ON PROPERTIES OF SOLUTIONS OF SEMILINEAR SECOND-ORDER PARABOLIC EQUATIONS

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We consider semilinear second-order parabolic equations whose principal parts may have either divergence or nondivergence form and whose nonlinear terms satisfy conditions of Bernstein-Dini type. We study the qualitative properties of the classical solutions of nondivergence equations and generalized solutions of equations with divergent principal parts: the behavior of solutions in various unbounded domains and near the boundaries of domains, removability of singularities of solutions, vanishing of solutions in unbounded domains, in particular solutions of compact support and uniqueness and continuous dependence on the boundary conditions for solutions of the exterior initial/boundary problem. Bibliography: 21 titles.

CONTENTS

INTRODUCTION

In this paper we study the qualitative properties of solutions of semilinear second-order parabolic equations of the nondivergence and divergence forms

$$Lu \equiv a_{ij}(t,x)u_{x_ix_j} - u_t = f(t,x,u), \tag{1}$$

$$\mathfrak{L}u \equiv (a_{ij}(t,x)u_{x_j})_{x_i} - u_t = a_0|u|^q u, \quad a_0 = \text{const } > 0, \tag{2}$$

where $(t,x) = (t,x_1,\dots,x_n) \in \mathbf{R}^{n+1}$ and the coefficients $a_{ij} : \mathbf{R}^{n+1} \to \mathbf{R}$ are bounded measurable functions satisfying the following conditions for all $(t,x) \in \mathbf{R}^{n+1}$: $a_{ij}(t,x) = a_{ji}(t,x)$ for $i,j = 1,\dots,n$, and there exists a number $\lambda \geq 1$ such that the inequalities

$$\lambda^{-1}|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \lambda|\xi|^2 \quad \forall \xi \in \mathbf{R}^n \tag{3}$$

(summation from 1 to n over repeated subscripts in any monomial is understood). As corollaries of the "parabolic considerations" we establish certain results for solutions of the semilinear uniformly elliptic equation

$$L_0 u = a_{ij}(x)u_{x_ix_j} = f(x,u). \tag{4}$$

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Throughout the first part of this paper it is assumed that the function $f : \mathbf{R}^{n+1} \times \mathbf{R} \to \mathbf{R}$ in (1) is a measurable locally bounded function and that for all $(t, x) \in \mathbf{R}^{n+1}$

$$f(t,x,u_1) \le f(t,x,u_2) \text{ for } u_1 \le u_2, \ \ u_1, u_2 \in \mathbf{R}, \tag{5}$$

$$f(t,x,0) = 0, \tag{6}$$

$$|f(t,x,u)| \ge \varphi(|u|) \ \forall u \in \mathbf{R}, \tag{7}$$

where

$$\left.\begin{array}{c}\varphi : [0,+\infty) \to [0,+\infty) \text{ is a nondecreasing continuous function,} \\ \varphi(0) = 0, \ \ \varphi(\rho) > 0 \text{ for } \rho > 0\end{array}\right\} \tag{8}$$

(analogous conditions are assumed to be satisfied for the functions $f(x, u)$ of (4), the only difference being that the dependence on t is eliminated). In Eq. (2), to which the second part of this paper is devoted, we assume that $-1 < q = \text{const} \neq 0$. Eq. (2) is called *superlinear* if $q > 0$ and *sublinear* if $-1 < q < 0$.

The elliptic equation (4) has been studied by many authors. The behavior of solutions of this equation in unbounded domains and the question of removability of singularities of its solutions has been studied by H. Brézis and L. Véron [1] for the case when the function φ of (7) has the form

$$\varphi(\rho) = a_0 \rho^{1+q}, \quad a_0 = \text{const } > 0, \tag{9}$$

with $0 \neq q = \text{const} > -1$, by L. Véron [2] for the Laplacian $L_0 = \Delta$, and by V. A. Kondrat'ev and E. M. Landis [3; 4] for a general linear operator L_0 of nondivergence structure $L_0 = a_{ij}(x)\dfrac{\partial^2}{\partial x_i \partial x_j}$ and divergence structure $L_0 = \dfrac{\partial}{\partial x_i}\Big(a_{ij}(x)\dfrac{\partial}{\partial x_j}\Big)$. The questions of the existence and asymptotic behavior of solutions of Eq. (4) in the case of the operator $L_0 = \Delta$ have been studied in the papers of N. Kawano and M. Naito [5] for $f(x,u) = \Phi(x)|u|^q u$, where Φ is a positive function and $q > -1$, by R. Osserman [6] under assumptions on the nonlinearity $f(x,u) = f(u)$ close to those of § 6 below, and also for nonlinearities of a different type by S. I. Pokhozhaev [7], O. A. Oleinik [8], and in the case of the operator $L_0 = \dfrac{\partial}{\partial x_i}\Big(a_{ij}(x)\dfrac{\partial}{\partial x_j}\Big)$ by I. Kametaka and O. A. Oleinik [9].

As was shown in [1; 3–5] the properties of solutions of Eq. (4) in which the right-hand side satisfies conditions (5), (6), (7), and (9) for $q > 0$ differ from those possessed by the solutions of this equation for $-1 < q < 0$. This qualitative difference in the properties of solutions manifests itself both in the asymptotic behavior of solutions in unbounded domains and in the study of the question of removability of singularities of solutions (in the latter case a singular point is removable if $q \ge 2/(n-2)$, $n \ge 3$, and is not removable if $-1 < q < 2/(n-2)$).

The paper of H. Brézis and A. Friedman [10] is devoted to the parabolic equation (2) with the operator $\mathcal{L} = \Delta - \dfrac{\partial}{\partial t}$. In this paper the initial/boundary-value problem and the Cauchy problem for this equation were studied with singular initial data, and in particular it was established that for $q \ge 2/n$ Eq. (2) has the property of "removability" of a singular point. This property, however, fails in the case when $-1 < q < 2/n$. The asymptotic behavior of solutions of Eq. (2) under the same assumption with respect to the operator $\mathcal{L}$ were studied in the papers of S. Kamin, L. A. Peletier [11], and R. Kajikiya [12].

In the present paper we assume the existence of classical solutions of a nondivergence equation (1) and generalized solutions of the divergence equation (2), and we study the behavior of solutions in various unbounded domains and near the boundaries of domains, the removability of singular points of solutions, the vanishing of solutions in unbounded domains (including solutions of compact support), and the uniqueness and continuous dependence on boundary conditions of a solution of the exterior initial/boundary-value problem. For Eq. (1) we use a study of barrier functions to establish a difference in the properties of solutions similar to the difference that holds for solutions of elliptic equations with $q > 0$ and $-1 < q < 0$. This difference is characterized in terms of convergence and divergence of certain integrals of φ at 0 and $+\infty$ (the so-called conditions of Dini-Bernshtein type, which are satisfied, in particular, by the function (9) for $q > 0$ or $-1 < q < 0$). In the case of Eq. (2) we apply the classical technique of J. Moser to establish integral estimates of the solutions and we show that, as in the case of an elliptic equation, solutions of superlinear and sublinear parabolic equations behave differently.

This paper contains complete proofs of the results announced in part in [13].

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PART I: THE NONDIVERGENCE PARABOLIC EQUATION

§ 1. ASSUMPTIONS AND NOTATION

In the first part of this paper we study properties of solutions of Eq. (1) with respect to which we assume throughout that conditions (3), (5), (6), (7), and (8) hold. We assume that the function φ satisfies one of the five conditions of Bernstein-Dini type: (A,a), (A,b), (A,B), (B,c), (B,d), where

$$(A)\quad \int^{+\infty} (\rho(\varphi(\rho))^{-1/2}\,d\rho < +\infty \text{ (and therefore } \int^{+\infty} (\varphi(\rho))^{-1}\,d\rho < +\infty);$$

$$(a)\quad \int_{+0} (\rho(\varphi(\rho))^{-1/2}\,d\rho = +\infty, \quad \int_{+0} (\varphi(\rho))^{-1}\,d\rho = +\infty;$$

$$(b)\quad \int_{+0} (\rho(\varphi(\rho))^{-1/2}\,d\rho = +\infty, \quad \int_{+0} (\varphi(\rho))^{-1}\,d\rho < +\infty;$$

$$(B)\quad \int_{+0} (\rho(\varphi(\rho))^{-1/2}\,d\rho < +\infty \text{ (and therefore } \int_{+0} (\varphi(\rho))^{-1}\,d\rho < +\infty);$$

$$(c)\quad \int^{+\infty} (\rho(\varphi(\rho))^{-1/2}\,d\rho = +\infty, \quad \int^{+\infty} (\varphi(\rho))^{-1}\,d\rho = +\infty;$$

$$(d)\quad \int^{+\infty} (\rho(\varphi(\rho))^{-1/2}\,d\rho = +\infty, \quad \int^{+\infty} (\varphi(\rho))^{-1}\,d\rho < +\infty^{*}.$$

A solution of Eq. (1) in a domain $D \subset \mathbf{R}^{n+1}$ is understood in the classical sense, i.e., a function $u = u(t,x) \in C^{1,2}(D)$ such that (1) becomes an identity for all points $(t,x) \in D$ when $u(t,x)$ is substituted into it; $C^{1,2}$ denotes the space of continuous functions $u(t,x)$ that have continuous partial derivatives u_t, u_{x_i}, and $u_{x_i x_j}$, $i,j = 1,\dots,n$. Solutions of the differential inequalities that occur below are understood to be classical solutions that are defined in analogy with solutions of Eq. (1).

We remark that if the function u is a solution of Eq. (1), then, as follows from (5), (6), and (7), it is a solution of the inequality

$$Lu \cdot \operatorname{sgn} u \geq \varphi(|u|), \tag{1.1}$$

where

$$\operatorname{sgn} u = \begin{cases} -1, & \text{if } u < 0; \\ 0 & \text{if } u = 0; \\ 1, & \text{if } u > 0. \end{cases}$$

We always assume, without specifying it each time, that the nonlinearity $f(t,x,u)$ is such that for any function $u(t,x) \in C^{1,2}$ the function $f(t,x,u(t,x))$ is measurable. We adhere to the following notation: $\bar{D}$ is the closure of the domain $D \subset \mathbf{R}^{n+1}$ or $D \subset \mathbf{R}^n$, $\partial D = \bar{D} \setminus D$ is the boundary of the domain D, $B_R(x^0) = \{x \in \mathbf{R}^n : |x - x^0| < R\}$ is the ball of radius $R > 0$ with center at the point $x^0 \in \mathbf{R}^n$, where $|x| = \sqrt{x_i \cdot x_i}$ is the length of $x = (x_1,\dots,x_n) \in \mathbf{R}^n$, $P_R(x^0) = \{x \in \mathbf{R}^n : |x_i - x_i^0| < R,\ i = 1,\dots,n\}$ is the cube with edge of length $2R > 0$ and center of symmetry at the point $x^0 \in \mathbf{R}^n$, $Q_R^H(T,x^0) = (T-H,T] \times B_R(x^0)$

* We note that the conditions imposed on the function φ exclude linear parabolic equations from consideration.

is a cylinder in $\mathbf{R}^{n+1}$ of height $H > 0$ with spherical base, $G_R^H(T, x^0) = (T - H, T] \times P_R(x^0)$ is the bar (parallelepiped) in $\mathbf{R}^{n+1}$ of height $H > 0$ with cubic base. The *top* of the domain $D \subset \mathbf{R}^{n+1}$ (cf. [14, pp. 166–167]) is defined as the set $\gamma(D) = \{(t,x) \in \partial D : \exists h > 0,\ Q_h^h(t,x) \subset \bar{D} \text{ and } Q_h^h(t+h,x) \cap D = \varnothing\}$. $\Gamma(D) = \partial D \setminus \gamma(D)$ is the parabolic (singular) boundary of the domain $D \subset \mathbf{R}^{n+1}$.

We shall need a maximum principle for solutions of semilinear inequalities in an arbitrary bounded domain $D \subset \mathbf{R}^{n+1}$, which we state as follows.

Theorem 1.0 (maximum principle). *Let $D \subset \mathbf{R}^{n+1}$ be a bounded domain and $f(t,x,u)$ a measurable and locally bounded function in $\bar{D} \times \mathbf{R}$ satisfying* (5). *Let u_1 and u_2 be continuous solutions in $\bar{D}$ of the inequalities $Lu_1 \le f(t,x,u_1)$ and $Lu_2 \ge f(t,x,u_2)$ for all $(t,x) \in \bar{D} \setminus \Gamma(D)$, where L is the operator of* (1), *and suppose $u_1 \ge u_2$ on $\Gamma(D)$. Then $u_1 \ge u_2$ everywhere in $\bar{D}$.*

This theorem is a consequence of the classical maximum principle for solutions of linear inequalities—super and/or subparabolic functions (cf., for example, [14, Ch. 3, § 2]).

We shall also use the following elementary proposition.

Proposition 1.1. *Suppose condition* (8) *holds. Then* a) *for $\beta = +\infty$ and any $\alpha \in [0, +\infty)$ and also for $\alpha = 0$ and any $\beta \in (0, +\infty]$ the integrals $\int_\alpha^\beta \left(\int_\alpha^\rho \varphi(\zeta)\, d\zeta \right)^{-1/2} d\rho$ and $\int_\alpha^\beta (\rho(\varphi(\rho))^{-1/2}\, d\rho$ either both converge or both diverge;* b) *for $\beta = +\infty$ and any $\alpha \in (0, +\infty)$ and also for $\alpha = 0$ and any $\beta \in (0, +\infty)$ the convergence of the integral $\int_\alpha^\beta (\rho(\varphi(\rho))^{-1/2}\, d\rho$ implies the convergence of the integral $\int_\alpha^\beta (\varphi(\rho))^{-1}\, d\rho$. The converse of* b) *is false.*

§ 2. THE CASE OF CONDITION (A). THE FUNDAMENTAL THEOREM

In this section and the one following we assume that condition (A) holds.

Lemma 2.1. (The barrier function). *Under the assumption of condition (A, a) there exist strictly decreasing continuous bijections $r : (0, +\infty) \to (0, +\infty)$ and $h : (0, +\infty) \to (0, +\infty)$ depending on n, λ, and φ and on n and φ respectively (in particular $\lim_{\alpha \to +\infty} r(\alpha) = \lim_{\beta \to +\infty} h(\beta) = 0$ and $\lim_{\alpha \to +0} r(\alpha) = \lim_{\beta \to +0} h(\beta) = +\infty$) such that for any point $(T, x^0) \in \mathbf{R}^{n+1}$ and any numbers $\alpha > 0$ and $\beta > 0$ there exists a function $V(t,x) \equiv V_{\alpha,\beta}(T, x^0, t, x) \in C^{1,2}(G_{r(\alpha)}^{h(\beta)}(T, x^0))$, $V > 0$, with the following properties:*

$$LV \le \varphi(V) \ \ in \ \ G_{r(\alpha)}^{h(\beta)}(T, x^0), \tag{2.1}$$

$$V(t,x) \to +\infty \ \ as \ \ (t,x) \to \Gamma\big(G_{r(\alpha)}^{h(\beta)}(T, x^0)\big), \tag{2.2}$$

$$V(t, x^0) \equiv V_{\alpha,\beta}(T, x^0, T, x^0) = n\alpha + \beta. \tag{2.3}$$

Proof. We shall seek a function $V(t,x)$ of the form

$$V(t,x) = w(t) + \sum_{i=1}^n v(x_i - x_i^0), \tag{2.4}$$

assuming that $w(t) \ge \beta$, $v(y) \ge \alpha$, and $v''(y) \ge 0$; the domains of the one-dimensional variables t and y will be found below.

