^2c^2}{dz^2} - \frac{1}{16}\frac{(dc^2/dz)^2}{c^2} + \frac{\gamma^2 g^2}{4c^2} = -Q. \tag{9}$$

Due to its autonomy, this equation is once integrated and reduced finally to quadratures:

$$z + a_3 = \pm\frac{1}{\gamma g}\int \frac{cdc}{\sqrt{a_1c^2 + a_2c + 1}}, \tag{10}$$

where $a_1 = -4Q/\gamma^2 g^2$, a_2 and a_3 are arbitrary dimensional constants. The integral in (10) can be calculated for any signs of these coefficients. It is natural here to assume that $a_3 = 0$ without any loss of generality. We consider the possible profiles of sound speed ensuring a nonreflected wave propagation in an inhomogeneous atmosphere.

First we separate out a class of solutions for which $Q = a_1 = 0$. In this case, Eq. (9) is reduced to the classical wave equation describing traveling waves $W(t \pm Z)$. In original variables, the wave has a variable amplitude and phase, but its temporal form remains unchanged

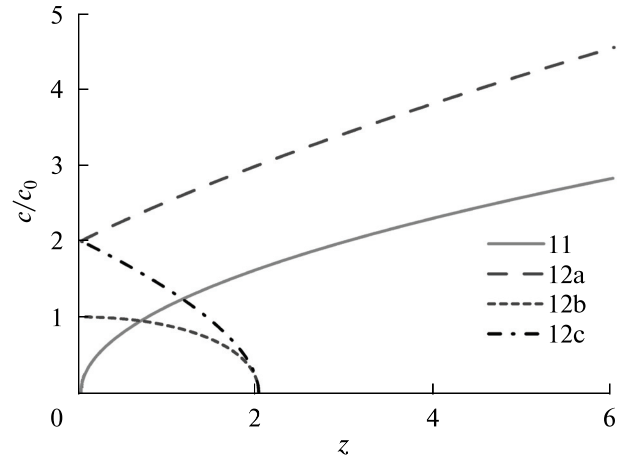

Fig. 1. Nonreflected profiles of sound speed for the case $Q = 0$; the numbers denote the numbers of formulas.

during the propagation. There are a few of these profiles. The first of them ($a_2 = 0$) is

$$c(z) = \sqrt{2\gamma g z}, \tag{11}$$

which corresponds to a polytropic atmosphere. The function $c(z)$ is defined on the semiaxis $(0, \infty)$.The next profiles for $a_2 = \pm(1/c_0)$ are described by inverse functions

$$\frac{3\gamma g}{2c_0^2}z = \sqrt{1 \pm \frac{c}{c_0}}\left(\frac{c}{c_0} \pm 2\right), \tag{12a}$$

$$\frac{3\gamma g}{2c_0^2}z = \sqrt{1 - \frac{c}{c_0}}\left(\frac{c}{c_0} + 2\right), \tag{12б}$$

$$\frac{3\gamma g}{2c_0^2}z = \sqrt{1 + \frac{c}{c_0}}\left(2 - \frac{c}{c_0}\right). \tag{12в}$$

Profile (12a) is also defined on the axis; it is monotonic and begins with $2c_0$. The two remaining profiles are defined on the whole range of z values and have limitations with respect to sound speed: up to c_0 in (12b) and up to $2c_0$ in (12c). All these profiles in dimensionless coordinates c/c_0, $Z = (3\gamma g/2c_0^2)z$ are shown in Fig. 1. For these profiles, $Q = 0$ and the upward propagation of waves is without dispersion (as a solution of a "pure" wave equation); we also call these "dispersion-free" profiles.

The next class of solutions can be obtained for nonzero values of Q when Eq. (7) is a Klein–Gordon equation, the solutions of which are dispersive waves (see the next section). For example, if $a_2 = 0$, we have

$$\left(\frac{\gamma g z}{c_0^2}\right)^2 - \text{sign}(Q)\left(\frac{c}{c_0}\right)^2 = 1, \quad Q = -\frac{\gamma^2 g^2}{4c_0^2}\text{sign}(a_1), \tag{13}$$

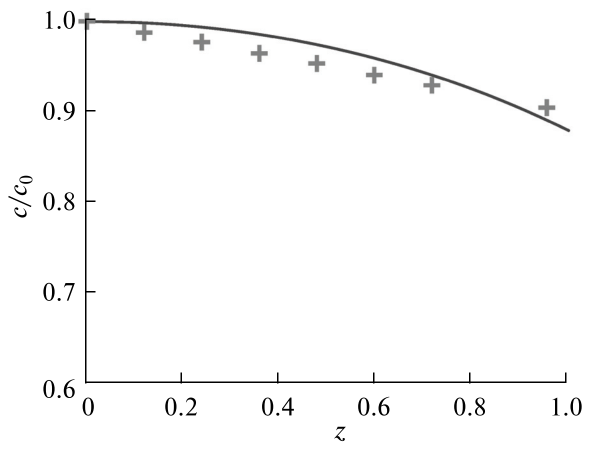

Fig. 2. Comparison of the nonreflected profile of sound speed with the real ground-level profile.

