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The Atiyah class, Hochschild cohomology and the Riemann–Roch theorem

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Abstract

We develop a formalism involving Atiyah classes of sheaves on a smooth manifold, Hochschild chain and cochain complexes. As an application we prove a version of the Riemann–Roch theorem.

Introduction

The present paper grew out of the question posed to the author by B. Feigin: 'Why does the Todd class look like the invariant volume form on a Lie group?' Our answer is contained in the proof of Proposition 6.

In Section 1 we develop a formalism describing Atiyah classes, Hochschild (co)homology and relationships between them. We introduce a global analogue of the Hochschild–Kostant–Rosenberg isomorphism [**9**]. This construction has appeared in the literature; see [**7**, **14**, **15**]. Our definition of the global HKR isomorphism appeared in the preprint of the present paper and was used in [**4**, **13**].

Let $\mathrm{D}(X)$ denote the derived category of sheaves of $\mathcal{O}$-modules on a smooth variety X. One can consider the Atiyah class as a morphism (in the category of endofunctors of $\mathrm{D}(X)$) from the identity functor to the functor of tensoring by the cotangent bundle shifted by one (See [**5**]):

$$\mathrm{at} : \mathrm{id} \longrightarrow \cdot \otimes \Omega^1[1]. \tag{1}$$

One may think about (1) as an action of the object $T[-1]$ dual to $\Omega^1[1]$ on the identity functor of $\mathrm{D}(X)$. Iterating this action, one can make the tensor power of $T[-1]$ (in fact, the symmetric power) act on the identity functor of $\mathrm{D}(X)$. This is an heuristic way to look at the map $\mathbf{I}$ in (20) that relates the Atiyah class with the Hochschild cochain complex.

Most of the constructions of Section 1 are applicable in a more general situation. Let D be a closed, symmetric, monoidal category with a compatible triangulation (see [**11**]). That is, D is a triangulated category with a symmetric product given by a functor $\otimes : D \times D \to D$ which is exact in each variable (with some compatibility properties that we do not need). In this setting we introduce the *category of Kähler differentials* of D as follows. The objects of this category are pairs (M, α), where M is an object of D and α is a morphism from the identity functor to the functor of tensoring by M:

$$\alpha : \mathrm{id} \longrightarrow \cdot \otimes M,$$

such that, for any E and F in D,

$$\alpha(E \otimes F) = \alpha(E) \otimes \mathrm{id} + \mathrm{id} \otimes \alpha(F).$$

Morphisms in the category of Kähler differentials are morphisms between the objects M which respect the morphisms α. Assume that the category of Kähler differentials has an initial object

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given by $(\Omega^1[1], \mathrm{At})$ (for suitable Ω^1 in D). In this case we will refer to the object Ω^1 as the *cotangent complex* of D and the morphism of functors At as the *Atiyah class*.

Consider the morphism

$$\mathrm{At} : \Omega^1[1] \longrightarrow \Omega^1 \otimes \Omega^1[2].$$

One can show that it defines a structure of a Lie coalgebra (in D) on $\Omega[1]$. Therefore, the dual object $T[-1]$ has a structure of a Lie algebra which acts on the identity functor of D. By analogy with the usual Lie-algebra theory, one may define the enveloping algebra of $T[-1]$ (an associative algebra in D) and its action extending the action of the $T[-1]$.

The basic example of such a situation is the derived category of representations of a Lie algebra $\mathfrak{g}$. In this case, we have $\Omega^1[1] = \mathfrak{g}^\vee$; that is, the (shifted) cotangent complex is the coadjoint representation $\mathfrak{g}^\vee$. For a representation V the Atiyah class is given by the map

$$V \longrightarrow V \otimes \mathfrak{g}^\vee$$

induced by the action of $\mathfrak{g}$ on V. The dual object is $\mathfrak{g}$ itself, and the Lie algebra structure on it is the usual one. Its enveloping algebra is the universal enveloping algebra of $\mathfrak{g}$ in the usual sense, equipped with the adjoint action.

Another example is the subject of the first part of the paper. The category D is the derived category of sheaves of $\mathcal{O}$-modules on a smooth variety with the usual tensor product. Comparing with the basic example one sees that the Hochschild cochain complex corresponds to the universal enveloping algebra, the Hochschild chain complex corresponds to formal functions on the group in the neighbourhood of the unit and so on. The reader may find this analogy helpful.

In the second part of the paper we prove the Riemann–Roch theorem as an application of the techniques developed in the first part.

Essentially, we follow [**12**]. However, instead of explicit calculations with the Čech cocycles we work in the derived category and use our algebraic-differential calculus. The proof consists of two parts. In the first part we reduce the theorem to a calculation of the dual class of the diagonal in the terminology of [**12**]. In the second part we perform the calculation.

NOTATION. By X we denote a smooth algebraic variety over a field k of characteristic 0 or bigger than $\dim X$; everything works for the complex-analytic case as well.

By $\mathrm{D}(X)$ we denote the derived category of perfect complexes of sheaves of $\mathcal{O}$-modules on X.

The symbol Δ always means the diagonal and the diagonal embedding.

We use p_i to denote projection onto the ith factor of X^n.

For $E \in \mathrm{D}(X)$, by E_Δ we denote $\Delta_* E = p_1^* E \otimes \mathcal{O}_\Delta = p_2^* E \otimes \mathcal{O}_\Delta$, where Δ is the diagonal in $X \times X$.

For $E \in \mathrm{D}(X)$, by $E^\vee$ we denote the dual object $\underline{R\mathrm{Hom}}^\bullet(E, \mathcal{O})$.

By Tr we denote the canonical morphism $E \otimes^L E^\vee \to \mathcal{O}$.

By Ω^i we denote the bundle of exterior forms, and $\Lambda^i T$ is dual to Ω^i.

We use ω to denote the bundle of exterior forms of top degree.

1. *Algebraic-differential calculus*

1.1. *The Atiyah class*

Let $\Delta \subset X \times X$ be the diagonal and let I denote the ideal sheaf of Δ. Then, by definition, $\mathcal{O}_\Delta = \mathcal{O}_{X\times X}/I$ and $\Omega_\Delta = I/I^2$. The two-step filtration on $\mathcal{O}_{X\times X}/I^2$ by powers of I gives rise to the exact sequence

$$0 \longrightarrow \Omega^1_\Delta \longrightarrow \mathcal{O}_{X\times X}/I^2 \longrightarrow \mathcal{O}_\Delta \longrightarrow 0. \tag{2}$$

Since the terms of the sequence (2) are supported on the diagonal, one may consider (2) as a sequence of sheaves of $\mathcal{O}_X$–$\mathcal{O}_X$-bimodules on X. The two $\mathcal{O}_X$-module structures coincide on Ω^1_Δ and $\mathcal{O}_X$, but are different on the middle term.

Let E be a sheaf of $\mathcal{O}$-modules or a complex of such sheaves on X. Take its tensor product with (2) with respect to the left $\mathcal{O}$ module structure, and consider it as a right $\mathcal{O}$ module. In other words, tensor (2) by p_1^*E and take the direct image p_{2*}. Because all terms in (2) are locally free left $\mathcal{O}$-modules, this operation is exact and one gets an exact sequence

$$0 \longrightarrow E \otimes \Omega^1 \longrightarrow \mathrm{J}^1(E) \longrightarrow E \longrightarrow 0. \tag{3}$$

Here $\mathrm{J}^1(E)$ denotes $E \otimes \mathcal{O}/I^2$ with the right $\mathcal{O}$-module structure and is called the *sheaf of the first jets*.