Taking account of (1), (3), (8), and (2.4) and the assumptions just made, we have

$$\begin{aligned} LV - \varphi(V) &= a_{ij}(t,x) V_{x_i x_j} - V_t - \varphi(V) = \sum_{i=1}^n a_{ii}(t,x) v''(x_i - x_i^0) - w'(t) - \varphi\Big(w(t) + \sum_{i=1}^n v(x_i - x_i^0)\Big) \\ &\le \sum_{i=1}^n \big(\lambda v''(x_i - x_i^0) - (n+1)^{-1} \varphi(v(x_i - x_i^0))\big) - \big(w'(t) + (n+1)^{-1} \varphi(w(t))\big). \end{aligned} \tag{2.5}$$

Keeping (8) in mind, we now choose the function $v(y)$ so as to be a solution of the Cauchy problem for the ordinary differential equation

$$\lambda(n+1)v''(y) = \varphi(v(y)), \tag{2.6}$$
$$v(0) = \alpha, \tag{2.7}$$
$$v'(0) = 0, \tag{2.8}$$

and the function $w(t)$ so as to be a solution of the Cauchy problem

$$(n+1)w'(t) + \varphi(w(t)) = 0, \tag{2.9}$$
$$w(T) = \beta. \tag{2.10}$$

Then the functions $v(y)$ and $w(t)$ respectively are determined as implicit functions from the following formulas:

$$\Big(\frac{\lambda(n+1)}{2}\Big)^{\frac{1}{2}} \int_\alpha^{v(y)} \Big(\int_\alpha^\rho \varphi(\zeta)\,d\zeta\Big)^{-\frac{1}{2}} d\rho \equiv \Big(\frac{\lambda(n+1)}{2}\Big)^{\frac{1}{2}} \int_0^{v(y)-\alpha} \Big(\int_0^\rho \varphi(\zeta+\alpha)\,d\zeta\Big)^{-\frac{1}{2}} d\rho = y, \tag{2.11}$$

$$(n+1)\int_\beta^{w(t)} \frac{d\rho}{\varphi(\rho)} \equiv (n+1)\int_0^{w(t)-\beta} \frac{d\rho}{\varphi(\rho+\beta)} = T - t. \tag{2.12}$$

By Eq. (2.8) the function $v(y)$ in (2.11) is an even function defined for y in the interval $(-r(\alpha), r(\alpha))$, where

$$r(\alpha) = \Big(\frac{\lambda(n+1)}{2}\Big)^{1/2} \int_\alpha^{+\infty} \Big(\int_\alpha^\rho \varphi(\zeta)\,d\zeta\Big)^{-1/2} d\rho, \tag{2.13}$$

and it follows from condition (A) and Proposition 1.1, a) that $r(\alpha) < +\infty$. We further find, by (2.11), that $v(y) \geq \alpha$ for all $y \in (-r(\alpha, r(\alpha))$, and we conclude from (2.13) that $v(y) \to +\infty$ as $y \to \pm r(\alpha)$. Moreover it follows from Eq. (2.6) that $v''(y) \geq 0$ (so that the assumptions made regarding $v(y)$ at the beginning of the proof are justified).

The function $w(T)$ in (2.12) is defined on the half-open interval $(T - h(\beta), T]$, where

$$h(\beta) = (n+1)\int_\beta^{+\infty} \frac{d\rho}{\varphi(\rho)}, \tag{2.14}$$

and $h(\beta) < +\infty$ by condition (A) and Proposition 1.1, b). It therefore follows from (2.12) that $w(t) \geq \beta$ for all $t \in (T - h(\beta), T]$ (and this justifies completely the assumptions at the beginning of the proof), and by (2.14) we have $w(t) \to +\infty$ as $t \to T - h(\beta)$.

Summarizing what has been said above, we conclude that the function $V(t,x)$ of (2.4) is defined on the bar $G_{r(\alpha)}^{h(\beta)}(T, x^0)$ and, as follows from (2.6) and (2.9), has a continuous first derivative with respect to t and is twice continuously differentiable with respect to x, while $V(t,x) \geq n\alpha + \beta$. Moreover inequality (2.1) follows from (2.5), (2.6), and (2.9); assertion (2.2) follows from the corresponding properties of the functions $w(t)$ and $v(y)$ for $y = x_i - x_i^0$; Eq. (2.3) follows from (2.4) (2.7), and (2.10).

Finally, by (8), the first equality in hypothesis (a), Proposition 1.1, a), and formula (2.13) we have $\lim\limits_{\alpha\to+0} r(\alpha) = +\infty$, and by (2.14) and the second equality in hypothesis (a) we have $\lim\limits_{\beta\to+0} h(\beta) = +\infty$. The remaining properties of the functions $r(\cdot)$ and $h(\cdot)$ mentioned in the statement of the lemma follow from the explicit representations (2.13) and (2.14). The lemma is now proved.

Remark 1. a) If we assume condition (A, b) holds, the preceding lemma undergoes the following changes: the function $r(\cdot)$ of (2.13) maps $(0, +\infty)$ onto $(0, +\infty)$ and the function $h(\cdot)$ of (2.14) maps $[0, +\infty)$ onto $(0, h(0)]$ (so that $\lim\limits_{\alpha\to+\infty} r(\alpha) = \lim\limits_{\beta\to+\infty} h(\beta) = 0$, $\lim\limits_{\alpha\to+0} r(\alpha) = +\infty$, and $\lim\limits_{\beta\to+0} h(\beta) = h(0)$, and in this case $h(0) < +\infty$ by hypotheses (A) and (b)). Throughout Lemma 2.1 one can set $\beta = 0$.

b) Under the assumption (A, B) the functions $r(\cdot)$ and $h(\cdot)$ map $[0, +\infty)$ onto $(0, r(0)]$ and $(0, h(0)]$ respectively (and consequently $\lim_{\alpha\to+\infty} r(\alpha) = \lim_{\beta\to+\infty} h(\beta) = 0$, $\lim_{\alpha\to+0} r(\alpha) = r(0)$, and $\lim_{\beta\to+0} h(\beta) = h(0)$, where $r(0) < +\infty$ and $h(0) < +\infty$ by (A) and (B) and Proposition 1.1, a)). Throughout Lemma 2.1 one can set $\alpha = \beta = 0$. When this is done, $V(t, x) \equiv V_{0,0}(T, x^0, t, x) \geq 0$ in $G_{r(0)}^{h(0)}(T, x^0)$.

Lemma 2.2. *Suppose the function u is a solution of the inequality* (1.1) *in the cylinder $Q_R^H(T, x^0)$ such that* a) *$|u(T, x^0)| > n\alpha + \beta$ with $\alpha > 0$ and $\beta > 0$ under the assumption (A, a);* b) *$|u(T, x^0)| > n\alpha > 0$ under assumption (A, b);* c) *$|u(T, x^0)| > 0$ under assumption (A, B). Then at least one of the following two inequalities holds in the respective cases:* a) *$R \leq \sqrt{n} r(\alpha)$, $H \leq h(\beta)$;* b) *$R \leq \sqrt{n} r(\alpha)$, $H \leq h(0)$;* c) *$R \leq \sqrt{n} r(0)$, $H \leq h(0)$, where the functions $r(\cdot)$ and $h(\cdot)$ are defined respectively by Eqs.* (2.13) *and* (2.14).

Proof. a) Suppose neither of the equalities given in the lemma holds: $R > \sqrt{n} r(\alpha)$ and $H > h(\beta)$, and for definiteness suppose

$$u(T, x^0) > n\alpha + \beta. \tag{2.15}$$

Then $Q_R^H(T, x^0) \supset G_{r(\alpha)}^{h(\beta)}(T, x^0)$ and the function $z(t, x) = u(t, x) - V(t, x)$, where $V(t, x) = V_{\alpha,\beta}(T, x^0, t, x)$ is the function of Lemma 2.1, is defined on the bar $G_{r(\alpha)}^{h(\beta)}(T, x^0)$. As follows from (2.15) and (2.3),

$$z(T, x^0) > 0, \tag{2.16}$$

and by (2.2) and the boundedness of the continuous function u on the bar $\bar{G}_{r(\alpha}^{h(\beta)}(T, x^0)$ we have

$$z(t, x) \to -\infty \text{ as } (t, x) \to \Gamma\big(G_{r(\alpha)}^{h(\beta)}(T, x^0)\big). \tag{2.17}$$

Now consider the connected component D of the set $\{(t, x) \in G_{r(\alpha)}^{h(\beta)}(T, x^0) : z(t, x) > 0\}$ containing the point (T, x^0). It follows from (2.16) that D is a nonempty bounded open set and $u > 0$ in D. It is clear that $\Gamma(D) \subset \{(t, x) \in G_{r(\alpha)}^{h(\beta)}(T, x^0) : z(t, x) = 0\}$, and it follows from (2.17) that $\Gamma(D)$ is a nonempty compact set. Thus by (2.1) and (1.1) we have the following inequalities in $\bar{D} \setminus \Gamma(D)$: $LV \leq \varphi(V)$ and $Lu \geq \varphi(u)$. Moreover $V = u$ on $\Gamma(D)$. By the maximum principle (Theorem 1.0) we obtain the inequality $V \geq u$ in $\bar{D}$ and, in particular, the inequality $z(T, x^0) \leq 0$, contradicting (2.16).

For the case when $-u(T, x^0) > n\alpha + \beta$ the proof of the lemma is similar, since we then set $z = -u - V$ and apply the inequality $L(-u) \geq \varphi(-u)$ in the corresponding connected component.

b) and c). The proof in these cases coincides with the one just given if we take account of Remark 1 and assume that $\beta = 0$ and $\alpha = \beta = 0$ respectively when conditions (A, b) and (A, B) hold.

The main result of this section is the following theorem, which establishes a connection between the dimensions of the cylinder in which a solution exists and the value of that solution at the center of the top of the cylinder.

Theorem 2.3. *Suppose the function u is a solution of inequality* (1.1) *in the cylinder $Q_R^H(T, x^0)$. Then we have the following inequalities:* a) *under assumption (A, a)*

$$|u(T, x^0)| \leq nr^{-1}(R/\sqrt{n}) + h^{-1}(H) \leq 2\max(nr^{-1}(R/\sqrt{n}), h^{-1}(H));$$

b) *under assumption (A, b)*

$$|u(T, x^0)| \leq nr^{-1}(R/\sqrt{n}) + h^{-1}(H) \text{ for } H \leq h(0),$$
$$|u(T, x^0)| \leq nr^{-1}(R/\sqrt{n}) \text{ for } H > h(0);$$

c) *under assumption (A, B)*

$$|u(T, x^0)| \leq nr^{-1}(R/\sqrt{n}) + h^{-1}(H) \text{ for } R \leq \sqrt{n} r(0) \text{ and } H \leq h(0),$$
$$|u(T, x^0)| \leq h^{-1}(H) \text{ for } R > \sqrt{n} r(0) \text{ and } H \leq h(0),$$
$$|u(T, x^0)| \leq nr^{-1}(R/\sqrt{n}) \text{ for } R \leq \sqrt{n} r(0) \text{ and } H > h(0),$$

and the following equality holds:

$$u(T,x^0)=0 \text{ for } R>\sqrt{n}r(0) \text{ and } H>h(0),$$

where $r^{-1}(\cdot)$ *and* $h^{-1}(\cdot)$ *denote the functions inverse to* $r(\cdot)$ *and* $h(\cdot)$ *respectively, which are defined in* (2.13) *and* (2.14).

Proof. a) It follows from the properties of the functions $r(\cdot)$ and $h(\cdot)$ shown in Lemma 2.1 that there exist $\alpha_0>0$ and $\beta_0>0$ such that for any $\varepsilon>0$ we have

$$\sqrt{n}r(\alpha_0+\varepsilon)<R=\sqrt{n}r(\alpha_0),\quad h(\beta_0+\varepsilon)<H=h(\beta_0). \tag{2.18}$$

From this we deduce that

$$|u(T,x^0)|\le n(\alpha_0+\varepsilon)+(\beta_0+\varepsilon). \tag{2.19}$$

Indeed, if inequality (2.19) does not hold, Lemma 2.2, a) implies that at least one of the following inequalities holds:

$$R\le\sqrt{n}r(\alpha_0+\varepsilon),\quad H\le h(\beta_0+\varepsilon),$$

and this contradicts (2.18). It thus follows from (2.19) and (2.18) that

$$|u(T,x^0)|\le n(r^{-1}(R/\sqrt{n})+\varepsilon)+(h^{-1}(H)+\varepsilon),$$

from which, taking account of the arbitrariness of ε, we obtain the desired inequality.

b) By Remark 1, a) there exist $\alpha_0>0$ and $\beta_0\ge 0$ such that for $\varepsilon>0$

$$\sqrt{n}r(\alpha_0+\varepsilon)<R=\sqrt{n}r(\alpha_0), \tag{2.20}$$

$$h(\beta_0+\varepsilon)<H=h(\beta_0) \text{ provided } H\le h(0). \tag{2.21}$$

Hence for $H>h(0)$

$$|u(T,x^0)|\le n(\alpha_0+\varepsilon)=n(r^{-1}(R/\sqrt{n})+\varepsilon),$$

and for $H\le h(0)$

$$|u(T,x^0)|\le n(\alpha_0+\varepsilon)+(\beta_0+\varepsilon)=n(r^{-1}(R/\sqrt{n})+\varepsilon)+(h^{-1}(H)+\varepsilon),$$

since if these inequalities do not hold, then by Lemma 2.2 (part b) or a) respectively) we arrive in (2.20) and (2.21) at a contradiction with the inequalities $R\le\sqrt{n}r(\alpha_0+\varepsilon)$, $H\le h(0)$, or $R\le\sqrt{n}r(\alpha_0+\varepsilon)$, $H\le h(\beta_0+\varepsilon)$.

c) In this case the first and third inequalities and the last equality are proved following the same outline as in parts a) and b). Only the second inequality requires justification. Thus, let $R>\sqrt{n}r(0)$ and $H\le h(0)$. We find $\beta_0\ge 0$ such that $h(\beta_0+\varepsilon)<H=h(\beta_0)$ and in Lemma 2.1 we set $\alpha=0$ and $\beta=\beta_0+\varepsilon>0$, where $\varepsilon>0$ (this is possible by assumption (B)). In this way we construct a bar $G^{h(0)}_{r(0)}(T,x^0)$ and a function $V(t,x)=V_{0,\beta}(T,x^0,t,x)$ defined in it with the properties (2.1), (2.2), and $V(T,x^0)=\beta$. We then obtain an assertion analogous to Lemma 2.2: if the function u is a solution of inequality (1.1) in the cylinder $Q^H_R(T,x^0)$ such that $|u(T,x^0)|>\beta>0$, then at least one of the inequalities $R\le\sqrt{n}r(0)$ and $H\le h(\beta)$ holds. From this assertion and the choice of β_0 we conclude that

$$|u(T,x^0)|\le h^{-1}(H)+\varepsilon.$$

The theorem is now proved.

From this theorem we can obtain various kinds of information about the properties of solutions of inequality (1.1) (and consequently about properties of solutions of Eq. (1) as well). Corollaries of this theorem are given in §§ 3 and 4.

It is not difficult to verify that condition (A, a) holds for φ of the form (9) for $q > 0$. In this case for the functions (2.13) and (2.14) we obtain the following expressions:

$$r(\alpha) = \Big(\frac{\lambda(n+1)(2+q)}{2a_0}\Big)^{1/2} \int_\alpha^{+\infty} (\rho^{2+q} - \alpha^{2+q})^{-1/2}\, d\rho, \ \ \alpha > 0, \ \ h(\beta) = \frac{n+1}{a_0 q} \cdot \beta^{-q}, \ \ \beta > 0.$$

We now estimate the quantity $r(\alpha)$ from above and below. Thus for $\rho \geq \alpha$

$$\int_\alpha^\rho \zeta^{1+q}\, d\zeta = (\rho - \alpha) \int_0^1 (y(\rho-\alpha)+\alpha)^{1+q}\, dy \geq (\rho-\alpha)\int_{1/2}^1 (y(\rho-\alpha)+\alpha)^{1+q}\, dy \geq \frac{\rho-\alpha}{2} \cdot \Big(\frac{\rho+\alpha}{2}\Big)^{1+q},$$

and we have for $\theta = \dfrac{\rho - \alpha}{2}$

$$r(\alpha) \leq 2\Big(\frac{\lambda(n+1)}{2a_0}\Big)^{1/2} \int_0^{+\infty} (\theta(\theta+\alpha)^{1+q})^{-1/2}\, d\theta \leq$$

$$2\Big(\frac{\lambda(n+1)}{2a_0}\Big)^{1/2} \Big(\int_0^\alpha (\theta\alpha^{1+q})^{-1/2}\, d\theta + \int_\alpha^{+\infty} (\theta^{2+q})^{-1/2}\, d\theta\Big) = 2\sqrt{2}(1+q^{-1})\Big(\frac{\lambda(n+1)}{a_0}\Big)^{1/2} \alpha^{-q/2}.$$

It follows from the inequalities $\rho^{2+q} - \alpha^{2+q} \leq \rho^{2+q}$ that

$$r(\alpha) \geq \frac{2}{q}\Big(\frac{\lambda(n+1)(2+q)}{2a_0}\Big)^{1/2} \cdot \alpha^{-q/2}.$$

Thus

$$a_1 R^{-2/q} \leq r^{-1}(R) \leq a_2 R^{-2/q}, \quad R > 0,$$

where a_1 and a_2 are positive constants depending only on n, λ, a_0, and q.

Therefore in the case of a function φ of the form (9) with $q > 0$ Theorem 2.3, a) has the following appearance:

Theorem 2.4. *There exists a positive constant c depending only on n, λ, a_0, and q such that if the function u is a solution of inequality* (1.1) *in the cylinder $Q_R^H(T, x^0)$, then the inequality*

$$|u(T, x^0)| \leq c(R^{-2/q} + H^{-1/q})$$

holds at the center of the top of the cylinder.