and, depending on the sign, Q yields a hyperbola or an ellipse. The general solution with all the constants being nonzero is written in a more cumbersome form

$$\pm 2\gamma g z = \frac{2\sqrt{M}}{a_1} - \frac{a_2}{a_1}$$

$$\times \begin{cases} \frac{1}{\sqrt{a_1}} \ln\left(2\sqrt{a_1 M} + N\right), & a_1 > 0, \ \Delta \neq 0, \\ \frac{1}{\sqrt{a_1}} \ln N, & a_1 > 0, \ \Delta = 0, \\ -\frac{1}{\sqrt{-a_1}} \arcsin\left(\frac{N}{\sqrt{-\Delta}}\right), & a_1 < 0, \ \Delta < 0, \end{cases} \tag{14}$$

where

$$M = a_1 c^2 + a_2 c + 1, \quad N = 2a_1 c + a_2, \quad \Delta = 4a_1 - a_2^2.$$

We emphasize that, in the family of "dispersion" profiles (13) and (14), all but the last in (14) are obtained when $Q < 0$. In general, they describe both monotonic and nonmonotonic profiles that are qualitatively similar to those given in (11)–(13). We will see below that the sign of Q affects the character of the AGW dispersion.

4. THE WAVE FIELD IN A NONREFLECTION ATMOSPHERE

In the general case, the elementary wave solution of Klein–Gordon equation (7) has the form

$$\chi(t, z) = A(z) \exp\left[i\left(\omega t - K\int \frac{dz}{c(z)}\right)\right] \tag{15}$$

with the dispersion relation

$$K = \pm\sqrt{\omega^2 + Q}. \tag{16}$$

Depending on the sign of Q, both positive and negative dispersion are possible.

Using elementary solutions (15), we can obtain integral expressions for waves of an arbitrary form. If $Q = 0$, the wave is written in the simplest form

$$\chi(t, z) = A(z) W[t - Z(z)] = \frac{G}{\sqrt{c(z)}} \times \exp\left(-\frac{\gamma g}{2}\int \frac{dz}{c^2(z)}\right) W\left[t - \int \frac{dz}{c(z)}\right]. \tag{17}$$

where G is an arbitrary constant and W is an arbitrary function.

Using (17), we can find all the components of the wave field. Indeed, the particle velocity V, which certainly is an integral of (17), can be expressed in a simpler form if we use [10]

$$V = -\frac{1}{\omega^2}\left[c(z)^2 \frac{\partial \chi}{\partial z} + \gamma g \chi\right]. \tag{18}$$

The wave perturbation of pressure also is uniquely determined through χ

$$p' = \frac{i\rho_0}{\omega^3}\left[g c^2(z) \frac{\partial \chi}{\partial z} + \left(\gamma g^2 - \omega^2 c^2\right)\chi\right]. \tag{19}$$

To investigate the possibility of propagation over super long distances in the case of nonreflected profiles of sound speed, we calculate the energy flux density in the vertical direction [10]:

$$\Pi = \frac{1}{2}[p' V^* + V p'^*], \tag{20}$$

where (*) denotes complex conjugation. Inserting (18) and (19) into (20), we obtain

$$\Pi(z) = -\frac{1}{2}\frac{G^2}{\omega^3} \frac{\rho_0(z) c^2(z)}{\exp\left(\gamma g \int dz/c^2\right)}. \tag{21}$$

In view of $c^2(z) = \gamma p_0(z)/\rho_0(z)$, we find that the energy flux is independent of z and, consequently, is retained. As a result, the wave can propagate to high altitudes without energy loss.

5. CONCLUSIONS

The vertical propagation of waves in an inhomogeneous atmosphere is investigated in the framework of a linear model. We found a family of sound speed profiles for which the wave field can be represented as a traveling wave in spite of the atmospheric heterogeneity. The wave energy flux on these nonreflected profiles is retained, and this proves the possibility of energy transfer to high altitudes. The number of these profiles is sufficiently large (around 10), which makes it possible to approximate real vertical distributions of sound speed in the Earth's and the solar atmospheres by piecewise nonreflected profiles. Figure 2 shows, in particular, that the nonreflected profile of sound speed calculated by formula (12b) is very consistent with the real profile of the Earth's standard atmosphere (here, the sound speed is normalized to the sound speed at the Earth's surface $c_0 = 330$ m/s and the height is normalized to that of the standard atmospheric height of $H_0 = 80$ km). In addition, the number of these profiles can be extended if the coefficients in the Klein–Gordon equation vary smoothly (as suggested by a reviewer of this paper). It is not improbable that their contribution to the energy balance may be significantly different from the currently accepted value. In addition, the approximations of real profiles in geophysics will simplify the wave dynamics calculations, reducing them to the solution of algebraic equations in the junction of nonreflected profiles.

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