

Online Teaching of Calculus Using *WebMathematica*

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Abstract: The paper addresses the on-line teaching of Calculus using *webMathematica* interactive electronic tutorials developed by the author. The tutorials are available on the web site <http://wm.iedu.ru>.

It is obvious that e-learning technologies need new pedagogy. It is usually called e-pedagogy. We share and realize the main pedagogical principle of *webMathematica* based learning. The principle is laid out as follows. *To teach mathematics not calculation or math \neq calculating.*

Key words: Calculus, *webMathematica*, interactive tutorials.

Introduction

Mathematical disciplines in general and Calculus, in particular, require the development of problem solving skills. Without solving problems, the assimilation of the discipline by students cannot be regarded as deep and effective. This is, certainly, valid for the online teaching. While it is simple to present the background of the mathematical discipline on the Internet by putting an electronic version of the textbook on the website, problem solving should be interactive. The web site should be supplied with instruments which students can use to perform remote calculations. At the same time, the teacher should be able to assess them on-line by using webinars, chats, forums, blogs, and so on. We use *webMathematica* [1] as a software system that provides computing and online teaching of mathematics on the Internet.

WebMathematica is a web technology for doing symbolic, graphic and numerical remote calculations based on scientific and educational interactive electronic documents hosted on the Java server. The distant students use only web browsers such as Internet Explorer or Netscape for learning and doing calculations.

WebMathematica can be regarded as a web interface for the computational engine (the kernel) of the computer algebra system (CAS)

called *Mathematica*. Note that there may be several *Mathematica* kernels on the server. But *Mathematica* is deeply hidden from users who only need to know the *Mathematica* input syntax for the mathematical formulae.

Impact of computer algebra systems on teaching of mathematics

Computer algebra systems have been used in education right after their appearance in the 1980s. Educators used them both for lectures [2] and for computer labs [3] to enhance the teaching of Calculus and related courses. *Mathematica* is attractive for teaching because its users work with electronic documents called Notebooks in which text and executable commands coexist. This is very important since educators can easily write interactive electronic tutorials for students. These tutorials make mathematics more exciting to students by getting them personally involved in experimentation and discovery.

The influence of computer algebra systems on the content and teaching methods of mathematical disciplines is carried out in several directions. Computer algebra systems provide more detailed and profound discussion of the basic concepts and principles of the mathematical disciplines than is possible within traditional techniques of teaching. This is due to savings in the time that instructors usually devote to perform routine and cumbersome calculations and to draw function graphs when delivering lectures in classes. CAS can do this job quickly, accurately, and aesthetically appealingly. At the same time, instructors can dwell on numerous examples of direct interest to students and problems which are relevant for the “real world”. As a consequence, the exposition of disciplines becomes more accessible and compelling and students have fun learning.

If, in addition, students can attend computer classes at any time or they have CAS installed on their personal computers or laptops, they have a good opportunity to study the theory and to solve problems with the aid of arbitrarily patient mentors which are interactive tutorials.

We refer the reader to paper [4] for further details.

Why *webMathematica*

WebMathematica was chosen by us for the following reasons. The author intensively used CAS *Mathematica* for teaching students in computer classes for a long time. To do this, the electronic tutorials containing Calculus background, exercises, and programs written in a program language “*Mathematica*” were developed. Once Wolfram Research released *webMathematica*, it was natural and not difficult to translate *Mathematica* tutorials into *webMathematica* ones.

It was timely, since my colleagues not familiar with *Mathematica* met with certain difficulties when teaching Calculus in computer classes. The same could be said for students. The matter is that students using *Mathematica* make many mistakes that can only be corrected by the experienced user of the system. Here are some of them.

Initialization mistakes. There is a section called Programs in each of the author's *Mathematica* Calculus tutorials, a section which contains the author's programs providing the calculations typical for that tutorial. Students must run (initialize) these programs before doing any calculation. But they often forget to make initialization although there is a red warning text at the beginning of each tutorial.

In addition, it is recommended that students should solve the problems they received in a separate Notebook of *Mathematica*. So often the students have two working electronic documents open at one and the same moment. One is the tutorial with the programs and the other the Notebook with the students' data to calculate. To initialize the programs, it is necessary the tutorial is an active document. But students try to make the initialization when the tutorial is inactive. Programs do not work!

The third initialization error lies in the fact that students open their working Notebook by re-launching *Mathematica* so there appear two concurrent independent *Mathematica* processes at the same time. And while the initialization is done correctly in one process, the other process does not know anything about the programs.

Syntax mistakes. All of the *Mathematica* functions (commands) start with capital letters. Their arguments should be enclosed in square brackets, but not in parentheses as usual. This leads to numerous errors, since even trigonometric functions sine and cosine should be written unusually.