§ 3. COROLLARIES OF THE FUNDAMENTAL THEOREM

Let $r^{-1}(\cdot)$ and $h^{-1}(\cdot)$ be, as before, the inverses of the functions $r(\cdot)$ and $h(\cdot)$ defined in (2.13) and (2.14).

1. The behavior of solutions in unbounded domains. Suppose the function u is a solution of inequality (1.1) in one of the following domains: 1) $(0, +\infty) \times \mathbf{R}^n$; 2) $\{x_i > 0\} \equiv \{(t,x) \in \mathbf{R}^{n+1} : x_i > 0, 1 \leq i \leq n \text{ fixed}\}$; 3) $\{t > 0, x_i > 0\} \equiv \{(t,x) \in \mathbf{R}^{n+1} : t > 0, x_i > 0, 1 \leq i \leq n \text{ fixed}\}$; 4) $(0, +\infty) \times (\mathbf{R}^n \setminus \bar{\Omega})$, where $\Omega \subset \mathbf{R}^n$ is a bounded domain; 5) an unbounded domain $D \subset \mathbf{R}^{n+1}$ containing a curve l that goes to infinity and has the property that for any point $(t_\sigma, x_\sigma) \in l$ (here σ is the parameter on the curve l and (t_σ, x_σ) is the point corresponding to it on the curve) there exists a cylinder $Q_{R_\sigma}^{H_\sigma}(t_\sigma, x_\sigma) \subset D$, and $R_\sigma \to +\infty$ and $H_\sigma \to +\infty$ as $\sigma \to +\infty$.

Then we have the following conclusions.

Under assumption (A, a) we have: 1) $|u(t,x)| \leq h^{-1}(t)$, $(t,x) \in (0,+\infty) \times \mathbf{R}^n$ and consequently $u(t,x) \to 0$ as $t \to +\infty$ uniformly with respect to $x \in \mathbf{R}^n$; 2) $|u(t,x)| \leq n r^{-1}(x_i/\sqrt{n})$, $(t,x) \in \{x_i > 0\}$ and consequently $u(t,x) \to 0$ as $x_i \to +\infty$ uniformly with respect to $t \in \mathbf{R}^1$, $\hat{x}_i \equiv (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbf{R}^{n-1}$; 3) $|u(t,x)| \leq n r^{-1}(x_i/\sqrt{n}) + h^{-1}(t)$, $(t,x) \in \{t > 0, x_i > 0\}$, and therefore $u(t,x) \to 0$ as $t \to +\infty$ and $x_i \to +\infty$ uniformly with respect to $\hat{x}_i \in \mathbf{R}^{n-1}$; 4) $|u(t,x)| \leq n r^{-1}(d(x)/\sqrt{n}) + h^{-1}(t)$, $(t,x) \in$

$(0,+\infty) \times (\mathbf{R}^n \setminus \bar{\Omega})$, $d(x) = \operatorname{dist}(x, \partial\Omega)$ being the distance from the point x to the boundary $\partial\Omega$, and therefore $u(t,x) \to 0$ as $t \to +\infty$ and $|x| \to +\infty$; 5) $|u(t_\sigma, x_\sigma)| \le nr^{-1}(R_\sigma/\sqrt{n}) + h^{-1}(H_\sigma)$, $(t_\sigma, x_\sigma) \in l$, and consequently $u(t_\sigma, x_\sigma) \to 0$ for $(t_\sigma, x_\sigma) \in l$, $\sigma \to +\infty$.

Under assumption (A, b): 1) $|u(t,x)| \le h^{-1}(t)$ for $0 < t < h(0)$, $x \in \mathbf{R}^n$, and $u(t,x) = 0$ for $t \ge h(0)$, $x \in \mathbf{R}^n$; 2) $|u(t,x)| \le nr^{-1}(x_i/\sqrt{n})$, $(t,x) \in \{x_i > 0\}$, and consequently $u(t,x) \to 0$ as $x_i \to +\infty$ uniformly with respect to $(t, \hat{x}_i) \in \mathbf{R}^n$; 3) $|u(t,x)| \le nr^{-1}(x_i/\sqrt{n})$ for $t \ge h(0)$, $x_i > 0$, $\hat{x}_i \in \mathbf{R}^{n-1}$ and therefore $u(t,x) \to 0$ as $x_i \to +\infty$ uniformly with respect to $t \ge h(0)$ and $\hat{x}_i \in \mathbf{R}^{n-1}$; 4) $|u(t,x)| \le nr^{-1}(d(x)/\sqrt{n})$, $t \ge h(0)$, $x \in \mathbf{R}^n \setminus \bar{\Omega}$, and therefore $u(t,x) \to 0$ as $|x| \to +\infty$ uniformly with respect to $t \ge h(0)$; 5) $u(t_\sigma, x_\sigma) \to 0$ for $(t_\sigma, x_\sigma) \in l$ as $\sigma \to +\infty$.

Under assumption (A, B): 1) $|u(t,x)| \le h^{-1}(t)$ for $0 < t < h(0)$, $x \in \mathbf{R}^n$, and $u(t,x) = 0$ for $t \ge h(0)$, $x \in \mathbf{R}^n$; 2) $|u(t,x)| \le nr^{-1}(x_i/\sqrt{n})$ for $0 < x_i < \sqrt{n}r(0)$, $(t, \hat{x}_i) \in \mathbf{R}^n$ and $u(t,x) = 0$ for $x_i \ge \sqrt{n}r(0)$, $(t, \hat{x}_i) \in \mathbf{R}^n$; 3) $u(t,x) = 0$ for $t \ge h(0)$, $x_i \ge \sqrt{n}r(0)$, $\hat{x}_i \in \mathbf{R}^{n-1}$; 4) $u(t,x) = 0$ for $t \ge h(0)$ and those $x \in \mathbf{R}^n \setminus \bar{\Omega}$ for which $d(x) > \sqrt{n}r(0)$; 5) there exists σ_0 such that the equality $u(t_\sigma, x_\sigma) = 0$ holds for $(t_\sigma, x_\sigma) \in l$ and $\sigma \ge \sigma_0$.

Remark 2. In particular part 5) can be applied to domains that expand in various ways, for example to domains of the form

$$\{(t,x) \in \mathbf{R}^{n+1} : t > \Psi(|x|)\} \quad \text{or} \quad \{(t,x) \in \mathbf{R}^{n+1} : x_i > \Psi(|t| + |\hat{x}_i|)\},$$

where $\Psi(\rho)$ is an increasing function for $\rho \ge 0$, $\Psi(0) = 0$, and $\Psi(\rho) \to +\infty$ as $\rho \to +\infty$.

To prove these assertions in all the cases enumerated one must apply Theorem 2.3 to the cylinders noted above and then take account of the properties of the functions $r^{-1}(\cdot)$ and $h^{-1}(\cdot)$. We have, in the respective cases:

1) $Q^t_R(t,x)$, $(t,x) \in (0,+\infty) \times \mathbf{R}^n$ is a fixed point, and $R \mapsto +\infty$;

2) $Q^{t-t_0}_{x_i}(t,x)$, $(t,x) \in \{x_i > 0\}$ is a fixed point, and $t_0 \to -\infty$;

3) $Q^t_{x_i}(t,x)$, $(t,x) \in \{t > 0,\, x_i > 0\}$;

4) $Q^t_{d(x)}(t,x)$, $t > 0$, $x \notin \bar{\Omega}$;

5) $Q^{H_\sigma}_{R_\sigma}(t_\sigma, x_\sigma)$, $(t_\sigma, x_\sigma) \in l$.

2. The behavior of solutions near the boundaries of the domains. Theorem 2.3 also implies certain corollaries on the behavior of solutions of the inequality (1.1) near the boundaries of various domains. We shall give details for the simplest of these—the behavior of solutions near the boundary of a cylinder.

Let the function u be a solution of inequality (1.1) in the cylinder $(0,T] \times \Omega$ or in the domain $(0,T] \times (\mathbf{R}^n \setminus \bar{\Omega})$, where $\Omega \subset \mathbf{R}^n$ is a bounded domain. Let $d(x) = \operatorname{dist}(x, \partial\Omega)$ be the distance from the point x to $\partial\Omega$. Then we have the following inequalities.

a) under assumption (A, a)

$$|u(t,x)| \le nr^{-1}(d(x)/\sqrt{n}) + h^{-1}(t);$$

b) under assumption (A, b)

$$|u(t,x)| \le nr^{-1}(d(x)/\sqrt{n}) + h^{-1}(t) \ \text{ for } \ t \le h(0), \quad |u(t,x)| \le nr^{-1}(d(x)/\sqrt{n}) \ \text{ for } \ t > h(0);$$

c) under assumption (A, B)

$$\begin{aligned} |u(t,x)| &\le nr^{-1}(d(x)/\sqrt{n}) + h^{-1}(t) \ \text{ for } \ t \le h(0) \ \text{ and } \ d(x) \le \sqrt{n}r(0), \\ |u(t,x)| &\le h^{-1}(t) \ \text{ for } \ t \le h(0) \ \text{ and } \ d(x) > \sqrt{n}r(0), \\ |u(t,x)| &\le nr^{-1}(d(x)/\sqrt{n}) \ \text{ for } \ t > h(0) \ \text{ and } \ d(x) \le \sqrt{n}r(0). \end{aligned}$$

For the proof it suffices to consider a solution u in the cylinder $Q^t_{d(x)}(t,x)$ and then apply Theorem 2.3.

3. Uniqueness and continuous dependence on the boundary conditions of solutions of the exterior initial boundary problem. In this section we assume that f is a continuously differentiable function. In $(0,+\infty)\times(\mathbf{R}^n\setminus\bar{\Omega})$, where $\Omega\subset\mathbf{R}^n$ is a bounded domain, we consider the following boundary-value problem for Eq. (1):

$$u(0,x)=0 \text{ in } \mathbf{R}^n\setminus\Omega, \tag{3.1}$$

$$u(t,x)=\chi(t,x) \text{ on } [0,+\infty)\times\partial\Omega, \tag{3.2}$$

where $\chi(t,x)$ is a bounded function on $[0,+\infty)\times\partial\Omega$ satisfying the consistency condition $\chi(0,x)=0$ on $\partial\Omega$.

Theorem 3.1. *The solution of the exterior initial boundary problem* (3.1), (3.2) *for Eq.* (1) *is unique.*

Theorem 3.2. *Let the functions u_1 and u_2 be solutions of the problem* (1), (3.1), (3.2) *with boundary functions χ_1 and χ_2 respectively. Then for all $(t,x)\in[0,+\infty)\times(\mathbf{R}^n\setminus\Omega)$ we have*

$$|u_1(t,x)-u_2(t,x)|\le\sup_{[0,+\infty)\times\partial\Omega}|\chi_1-\chi_2|.$$

We shall prove this theorem assuming condition (A,a) holds. In the case of conditions (A,b) and (A,B) the proof is analogous. It is clear that Theorem 3.1 follows from this theorem.

Proof of Theorem 3.2. The function $z=u_1-u_2$ is a solution of the problem

$$Lz=C(t,x)z \text{ in } (0,+\infty)\times(\mathbf{R}^n\setminus\bar{\Omega}),\quad z(0,x)=0 \text{ in } \mathbf{R}^n\setminus\Omega,$$
$$z(t,x)=\chi_1(t,x)-\chi_2(t,x) \text{ on } [0,+\infty)\times\partial\Omega,$$

where

$$C(t,x)=\begin{cases}\frac{f(t,x,u_1(t,x))-f(t,x,u_2(t,x))}{u_1(t,x)-u_2(t,x)}, & \text{if } u_1(t,x)\neq u_2(t,x),\\ 0, & \text{if } u_1(t,x)=u_2(t,x),\end{cases} \tag{3.3}$$

and $C(t,x)\ge 0$, as follows from (5).

Let $R>0$ be so large that $\Omega\Subset B_R(0)$. We continue the functions u_1 and u_2 by setting them equal to zero for $t<0$, $x\in\mathbf{R}^n\setminus\bar{\Omega}$ (keeping the same symbols for the extended functions). Then the functions u_1 and u_2 are solutions of Eq. (1) in $\mathbf{R}^1\times(\mathbf{R}^n\setminus\bar{\Omega})$. Applying Theorem 2.3, a) to the solution u_k, $k=1,2$, in the cylinder $Q^{t-t_0}_{d(x)}(t,x)$, where $t>0$, $x\in\partial B_R(0)$, $d(x)=\operatorname{dist}(x,\partial\Omega)$, and $t_0<t$ is arbitrary, we arrive at the inequality

$$|u_k(t,x)|\le nr^{-1}(d(x)/\sqrt{n})+h^{-1}(t-t_0),\quad k=1,2,$$

from which, as $t_0\to-\infty$, we obtain the inequality

$$|u_k(t,x)|\le nr^{-1}(d(x)/\sqrt{n}),\quad k=1,2.$$

Thus for a fixed number $T>0$ we conclude that

$$\sup_{(0,T)\times\partial B_R(0)}|z|\le 2\max_{|x|=R}nr^{-1}(d(x)/\sqrt{n})\to 0 \text{ as } R\to+\infty. \tag{3.4}$$

Furthermore the functions $s^{\pm}(t,x)\equiv\pm z(t,x)+N\equiv\pm z(t,x)+\{\sup_{[0,T)\times\partial\Omega}|\chi_1-\chi_2|+\sup_{(0,T)\times\partial B_R(0)}|z|\}$ assume nonnegative values on the set $\{[0,T)\times(\partial\Omega\cup\partial B_R(0))\}\cup\{(\mathbf{R}^n\setminus\bar{\Omega})\cap B_R(0)\}$, which is the parabolic boundary of the domain

$$D\equiv(0,T)\times((\mathbf{R}^n\setminus\bar{\Omega})\cap B_R(0)).$$

We remark that in the domain D we have $Ls^{\pm}-C(t,x)s^{\pm}=\pm Lz-C(t,x)(\pm z)-C(t,x)N=-C(t,x)N\leq 0$ and consequently by the maximum principle we obtain the inequalities $s^{\pm}\geq 0$ in $D\cup\Gamma(D)$, or, what is the same, the inequality

$$|u_1(t,x)-u_2(t,x)|\leq N=\sup_{[0,T)\times\partial\Omega}|\chi_1-\chi_2|+\sup_{(0,T)\times\partial B_R(0)}|z|.$$

It now remains only to let first R and then T tend to infinity, taking account of (3.4) in the case of R. The theorem is now proved.

§ 4. THE REMOVABLE SINGULARITIES THEOREM

In this section it is essential that the function φ in estimate (7) have the form (9).

Theorem 4.1. *Let the coefficients of the operator L in Eq.* (1) *satisfy*

$$a_{ij}\in C^{1,2}(\mathbf{R}^{n+1}),\ \ i,j=1,\dots,n,\ \ \max_{1\leq i,j\leq n}\|a_{ijx_i}\|_{L^\infty(\mathbf{R}^{n+1})}<\infty,$$

and let the function f be continuously differentiable in $\mathbf{R}^{n+1}\times\mathbf{R}$. As always, we assume that conditions (3), (5), *and* (6) *hold. Suppose that estimate* (7) *holds for the function f with $\varphi(\rho)=a_0\rho^{1+q}$, $\rho\geq 0$, $a_0>0$, and $q>2/n$. If $Q\subset\mathbf{R}^{n+1}$ is a domain containing the point* $(0,0)$ *and the function u is a solution of Eq.* (1) *in $Q'\equiv Q\setminus\{(0,0)\}$, then this function can be defined at the point* $(0,0)$ *so as to be a solution of Eq.* (1) *in Q.*

Proof. We set $Z_1=(-\tau_0,0]\times B_{R_0}(0)$ and $Z=(0,T_0]\times B_{R_0}(0)$, where $\tau_0>0$, $T_0>0$, and $R_0>0$ are chosen sufficiently small that the cylinder $Z_0=Z_1\cup Z=(-\tau_0,T_0]\times B_{R_0}(0)$ is contained in the interior of the domain Q. We denote by $v=v(t,x)$ the solution of the following initial boundary problem for Eq. (1) in Z_0:

$$Lv=f(t,x,v)\ \text{ in }\ Z_0,\quad v=u\ \text{ on }\ \Gamma(Z_0),$$

where u is the solution of Eq. (1) in Q' in the hypothesis of the theorem (by the assumptions on the smoothness of the functions a_{ij} and f it follows from the results of [15, Ch. 6] that this problem has a unique solution).