Definition 1 ([**1**, **10**]). For a sheaf of $\mathcal{O}$-modules or a complex of such sheaves E on X, the class of extensions represented by (3) is called the *Atiyah class* $\mathrm{at}(E) \in \mathrm{Ext}^1(E, E \otimes \Omega^1)$ of E.

The Atiyah class is the only obstruction to the existence of a connection on a sheaf (see [**1**]).

Definition 2. A *connection* on a sheaf of $\mathcal{O}$-modules E is a splitting of the exact sequence (3), that is, a map $\nabla : E \to \mathrm{J}^1(E)$ whose composition with the projection $\mathrm{J}^1(E) \to E$ equals to the identity map.

The Atiyah class of an object in the derived category is defined in a way compatible with the definition above. Let At be the morphism in the derived category, represented by extension (2):

$$\mathrm{At} : \mathcal{O}_\Delta \longrightarrow \Omega^1_\Delta[1]. \tag{4}$$

Definition 3. For $E \in \mathrm{D}(X)$ the morphism in $\mathrm{D}(X)$

$$\mathrm{at}(E) : E \longrightarrow E \otimes^L \Omega^1[1]$$

given by $Rp_{2*}(\mathrm{At} \otimes^L p_1^*E)$ is the *Atiyah class* of E.

A trivial but important observation is that the Atiyah class is natural. That is, for any morphism $E \xrightarrow{m} F$, the diagram

$$\begin{tikzcd} E \arrow[r, "m"] \arrow[d, "\mathrm{at}(E)"'] & F \arrow[d, "\mathrm{at}(F)"'] \\ E \otimes^L \Omega^1[1] \arrow[r, "m\otimes \mathrm{id}"] & F \otimes^L \Omega^1[1] \end{tikzcd} \tag{5}$$

is commutative.

Here are two useful lemmas.

Lemma 1. *Let $E = (E^i, d^i : E^i \to E^{i+1})$ be a complex of sheaves of $\mathcal{O}$-modules. Assume given a connection ∇^i on E^i. Then, $\mathrm{at}(E)$ is represented by*

$$(\nabla d)^i \stackrel{\mathrm{def}}{=} (d^i \circ \nabla^i - \nabla^{i+1} \circ d^i) : \, E^i \to E^{i+1} \otimes \Omega^1.$$

Proof. It suffices to show that the Atiyah class of E is given by the extension

$$0 \longrightarrow E \otimes \Omega^1 \longrightarrow \operatorname{cone}(\nabla d) \longrightarrow E \longrightarrow 0. \tag{6}$$

Consider the short exact sequence of complexes

$$\begin{tikzcd}
 & \cdots & \cdots & \cdots & \\
0 \arrow[r] & E^{i+1} \otimes \Omega^1 \arrow[r] \arrow[u] & \mathrm{J}^1(E^{i+1}) \arrow[r] \arrow[u] & E^{i+1} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & E^i \otimes \Omega^1 \arrow[r] \arrow[u, "d^i \otimes \mathrm{id}"] & \mathrm{J}^1(E^i) \arrow[r] \arrow[u, "\mathrm{J}^1(d^i)"] & E^i \arrow[r] \arrow[u, "d^i"] & 0 \\
 & \cdots \arrow[u] & \cdots \arrow[u] & \cdots \arrow[u] &
\end{tikzcd} \tag{7}$$

which, by definition, represents the Atiyah class. The connections on the terms of the complex E give rise to the isomorphisms (splittings) $\mathrm{J}^1(E^i) = E^i \oplus (E^i \otimes \Omega^1)$. In terms of these splittings we have

$$\mathrm{J}^1(d^i) = \begin{pmatrix} d^i & (\nabla d)^i \\ 0 & d^i \otimes \mathrm{id} \end{pmatrix},$$

which means that (7) is equal to (6). This proves the lemma. $\square$

LEMMA 2. *Let E and F be objects of* $\mathrm{D}(X)$. *Then* $\mathrm{at}(E \otimes^L F) = \mathrm{at}(E) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{at}(F)$.

Proof. Let Δ and Δ_i ($i \in \{1, 2, 3\}$) be the submanifolds of X^3 defined by the equations $x_1 = x_2 = x_3$ and $x_j = x_k$ for $j, k \neq i$, respectively, where (x_1, x_2, x_3) is a point in X^3. Let I and I_i denote the ideal sheaves of Δ and Δ_i in X^3.

The sheaf I/I^2 is by definition the conormal bundle to Δ in X^3. Therefore, it is isomorphic to the direct sum of two copies of Ω^1_Δ. The sheaf $I/(I^2 + I_i)$ is the restriction of the conormal bundle to Δ_i. Hence, it is isomorphic to Ω^1_Δ. There are three projections: $\pi_i : I/I^2 \to I/(I^2 + I_i) = \Omega^1_\Delta$. Identify I/I^2 with the direct sum of two copies of Ω^1_Δ so that projections on the first and the second summands are equal to π_1 and π_2. Then π_3 is equal to $-\pi_1 - \pi_2$.

Consider the exact sequences

$$0 \longrightarrow I/(I^2 + I_i) \longrightarrow \mathcal{O}/(I^2 + I_i) \longrightarrow \mathcal{O}_\Delta \longrightarrow 0 \tag{8}$$

and the corresponding morphisms in the derived category

$$\alpha_i : \mathcal{O}_\Delta \longrightarrow I/(I^2 + I_i)[1] = \Omega^1_\Delta[1]. \tag{9}$$

By definition, $Rp_{3*}(\alpha_i \otimes^L (p_1^* E \otimes^L p_2^* F))$ is equal to $\mathrm{at}(E) \otimes^L \mathrm{id}$, $\mathrm{id} \otimes^L \mathrm{at}(F)$ and $\mathrm{at}(E \otimes^L F)$ for $i = 1, 2, 3$, respectively.