In the domain $Z_0'=Z_0\setminus\{(0,0)\}^*$ the function $z=u-v$ satisfies the equation

$$Lz=C(t,x)z \tag{4.1}$$

and the initial boundary conditions

$$z=0\ \text{ on }\ \Gamma(Z_0), \tag{4.2}$$

where the coefficient $C(t,x)$, which is nonnegative by (5), is defined in (3.3) if we assume that $u_1=u$ and $u_2=v$.

We shall prove that $z=0$ in Z_0'. It follows from the maximum principle for the solution of the problem (4.1) and (4.2) in Z_1' that $z=0$ in $(\bar{Z}_1)'$. It therefore suffices to prove that

$$z=0\ \text{ in }\ Z, \tag{4.3}$$

while taking account of the fact that

$$z=0\ \text{ on }\ (\Gamma(Z))'. \tag{4.4}$$

We carry out the proof of (4.3) by contradiction. Assume that there exists a point $(t^0,x^0)\in Z$ such that $z(t^0,x^0)\neq 0$. Without loss of generality we assume that

$$z(t^0,x^0)>0. \tag{4.5}$$

* Throughout the remainder of the proof the prime on a symbol for a set means that the point $(0,0)$ has been removed from the set.

It is known ([16, Ch. 1, Ch. 9], [17, § 7]) that under the hypotheses of the theorem there exists a classical fundamental solution E of the operator L in the fiber $\Pi = [-\tau_0, T_0] \times \mathbf{R}^n$—a function $E(t, x; \tau, \xi)$ defined for $(t, x; \tau, \xi) \in \Pi \times \Pi$, $(t, x) \neq (\tau, \xi)$, that is nonnegative, continuous in $\Pi \times \Pi$ for $t > \tau$, equal to zero in $\Pi \times \Pi$ for $t < \tau$, and has in particular the following property: for fixed $(\tau, \xi) \in [-\tau_0, T_0) \times \mathbf{R}^n$ the function E is continuously differentiable with respect to t, twice continuously differentiable with respect to x, and satisfies as a function of (t, x) the equation

$$LE(\cdot, \cdot; \tau, \xi) = 0 \text{ in } (\tau, T_0] \times \mathbf{R}^n. \tag{4.6}$$

Moreover, taking account of the fact that

$$Lu = (a_{ij}(t, x)u_{x_j})_{x_i} - a_{ijx_i}(t, x)u_{x_j} - u_t$$

and using the result of [17, §§ 7–8] (cf. also [18, Ch. 3]), we find positive constants K_1, K_2, k_1, and k_2 depending only on n, λ, $\tau_0 + T_0$, and $\|a_{ijx_i}\|_{L^\infty(\mathbf{R}^{n+1})}$, $i, j = 1, \ldots, n$, such that

$$K_1 E_1(t - \tau, x - \xi) \leq E(t, x; \tau, \xi) \leq K_2 E_2(t - \tau, x - \xi) \tag{4.7}$$

for all (t, x), $(\tau, \xi) \in \Pi$ with $t > \tau$, where

$$E_s(t, x) = \begin{cases} (4\pi k_s t)^{-n/2} \exp(-|x|^2/4k_s t), & \text{if } t > 0,\ x \in \mathbf{R}^n; \\ 0, & \text{if } (t, x) \in (-\infty, 0] \times \mathbf{R}^n \setminus \{(0, 0)\} \end{cases}$$

is a fundamental solution of the equation $k_s \Delta u = u_t$, $s = 1, 2$.

(Remark: Only the leftmost inequality of (4.7) is used in what follows.)

The main step in the proof of (4.3) which leads to a contradiction with (4.5) is to obtain the following assertion:

$$\forall \varepsilon > 0\, \exists\, 0 < \nu_\varepsilon < \tau_0 \text{ such that } z(t^0, x^0) \leq \varepsilon E(t^0, x^0; -\nu_\varepsilon, 0), \text{ and } \nu_\varepsilon \to 0 \text{ as } \varepsilon \to +0. \tag{4.8}$$

Indeed, if we assume that (4.8) is proved, then, letting $\varepsilon \to +0$ in (4.8), we obtain the inequality $z(t^0, x^0) \leq 0$, contradicting (4.5). Consequently $u = v$ in Z_0' and it remains only to set $u(0, 0) = v(0, 0)$.

Let us now prove (4.8). For $M > 0$ and $\sigma > 0$ we consider the following bounded domain in $\mathbf{R}^{n+1}$:

$$\Sigma_M(\sigma) = \{(t, x) \in (-\sigma, +\infty) \times \mathbf{R}^n : E_1(t + \sigma, x) > M\}.$$

Its boundary $S_M(\sigma) = \partial \Sigma_M(\sigma)$ is a (smooth) level surface of the function $E_1(t + \sigma, x)$:

$$\begin{aligned} S_M(\sigma) = \operatorname{cl} \{(t, x) \in (-\sigma, +\infty) \times \mathbf{R}^n : E_1(t + \sigma, x) = M\} = \\ \operatorname{cl} \{(t, x) \in (-\sigma, +\infty) \times \mathbf{R}^n : |x|^2 = -4k_1(t + \sigma) \ln[(4\pi k_1(t + \sigma))^{n/2} M]\}, \end{aligned}$$

where $\operatorname{cl}\{\ldots\}$ denotes the closure of the set $\{\ldots\}$.

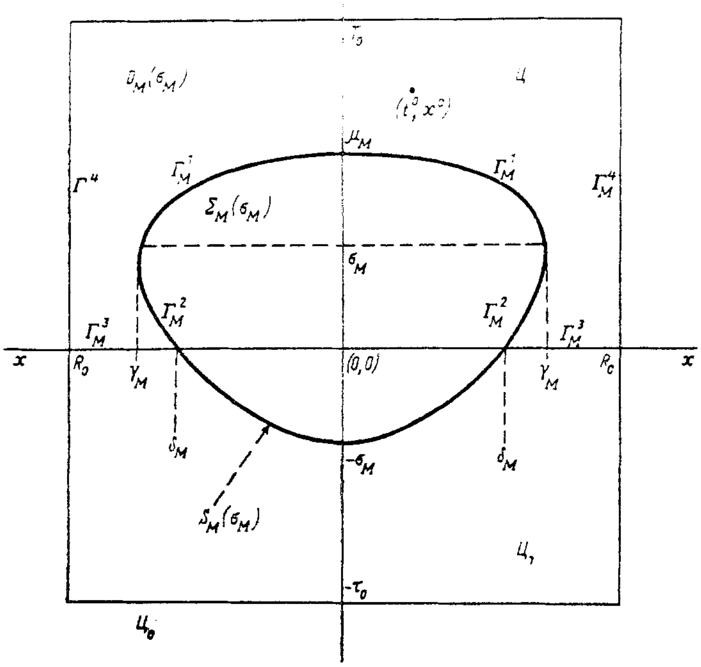
Setting

$$\sigma_M = (8\pi e k_1)^{-1} M^{-2/n}, \tag{4.9}$$

we consider the level surface $S_M(\sigma_M)$. We find the following: the surface $S_M(\sigma_M)$ is inscribed in the cylinder $[-\sigma_M, \mu_M] \times \bar{B}_{\gamma_M}(0)$, where $\mu_M = (2e - 1)\sigma_M$ and $\gamma_M = \sqrt{n/2\pi e} M^{-1/n}$, and intersects the plane $t = 0$ in the sphere $\partial B_{\delta_M}(0)$ of radius $\delta_M = \sqrt{(1 + \ln 2)/2}\, \gamma_M$ with center at the point $x = 0$; furthermore $(0, 0) \in \Sigma_M(\sigma_M)$ and as $M \to +\infty$ the surface $S_M(\sigma_M)$ contracts to the point $(0, 0)$ (the surface $S_M(\sigma_M)$ is depicted in the figure).

Now let $M > 0$ be an arbitrary sufficiently large number satisfying the following inequalities:

$$4\gamma_M \leq |x^0|,\ 2\mu_M \leq t^0,\ 2\sigma_M \leq \tau_0 \tag{4.10}$$



Figure

(these inequalities guarantee with some margin of safety, whose role will be clear in what follows, that the surface $S_M(\sigma_M)$ is contained entirely inside the cylinder Z_0). Setting

$$D_M(\sigma_M) = \mathbf{R}^{n+1} \setminus (S_M(\sigma_M) \cup \Sigma_M(\sigma_M)),$$
$$\Gamma_M^1 = \{(t,x) \in S_M(\sigma_M) : \sigma_M \le t \le \mu_M\}, \quad \Gamma_M^2 = \{(t,x) \in S_M(\sigma_M) : 0 \le t < \sigma_M\},$$
$$\Gamma_M^3 = \{(t,x) \in \Gamma(Z) : t = 0,\, \delta_M < |x| < R_0\}, \quad \Gamma^4 = \{(t,x) \in \Gamma(Z) : 0 \le t \le T_0,\, |x| = R_0\}$$

(see figure), we remark that

$$\Gamma_M^1 \cup \Gamma_M^2 \cup \Gamma_M^3 \cup \Gamma^4 = \Gamma(Z \cap D_M(\sigma_M)) \tag{4.11}$$

and that $(t^0, x^0) \in Z \cap D_M(\sigma_M)$ by (4.10).

Keeping in mind the application of the maximum principle, we compare the functions $z(t,x)$ and $\varepsilon E(t,x; -\sigma_M, 0)$, $\varepsilon > 0$, on the parabolic boundary (4.11). For $(t,x) \in \Gamma_M^1$ we conclude by (4.10) that the functions u and v are solutions of Eq. (1) in the cylinder $Q^t_{\gamma_M}(t,x)$, where $\mu_M \ge t \ge \sigma_M$, and consequently, applying Theorem 2.4, we obtain

$$|z(t,x)| \le |u(t,x)| + |v(t,x)| \le 2c[(\gamma_M)^{-2/q} + (\sigma_M)^{-1/q}] \le C_1 M^{2/nq}, \tag{4.12}$$

where the positive constant C_1 depends on n, λ, a_0, q, and k_1.

We now require that in addition to inequalities (4.10) the quantity M satisfy the inequality

$$C_1 M^{2/nq} \le \varepsilon K_1 M \tag{4.13}$$

(here we are taking account of the fact that $1 - \dfrac{2}{nq} = \dfrac{1}{q}\left(q - \dfrac{2}{n}\right) > 0$). We remark here that the quantity M so chosen will depend on ε and that on the portion $\Gamma_M^1 \subset S_M(\sigma_M)$ we have the equality

$$M = E_1(t + \sigma_M, x), \quad (t,x) \in \Gamma_M^1. \tag{4.14}$$

Carrying out analogous reasoning in $Q^{t+\sigma_M}_{|x|}(t,x)$ for $(t,x) \in \Gamma_M^2$ (by virtue of which $\sigma_M > t \ge 0$ and $\gamma_M \ge |x| \ge \delta_M$), we can assume that the quantity M is chosen so that

$$|z(t,x)| \le C_2 M^{2/nq} \le \varepsilon K_1 M = \varepsilon K_1 E_1(t + \sigma_M, x), \quad (t,x) \in \Gamma_M^2, \tag{4.15}$$

where the constant $C_2 > 0$ depends on the same quantities as C_1.

Finally, by (4.4), we have for $(t,x) \in \Gamma_M^3 \cup \Gamma^4$

$$|z(t,x)| = 0 \le \varepsilon K_1 E_1(t+\sigma_M, x). \tag{4.16}$$

Thus by (4.10)–(4.16) and (4.7) we find that for $\varepsilon > 0$ there exists $M_\varepsilon > 0$ (and $M_\varepsilon \to +\infty$ as $\varepsilon \to 0$) such that for $(t,x) \in \Gamma(Z \cap D_{M_\varepsilon}(\sigma_{M_\varepsilon}))$ we shall have

$$|z(t,x)| \le \varepsilon K_1 E_1(t+\sigma_{M_\varepsilon}, x) \le \varepsilon E(t,x;-\sigma_{M_\varepsilon},0). \tag{4.17}$$

Consider the connected component D of the set

$$\{(t,x) \in Z \cap D_{M_\varepsilon}(\sigma_{M_\varepsilon}) : z(t,x) > 0\},$$

containing the point (t^0, x^0). By (4.5) D is a nonempty bounded open set. We remark that $u(t,x)-v(t,x) = z(t,x) > 0$ for $(t,x) \in D$, and consequently the nonnegative coefficient $C(t,x)$ in (4.1) is well-defined in D.

It follows from (4.6) that

$$LE(\cdot,\cdot;-\sigma_{M_\varepsilon},0) = 0 \text{ in } D,$$

and so from (4.1) and (4.17) and the definition of the set D we find that

$$L(z - \varepsilon E(\cdot,\cdot;\sigma_{M_\varepsilon},0)) \ge 0 \text{ in } D, \quad z(t,x) \le \varepsilon E(t,x;-\sigma_{M_\varepsilon},0) \text{ for } (t,x) \in \Gamma(D).$$

Applying the maximum principle, we obtain the inequality

$$z(t,x) \le \varepsilon E(t,x;-\sigma_{M_\varepsilon},0) \ \forall (t,x) \in D \cup \Gamma(D),$$

from which, in particular, assertion (4.18) follows with $\nu_\varepsilon = \sigma_{M_\varepsilon}$ ($\nu_\varepsilon \to 0$ as $\varepsilon \to +0$ by virtue of (4.9)). The theorem is now proved.

Remark 3. In the case when $-1 < q < 2/n$ Theorem 4.1 does not hold. The existence of solutions with nonremovable singularities for this case is proved in Sec. 3 of [10].

§ 5. THE CASE OF CONDITION (*B*). VANISHING OF SOLUTIONS

Throughout this section we assume that condition (B) holds. We define two functions

$$s(a) = \left(\frac{\lambda(n+1)}{2}\right)^{1/2} \int_0^a \left(\int_0^\rho \varphi(\zeta)\,d\zeta\right)^{-1/2} d\rho, \quad a \in [0,+\infty), \tag{5.1}$$

$$g(b) = (n+1)\int_0^b \frac{d\rho}{\varphi(\rho)}, \quad b \in [0,+\infty). \tag{5.2}$$

By the convergence of the integrals at zero in condition (B) and Proposition 1.1 these functions are well defined.

Lemma 5.1 (the barrier function). a) *Under the assumption* (B,c) *for any point* $(T,x^0) \in \mathbf{R}^{n+1}$ *there exists a function*

$$U(t,x) \equiv U(T,x^0,t,x) \in C^{1,2}((-\infty,T] \times \mathbf{R}^n), \quad U \ge 0,$$

for which the following relations hold:

$$LU \le \varphi(U) \text{ in } (-\infty,T] \times \mathbf{R}^n, \tag{5.3}$$

$$U \ge \min\{s^{-1}(R), g^{-1}(H)\} \text{ on } \Gamma(G_R^H(T,x^0)) \ \forall R > 0,\ H > 0, \tag{5.4}$$

$$U(t,x^0) = U(T,x^0,T,x^0) = 0, \tag{5.5}$$

where $s^{-1} : [0,+\infty) \to [0,+\infty)$ *and* $g^{-1} : [0,+\infty) \to [0,+\infty)$ *are strictly increasing continuous bijections that are the inverses of the functions defined in* (5.1) *and* (5.2) *respectively.*

b) *Under assumption* (B,d) *the lemma undergoes the following modifications: the nonnegative function* $U(t,x) = U(T,x^0,t,x) \in C^{1,2}((T-g(+\infty),T]\times\mathbf{R}^n)$ *is a solution of inequality* (5.3) *in* $(T-g(+\infty),T]\times\mathbf{R}^n$, *where* $g(+\infty) < +\infty$, *and the following relations hold instead of* (5.4) :

$$U \geq s^{-1}(R) \ \text{on} \ (T-g(+\infty),T] \times \partial P_R(x^0) \quad \forall R > 0. \tag{5.6}$$

$$U(T-H,x) \geq g^{-1}(H) \quad \forall x \in \mathbf{R}^n, \ \text{if} \ 0 < H < g(+\infty),$$

$$U(t,x) \to +\infty \quad \forall x \in \mathbf{R}^n \ \text{as} \ t \to T - g(+\infty), \tag{5.7}$$

where $s^{-1} : [0,+\infty) \to [0,+\infty)$ *and* $g^{-1} : [0,g(+\infty)) \to [0,+\infty)$ *are the functions inverse to the functions* $s(\cdot)$ *and* $g(\cdot)$ *of* (5.1) *and* (5.2).

Proof. a) Since $\lim\limits_{a\to+0} s(a) = \lim\limits_{b\to+0} g(b) = 0$ and by assumption (c) and Proposition 1.1 we have a) $\lim\limits_{a\to+\infty} s(a) = \lim\limits_{b\to+\infty} g(b) = +\infty$, it follows that the functions $s : [0,+\infty) \to [0,+\infty)$ and $g : [0,+\infty) \to [0,+\infty)$ are strictly increasing continuous bijections. The functions $v(y)$ and $w(t)$ of one variable $y \in \mathbf{R}^1$ defined implicitly by the formulas

$$s(v(y)) = y, \quad g(w(t)) = T - t, \tag{5.8}$$

$t \in (-\infty,T]$, (compare with (2.11) and (2.12)) are respectively solutions of the problems (2.6), (2.7), (2.8) with $\alpha = 0$ and (2.9), (2.10) with $\beta = 0$, and $w(t) \geq 0$, $v(y) \geq 0$, $v(y)$ being an even function. Setting

$$U(t,x) = w(t) + \sum_{i=1}^{n} v(x_i - x_i^0),$$

just as in (2.5), we establish inequality (5.3). From (5.8) we find that $v(y) = s^{-1}(|y|)$, $y \in \mathbf{R}^1$; $w(t) = g^{-1}(T-t)$, $t \in (-\infty,T]$. Therefore with $t \in (-\infty,T]$, $|x_j - x_j^0| = R$, $j = 1,2,\ldots,n$, we have

$$U(t,x) \geq v(x_j - x_j^0) = s^{-1}(R),$$

and for $x \in \mathbf{R}^n$, $t = T - H$

$$U(t,x) = U(T-H,x) \geq w(T-H) = g^{-1}(H),$$

which leads to (5.4). Finally, the equality (5.5) follows from (2.7) and (2.10).

b) Here it should be mentioned that the function $w(t)$ must be regarded as defined on $t \in (T - g(+\infty),T]$, where $g(+\infty) < +\infty$ by the second inequality of (d), and also that $U(T-H,x) \geq g^{-1}(H) \to +\infty$ as $H \to g(+\infty)$. The lemma is now proved.

Remark 4. The case when condition (A,B) holds could have been studied within the framework of Lemma 5.1. Lemma 5.1 and Remark 1, b) would then have led to the same result, since $s(+\infty) = r(0)$ and $g(+\infty) = h(0)$.

Lemma 5.2. *Let the function* u *be a solution of inequality* (1.1) *in the bar* $\bar{G}_R^H(T,x^0)$ *such that*

a) *under assumption* (B,c)

$$|u| \leq \min\{s^{-1}(R), g^{-1}(H)\} \ \text{on} \ \Gamma(G_R^H(T,x^0));$$

b) *under assumption* (B,d)

$$|u| \leq \min\{s^{-1}(R), g^{-1}(H)\} \ \text{on} \ \Gamma(G_R^H(T,x^0)), \ \text{if} \ H < g(+\infty),$$

$$|u| \leq s^{-1}(R) \ \text{on} \ [T-H,T] \times \partial P_R(x^0), \ \text{if} \ H \geq g(+\infty),$$

where $s^{-1}(\cdot)$ *and* $g^{-1}(\cdot)$ *are the functions defined in Lemma* 5.1. *Then the equality* $u(T, x^0) = 0$ *holds at the center of the top of the bar* $G_R^H(T, x^0)$.

Proof. a) We use proof by contradiction. Without loss of generality we shall assume that

$$u(T, x^0) > 0. \tag{5.9}$$

Consider the connected component D of the set $\{(t, x) \in G_R^H(T, x^0) : u(t, x) - U(t, x) > 0\}$ containing the point (T, x^0), where $U(t, x) = U(T, x^0, t, x)$ is the function of Lemma 5.1. Then by (5.9) and (5.5) $D \neq \varnothing$ is a bounded open set and $u > 0$ in D. Since $LU \leq \varphi(U)$ in D by (5.3) and $Lu \geq \varphi(u)$ by (1.1), while it follows from (5.4), the hypothesis of the lemma, and the definition of the set D that $U \geq u$ on $\Gamma(D)$, we derive from the maximum principle (Theorem 1.0) the inequality $U \geq u$ in $D \cup \Gamma(D)$, and in particular the inequality $u(T, x^0) \leq U(T, x^0) = 0$, which contradicts (5.9).

b) For $H < g(+\infty)$ the proof of the lemma resembles the proof of part a). When $H \geq g(+\infty)$, by the boundedness of the function u in $\bar{G}_R^H(T, x^0)$, (5.6), and (5.7) there exists $0 \leq H_0 < g(+\infty)$ such that, when we take account of the inequality in the hypothesis of the lemma, we shall have

$$|u| \leq U \text{ on } \Gamma(G_R^{H_0}(T, x^0)).$$

Now carrying out the same reasoning in the bar $G_R^{H_0}(T, x^0)$ as in the proof of part a), we find that $u(T, x^0) = 0$. The lemma is now proved.

Theorem 5.3 (vanishing of solutions). *Let* $\Omega \subset \mathbf{R}^n$ *be a bounded domain, and let the function* u *be a solution of inequality* (1.1) *in* $(0, +\infty) \times (\mathbf{R}^n \setminus \bar{\Omega})$ *such that*

a) *under assumption* (B, c)

$$|u(t, x)| \leq \min\{s^{-1}(o(|x|)), g^{-1}(o(t))\}, \quad t \to +\infty, \quad |x| \to +\infty;$$

b) *under assumption* (B, d) *for any fixed* $t > g(+\infty)$

$$|u(t, x)| \leq s^{-1}(o(|x|)), \quad |x| \to +\infty,$$

where $s^{-1}(\cdot)$ *and* $g^{-1}(\cdot)$ *are the functions of Lemma* 5.1. *Then there exist* $R_0 > 0$ *and* $H_0 > 0$ *so large that* $u \equiv 0$ *on* $[H_0, +\infty) \times (\mathbf{R}^n \setminus B_{R_0}(0))$. *In particular, this theorem holds for bounded solutions* u.

Remark 5. An analogous assertion was obtained in studying the asymptotic behavior of the solutions in (§ 3) under the assumption of condition (A, B) in Sec. 4 without any assumptions about the behavior of the solution at infinity.

Proof of Theorem 5.3. a) Let $R_1 > 0$ be so large that $\Omega \subset B_{R_1}(0)$. For $0 < \varepsilon < 1/(1 + 2\sqrt{n})$ there exist $\rho_0 = \rho_0(\varepsilon) > R_1$ and $\tau_0 = \tau_0(\varepsilon) > 0$ such that

$$|u(t, x)| \leq \min\{s^{-1}(\varepsilon|x|), g^{-1}(\varepsilon t)\}, \tag{5.10}$$

provided $|x| \geq \rho_0$ and $t \geq \tau_0$. Let the point $(t^0, x^0) \in (0, +\infty) \times (\mathbf{R}^n \setminus \bar{\Omega})$ be such that $|x^0| \equiv (1 + \sqrt{n})R \geq R_0 \equiv (1 + \sqrt{n})\rho_0$ and $t^0 \equiv 2H \geq H_0 \equiv 2\tau_0$. Then $u(t^0, x^0) = 0$. Indeed, on the set

$$Y \equiv \{(t, x) \in (0, +\infty) \times (\mathbf{R}^n \setminus \bar{\Omega}) : R \leq |x| \leq (1 + 2\sqrt{n})R, \ H \leq t \leq t^0\}$$

by (5.10) we have the inequality

$$|u(t, x)| \leq \min\{s^{-1}(R), g^{-1}(H)\}, \quad (t, x) \in Y,$$

which holds in particular in $\bar{G}_R^H(t^0, x^0) \subset Q_{\sqrt{n}R}^H(t^0, x^0) \subset Y$. It then follows from Lemma 5.2, a) that $u(t^0, x^0) = 0$.

b) Setting $\tau_0 = g(+\infty)$ in the preceding proof, we find that in $\bar{G}_R^H(t^0, x^0) \subset Y$ we have the inequality

$$|u(t,x)| \le s^{-1}(R), \quad (t,x) \in \bar{G}_R^H(t^0, x^0),$$

from which it follows by Lemma 5.2, b) that $u(t^0, x^0) = 0$. The theorem is now proved.

The following theorem holds for each of the assumptions (B,c), (B,d), and (A,B). Let f be a continuously differentiable function.

Theorem 5.4 (solutions of compact support for the Cauchy problem). *If u is a globally bounded solution (i.e., bounded in $[0,+\infty) \times \mathbf{R}^n$) of the Cauchy problem for Eq. (1) with initial function $u(0,x) = u_0(x)$ of compact support, then u is of compact support.*

Proof (assuming (B,c)). Let $|u| \le M_0$ in $[0,+\infty) \times \mathbf{R}^n$. For M_0 we find $R_0 > 0$ and $H_0 > 0$ such that $M_0 \le s^{-1}(R_0)$ and $M_0 \le g^{-1}(H_0)$, so that because the functions $s^{-1}(\cdot)$ and $g^{-1}(\cdot)$ are increasing for $R \ge R_0$ and $H \ge H_0$, we shall have

$$|u(t,x)| \le M_0 \le \min\{s^{-1}(R), g^{-1}(H)\}$$

for all $(t,x) \in [0,+\infty) \times \mathbf{R}^n$. It follows from this that when $t^0 > H_0$, for any point $x^0 \in \mathbf{R}^n$ in the bar $\bar{G}_R^{t^0}(t^0, x^0)$, where $R \ge R_0$, the inequality

$$|u| \le \min\{s^{-1}(R), g^{-1}(t^0)\}$$

holds. Consequently, applying Lemma 5.2, a), we find that $u(t^0, x^0) = 0$.

We denote by $\Omega_0 = \operatorname{supp} u_0$ the support of the function $u_0(x)$ and we set $\Omega_0^{R_0} = \{x \in \mathbf{R}^n : \operatorname{dist}(x, \Omega_0) \le R_0\}$, where the quantity R_0 was chosen above. We now extend the function $u(t,x)$ by setting it equal to zero for $t < 0$ and $x \in \mathbf{R}^n \setminus \Omega_0$ (keeping the same symbol to denote the extended function). Then the function u so extended will be a solution of Eq. (1) in $\mathbf{R} \times (\mathbf{R}^n \setminus \Omega_0)$. For $0 \le t^1 \le H_0$ and $x^1 \in \mathbf{R}^n \setminus \Omega_0^{R_0}$ we construct the bar

$$\bar{G}^H_{\operatorname{dist}(x^1, \partial\Omega_0^{R_0})}(t^1, x^1), \tag{5.11}$$

in which the solution u is defined. It is clear that $\operatorname{dist}(x^1, \partial\Omega_0^{R_0}) > R_0$ and that H can be chosen so that $H > H_0$ (by the way in which the function u was extended for $t < 0$). Consequently in the bar (5.11) we have

$$|u| \le \min\{s^{-1}(\operatorname{dist}(x^1, \partial\Omega_0^{R_0})), g^{-1}(H)\}.$$

Now applying Lemma 5.2, a), we find that $u(t^1, x^1) = 0$.

In the case of assumptions (B,d) and (A,B) the proof is analogous.

In concluding this section we note that condition (B,c) holds for a function φ of the form (9) with $-1 < q < 0$. The functions $s^{-1}(\cdot)$ and $g^{-1}(\cdot)$ of Lemma 5.1, a) are expressed as follows in this case:

$$s^{-1}(R) = \Big(\frac{a_0|q|^2}{2\lambda(n+1)(2+q)}\Big)^{1/|q|} R^{2/|q|}, \quad R \in [0,+\infty),$$
$$g^{-1}(H) = \Big(\frac{a_0|q|}{n+1}\Big)^{1/|q|} H^{1/|q|}, \quad H \in [0,+\infty).$$

§ 6. THE SEMILINEAR ELLIPTIC EQUATION

In this section we give the basic results on the properties of solutions of the semilinear elliptic equation

$$Lu \equiv a_{ij}(x) u_{x_i x_j} = f(x,u), \tag{6.1}$$

where $x \in \mathbf{R}^n$, $a_{ij} = a_{ji}$ and the restrictions (3) and (5)–(8) (with the dependence of the functions on t excluded) are imposed on the functions that occur in Eq. (6.1). In regard to the function φ of (7) it is assumed that it satisfies one of the following three conditions of Bernshtein-Dini type:

$$(A) \qquad \int^{+\infty} (\rho\varphi(\rho))^{-1/2}\, d\rho < +\infty, \quad \int_{+0} (\rho(\varphi(\rho))^{-1/2}\, d\rho = +\infty;$$

$$(B) \qquad \int^{+\infty} (\rho(\varphi(\rho))^{-1/2}\, d\rho = +\infty, \quad \int_{+0} (\rho(\varphi(\rho))^{-1/2}\, d\rho < +\infty;$$

$$(C) \qquad \int_{+0}^{+\infty} (\rho(\varphi(\rho))^{-1/2}\, d\rho < +\infty.$$

Solutions of Eq. (6.1) in the domain $D \subset \mathbf{R}^n$ are understood in the classical sense, i.e., functions $u = u(x) \in C^2(D)$ that give an identity when substituted into Eq. (6.10); here C^2 is the space of continuous functions $u(x)$ possessing continuous partial derivatives u_{x_i}, $u_{x_i x_j}$, $i, j = 1, \ldots, n$.

We note that if the function u is a solution of Eq. (6.1), then it is also a classical solution of inequality (1.1), in which the operator L is taken from (6.1).

In studying the properties of solutions of Eq. (6.1) we shall make use of the following proposition.

Theorem 6.0 (the maximum principle). *Let $D \subset \mathbf{R}^n$ be a bounded domain and $f(x,u)$ a measurable locally bounded function on $\bar{D} \times \mathbf{R}$ for which* (5) *holds. Let u_1 and u_2 be continuous solutions of the inequalities $Lu_1 \leq f(x,u_1)$ and $Lu_2 \geq f(x,u_2)$ respectively for all $x \in D$, and let $u_1 \geq u_2$ on ∂D. Then $u_1 \geq u_2$ in $\bar{D}$.*

The proof can be found in [3].

1. The case of condition (A)**.** Suppose condition (A) holds, and let $r : (0,+\infty) \to (0,+\infty)$ be the strictly decreasing continuous bijection defined in (2.13) (cf. Proposition 1.1).

Lemma 6.1. *For any point $x^0 \in \mathbf{R}^n$ and any number $\alpha > 0$ there exists a function $V(x) \equiv V_\alpha(x^0, x) \in C^2(P_{r(\alpha)}(x^0))$, $V > 0$, satisfying the relations: $LV - \varphi(V) \leq 0$ in $P_{r(\alpha)}(x^0)$, $V(x) \to +\infty$ as $x \to \partial P_{r(\alpha)}(x^0)$, and $V(x^0) = V_\alpha(x^0, x^0) = n\alpha$.*

Theorem 6.2 (the connection between the radius of the ball in which the solution is defined and the value of the solution at the center of the ball). *If u is a solution of inequality* (1.1) *in the ball $B_R(x^0)$ then*

$$|u(x^n)| \leq nr^{-1}(R/\sqrt{n}),$$

where $r^{-1}(\cdot)$ is the function inverse to $r(\cdot)$.

From this fundamental theorem we obtain Theorems 6.3–6.5.

Theorem 6.3 (behavior of solutions in an unbounded domain). *If the function u is a solution of inequality* (1.1) *in the exterior of the compact set $\bar{\Omega}$, then*

$$|u(x)| \leq nr^{-1}((\operatorname{dist}(x, \partial\Omega))/\sqrt{n}), \quad x \in \mathbf{R}^n \setminus \bar{\Omega}.$$

It is clear from this inequality that $u(x) \to 0$ as $|x| \to +\infty$.

Theorem 6.4 (the behavior of solutions in a neighborhood of an isolated singular point). *Suppose the domain Ω contains 0 and u is a solution of inequality* (1.1) *in $\Omega \setminus \{0\}$. Then for $x \in \Omega \setminus \{0\}$ we have*

$$|u(x)| \leq nr^{-1}(|x|/\sqrt{n}).$$

Remark 6. The removable singularities theorem for a solution of the elliptic equation (6.1) in the case of a function φ of the form (9) with $q \geq 2/(n-2)$, $n \geq 3$, was proved in [1] (for $L = \Delta$) and in [3; 4] for $q > 2/(n-2)$, $n \geq 3$ (for a general linear operator L of divergence and nondivergence structure). The solutions of (6.1) for $L = \Delta$ and $0 < q < 2/(n-2)$ with isolated singularities were studied in [19].

Theorem 6.5 (uniqueness of the solution of the exterior problem). *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain and u_1 and u_2 solutions of Eq.* (6.1) *in $\mathbf{R}^n \setminus \bar{\Omega}$, and let $u_1 = u_2$ on $\partial\Omega$. Then $u_1 = u_2$ in $\mathbf{R}^n \setminus \Omega$.*

This last theorem follows from Theorems 6.3 and 6.0.