Consider the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & I/I^2 \arrow[r] \arrow[d, "\pi_i"'] & \mathcal{O}/I^2 \arrow[r] \arrow[d] & \mathcal{O}_\Delta \arrow[r] \arrow[d, "\shortparallel"'] & 0 \\
0 \arrow[r] & I/(I^2 + I_i) \arrow[r] & \mathcal{O}/(I^2 + I_i) \arrow[r] & \mathcal{O}_\Delta \arrow[r] & 0
\end{tikzcd} \tag{10}$$

Here the vertical arrows are the natural projections. Now, (10) gives the following commutative diagram in the derived category.

$$\begin{array}{ccc} \mathcal{O}_\Delta & \longrightarrow & I/I^2[1] \\ \Big\| & & \Big\downarrow{\scriptstyle \pi_i} \\ \mathcal{O}_\Delta & \xrightarrow{\alpha_i} & I/(I^2+I_i)[1] \end{array} \tag{11}$$

Applying $Rp_{3*}(- \otimes^L (p_1^*E \otimes^L p_2^*F))$ to the top row of (11) we get the morphism

$$E \otimes^L F \longrightarrow E \otimes^L F \otimes^L (\Omega^1 \oplus \Omega^1)[1], \tag{12}$$

where I/I^2 is identified with $\Omega^1_\Delta \oplus \Omega^1_\Delta$, as above. We denote the projections of $\Omega^1 \oplus \Omega^1$ onto the summands by π_1 and π_2 as well. The compositions of the extension (12) with the projections π_1, π_2 and $\pi_3 = -\pi_1 - \pi_2$ are equal to $Rp_{3*}(\alpha_i \otimes^L (p_1^*E \otimes^L p_2^*F))$. The latter are equal to $\mathrm{at}(E) \otimes^L \mathrm{id}$, $\mathrm{id} \otimes^L \mathrm{at}(F)$ and $\mathrm{at}(E \otimes^L F)$ for $i = 1, 2, 3$, respectively. This proves the lemma. □

1.2. *The Lie algebra $\mathcal{T}$*

Consider the Atiyah class of the cotangent bundle

$$\mathrm{at}(\Omega^1) : \Omega^1 \longrightarrow \Omega^1 \otimes \Omega^1[1]. \tag{13}$$

Proposition 1. (i) *The Atiyah class* $\mathrm{at}(\Omega^1)$ *is symmetric, that is invariant under the permutation of factors in* $\Omega^1 \otimes \Omega^1$.

(ii) *The Atiyah class* $\mathrm{at}(\Omega^1)$ *obeys the* Jacobi identity, *that is, the projection of* $\mathrm{at}(\Omega^1) \otimes \mathrm{id} \circ \mathrm{at}(\Omega^1)$ *onto the part of* $\Omega^1 \otimes \Omega^1 \otimes \Omega^1$ *invariant under permutations is equal to zero.*

Proof. Let Δ be the diagonal in $X \times X$, and let I be its ideal sheaf.

(i) By definition, the Atiyah class of Ω^1 is given by the image under Rp_{2*} of the exact sequence

$$0 \longrightarrow I/I^2 \otimes p_1^*\Omega^1 \longrightarrow \mathcal{O}/I^2 \otimes p_1^*\Omega^1 \longrightarrow \mathcal{O}/I \otimes p_1^*\Omega^1 \longrightarrow 0. \tag{14}$$

It is enough to show that this extension may be represented as the composition of some extension and the embedding of $(S^2\Omega^1)_\Delta$ into $(\Omega^1 \otimes \Omega^1)_\Delta = I/I^2 \otimes p_1^*\Omega^1$, as the symmetric part.

The sequence (14) is included in the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^2/I^3 & \longrightarrow & I/I^3 & \longrightarrow & I/I^2 & \longrightarrow & 0 \\ & & {\scriptstyle d}\Big\downarrow & & {\scriptstyle d}\Big\downarrow & & {\scriptstyle d}\Big\downarrow\wr & & \\ 0 & \longrightarrow & I/I^2 \otimes p_1^*\Omega^1 & \longrightarrow & \mathcal{O}/I^2 \otimes p_1^*\Omega^1 & \longrightarrow & \mathcal{O}/I \otimes p_1^*\Omega^1 & \longrightarrow & 0 \end{array} \tag{15}$$

Here vertical arrows are given by the exterior differential (see [6]). It follows that the extension given by the bottom row is composition of the one given by the top row and the exterior differential $I^2/I^3 \to I/I^2 \otimes p_1^*\Omega^1$. The latter composition is equal to the embedding of $(S^2\Omega^1)_\Delta = I^2/I^3$ in $(\Omega^1 \otimes \Omega^1)_\Delta$ as the symmetric part. Thus the Atiyah class is symmetric.

(ii) In addition to (15), consider the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^3/I^4 & \longrightarrow & I^2/I^3 & \longrightarrow & I^2/I^3 & \longrightarrow & 0 \\ & & {\scriptstyle d}\Big\downarrow & & {\scriptstyle d}\Big\downarrow & & {\scriptstyle d}\Big\downarrow\wr & & \\ 0 & \longrightarrow & S^2_\Delta\Omega^1 \otimes p_1^*\Omega^1 & \longrightarrow & I/I^3 \otimes p_1^*\Omega^1 & \longrightarrow & \Omega^1_\Delta \otimes p_1^*\Omega^1 & \longrightarrow & 0 \end{array} \tag{16}$$

Here vertical arrows are again given by the exterior differential (see [**6**]), the left vertical map being the embedding of the symmetric tensor product into the partially symmetrised tensor product. By the previous paragraph, the image under Rp_{2*} of the bottom row gives the extension $\mathrm{Sym}(\mathrm{at}(\Omega^1) \otimes \mathrm{id})$, where Sym denotes the projection onto the symmetric part of $\Omega^1 \otimes \Omega^1$.

Therefore, to prove the lemma it is enough to show that push-out of the composition of extensions given by the bottom rows of (15) and (16) by the projection onto the symmetric part of $S^2\Omega^1_\Delta \otimes p_1^*\Omega^1$ is trivial.

Combining (15) and (16), one obtains the following commutative diagram in the derived category of X^2.

$$\begin{tikzcd}
I/I^2 \arrow[r] \arrow[d, hook, "d"'] & I^2/I^3[1] \arrow[r] \arrow[d, "d"'] & I^3/I^4[2] \arrow[r, equal] \arrow[d, "d"'] & S^3\Omega^1_\Delta[2] \arrow[d, equal] \\
\mathcal{O}_\Delta \otimes p_1^*\Omega^1 \arrow[r] & \Omega^1 \otimes p_1^*\Omega^1[1] \arrow[r] & S^2\Omega^1_\Delta \otimes p_1^*\Omega^1[2] \arrow[r] & S^3\Omega^1_\Delta[2]
\end{tikzcd} \tag{17}$$

Here, the last arrow in the bottom row is the projection onto the symmetric part. Therefore, the composition of morphisms in the top row is equal to the composition of morphisms in the bottom row. However, composition of the first two arrows in the top row is zero because they represent the extension classes between successive quotients of the two-step filtration on I/I^4 given by powers of I. This proves the lemma. □

It follows from the above proposition that there is a structure of a Lie (super)algebra in the derived category $\mathrm{D}(X)$ on the shifted tangent bundle $T[-1]$. The map dual to (13) is the bracket. Denote this Lie algebra by $\mathcal{T}$:

$$[\ ,\ ] : \mathcal{T} \otimes^L \mathcal{T} \longrightarrow \mathcal{T}.$$

The following definition could be given in more general context, but we need it only for a special case.

Definition 4. We say that an algebra U in $\mathrm{D}(X)$ with unit $e \in \mathrm{Hom}(\mathcal{O}, U)$ and multiplication $m : U \otimes^L U \to U$ is the *enveloping algebra* of $\mathcal{T}$ if the following hold:

(i) There is a map $\imath : \mathcal{T} \to U$ such that

$$\imath \circ [\ ,\ ] = (m - m^\sigma) \circ \imath \otimes \imath,$$

where m^σ is multiplication in the reverse order with an appropriate sign.