2. The case of conditions (B) and (C). Solutions of compact support. Suppose condition (B) or (C) holds, and let $s : [0, +\infty) \to [0, s(+\infty))$ be the strictly increasing continuous bijection defined in (5.1), where $s(+\infty) = +\infty$ under assumption (B) and $s(+\infty) < +\infty$ under assumption (C) (cf. Proposition 1.1). The following result holds.

Lemma 6.6. *For any point $x^0 \in \mathbf{R}^n$ there exists a function $U(x) \equiv U(x^0, x) \in C^2(P_{s(+\infty)}(x^0))$, $U \geq 0$ satisfying the relations*

$$LU - \varphi(U) \leq 0 \quad in \quad P_{s(+\infty)}(x^0),$$

and $U \geq s^{-1}(R)$ on $\partial P_R(x^0)$ for all $R > 0$ if condition (B) holds, while

$$U(x) \to +\infty$$

as $x \to \partial P_{s(+\infty)}(x^0)$ if condition (C) holds, and

$$U(x^0) \equiv U(x^0, x^0) = 0,$$

where $s^{-1}(\cdot)$ is the function inverse to $s(\cdot)$.

Suppose condition (B) holds. Then Lemma 6.6 and Theorem 6.0 imply the following result.

Theorem 6.7. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain and u a solution of inequality* (1.1) *in $\mathbf{R}^n \setminus \bar{\Omega}$ such that $|u(x)| \leq s^{-1}(o(|x|))$ as $|x| \to +\infty$. Then $u \equiv 0$ outside a ball $B_{R_0}(0)$ of sufficiently large radius R_0.*

Now suppose that condition (C) holds. Then Lemma 6.6. implies the following results.

Theorem 6.8. *If u is a solution of inequality* (1.1) *in $\mathbf{R}^n \setminus \bar{\Omega}$, where Ω is a bounded domain in $\mathbf{R}^n$, then $u \equiv 0$ outside the ball with center at zero and sufficiently large radius.*

Theorem 6.9. *If u is a solution of Eq.* (6.1) *in $\mathbf{R}^n$, then $u \equiv 0$ in $\mathbf{R}^n$.*

PART II

THE DIVERGENCE PARABOLIC EQUATION

§ 7. FUNDAMENTAL LEMMAS. ESTIMATES FOR SOLUTIONS

In the second part of this paper we study the properties of solutions of Eq. (2), in which $a_{ij}(t,x) = a_{ji}(t,x)$ $(i,j = 1,\dots,n)$ are bounded measurable functions satisfying (3) and the exponent $q =$ const is either positive or satisfies $-1 < q < 0$. In addition to the notation introduced in Part I, we shall use the following: $C(D)$ is the space of continuous functions in the domain D; $L^p(D)$ (resp. $L^p_{\text{loc}}(D)$) is the space of measurable functions in the domain D for which the pth power of the absolute value is Lebesgue integrable (resp. locally Lebesgue integrable), $p \geq 1$; $W_2^{1,1}(D)$ (resp. $W_{2,\text{loc}}^{1,1}(D)$) is the space consisting of elements $u(t,x) \in L^2(D)$ (resp. $u(t,x) \in L^2_{\text{loc}}(D)$) having generalized Sobolev derivatives u_t, $u_{x_i} \in L^2(D)$ (resp. u_t, $u_{x_i} \in L^2_{\text{loc}}(D)$), $i = 1,\dots,n$; $\overset{0}{W}{}_2^1(D)$ $(D \subset \mathbf{R}^n)$ is the Banach space of elements $u(x) \in L^2(D)$ having generalized Sobolev derivatives $u_{x_i} \in L^2(D)$, $i = 1,\dots,n$; a dense subset of this space is formed by the set of infinitely differentiable functions with support in D, and the norm is defined as $\|u\|_{\overset{0}{W}{}_2^1(D)} = \left(\int_D u_{x_i} u_{x_i}\, dx\right)^{1/2}$;

$u_x = (u_{x_1}, \dots, x_{x_n})$ is the gradient of the function u on $x \in \mathbf{R}^n$; mes denotes $(n+1)$-dimensional Lebesgue measure; $\mathbf{N}$ is the set of natural numbers; and a. e. means almost everywhere, almost every, etc.

A *solution* of Eq. (2) in a bounded domain $D \subset \mathbf{R}^{n+1}$ is a function

$$u = u(t,x) \in W_2^{1,1}(D) \cap C(D),$$

satisfying the integral identity

$$-\iint\limits_D (a_{ij} u_{x_j} \psi_{x_i} + u_t \psi)\, dx\, dt = a_0 \iint\limits_D |u|^q u \psi \, dx\, dt \tag{7.1}$$

for any test function $\psi = \psi(t,x) \in W_2^{1,1}(D)$ which belongs to the space $\overset{0}{W}{}_2^1(D_t) : \psi(t,\cdot) \in \overset{0}{W}{}_2^1(D_t)$ as a function of x for almost every t such that $D_t = \{(t,x) \in D : t = \text{const}\} \neq \varnothing$.

If $\tilde{D} \subset \mathbf{R}^{n+1}$ is an unbounded domain, a solution of Eq. (2) in the domain $\tilde{D}$ is defined as a function

$$u = u(t,x) \in W_{2,\text{loc}}^{1,1}(\tilde{D}) \cap C(\tilde{D}),$$

for which the integral identity (7.1) holds for all the test functions ψ mentioned above in any bounded subdomain $D \Subset \tilde{D}$.

We now state a weak maximal principle for a solution of semilinear inequalities in a form suitable for our use. Let $f(t,x,u)$ be a measurable locally bounded function in $\mathbf{R}^{n+1} \times \mathbf{R}^1$ such that for any bounded domain $D \subset \mathbf{R}^{n+1}$ and any function $u(t,x) \in W_2^{1,1}(D) \cap C(D)$ the function $f(t,x,u(t,x))$ belongs to the space $L^2(D)$. We shall say that the function $u = u(t,x)$ is a solution of the inequality $\mathcal{L}u \geq f(t,x,u)$ (where $\mathcal{L}$ is the operator of (2)) in a bounded domain $D \subset \mathbf{R}^{n+1}$ if $u \in W_2^{1,1}(D) \cap C(D)$ and the integral inequality

$$-\iint\limits_D (a_{ij} u_{x_j} \psi_{x_i} + u_t \psi)\, dx\, dt \geq \iint\limits_D f(t,x,u) \psi \, dx\, dt$$

holds for all the test functions ψ mentioned above with $\psi \geq 0$ almost everywhere in D. A solution of the inequality $\mathcal{L}u \leq f(t,x,u)$ is defined similarly.

Theorem (weak maximum principle). *Let the function $u(t,x) \in W_2^{1,1}(D) \cap C(\bar{D})$ be a solution of the inequality $\mathcal{L}u \geq 0$ in a bounded domain $D \subset \mathbf{R}^{n+1}$, and let $u(t,x) \leq 0$ for $(t,x) \in \Gamma(D)$. Then $u(t,x) \leq 0$ for $(t,x) \in \bar{D}$.*

This theorem implies the following corollary.

Theorem 7.0 (weak maximum principle for semilinear inequalities). *Let $D \subset \mathbf{R}^{n+1}$ be a bounded domain, and let the functions $u_1, u_2 \in W_2^{1,1}(D) \cap C(\bar{D})$ be solutions of the inequalities $\mathcal{L}u_1 \leq f(t,x,u_1)$ and $\mathcal{L}u_2 \geq f(t,x,u_2)$ in the domain D, while the function $f(t,x,u)$ satisfies (5). If $u_1 \geq u_2$ on $\Gamma(D)$, then $u_1 \geq u_2$ in $\bar{D}$.*

Proof. Suppose the theorem is false, i.e., that there exists a point $(t^0, x^0) \in \bar{D} \setminus \Gamma(D)$ for which

$$u_1(t^0, x^0) < u_2(t^0, x^0), \tag{7.2}$$

Consider the connected component G of the set $\{(t,x) \in \bar{D} : u_1(t,x) < u_2(t,x)\}$ containing the point (t^0, x^0). Set $z = u_2 - u_1$. Then $z = 0$ on $\Gamma(G)$ and $z > 0$ in G. The function z is a solution of the inequality

$$\mathcal{L}z = C(t,x) z \geq 0 \text{ in } G,$$

where the coefficient (3.3), which is nonnegative by (5), is well-defined in G (since $z > 0$ in G). Applying the preceding theorem, we obtain the inequality $z \leq 0$ in G, contradicting (7.2).

The estimates given below for solutions are valid without any restrictions on the exponent $q > -1$. The solutions of Eq. (2) are considered in bounded domains and understood in the sense of Definition (7.1). In what follows we assume that if the center of the top is not explicitly shown in the notation for a cylinder Q_R^H, then it is understood that $Q_R^H = Q_R^H(T, x^0)$; we adopt an analogous convention for balls $B_R = B_R(x^0)$.

In the following lemma we use an idea of J. Moser [20].

Lemma 7.1. *Suppose the function u is a solution of Eq. (2) in the cylinder Q_R^H, and let $p \geq 2$. Then for the numbers $h \in (0, H)$ and $r \in (0, R)$ we have the inequalities*

$$\iint_{Q_{R-r}^{H-h}} |(u|u|^{\frac{1}{2}p-1})_x|^2\,dx\,dt \leq c_1 \frac{p^2}{(p-1)^2}(h^{-1} + r^{-2}) \iint_{Q_R^H} |u|^p\,dx\,dt, \tag{7.3}$$

$$\max_{t\in[T-(H-h),T]} \int_{B_{R-r}} |u(t,x)|^p\,dx \leq c_1 \frac{p^2}{(p-1)^2}(h^{-1} + r^{-2}) \iint_{Q_R^H} |u|^p\,dx\,dt, \tag{7.4}$$

where the positive constant c_1 depends only on the λ of (3).

Proof. For $R > 0$ and a number $r \in (0, R)$ we define the truncating function $\zeta_{R,r}(\rho)$, $\rho \geq 0$, by the formula

$$\zeta_{R,r}(\rho) = \begin{cases} 1, & \text{if } 0 \leq \rho \leq R - r; \\ \frac{1}{r}(R - \rho) & \text{if } R - r < \rho < R: \\ 0, & \text{if } \rho \geq R, \end{cases} \tag{7.5}$$

and we set

$$\psi(t, x) = \frac{1}{2} p u(t, x) |u(t, x)|^{p-2} \Phi^2(t, x),$$

where

$$\Phi(t, x) = \zeta_{H,h}(T - t) \cdot \zeta_{R,r}(|x - x^0|).$$

Then

$$\psi \in W_2^{1,1}(Q_R^H) \cap C(Q_R^H), \quad \psi(t, \cdot) \in \overset{0}{W}{}_2^1 (B_R)$$

for almost every $t \in (T - H, T]$, and by (7.1) we have

$$-\iint_{Q_R^H} (a_{ij} u_{x_j} \psi_{x_i} + u_t \psi)\,dx\,dt = \frac{1}{2} a_0 p \iint_{Q_R^H} |u|^{p+q} \Phi^2\,dx\,dt \geq 0. \tag{7.6}$$

We set

$$w = u|u|^{\frac{1}{2}p-1}; \quad w^2 = |u|^p. \tag{7.7}$$

Remarking that

$$u_t \psi = \frac{1}{2}(w^2)_t \Phi^2, \quad a_{ij} u_{x_j} \psi_{x_i} = 2\frac{p-1}{p} \Phi^2 \cdot a_{ij} w_{x_i} w_{x_j} + 2w\Phi \cdot a_{ij} w_{x_j} \Phi_{x_i},$$

we find by (7.6) that

$$\frac{1}{2} \iint_{Q_R^H} (w^2)_t \Phi^2\,dx\,dt + 2\frac{p-1}{p} \iint_{Q_R^H} a_{ij} w_{x_i} w_{x_j} \Phi^2\,dx\,dt \leq -2 \iint_{Q_R^H} a_{ij} w_{x_j} \Phi_{x_i} w \Phi\,dx\,dt. \tag{7.8}$$

Applying the Schwarz inequality

$$|a_{ij} \xi_i \eta_j| \leq \sqrt{a_{ij} \xi_i \xi_j} \cdot \sqrt{a_{ij} \eta_i \eta_j}, \tag{7.9}$$

and then Cauchy's inequality

$$2|ab| \le \varepsilon a^2 + \varepsilon^{-1} b^2, \quad \varepsilon > 0, \tag{7.10}$$

with $\varepsilon = \dfrac{p-1}{p} < 1$, we have

$$2|a_{ij} w_{x_j} \Phi_{x_i} w \Phi| \le 2(a_{ij} w_{x_i} w_{x_j} \Phi^2)^{1/2} (a_{ij} \Phi_{x_i} \Phi_{x_j} w^2)^{1/2} \le \frac{p-1}{p} a_{ij} w_{x_i} w_{x_j} \Phi^2 + \frac{p}{p-1} a_{ij} \Phi_{x_i} \Phi_{x_j} w^2,$$

so that by (7.8)

$$\frac{p}{2(p-1)} \iint\limits_{Q_R^H} (w^2)_t \Phi^2 \, dx\, dt + \iint\limits_{Q_R^H} a_{ij} w_{x_i} w_{x_j} \Phi^2 \, dx\, dt \le \frac{p^2}{(p-1)^2} \iint\limits_{Q_R^H} a_{ij} \Phi_{x_i} \Phi_{x_j} w^2 \, dx\, dt.$$

Using assumption (3), adding to both sides of the last inequality the expression

$$\frac{p}{p-1} \iint\limits_{Q_R^H} w^2 \Phi \Phi_t \, dx\, dt = \frac{p}{2(p-1)} \iint\limits_{Q_R^H} w^2 (\Phi^2)_t \, dx\, dt,$$

and taking account of the inequality $\lambda \ge 1$, we find after simple computations that

$$\frac{\lambda p}{2(p-1)} \iint\limits_{Q_R^H} (w^2 \Phi^2)_t \, dx\, dt + \iint\limits_{Q_R^H} |w_x|^2 \Phi^2 \, dx\, dt \le \frac{\lambda^2 p^2}{(p-1)^2} \iint\limits_{Q_R^H} w^2 (|\Phi_x|^2 + |\Phi \Phi_t|) \, dx\, dt.$$

Furthermore, using the reasoning in the proof of Theorem 3 of [20], keeping in mind (7.7) and the fact that $\Phi = 0$ for $t = T - H$ and

$$\Phi = 1 \text{ in } Q_{R-R}^{H-h} \text{ and } |\Phi_x|^2 + |\Phi \Phi_t| \le r^{-2} + h^{-1},$$

we arrive at inequalities (7.3) and (7.4).

Lemma 7.2. *Let the function u be a solution of Eq. (2) in the cylinder Q_R^H, $p \ge 2$; and let $0 < \tau_m < H$ and $0 < \rho_m < R$ be strictly monotonic sequences of numbers, $m \in \mathbf{N}$. Set $Q_m = Q_{\rho_m}^{\tau_m}(T, x^0)$, $m \in \mathbf{N}$. Then in the case of increasing sequences τ_m, ρ_m we have the inequality*

$$\frac{1}{2} p a_0 \iint\limits_{Q_m} |u|^{p+q} \, dx\, dt \le c_2 K_1(m) \iint\limits_{Q_{m+2}} |u|^p \, dx\, dt, \quad m \in \mathbf{N}, \tag{7.11}$$

where

$$K_1(m) = (\tau_{m+2} - \tau_{m+1})^{-1} + (\rho_{m+2} - \rho_{m+1})^{-2} + (\rho_{m+1} - \rho_m)^{-2},$$

and in the case of decreasing sequences τ_m and ρ_m the following inequality holds:

$$\frac{1}{2} p a_0 \iint\limits_{Q_{m+2}} |u|^{p+q} \, dx\, dt \le c_2 K_2(m) \iint\limits_{Q_m} |u|^p \, dx\, dt, \quad m \in \mathbf{N}, \tag{7.12}$$

where

$$K_2(m) = (\tau_m - \tau_{m+1})^{-1} + (\rho_m - \rho_{m+1})^{-2} + (\rho_{m+1} - \rho_{m+2})^{-2};$$

and the constant c_2 depends only on λ.