(ii) The map defined as the composition of embedding

$$S^*\mathcal{T} \stackrel{\mathrm{def}}{=} \bigoplus_i \Lambda^i T[-i] \hookrightarrow \bigoplus_i \mathcal{T}^{\otimes i}$$

and multiplication

$$\mathcal{T} \otimes \ldots \otimes \mathcal{T} \xrightarrow{\imath \otimes \ldots \otimes \imath} U \otimes \ldots \otimes U \longrightarrow U$$

is an isomorphism.

Note that the enveloping algebra exists and is unique. The second condition of Definition 4 gives the underlying object of $\mathrm{D}(X)$, and the first condition defines the multiplication on it. In fact, the multiplication is given by the Campbell–Hausdorff formula (see [**2**]), which describes the co-product on the formal function ring in exponential coordinates that is dual to the product on the universal enveloping algebra in terms of the Poincaré–Birkhoff–Witt isomorphism.

1.3. *Hochschild cohomology*

We define the object $\mathcal{U}$ in $\mathrm{D}(X)$ (*Hochschild cochain complex*) by

$$\mathcal{U} \stackrel{\mathrm{def}}{=} Rp_{1*}\, \underline{R\mathrm{Hom}}^{\bullet}{}_{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta),$$

where Δ is the diagonal in $X \times X$. The object $\mathcal{U}$ is endowed with the canonical structure of an algebra in $\mathrm{D}(X)$.

Let $\pi : \Omega^1_{X\times X} \to p_1^*\Omega^1_X$ be the natural projection. Let

$$\imath : \mathcal{T} \longrightarrow \mathcal{U} \tag{18}$$

denote the map defined as the contraction with

$$(\mathrm{id} \otimes \pi) \circ \mathrm{at}(\mathcal{O}_\Delta) \in \mathrm{Ext}^1(\mathcal{O}_\Delta, \mathcal{O}_\Delta \otimes p_1^*\Omega^1). \tag{19}$$

THEOREM 1. *The algebra $\mathcal{U}$ together with the map $\imath$ is the enveloping algebra of $\mathcal{T}$ in the sense of Definition* 4.

Proof. We prove that the first condition of Definition 4 is satisfied. To simplify the notation we omit the projection π.

One needs to show that the symmetric part of the composition

$$\mathcal{O}_\Delta \xrightarrow{\mathrm{at}(\mathcal{O}_\Delta)} \mathcal{O}_\Delta \otimes p_1^*\Omega^1[1] \xrightarrow{\mathrm{at}(\mathcal{O}_\Delta)\otimes\mathrm{id}} \mathcal{O}_\Delta \otimes p_1^*\Omega^1 \otimes p_1^*\Omega^1[2]$$

is equal to the composition

$$\mathcal{O}_\Delta \xrightarrow{\mathrm{at}(\mathcal{O}_\Delta)} \mathcal{O}_\Delta \otimes p_1^*\Omega^1[1] \xrightarrow{\mathrm{id}\otimes\mathrm{at}(\Omega^1)} \mathcal{O}_\Delta \otimes p_1^*\Omega^1 \otimes p_1^*\Omega^1[2].$$

Consider the following diagram.

$$\begin{array}{ccc} \mathcal{O}_\Delta & \xrightarrow{\mathrm{at}(\mathcal{O}_\Delta)} & \mathcal{O}_\Delta \otimes p_1^*\Omega^1[1] \\ {\scriptstyle \mathrm{at}(\mathcal{O}_\Delta)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{at}(\mathcal{O}_\Delta\otimes p_1^*\Omega^1)} \\ \mathcal{O}_\Delta \otimes p_1^*\Omega^1[1] & \xrightarrow{\mathrm{at}(\mathcal{O}_\Delta)\otimes\mathrm{id}} & (\mathcal{O}_\Delta \otimes p_1^*\Omega^1) \otimes p_1^*\Omega^1[2] \end{array}$$

It is (super)commutative by (5). By Lemma 2 we have the equality $\mathrm{at}(\mathcal{O}_\Delta \otimes p_1^*\Omega^1) = \mathrm{at}(\mathcal{O}_\Delta) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{at}(p_1^*\Omega^1)$. This proves the first condition.

To prove the second condition it suffices to show that the morphism from the second part of Definition 4 is an isomorphism at any geometric point of X. Because the construction of $\mathcal{U}$ is natural with respect to open embeddings, it is sufficient to prove it for the spectrum of a local ring. Moreover, one can replace the local ring with its completion, because completion is flat. Thus, one may assume that $X = \mathrm{Spec}\, k[[x_1, \ldots, x_n]]$. However, in this case it could be solved by direct calculation, making use of the Koszul resolution. Also, Proposition2 below could be applied. $\square$

We denote the isomorphism from the second part of Definition 4 by

$$\mathbf{I} : \bigoplus_i \Lambda^i T[-i] \xrightarrow{\sim} \mathcal{U}. \tag{20}$$

In [9] the case of a smooth affine manifold $X = \mathrm{Spec}(A)$ was considered. There the cohomology of $\mathcal{U}$, which is to say, $\mathrm{Ext}^i_{A\otimes A}(A, A)$ by means of the *standard resolution* (see [3]).

The standard resolution $B = (B_i, d_i)$, $i \geqslant 0$, of A as an $A \otimes A$-module is as follows: B_n is a free $A \otimes A$-module generated by tensor power $A^{\otimes n}$ over the base field, and the differential is given by

$$d(a_1 \otimes a_2 \otimes \ldots \otimes a_{n-1} \otimes a_n)$$
$$= a_1(a_2 \otimes \ldots \otimes a_n) - (a_1 a_2 \otimes \ldots \otimes a_n) + \ldots + (-1)^n (a_1 \otimes a_2 \otimes \ldots \otimes a_{n-1}) a_n. \tag{21}$$

The Hochschild–Kostant–Rosenberg isomorphism (see [9]) is given by

$$\begin{gathered}\Lambda^i T \to \operatorname{Hom}_{A\otimes A}(S_i, A) = \operatorname{Ext}^i_{A\otimes A}(A, A)\\ \partial_1 \wedge \ldots \wedge \partial_i : a_1 \otimes \ldots \otimes a_i \longmapsto \sum_{\sigma \in \Sigma_i} (-1)^{\operatorname{sign}\sigma} \partial_{\sigma(1)} a_1 \cdot \ldots \cdot \partial_{\sigma(i)} a_i.\end{gathered} \tag{22}$$

PROPOSITION 2. *For a smooth affine manifold $X = \operatorname{Spec}(A)$ the isomorphism* $\mathbf{I}$ *coincides with* (22).

Proof. The standard resolution is a resolution of the structure sheaf of the diagonal $\mathcal{O}_\Delta$. The terms of the standard resolution are free modules, and hence they are equipped with canonical connections. Applying Lemma 1 to the standard resolution, one obtains the following expression for the Atiyah class of $\mathcal{O}_\Delta$:

$$\operatorname{at}(\mathcal{O}_\Delta) : (a_1 \otimes a_2 \otimes \ldots \otimes a_{n-1} \otimes a_n)$$
$$\longmapsto da_1(a_2 \otimes \ldots \otimes a_{n-1} \otimes a_n) + (-1)^n (a_1 \otimes a_2 \otimes \ldots \otimes a_{n-1}) da_n,$$

where d is the exterior differential. To complete the proof, substitute the above formula into (19). □

1.4. *Hochschild homology*

Let $\mathcal{F}$ denote the object in $\mathrm{D}(X)$ (*Hochschild chain complex*) defined by

$$\mathcal{F} \stackrel{\text{def}}{=} Rp_{1*}(\mathcal{O}_\Delta \otimes^L \mathcal{O}_\Delta), \tag{23}$$

where Δ is the diagonal in $X \times X$.

The canonical action of $\mathcal{U}$ on $\mathcal{F}$

$$\mathcal{D} : \mathcal{U} \otimes^L \mathcal{F} \longrightarrow \mathcal{F} \tag{24}$$

is given by

$$\mathcal{U} = \underline{R\operatorname{Hom}}^\bullet(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \xrightarrow{-\otimes \mathrm{id}} \underline{R\operatorname{Hom}}^\bullet(\mathcal{O}_\Delta \otimes^L \mathcal{O}_\Delta, \mathcal{O}_\Delta \otimes^L \mathcal{O}_\Delta) = \underline{R\operatorname{Hom}}^\bullet(\mathcal{F}, \mathcal{F}).$$

Note that the composition of this morphism with $\imath$, equation (18), gives an action of $\mathcal{T}$ on $\mathcal{F}$:

$$\mathcal{T} \otimes^L Rp_{1*}(\mathcal{O}_\Delta \otimes^L \mathcal{O}_\Delta) \xrightarrow{\mathcal{D}} Rp_{1*}(\mathcal{O}_\Delta \otimes^L \mathcal{O}_\Delta), \tag{25}$$

which is equal to (19) tensored by $\mathcal{O}_\Delta$.

The canonical morphism $\mathcal{O}_\Delta \otimes^L \mathcal{O}_\Delta \to \mathcal{O}_\Delta \otimes \mathcal{O}_\Delta = \mathcal{O}_\Delta$ gives rise to the morphism

$$\varepsilon : \mathcal{F} \longrightarrow \mathcal{O}_\Delta. \tag{26}$$

PROPOSITION 3. *The composition of the maps $\mathcal{D}$ and ε defines a perfect pairing*

$$\mathcal{U} \otimes^L \mathcal{F} \longrightarrow \mathcal{O}. \tag{27}$$

Proof. The statement is local and may be proved by the same considerations as in the second part of the proof of Theorem 1. □

It follows from the proposition that $\mathcal{F}$ is dual to $\mathcal{U}$. Denote the isomorphism dual to $\mathbf{I}$ by

$$\mathbf{E}:\left(\bigoplus_i \Lambda^i T[-i]\right)^\vee = \bigoplus_i \Omega^i[i] \xleftarrow{\sim} \mathcal{F}.$$

Let $L^n \in \mathrm{Hom}(\bigoplus_i \Omega^i[i], (\bigoplus_i \Omega^i[i]) \otimes \Omega^1[1])$ denote the morphism defined as follows: its restriction to $\mathcal{O}$ is zero and its restriction to Ω^1 is given by the composition

$$L^n : \Omega^1 \xrightarrow{\mathrm{at}^n(\Omega^1)} \Omega^1 \otimes (\Omega^1)^{\otimes i}[i] \xrightarrow{\mathrm{id}\otimes\wedge} \Omega^1 \otimes \Omega^n[n] = \Omega^n \otimes \Omega^1[n]$$

(compare (36)); L^n is extended to all of $\bigoplus_i \Omega^i[i]$ by the Leibniz rule with respect to the wedge product.

Let

$$\mathbf{L}: \quad \bigoplus_i \Omega^i[i] \longrightarrow \left(\bigoplus_i \Omega^i[i]\right) \otimes \Omega^1[1]$$

denote the morphism defined by the formula

$$\mathbf{L} = \sum l_n L^n, \tag{28}$$

where $\sum l_n z^n = z/(e^z-1)$.

The following theorem provides a description of the action of $\mathcal{T} \subset \mathcal{U}$ on $\mathcal{F}$ that allows us to obtain the action of all of $\mathcal{U}$.

THEOREM 2. *The diagram*

$$\begin{array}{ccc} \mathcal{T}\otimes\mathcal{F} & \xrightarrow{\mathcal{D}} & \mathcal{F} \\ {\scriptstyle \mathrm{id}\otimes\mathbf{E}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{E}} \\ T\otimes\bigoplus_i \Omega^i[i][-1] & \xrightarrow{\mathbf{L}} & \bigoplus_i \Omega^i[i] \end{array}$$

is commutative.

Proof. The map adjoint to the action of $\mathcal{T}$ on $\mathcal{F}$ with respect to (27) is simply the (right) multiplication on $\mathcal{U}$. Thus the question is reduced to the problem of describing the multiplication in terms of the isomorphism $\mathbf{I}$. This is a purely combinatorial question and the answer may be found in any handbook on Lie algebras (for example, [**2**]). In dual terms, the problem is to write down the left invariant fields in exponential coordinates. □

In [**9**], in the case of a smooth affine variety $X = \mathrm{Spec}(A)$, the isomorphism dual (with respect to (27)) to (22) was computed in terms of the standard resolution S:

$$\begin{gathered} S_i \bigotimes_{A\otimes A} A = \mathrm{Tor}^i_{A\otimes A}(A,A) \longrightarrow \Omega^i \\ a_0(a_1\otimes\ldots\otimes a_i) \longmapsto a_0\, da_1\wedge\ldots\wedge da_i. \end{gathered} \tag{29}$$

PROPOSITION 4. *For a smooth affine manifold $X = \mathrm{Spec}(A)$ the isomorphism $\mathbf{E}$ coincides with the isomorphism (29).*

Proof. The proof follows from the proof of Proposition 2 and the definition of $\mathbf{E}$. □

There is a multiplication on $\mathcal{F}$ defined by the composition

$$(\mathcal{O}_\Delta \otimes^L \mathcal{O}_\Delta) \otimes^L (\mathcal{O}_\Delta \otimes^L \mathcal{O}_\Delta) \xrightarrow{\mathrm{id}\otimes\sigma\otimes\mathrm{id}} \mathcal{O}_\Delta \otimes^L \mathcal{O}_\Delta \otimes^L \mathcal{O}_\Delta \otimes^L \mathcal{O}_\Delta \xrightarrow{\varepsilon\otimes\varepsilon} \mathcal{O}_\Delta \otimes^L \mathcal{O}_\Delta,$$

where σ is the permutation of factors. One can show that it corresponds to the usual multiplication on differential forms under isomorphism E. We do not need this fact.

2. *The Riemann–Roch theorem*

2.1. *Serre duality*

In the following theorem we list the necessary facts concerning Serre duality. The first and the second statements are basic; see [**5**, **8**]. The third follows from the very definition, see, for example, [**12**].

THEOREM 3 (Serre duality). (i) *For X proper, there is a map $\int : H^{\dim X}(\omega) \to k$ such that for any $E \in \mathrm{D}(X)$ the composition of maps*

$$H^i(E) \otimes H^{\dim X - i}(E^\vee \otimes^L \omega) \xrightarrow{\mathrm{Tr}} H^{\dim X}(\omega) \xrightarrow{\int} k \tag{30}$$

gives a perfect (super)symmetric pairing.