Proof. It suffices to prove inequality (7.11) since (7.12) can then be obtained from (7.11) by relabeling. Set

$$\chi_m \equiv \chi_m(|x-x^0|) = \zeta_{\rho_{m+1},\rho_{m+1}-\rho_m}(|x-x^0|)$$

and

$$\psi = \frac{1}{2}pu|u|^{p-2}\chi_m,$$

where the function $\zeta_{R,r}(\rho)$ is defined in (7.5). Since the function ψ in (7.1) is not assumed to vanish on the bottom of the cylinder Q_R^H (i.e., for $t = T-H$), we find by (7.1) that

$$-\int\limits_{T-\tau_{m+1}}^{T}\int\limits_{B_R}(a_{ij}u_{x_j}\psi_{x_i}+u_t\psi)\,dx\,dt = \frac{1}{2}pa_0\int\limits_{T-\tau_{m+1}}^{T}\int\limits_{B_R}|u|^{p+q}\chi_m\,dx\,dt \geq \frac{1}{2}pa_0\iint\limits_{Q_m}|u|^{p+q}\,dx\,dt. \tag{7.13}$$

We now find an upper bound for the left-hand side of (7.13). Using the notation of (7.7), and taking account of the relations

$$u_t\psi = \frac{1}{2}(w^2\chi_m)_t, \quad a_{ij}u_{x_j}\psi_{x_i} = 2\frac{p-1}{p}\chi_m\cdot a_{ij}w_{x_i}w_{x_j} + w\cdot a_{ij}w_{x_j}(\chi_m)_{x_i},$$

we obtain

$$-\int\limits_{T-\tau_{m+1}}^{T}\int(a_{ij}u_{x_j}\psi_{x_i}+u_t\psi)\,dx\,dt = -2\frac{p-1}{p}\int\limits_{T-\tau_{m+1}}^{T}\int\chi_m\cdot a_{ij}w_{x_i}w_{x_j}\,dx\,dt$$

$$-\int\limits_{T-\tau_{m+1}}^{T}\int w\cdot a_{ij}w_{x_j}(\chi_m)_{x_i}\,dx\,dt - \frac{1}{2}\int w^2(T,x)\chi_m\,dx + \frac{1}{2}\int w^2(T-\tau_{m+1},x)\chi_m\,dx \equiv A$$

(here the integration with respect to x extends over the support of the function χ_m). Now remarking that $\chi_m \leq 1$ and applying assumption (3) to the integrand in the first integral of the last equality and inequalities (7.9), (3), and (7.10) (with $a = |w_x|$, $b = |(\chi_m)_x|\cdot|w|$, $\varepsilon = 2$) to the integrand in the second integral, and taking account of the fact that the third integral is nonpositive, we find that

$$A \leq 3\lambda\iint\limits_{Q_{m+1}}|w_x|^2\,dx\,dt + \lambda\iint\limits_{Q_{m+1}}|(\chi_m)_x|^2|w|^2\,dx\,dt + \frac{1}{2}\int\limits_{B_{\rho_{m+1}}}w^2(T-\tau_{m+1},x)\,dx. \tag{7.14}$$

Since $|(\chi_m)_x| \leq (\rho_{m+1}-\rho_m)^{-1}$, estimating the first and third terms of the right-hand side of (7.14) using inequalities (7.3) and (7.4) with $R = \rho_{m+2}$, $r = \rho_{m+2}-\rho_{m+1}$, $H = \tau_{m+2}$, $h = \tau_{m+2}-\tau_{m+1}$, we obtain an inequality which together with (7.13) gives inequality (7.11). The lemma is now proved.

The following estimate holds for the maximum modulus of a solution of Eq. (2) in terms of the L^p norm of the solution:

Lemma 7.3 (similar to a theorem of J. Moser [20]). *Let the function u be a solution of Eq. (2) in the cylinder Q_R^H and $p \geq 2$. Then for $h \in (0,H)$ and $r \in (0,R)$ we have the inequality*

$$\max_{Q_{R-r}^{H-h}}|u|^p \leq c_3\cdot J\cdot(h^{-1}+r^{-2})^{k/(k-1)}(\operatorname{mes}Q_R^H)^{-1}\iint\limits_{Q_R^H}|u|^p\,dx\,dt, \tag{7.15}$$

where c_3 is a positive constant depending only on n and λ and

$$J \equiv J(R,r,H,h,n) = \left\{\left(\frac{R}{R-r}\right)^n\frac{H}{H-h}(R^2+H)\left(\frac{(1+R)^4}{R^2}+\frac{(1+H)^2}{H}\right)\right\}^{k/(k-1)},$$

$$k = \begin{cases}1+\frac{2}{n}, & \text{if } n>2;\\ \frac{5}{3}, & \text{if } n=1,2.\end{cases}$$

We note that the explicit dependence of the constant in inequality (7.15) on the quantity $h^{-1}+r^{-2}$ is used in an essential manner to establish the main results for solutions of Eq. (2) in the case when $-1<q<0$.

The proof of Lemma 7.3 is carried out by applying the iteration technique of J. Moser [20]. For the sake of completeness we shall present it here.

Setting

$$D_R^H(w)=H^{-1}R^{-n}\iint\limits_{Q_R^H} w^2\,dx\,dt$$

for a function $w=w(t,x)$ defined in the cylinder Q_R^H, we use the following known lemma, which is a corollary of the Sobolev imbedding theorem:

Lemma 7.4. *The inequality*

$$(D_R^H(w^k))^{1/k}\le c_4\Big(H^{-1}R^{2-n}\iint\limits_{Q_R^H}|w_x|^2\,dx\,dt+R^{-n}\cdot\max_{t\in[T-H,T]}\int\limits_{B_R} w^2(t,x)\,dx\Big)$$

holds for any function w for which these integrals exist. The constant c_4 depends only on n, and the quantity k is defined in Lemma 7.3.

For a solution u of Eq. (2) in the cylinder Q_R^H and $p\ge 2$ we set

$$u_\nu=u|u|^{\frac{1}{2}pk^\nu-1};\quad u_\nu^2=(|u|^p)^{k^\nu},\quad \nu=0,1,2,3,\ldots,$$

and we remark that $(u_\nu)^k=u_{\nu+1}$. We define a sequence of cylinders as follows: $Q_\nu=Q_{R-r_\nu}^{H-h_\nu}$, where

$$h_\nu=\Big(1-\frac{1}{1+(2^\nu-1)H}\Big)h,\quad r_\nu=\Big(1-\frac{1}{1+(2^\nu-1)R}\Big)r,\quad \nu=0,1,2,3,\ldots,$$

so that $R-r<R-r_\nu\le R$, $H-h<H-h_\nu\le H$, and $Q_0=Q_R^H$, $Q_\nu\to Q_{R-r}^{H-h}$ as $\nu\to+\infty$.

We apply Lemma 7.4 (setting $w=u_\nu$, $R=R-r_{\nu+1}$, $H=H-h_{\nu+1}$) and then inequalities (7.3) and (7.4) with

$$R=R-r_\nu,\ \ H=H-h_\nu,\ \ h=h_{\nu+1}-h_\nu,\ \ r=r_{\nu+1}-r_\nu,\ \ p=pk^\nu;$$

$$(D_{R-r_{\nu+1}}^{H-h_{\nu+1}}(u_{\nu+1}))^{1/k}=(D_{R-r_{\nu+1}}^{H-h_{\nu+1}}((u_\nu))^{1/k}\le c_4\Big(\frac{R-r_\nu}{R-r_{\nu+1}}\Big)^n\frac{H-h_\nu}{H-h_{\nu+1}}(R-r_\nu)^{-n}(H-h_\nu)^{-1}\times$$
$$\times\Big\{(R-r_{\nu+1})^2\iint\limits_{Q_{\nu+1}}|(u|u|^{\frac{1}{2}pk^\nu-1})_x|^2\,dx\,dt$$
$$+(H-h_{\nu+1})\cdot\max_{t\in[T-(H-h_{\nu+1}),T]}\int\limits_{B_{R-r_{\nu+1}}}|u(t,x)|^{pk^\nu}\,dx\Big\}\le$$
$$c_1c_4\Big(\frac{R}{R-r}\Big)^n\frac{H}{H-h}(R^2+H)\frac{(pk^\nu)^2}{(pk^\nu-1)^2}\{(h_{\nu+1}-h_\nu)^{-1}+(r_{\nu+1}-r_\nu)^{-2}\}D_{R-r_\nu}^{H-h_\nu}(u_\nu).$$

Now extracting the k^νth root in this inequality and taking into account the relations

$$(r_{\nu+1}-r_\nu)^{-2}\le 4^{\nu+1}\frac{(1+R)^4}{R^2}r^{-2},\quad (h_{\nu+1}-h_\nu)^{-1}\le 4^{\nu+1}\frac{(1+H)^2}{H}h^{-1},$$
$$\frac{(pk^\nu)^2}{(pk^\nu-1)^2}\le 4,\ \text{ since } pk^\nu\ge p\ge 2,$$

and setting

$$\theta_\nu = (D_{R-r_\nu}^{H-h_\nu}(u_\nu))^{1/k^\nu}, \quad J_1 = \left(\frac{R}{R-r}\right)^n \frac{H}{H-h}(R^2+H)\left\{\frac{(1+R)^4}{R^2} + \frac{(1+H)^2}{H}\right\},$$

we obtain the following recurrence relation:

$$\theta_{\nu+1} \le 4^{(\nu+1)/k^\nu}\{c_5(n,\lambda)\cdot J_1\cdot(h^{-1}+r^{-2})\}^{1/k^\nu}\cdot\theta_\nu.$$

Now carrying out the corresponding iteration process, we find that

$$4^{\sum_{j=0}^{\infty}\frac{j+1}{k^j}}\{c_5(n,\lambda)\cdot J_1\cdot(h^{-1}+r^{-2})\}^{\sum_{j=0}^{\infty}\frac{1}{k^j}}\cdot\theta_0 \ge \theta_{\nu+1} \ge$$

$$\left(\frac{1}{HR^n}\right)^{1/k^{\nu+1}}\left\{\iint_{Q_{R-r}^{H-h}}(|u|^p)^{k^{\nu+1}}\,dx\,dt\right\}^{1/k^{\nu+1}} \underset{\nu\to\infty}{\longrightarrow} \max_{Q_{R-r}^{H-h}}|u|^p.$$

It now remains only to remark that $\sum_{j=0}^{\infty}\frac{1}{k^j} = \frac{k}{k-1}$. The lemma is now proved.

§ 8. THE SUPERLINEAR EQUATION

Throughout this section we assume that the exponent q in Eq. (2) is positive.

Theorem 8.1. *Let the function u be a solution of Eq. (2) in the cylinder $Q_1^1 \equiv (T-1,T]\times B_1(x^0)$. Then at the center of the top of the cylinder we have the inequality*

$$|u(T,x^0)| \le \alpha_0, \tag{8.1}$$

where the positive constant α_0 depends only on n, λ, a_0 and q.

Proof. Let $\varepsilon > 0$ be a number satisfying the condition

$$(1+\varepsilon)^3 = 1 + q/2 \tag{8.2}$$

(so that ε depends only on q). In what follows we shall use the notation of Lemma 7.2. With this notation we set

$$\rho_m = \frac{6}{\pi^2}\sum_{j=1}^{m}\frac{1}{j^2}, \quad \tau_m = \rho_m^2, \quad m\in\mathbf{N},$$

so that the increasing sequences ρ_m and τ_m are bounded: $0<\rho_m<1$, $0<\tau_m<1$ and $\rho_m\uparrow 1$ and $\tau_m\uparrow 1$ as $m\uparrow\infty$.

We remark that

$$\rho_{m+1}-\rho_m \ge d_1 m^{-3}, \quad \tau_{m+1}-\tau_m \ge d_2 m^{-6},$$

where d_1 and d_2 are certain positive constants; therefore

$$\max\{(\tau_{2m+1}-\tau_{2m})^{-1}, (\rho_{2m+1}-\rho_{2m})^{-2}, (\rho_{2m}-\rho_{2m-1})^{-2}\} \le dm^6, \tag{8.3}$$

where d is a positive absolute constant.

We set

$$M_m = \iint_{Q_m}|u|^2\,dx\,dt, \quad m\in\mathbf{N},$$

where the cylinder Q_m is defined in Lemma 7.2.

The theorem is proved by contradiction. Suppose the inequality

$$|u(T,x^0)| > \alpha_0 \tag{8.4}$$

holds, in which the number $\alpha_0 = \alpha_0(n,\lambda,a_0,q) > 0$ in (8.1) will be chosen below. Since the function u is also a solution of Eq. (2) in the cylinder Q_1, applying Lemma 7.3 with $p=2$, we obtain

$$M_1 = \iint\limits_{Q_1} |u|^2\,dx\,dt \geq c_0^2 \max_{Q_{r_0}^{r_0^2}} |u|^2 \geq c_0^2|u(T,x^0)|^2 > (c_0\alpha_0)^2, \tag{8.5}$$

where $r_0 = \frac{1}{2}\rho_1$, and the constant c_0 depends only on n and λ.

We now choose α_0 so that the equality

$$(c_0\alpha_0)^2 = \exp\{(1+\varepsilon)^{k_0+1}\} \tag{8.6}$$

holds, where the constant $k_0 = k_0(n,\lambda,a_0,q) > 0$ will be found below. We shall prove by induction that

$$M_{2m-1} > \exp\{(1+\varepsilon)^{k_0+2m-1}\}, \quad m \in \mathbf{N}; \tag{8.7}$$

by (8.2) this will lead to a contradiction with the fact that $u \in L^2(Q_1^1)$. Indeed, when $m=1$, inequality (8.7) holds by (8.4), (8.5), and (8.6). Suppose there is an index $m_0 \in \mathbf{N}$ such that

$$M_{2m_0-1} > \exp\{(1+\varepsilon)^{k_0+2m_0-1}\}, \tag{8.8}$$

yet

$$M_{2m_0+1} \leq \exp\{(1+\varepsilon)^{k_0+2m_0+1}\}. \tag{8.9}$$

We shall show that this is impossible.

Applying inequality (7.11) of Lemma 7.2 with $m = 2m_0-1$, we arrive at the inequality

$$a_0 \iint\limits_{Q_{2m_0-1}} |u|^{2+q}\,dx\,dt \leq c_2 K_1(2m_0-1) M_{2m_0+1}, \tag{8.10}$$

in which, by (8.3),

$$K_1(2m_0-1) \leq 3dm_0^6, \tag{8.11}$$

and c_2 is the constant from Lemma 7.2, which depends on λ.

We now apply Jensen's inequality [21, p. 84] to the left-hand side of inequality (8.10):

$$\Big(\iint\limits_D |u(t,x)|\,dx\,dt\Big)^\nu \leq (\operatorname{mes} D)^{\nu-1} \iint\limits_D |u(t,x)|^\nu\,dx\,dt, \quad \nu \geq 1,$$

and, taking account of the fact that $q>0$, we obtain

$$\iint\limits_{Q_{2m_0-1}} |u|^{2+q}\,dx\,dt = \iint\limits_{Q_{2m_0-1}} (|u|)^2)^{1+\frac{q}{2}}\,dx\,dt \geq$$

$$(\operatorname{mes} Q_{2m_0-1})^{-\frac{q}{2}} \Big(\iint\limits_{Q_{2m_0-1}} |u|^2\,dx\,dt\Big)^{1+\frac{q}{2}} \geq (\operatorname{mes} Q_1^1)^{-\frac{q}{2}} (M_{2m_0-1})^{1+\frac{q}{2}}. \tag{8.12}$$

Thus, combining inequalities (8.10), (8.11), and (8.12), we find that

$$(M_{2m_0-1})^{1+\frac{q}{2}} \leq \frac{3c_2 d}{a_0} (\operatorname{mes} Q_1^1)^{\frac{q}{2}} m_0^6 M_{2m_0+1} \equiv c_6(n,\lambda,a_0,q) m_0^6 M_{2m_0+1}. \tag{8.13}$$

From inequalities (8.13), (8.8), and (8.9), taking account of (8.2), we find that

$$\exp\{(1+\varepsilon)^{k_0+2m_0+2}\} = (\exp\{(1+\varepsilon)^{k_0+2m_0-1}\})^{1+\frac{q}{2}} < (M_{2m_0-1})^{1+\frac{q}{2}} \leq c_6 m_0^6 M_{2m_0+1} \leq c_6 m_0^6 \exp\{(1+\varepsilon)^{k_0+2m_0+1}\},$$

from which we find the following inequality for the index m_0

$$\exp\{\varepsilon(1+\varepsilon)^{k_0+2m_0+1}\} < c_6 m_0^6. \tag{8.14}$$

We now choose and keep fixed a number $k_0 = k_0(n, \lambda, a_0, q) > 0$ from (8.6) such that the inequality

$$\varepsilon(1+\varepsilon)^{k_0+1} \geq (1+\varepsilon)^{-2m} \ln\{m^6 \cdot \max(e, c_6)\}$$

holds for all $m \in \mathbf{N}$. With such a choice of k_0 in (8.6) we arrive at a contradiction with (8.14). Therefore inequality (8.7) must hold. This proves inequality (8.1) with α_0 from (8.6) depending only on n, λ, a_0 and q by virtue of the choice of k_0. The theorem is now proved.

Theorem 8.2 (the connection between the dimensions of the cylinder in which the solution is defined and the value of the solution at the center of the top of the cylinder). *There exists a positive constant c depending only on n, λ, a_0, and q such that if the function u is a solution of Eq. (2) in the cylinder $Q_R^H \equiv Q_R^H(T, x_0)$, the following inequality holds:*

$$|u(T, x^0)| \leq c\{\min(H, R^2)\}^{-1/q} \leq c(H^{-1/q} + R^{-2/q}). \tag{8.15}$$

Proof. We first assume that $H = R^2$. Eq. (2) is invariant under the following transformations of variables and functions:

$$t \mapsto R^2(t' - T) + T, \quad x \mapsto R(x' - x^0) + x^0, \quad u \mapsto R^{-2/q} u'.$$

Under this transformation the cylinder $Q_R^{R^2}$ maps onto the cylinder Q_1^1 and the function

$$u'(t', x') = R^{\frac{2}{q}} u(R^2(t' - T) + T, R(x' - x^0) + x^0)$$

is a solution of Eq. (2) in the cylinder Q_1^1. Applying Theorem 8.1 to the function u', we find that $R^{2/q}|u(T, x^0)| = |u'(T, x^0)| \leq \alpha_0 \equiv c$, from which the desired estimate (8.15) now follows.

In the general case, since the function u, as a solution of Eq. (2) in Q_R^H is a solution of (2) in the cylinder $Q_{R_0}^{R_0^2}$, where $R_0 = \sqrt{\min(H, R^2)}$, we obtain (8.15) from the previous reasoning.

In the situations analogous to those of part I this theorem can be used to obtain information on the properties of solutions of Eq. (2).

1. The behavior of solutions in unbounded domains. Let the function u be a solution of Eq. (2) in one of the unbounded domains exhibited in § 3, Sec. 1. Then we have the following respective results: 1) $|u(t,x)| \leq ct^{-1/q}$, $(t,x) \in (0, +\infty) \times \mathbf{R}^n$, and consequently $u(t,x) \to 0$ as $t \to +\infty$ uniformly with respect to $x \in \mathbf{R}^n$; 2) $|u(t,x)| \leq c(x_i)^{-2/q}$, $(t,x) \in \{x_i > 0\}$, and consequently $u(t,x) \to 0$ as $x_i \to +\infty$ uniformly with respect to $(t, \hat{x}_i) \in \mathbf{R}^n$; 3) $|u(t,x)| \leq c\{\min(t, x_i^2)\}^{-1/q}$, $(t,x) \in \{t > 0, x_i > 0\}$, and therefore $u(t,x) \to 0$ as $t \to +\infty$ and $x_i \to +\infty$ uniformly with respect to $\hat{x}_i \in \mathbf{R}^{n-1}$; 4) $|u(t,x)| \leq c\{\min(t, \operatorname{dist}(x, \partial\Omega))\}^{-1/q}$, $(t,x) \in (0, +\infty) \times (\mathbf{R}^n \setminus \bar{\Omega})$, and therefore $u(t,x) \to 0$ as $t \to +\infty$ and $|x| \to +\infty$; 5) $|u(t_\sigma, x_\sigma)| \leq c\{\min(H_\sigma, R_\sigma^2)\}^{-1/q}$, $(t_\sigma, x_\sigma) \in l$, and consequently $u(t_\sigma, x_\sigma) \to 0$ as $(t_\sigma, x_\sigma) \in l$, $\sigma \to +\infty$.

2. Behavior of solutions near the boundary of a cylinder. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain, and let the function u be a solution of Eq. (2) in the cylinder $(0, T] \times \Omega$ or the unbounded domain $(0, T] \times (\mathbf{R}^n \setminus \bar{\Omega})$. Then the following inequality holds:

$$|u(t,x)| \leq c\{\min(t, \operatorname{dist}^2(x, \partial\Omega))\}^{-1/q}.$$

3. Uniqueness and continuous dependence on boundary conditions for the solution of the exterior initial boundary problem. In $(0,+\infty) \times (\mathbf{R}^n \setminus \bar{\Omega})$, where $\Omega \subset \mathbf{R}^n$ is a bounded domain, we consider the exterior initial boundary problem (3.1), (3.2) for Eq. (2), where $\chi(t,x)$ is a bounded function on $[0,+\infty) \times \partial\Omega$. The following theorem is established on the basis of Theorems 8.2 and 7.0 in analogy with Theorem 3.2.

Theorem 8.3. *If the functions u' and $u'' \in C([0,+\infty) \times (\mathbf{R}^n \setminus \Omega))$ are solutions of the problem (2), (3.1), (3.2) with boundary functions χ' and χ'' respectively, then for $(t,x) \in [0,+\infty) \times (\mathbf{R}^n \setminus \Omega)$ the following inequality holds:*

$$|u'(t,x) - u''(t,x)| \leq \sup_{[0,+\infty)\times\partial\Omega} |\chi' - \chi''|.$$

It follows from this theorem that the solution of the exterior initial boundary problem is unique.

4. The removable singularities theorem.

Theorem 8.4. *Let the coefficients a_{ij}, $i,j = 1,\dots,n$, of the operator $\mathfrak{L}$ be continuously differentiable functions in $\mathbf{R}^{n+1}$. If $Q \subset \mathbf{R}^{n+1}$ is a domain containing the point $(0,0)$, $q > 2/n$, and u is a solution of Eq. (2) in $Q \setminus \{(0,0)\}$, the function u can be defined at the point $(0,0)$ so as to be a solution of Eq. (2) in Q.*

On the basis of Theorems 8.2 and 7.0 this theorem can be proved in analogy with Theorem 4.1. When this is being done, the solvability of the corresponding initial boundary problem for Eq. (2) follows from the results of [16, Ch. 7], and the existence of the fundamental solution $E(t,x;\tau,\xi)$ of the operator $\mathfrak{L}$ and estimate (4.7) with K_1, K_2, k_1, and k_2 depending only on n, λ, and $\tau_0 + T_0$ follows from [17, §§ 7–8] (cf. also [18]).

§ 9. THE SUBLINEAR EQUATION

In this section we assume that $-1 < q < 0$ in Eq. (2).

Theorem 9.1. *There exists a positive constant β_0 depending only on n, λ, a_0, and q such that if the function u is a solution of Eq. (2) in the cylinder $Q_1^1 = (T-1,T] \times B_1(x^0)$ and*

$$\max_{Q_1^1} |u| \leq \beta_0, \tag{9.1}$$

then the equality $u(T,x^0) = 0$ holds.

Proof. Fix an arbitrary $p \geq 3$. Let $\varepsilon > 0$ be a number satisfying the condition

$$(1+\varepsilon)^{-4} = 1 + q/p \tag{9.2}$$

(so that ε depends only on q). From now on we shall use the notation of Lemma 7.2. In this notation we set

$$\rho_m = \frac{1}{2}\left(1 + \frac{6}{\pi^2}\sum_{j=m}^{\infty}\frac{1}{j^2}\right), \quad \tau_m = \rho_m^2, \quad m \in \mathbf{N},$$

so that the decreasing sequences ρ_m and τ_m are bounded:

$$\frac{1}{2} < \rho_m \leq 1, \quad \frac{1}{4} < \tau_m \leq 1, \tag{9.3}$$

$\rho_1 = \tau_1 = 1$; $\rho_m \downarrow 1/2$, $\tau_m \downarrow 1/4$ as $m \uparrow \infty$.

Remarking that

$$\rho_m - \rho_{m+1} = \frac{3}{\pi^2}m^{-2}, \quad \tau_m - \tau_{m+1} \geq \frac{9}{\pi^4}m^{-4},$$

we obtain the following inequalities:

$$\max\{(\tau_{3m}-\tau_{3m+1})^{-1},\,(\rho_{3m}-\rho_{3m+1})^{-2}\}\le dm^4, \tag{9.4}$$

$$\max\{(\tau_{3m-2}-\tau_{3m-1})^{-1},\,(\rho_{3m-2}-\rho_{3m-1})^{-2},(\rho_{3m-1}-\rho_{3m})^{-2}\}\le dm^4, \tag{9.5}$$

where d is a certain positive absolute constant.

We set $Q_\infty = Q_{1/2}^{1/4}$ and

$$M_m=\max_{Q_m}|u|,\quad m\in\mathbf{N}\cup\{\infty\},$$

where the cylinder Q_m is defined in Lemma 7.2.

We now define the number β_0 in (9.1) as follows:

$$\beta_0=\exp\{-(1+\varepsilon)^{k_0+1}\}, \tag{9.6}$$

where the quantity $k_0=k_0(n,\lambda,a_0,q)>0$ will be chosen below. Starting from this, we prove by induction that

$$M_{3m-2}\le\exp\{-(1+\varepsilon)^{k_0+3m-2}\},\quad m\in\mathbf{N}. \tag{9.7}$$

When $m=1$, inequality (9.7) holds by (9.1), (9.6), and the equality $Q_1=Q_1^1$. Assume that there exists an index $m_0\in\mathbf{N}$ such that

$$M_{3m_0-2}\le\exp\{-(1+\varepsilon)^{k_0+3m_0-2}\}, \tag{9.8}$$

yet

$$M_{3m_0+1}>\exp\{-(1+\varepsilon)^{k_0+3m_0+1}\}. \tag{9.9}$$

Since $p+q>2$, applying inequality (7.15) with

$$R=\rho_{3m_0},\quad r=\rho_{3m_0}-\rho_{3m_0+1},\quad H=\tau_{3m_0},\quad h=\tau_{3m_0}-\tau_{3m_0+1},$$

and then inequality (7.12) with $m=3m_0-2$ and taking account of (9.4), we find that

$$(M_{3m_0+1})^{p+q}=\max_{Q_{3m_0+1}}|u|^{p+q}\le c_3J_{m_0}(\operatorname{mes}Q_{3m_0})^{-1}((\tau_{3m_0}-\tau_{3m_0+1})^{-1}+$$

$$(\rho_{3m_0}-\rho_{3m_0+1})^{-2})^{\frac{k}{(k-1)}}\iint\limits_{Q_{3m_0}}|u|^{p+q}\,dx\,dt\le c_3J_{m_0}(\operatorname{mes}Q_\infty)^{-1}(2dm_0^4)^{\frac{k}{(k-1)}}\frac{2c_2}{pa_0}K_2(3m_0-2)\iint\limits_{Q_{3m_0-2}}|u|^p\,dx\,dt$$

$$\le\frac{2c_2c_3\operatorname{mes}Q_1}{pa_0\operatorname{mes}Q_\infty}J_{m_0}(2dm_0^4)^{k/(k-1)}K_2(3m_0-2)(M_{3m_0-2})^p, \tag{9.10}$$

where the quantity J_{m_0} is given in terms of the J of (7.15) by

$$J_{m_0}\equiv J(\rho_{3m_0},\rho_{3m_0}-\rho_{3m_0+1},\tau_{3m_0},\tau_{3m_0}-\tau_{3m_0+1},n)\le(2^{n+9})^{k/(k-1)}\equiv c_7(n)$$

by virtue of (9.3), and

$$K_2(3m_0-2)=(\tau_{3m_0-2}-\tau_{3m_0-1})^{-1}+(\rho_{3m_0-2}-\rho_{3m_0-1})^{-2}-(\rho_{3m_0-1}-\rho_{3m_0})^{-2}\le 3dm_0^4$$

by (9.5). The constants c_2 and c_3 in Lemmas 7.2 and 7.3 depend respectively on λ and on n and λ.

Thus, taking the pth root in (9.10), we obtain the inequality

$$(M_{3m_0+1})^{1+\frac{q}{p}}\le\left\{\frac{6dc_2c_3c_7\operatorname{mes}Q_1}{pa_0\operatorname{mes}Q_\infty}(2d)^{\frac{k}{k-1}}\right\}^{\frac{1}{p}}m_0^\gamma M_{3m_0-2}\equiv c_8(n,\lambda,a_0)m_0^\gamma M_{3m_0-2}, \tag{9.11}$$

where $\gamma=\dfrac{4}{p}\Big(\dfrac{k}{k-1}+1\Big)$.

By (9.11), (9.8), and (9.9), taking account of (9.2), we have

$$\exp\{-(1+\varepsilon)^{k_0+3m_0-3}\} = (\exp\{-(1+\varepsilon)^{k_0+3m_0+1}\})^{1+\frac{q}{p}} < (M_{3m_0+1})^{1+\frac{q}{p}} \\ \leq c_8 m_0^\gamma M_{3m_0-2} \leq c_8 m_0^\gamma \exp\{-(1+\varepsilon)^{k_0+3m_0-2}\},$$

from which we find that

$$\exp\{\varepsilon(1+\varepsilon)^{k_0+3m_0-3}\} < c_8 m_0^\gamma. \tag{9.12}$$

From this point on we reason as in (8.14) (with obvious modifications). In this way we find and hold fixed the quantity $k_0 = k_0(n, \lambda, a_0, q) > 0$ of (9.6) such that inequality (9.12) will be violated. This proves (9.7).

Taking account of (9.2) and passing to the limit as $m \to \infty$ in (9.7), we find that

$$\max_{Q_\infty} |u| = M_\infty \leq 0,$$

from which, in particular, it follows that $u(T, x^0) = 0$. The theorem is now proved.

Theorem 9.2. *If the function u is a solution of Eq. (2) in the cylinder $Q_R^H \equiv Q_R^H(T, x^0)$ and*

$$\max_{Q_R^H} |u| \leq \beta_0 \min(H^{-1/q}, R^{-2/q}),$$

where β_0 is the constant from Theorem 9.1, then $u(T, x^0) = 0$.

Proof. We first assume that $H = R^2$. We carry out a change of variables and functions:

$$x' = \frac{1}{R}(x - x^0) + x^0, \quad t' = \frac{1}{R^2}(t - T) + T, \quad u' = R^{\frac{2}{q}} u.$$

Under this transformation the cylinder $Q_R^{R^2}$ maps onto the cylinder Q_1^1 and the function $u'(t', x') = R^{2/q} u(t, x)$ will be a solution of Eq. (2) in the cylinder Q_1^1 such that $\max_{Q_1^1} |u'| \leq \beta_0$. Applying Theorem 9.1 to the function u', we find that $R^{2/q} u(T, x^0) = u'(T, x^0) = 0$.

In the general case if $R_0 = \sqrt{\min(H, R^2)}$, then by the hypothesis of the theorem we have

$$\max_{Q_{R_0}^{R_0^2}} |u| \leq \max_{Q_R^H} |u| \leq \beta_0 (R_0)^{-2/q}.$$

Consequently $u(T, x^0) = 0$. The theorem is now proved.

Theorem 9.3 (vanishing of the solution). *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain, and let the function u be a solution of Eq. (2) in the unbounded domain $(0, +\infty) \times (\mathbf{R}^n \setminus \bar{\Omega})$ such that*

$$|u(t, x)| = o(\min(t^{-1/q}, |x|^{-2/q})), \quad t \to +\infty, \quad |x| \to +\infty.$$

Then there exist $R_0 > 0$ and $H_0 > 0$ so large that $u \equiv 0$ on $[H_0, +\infty) \times (\mathbf{R}^n \setminus B_{R_0}(0))$. In particular this theorem holds for bounded solutions.

Proof. Let $R_1 > 0$ be such that $\Omega \subset B_{R_1}(0)$. For $0 < \varepsilon \leq \beta_0 9^{1/q}$ there exist $\rho_0 = \rho_0(\varepsilon) > R_1$ and $\tau_0 = \tau_0(\varepsilon) > 0$ such that

$$|u(t, x)| \leq \varepsilon \min(t^{-1/q}, |x|^{-2/q})$$

for $t \geq \tau_0$ and $|x| \geq \rho_0$. Consequently if $|x^0| \equiv 2R \geq R_0 \equiv 2\rho_0$, $t^0 \equiv 2H \geq H_0 \equiv 2\tau_0$, then on the set $\{(t, x) \in (0, +\infty) \times (\mathbf{R}^n \setminus \bar{\Omega}) : R \leq |x| \leq 3R,\ H \leq t \leq 9H\}$ we have the inequality

$$|u| \leq \beta_0 9^{1/q} \min\{(9H)^{-1/q}, (3R)^{-2/q}\} \leq \beta_0 \min(H^{-1/q}, R^{-2/q}),$$

which holds in particular in the cylinder $Q_R^H(t^0, x^0)$, which is contained in the set in question. Applying Theorem 9.2, we obtain the equality $u(t^0, x^0) = 0$. The theorem is now proved.

The proof of the following theorem is analogous to the proof of Theorem 5.4.

Theorem 9.4. *A bounded solution $u \in C([0, +\infty) \times \mathbf{R}^n)$ of the Cauchy problem for Eq.* (2) *with an initial function $u(0, x) = u_0(x)$ of compact support has compact support in $[0, +\infty) \times \mathbf{R}^n$.*

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