(ii) *There exists a morphism in the derived category $\mathrm{D}(X)(X \times X)$*

$$\mathrm{can} : \mathcal{O}_\Delta \longrightarrow \mathcal{O} \boxtimes \omega[\dim X] \tag{31}$$

called the canonical extension, such that, for X proper, $E, F \in \mathrm{D}(X)$ and $m \in \mathrm{Hom}(E, F)$, the composition

$$\mathcal{O}_{X\times X} \xrightarrow{m} (F \boxtimes E^\vee)_\Delta \xrightarrow{\mathrm{can}\otimes(F\boxtimes E^\vee)} F \boxtimes (E^\vee \otimes^L \omega)[\dim X]$$

is equal to

$$\begin{aligned} H^*(m) \in \mathrm{Hom}(H^*(E), H^*(F)) = H^*(F) \otimes H^*(E)^\vee &\overset{(30)}{=} H^*(F) \otimes H^{\dim X - *}(E^\vee \otimes^L \omega) \\ &= H^*(F \boxtimes (E^\vee \otimes^L \omega)). \end{aligned}$$

(iii) *For $X = \mathrm{Spec}\, k\,[x_1, \dots, x_n]$, consider the Koszul resolution $K^\bullet$ of the diagonal (see [**3**]). Then the canonical extension is represented by the natural isomorphism $K_n = \mathcal{O} \boxtimes \omega$.*

For $E \in \mathrm{D}(X)$ let $\mathbf{K}$ denote the composition

$$\mathcal{O}_{X\times X} \xrightarrow{\mathbb{1}} E \boxtimes E^\vee \otimes^L \mathcal{O}_\Delta \xrightarrow{\mathrm{can}\otimes E\boxtimes E^\vee} E \boxtimes (E^\vee \otimes^L \omega)[\dim X], \tag{32}$$

where the first arrow is given by the identity operator $\mathcal{O} \to E \otimes E^\vee$. By the second statement of the theorem, $\mathbf{K} \in H^*(E) \otimes H^*(E^\vee \otimes \omega) = H^*(E) \otimes H^*(E)^\vee = \mathrm{End}\, H^*(E)$ is equal to the identity operator.

By the first statement of the theorem the trace of the restriction of $\mathbf{K}$ to the diagonal $\Delta^*\mathbf{K} \in H^*(E \otimes E^\vee \otimes \omega)$ followed by $\int$ is equal to the supertrace of the identity operator on $H^*(E)$, that is, to the *Euler characteristic*:

$$\chi(E) \overset{\mathrm{def}}{=} \sum_i (-1)^i \dim H^i(E).$$

To state the Riemann–Roch theorem we need to factor the morphism $\int \Delta^*\mathbf{K} \in H^*(E \otimes E^\vee \otimes \omega)$. Restricting (32) to the diagonal and taking the trace, we obtain

$$\mathcal{O}_X \xrightarrow{\mathbb{1}\otimes\mathcal{O}_\Delta} E \otimes^L E^\vee \otimes^L \mathcal{F} \xrightarrow{\mathrm{id}\otimes(\mathrm{can}\otimes\mathcal{O}_\Delta)} E \otimes^L E^\vee \otimes^L \omega[\dim X] \xrightarrow{\mathrm{Tr}} \omega[\dim X], \tag{33}$$

where $\mathcal{F}$ is defined by (23). Interchanging the last two arrows we obtain

$$\mathcal{O}_X \xrightarrow{\mathbb{1}\otimes\mathcal{O}_\Delta} E \otimes^L E^\vee \otimes^L \mathcal{F} \xrightarrow{\mathrm{Tr}\otimes\mathrm{id}} \mathcal{F} \xrightarrow{\mathrm{can}\otimes\mathcal{O}_\Delta} \omega[\dim X].$$

We introduce the following notation: let

$$\mathrm{Ch}(E) : \mathcal{O}_X \xrightarrow{\mathbb{1}\otimes\mathcal{O}_\Delta} E \otimes^L E^\vee \otimes^L \mathcal{F} \xrightarrow{\mathrm{Tr}\otimes\mathrm{id}} \mathcal{F}, \tag{34}$$

and let

$$\mathrm{Td} : \mathcal{F} \xrightarrow{\mathrm{can}\otimes\mathcal{O}_\Delta} \omega[\dim X]. \tag{35}$$

THEOREM 4 (Riemann–Roch theorem). *For $E \in \mathrm{D}(X)$*

$$\chi(E) = \int \mathrm{Td} \circ \mathrm{Ch}(E).$$

The classes Ch and Td are calculated explicitly in Propositions 5 and 6 below.

2.2. *The Chern character*

For E in $\mathrm{D}(X)$, let

$$\mathrm{at}^i(E) : E \longrightarrow E \otimes^L (\Omega^1)^{\otimes i}[i] \xrightarrow{\mathrm{id}\otimes\wedge} E \otimes^L \Omega^i[i], \tag{36}$$

where the first arrow is the i-fold of $\mathrm{at}(E)$ with itself, and the second is the usual multiplication in Ω^*. Let

$$\wedge\,\mathrm{at}^i(E) : \quad E \otimes^L \Omega^* \xrightarrow{\mathrm{at}^i(E)\otimes\mathrm{id}} E \otimes^L \Omega^i \otimes \Omega^*[i] \xrightarrow{\mathrm{id}\otimes\wedge} E \otimes^L \Omega^{i+*}[i].$$

We define

$$\exp(\mathrm{at}(E)) \in \bigoplus_i \mathrm{Ext}^i(E, E \otimes^L \Omega^i[i])$$

by the formula $\exp(\mathrm{at}(E)) = \sum \mathrm{at}^i(E)/i!$ and

$$\wedge \exp(\mathrm{at}(E)) : E \otimes^L \bigoplus_i \Omega^i[i] \longrightarrow E \otimes^L \bigoplus_i \Omega^i[i]$$

by the formula $\wedge \exp(\mathrm{at}(E)) = \wedge \sum \mathrm{at}^i(E)/i!$.

For $i = 1, 2$ we define the isomorphism a_i as the composition of isomorphisms

$$\begin{aligned} a_i : E_\Delta \otimes^L \mathcal{O}_\Delta &= (p_i^* E \otimes^L \mathcal{O}_\Delta) \otimes^L \mathcal{O}_\Delta \\ &= p_i^* E \otimes^L (\mathcal{O}_\Delta \otimes^L \mathcal{O}_\Delta) = p_i^* E \otimes^L \mathcal{F}_\Delta = (E \otimes^L \mathcal{F})_\Delta. \end{aligned}$$

Let

$$\mathbf{E}E_i : Rp_{1*}(E_\Delta \otimes^L \mathcal{O}_\Delta) \overset{a_i}{=} E \otimes^L \mathcal{F} \overset{\mathrm{id}\otimes\mathbf{E}}{=} E \otimes^L \bigoplus_j \Omega^j[j]$$

denote the composition of the image of a_i under Rp_{1*} with $\mathbf{E}$.

The following lemma could be an alternative definition of the Atiyah class.

LEMMA 3. *In the notation introduced above we have*

$$\mathbf{E}E_1 = (\mathrm{id} \otimes \wedge \exp(\mathrm{at}(E))) \circ \mathbf{E}E_2. \tag{37}$$

Proof. By analogy with (19) we define an action of $\mathcal{T}$ on $Rp_{1*}(E_\Delta \otimes^L \mathcal{O}_\Delta)$:

$$\mathcal{T} \otimes^L Rp_{1*}(E_\Delta \otimes^L \mathcal{O}_\Delta) \longrightarrow Rp_{1*}(E_\Delta \otimes^L \mathcal{O}_\Delta), \tag{38}$$

as the action of $(\mathrm{id}\otimes\pi)\circ\mathrm{at}(E_\Delta)\in\mathrm{Ext}^1(E_\Delta,E_\Delta\otimes^L p_1^*\Omega^1)$ on E_Δ (where $\pi:\Omega^1_{X\times X}\to p_1^*\Omega^1_X$ is the projection) followed by the restriction to the diagonal.

Substituting (38) into the definition of the isomorphisms a_i and using the definition of the action of $\mathcal{T}$ on $\mathcal{F}$ in (25) and Lemma 2, one obtains the commutative diagrams

$$\begin{tikzcd} \mathcal{T}\otimes^L Rp_{1*}(E_\Delta\otimes^L\mathcal{O}_\Delta) \arrow[r, "(38)"] \arrow[d, "\mathrm{id}\otimes Rp_{1*}(a_1)"'] & Rp_{1*}(E_\Delta\otimes^L\mathcal{O}_\Delta) \arrow[d, "Rp_{1*}(a_1)"'] \\ \mathcal{T}\otimes^L(E\otimes^L\mathcal{F}) \arrow[r, "\mathcal{D}"] & E\otimes^L\mathcal{F} \end{tikzcd} \tag{39}$$

where $\mathcal{D}$ denotes the morphism (25), and

$$\begin{tikzcd} \mathcal{T}\otimes^L Rp_{1*}(E_\Delta\otimes^L\mathcal{O}_\Delta) \arrow[r, "(38)"] \arrow[d, "\mathrm{id}\otimes Rp_{1*}(a_2)"'] & Rp_{1*}(E_\Delta\otimes^L\mathcal{O}_\Delta) \arrow[d, "Rp_{1*}(a_2)"'] \\ \mathcal{T}\otimes^L(E\otimes^L\mathcal{F}) \arrow[r, "\mathcal{D}+\mathrm{at}(E)"] & E\otimes^L\mathcal{F} \end{tikzcd} \tag{40}$$

The latter diagram is commutative because the projection of $\mathrm{at}(p_i^*E)$ to $p_i^*\Omega^1$ is equal to $p_i^*\,\mathrm{at}(E)$, while the projection to the complementary bundle is zero.

Arguments, as in Theorem 1, show that action of $\mathcal{T}$, equation (38), extends to an action of $\mathcal{U}$:

$$\mathcal{U}\otimes^L Rp_1^*(E_\Delta\otimes^L\mathcal{O}_\Delta)\longrightarrow Rp_1^*(E_\Delta\otimes^L\mathcal{O}_\Delta). \tag{41}$$

Let

$$\varepsilon_E: Rp_1^*(E_\Delta\otimes^L\mathcal{O}_\Delta)\longrightarrow E$$

denote the morphism induced by the canonical morphism $E_\Delta\otimes^L\mathcal{O}_\Delta\to E_\Delta$. Combining (41) with ε_E, one obtains the morphism

$$\mathcal{U}\otimes^L Rp_1^*(E_\Delta\otimes^L\mathcal{O}_\Delta)\longrightarrow E$$

and, dually, the morphism

$$Rp_1^*(E_\Delta\otimes^L\mathcal{O}_\Delta)\longrightarrow E\otimes^L\mathcal{U}^\vee \overset{\mathrm{id}\otimes\mathbf{E}}{=} E\otimes^L\bigoplus_i\Omega^i[i]. \tag{42}$$

It follows from the definition of $\mathbf{E}$ and (39) that the composition (42) is equal to the left-hand side of (37). It follows from (40) that it is equal to the right-hand side of (37). This proves the lemma. ∎

Definition 5. For $E\in\mathrm{D}(X)$, the *Chern character* is defined by the formula

$$\mathrm{ch}(E)=\mathrm{Tr}\sum_i \mathrm{at}^i(E)/i!\in\bigoplus_i H^i(\Omega^i).$$

Proposition 5. *For $E\in\mathrm{D}(X)$ the composition of the map $\mathrm{Ch}(E)$ given by (34) and the isomorphism $\mathbf{E}$ is equal to $\mathrm{ch}(E)$.*

Proof. It is clear from Definition 5 that it is sufficient to show that the composition of the first arrow in (34) with the map $\mathbf{E}$ is equal to $\mathrm{sexp}(\mathrm{at}(E))$.

For $i,j=1,2$ we have isomorphisms $I_{i,j}$ defined as the compositions

$$I_{i,j}: Rp_{1*}((E\otimes^L E^\vee)_\Delta\otimes^L\mathcal{O}_\Delta)=Rp_{1*}((p_i^*E\otimes^L p_j^*E^\vee)\otimes^L\mathcal{O}_\Delta)=(E\otimes^L E^\vee)\otimes^L\mathcal{F}.$$

The map in question is the composition of the identity section $\mathbb{1}: \mathcal{O}_{X\times X} \to (E\otimes^L E^\vee)_\Delta$ restricted to the diagonal with the isomorphism $I_{1,2}$:

$$\mathcal{O}_X \xrightarrow{\mathbb{1}\otimes\mathcal{O}_\Delta} Rp_{1*}((E\otimes^L E^\vee)_\Delta \otimes^L \mathcal{O}_\Delta) \xrightarrow{I_{1,2}} E\otimes^L E^\vee \otimes^L \mathfrak{F}. \tag{43}$$

The identity section of $(E\otimes^L E^\vee)_\Delta$ is equal to the composition

$$\mathcal{O}_{X\times X} \xrightarrow{1} \mathcal{O}_\Delta \xrightarrow{p_1^*\mathbb{1}\otimes\mathcal{O}_\Delta} p_1^*(E\otimes^L E^\vee)\otimes^L \mathcal{O}_\Delta = (E\otimes^L E^\vee)_\Delta, \tag{44}$$

where $\mathbb{1}: \mathcal{O}_X \to E\otimes^L E^\vee$ is the identity and $1: \mathcal{O}_{X\times X} \to \mathcal{O}_\Delta$ is the canonical map. Restricting to the diagonal and applying Rp_{1*}, one obtains

$$\mathcal{O}_X \xrightarrow{1\otimes\mathcal{O}_\Delta} \mathfrak{F} \xrightarrow{\mathbb{1}\otimes\mathrm{id}} E\otimes^L E^\vee \otimes^L \mathfrak{F}, \tag{45}$$

using $I_{1,1}: Rp_{1*}((E\otimes^L E^\vee)_\Delta \otimes^L \mathcal{O}_\Delta) = (E\otimes^L E^\vee)\otimes^L \mathfrak{F}$. Thus the composition

$$\mathcal{O}_X \xrightarrow{\mathbb{1}\otimes\mathcal{O}_\Delta} Rp_{1*}((E\otimes^L E^\vee)_\Delta \otimes^L \mathcal{O}_\Delta) \xrightarrow{I_{1,1}} E\otimes^L E^\vee \otimes^L \mathfrak{F} \tag{46}$$

is equal to $\mathbb{1}\otimes 1$, where $1: \mathcal{O} \to \mathfrak{F}$ is the natural embedding.

Applying Lemma 3 to (43) and (46), one proves the statement. □

2.3. *The Todd class*

PROPOSITION 6. *Let*

$$\mathrm{td} = \exp\left(\sum t_i \operatorname{ch}(\Omega^1)\right) \in \bigoplus_i H^i(\Omega^i), \tag{47}$$

where $\sum t_i z^i = \log(z/(e^z-1))$. *Then the class* Td *from* (35) *may be expressed as the composition*

$$\mathrm{Td}: \mathfrak{F} \xrightarrow{\mathbf{E}} \bigoplus_i \Omega^i[i] \xrightarrow{\wedge\,\mathrm{td}} \bigoplus_i \Omega^i[i] \twoheadrightarrow \omega[\dim X],$$

where the last arrow is the projection onto the differential forms of top degree.

Proof. Applying (5) to the canonical map (31) and composing with the projection $\pi: \Omega^1_{X\times X} \to p_1^*\Omega^1_X$, one obtains the following commutative diagram.

$$\begin{array}{ccc} \mathcal{O}_\Delta & \xrightarrow{\mathrm{can}} & \mathcal{O}\boxtimes\omega[\dim X] \\ {\scriptstyle \pi\circ\mathrm{at}(\mathcal{O}_\Delta)}\downarrow & & \downarrow{\scriptstyle \mathrm{at}(\mathcal{O})} \\ \mathcal{O}_\Delta\otimes p_1^*\Omega^1 & \xrightarrow{\mathrm{can}\otimes\mathrm{id}} & \Omega^1\boxtimes\omega[\dim X+1] \end{array} \tag{48}$$

The right vertical arrow is zero, using Lemma 2 and the fact that the Atiyah class of $\mathcal{O}$ is trivial. Therefore, so is the composition of the bottom and the left vertical arrows. Tensoring by $\mathcal{O}_\Delta$ one finds (using notation of (25)) that the composition

$$\mathrm{Td}\circ\mathfrak{D}: \mathfrak{F} \xrightarrow{\mathfrak{D}} \mathfrak{F}\otimes\Omega^1[1] \xrightarrow{\mathrm{Td}\otimes\mathrm{id}} \omega\otimes\Omega^1[\dim X+1] \tag{49}$$

is equal to zero.

Applying isomorphism $\mathbf{E}$ to (49) and making use of notations of Theorem 2, we obtain that

$$\bigoplus_i \Omega^i[i] \xrightarrow{\mathbf{L}} \left(\bigoplus_i \Omega^i[i]\right)\otimes\Omega^1[1] \xrightarrow{x\otimes\mathrm{id}} \omega\otimes\Omega^1[\dim X+1].$$

It follows from Lemma 4 below that vanishing of (49) determines Td up to a scalar factor. This factor (which is 1) can be determined from the local considerations by means of the third part of Theorem 3. This completes the proof. □

Lemma 4. *Suppose that $x : \bigoplus_i \Omega^i[i] \to \omega[\dim X]$ is a morphism such that the composition*

$$\bigoplus_i \Omega^i[i] \xrightarrow{\mathbf{L}} \left(\bigoplus_i \Omega^i[i]\right) \otimes \Omega^1[1] \xrightarrow{x\otimes \mathrm{id}} \omega \otimes \Omega^1[\dim X + 1]$$

is zero. Then, up to a factor it is given by the composition of $\wedge\,\mathrm{td}$ (see (47)) with the projection onto the differential forms of top degree.

Proof. Denote by $\overline{\cdot}$ the anti-involution on $\bigoplus_i \Omega^i[i]$ that multiplies Ω^i by $(-1)^{i(i-1)/2}$. We define a non-degenerate pairing on $\bigoplus_i \Omega^i[i]$ by the formula

$$\langle , \rangle : \left(\bigoplus_i \Omega^i[i]\right) \otimes \left(\bigoplus_i \Omega^i[i]\right) \xrightarrow{\overline{\cdot}\wedge\cdot} \bigoplus_i \Omega^i[i] \twoheadrightarrow \omega[\dim X], \tag{50}$$

where the last arrow is the projection onto the forms of top degree. There exists a unique $y \in \bigoplus_i H^i(\Omega^i)$ such that $\langle \overline{y}, \cdot\rangle = x$.

Let $\mathbf{L}^+$ denote the map adjoint to $\mathbf{L}$ with respect to $\langle\, ,\, \rangle$. This means that the diagram

$$\begin{CD}
\left(\bigoplus_i \Omega^i[i]\right) \otimes \left(\bigoplus_i \Omega^i[i]\right) @>{\mathrm{id}\otimes\mathbf{L}}>> \left(\bigoplus_i \Omega^i[i]\right) \otimes \left(\bigoplus_i \Omega^i[i]\right) \otimes \Omega^1[1]\\
@V{\mathbf{L}^+\otimes\mathrm{id}}VV @VV{\langle,\rangle\otimes\mathrm{id}}V\\
\left(\bigoplus_i \Omega^i[i]\right) \otimes \left(\bigoplus_i \Omega^i[i]\right) \otimes \Omega^1[1] @>{\langle,\rangle\otimes\mathrm{id}}>> \omega\,[\dim X] \otimes \Omega^1[1]
\end{CD} \tag{51}$$

is commutative. It follows from Theorem 2, by direct calculation, that $\mathbf{L}^+ = \wedge\overline{\mathrm{td}} \circ \mathbf{L} \circ \wedge\overline{\mathrm{td}}^{-1}$. (That is, td is analogous to the left-invariant volume form on a Lie group.)

By the hypothesis of the lemma, $0 = \langle \overline{y}, \mathbf{L}\,\cdot\rangle = \langle \mathbf{L}^+\overline{y},\, \cdot\,\rangle = \langle \overline{\mathrm{td}} \wedge \overline{\mathbf{L}\,\mathrm{td}^{-1}\wedge y},\, \cdot\,\rangle$. Since the pairing is non-degenerate and $\overline{\mathrm{td}}$ is invertible, it follows that $\mathbf{L}(\mathrm{td}^{-1}\wedge y) = 0$.

Thus it remains to prove that the only sections $s : \mathcal{O} \to \bigoplus_i \Omega^i[i]$ for which the composition

$$\mathcal{O} \xrightarrow{s} \bigoplus_i \Omega^i[i] \xrightarrow{\mathbf{L}} \bigoplus_i \Omega^i[i] \otimes \Omega^1[1] \tag{52}$$

vanishes are those which factor through $\mathcal{O} \hookrightarrow \bigoplus_i \Omega^i[i]$. It follows from formula (28) that the component of $\mathbf{L}$ in $\mathrm{Hom}(\Omega^i[i], \Omega^{i-1}[i-1] \otimes \Omega^1[1])$ for $i > 0$ is equal to the embedding $\Omega^i \hookrightarrow \Omega^{i-1} \otimes \Omega^1$ as the skew-symmetric part. This means that (52) cannot vanish if s has components in $\Omega^{>0}$. □

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