Operation mistakes. We mentioned already that students should solve their problems using the separate Notebook. Before doing calculations to solve the problem, students read the tutorial and run the commands the tutorial contains. There are global definitions among the commands, the definitions that prescribe values to symbols. These values are valid during the *Mathematica* session unless the user redefines the values or cancel them. It is recommended that students do not run the commands of the tutorial while working with the Notebook in which they solve their problems. Otherwise the global definitions of the tutorial become valid for the student's Notebook. This is no good and leads to mistakes. Some students neglect this recommendation!

WebMathematica completely eliminated all of these kinds of obstacles. However, for a more successful and widespread use of *webMathematica*, it is desirable to develop program tools that would allow the users to enter the mathematical formulae in the usual mathematical form, i.e. with special symbols for quotients, powers, roots, sums, derivatives, integrals, series, etc.

Another reason for adopting *webMathematica* was the spread of e-learning, which required the providing of an access to interactive educational resources for the distant students. Currently, the distant students of the Higher School of Economics use the educational portal <http://miem.iedu.ru/> to gain an access to *webMathematica* as well as to receive the problems to solve, teacher comments, and assessments.

The *webMathematica* operation also showed that full-time students readily use it to prepare for tests and exams. In fact, the access of students to their teachers is limited due to the lack of teacher's time, while *webMathematica* allows unlimited time to practice and to understand basic concepts of Calculus.

Some other issues of the *webMathematica* application for the learning of mathematics are discussed in [5].

BRIEF REVIEW OF *WEBMATHEMATICA*

We recommend the reader who knows nothing about *webMathematica* to visit the web site <http://library.wolfram.com/explorations/> to become familiar with it. There one can interact with a number of examples of *webMathematica* applications.

WebMathematica makes special web pages that contain HTML-forms with the input fields for entering the user's information. These forms can be supplied with radio buttons and pop up menus. The pages are sent to the user upon request.

WebMathematica, installed on the Java server, uses two standard Java technologies: Java servlets and Java server pages. But the developer of *webMathematica* tutorials does not need to know anything about Java except of several special tags that work with *Mathematica*. The *Mathematica* commands and programs are placed between `<msp:evaluate>` and `</msp:evaluate>` tags. *WebMathematica* sends these commands for executing to the *Mathematica* kernel. The results of calculations are inserted into the generated new web page.

Now we must mention some things which have important implications for the authors of interactive tutorials. The `<msp:evaluate>` tag extracts one of the *Mathematica* kernels from the kernel pool, this kernel makes calculations and then the `</msp:evaluate>` tag clears the kernel and sends it back to the pool. So, no information about the calculations remains in the kernel. This makes it impossible to use the usual *Mathematica* mechanism of assigning global values to the variables. Instead, it is necessary to declare such a variable as a session variable. The value of the session variable is stored on the server or on the user's computer.

After the user fills in the input fields with mathematical formulae, he presses the Calculate button, the page goes back to the server, the

formulae are calculated, and the results of calculation are pasted onto a new web page resent to the user.

It should be noted that it is impossible to get direct access to the computing engine of *Mathematica* with the help of *webMathematica*. The user can execute only those commands that are contained in the body of the HTML-forms. Thus, only a relatively small set of the *Mathematica* commands are available for the user on each web page. This significantly limits the flexibility of *webMathematica* and forces the instructor to invent new teaching methods to overcome these limitations.

Benefits of Teaching with *WebMathematica*

We can note the following benefits of teaching mathematical sciences, the benefits which *webMathematica* opens. In first place we have the high-quality static and dynamic two- and three-dimensional graphics. These graphics can be animated and rotated. Graphical objects constitute the core of visualization in mathematics: *an adequate image is worth a thousand words*. It is true that the formalistic tendencies of the 20th century forced mathematicians to neglect the importance of visualization. *The more formal the better* was the slogan of that century. Indeed, the figure is not a proof although sometimes the visual image makes it significantly easier to understand the rigorous proof. But limitations of the formalistic methods and appearance of computers brought the renewal of the interest to visualization both from the mathematical researches and educators.

Further, we can mention the unmistakable automation of cumbersome symbolic and numeric calculations. Unfortunately, the lecturers and students waste the great part of learning time for routine calculations. Once and once again they compute derivatives and antiderivatives, expansions and series, limits and asymptotes, and so on. These calculations take place after the above mentioned concepts were taught. So the lecturers and students are busy with the endless repetition of one and the same work. We suppose that it would be better to use the computer algebra systems or *webMathematica* to calculate saving time for better understanding.

Then, we must mention the interactivity, i.e. solving the problems for which data are not known by the authors of electronic tutorials. Finally, it can be noted that the verification of calculations performed with pencil and paper with *webMathematica* is very useful for students.

The following principle of using *webMathematica* is adopted in the interactive electronic tutorials developed by the author. It is clear that it would be pointless to write programs that completely automate calculations made by students in order to obtain the answer to the problems they solve although this is possible. Therefore, interactive tutorials should provide detailed examples of solving problems with calculations that are performed both manually and by using *webMathematica*. Students repeat these

calculations with respect to their problems either on the server or by using the traditional pen and paper technology.

Interactive electronic Calculus tutorials

Our online teaching of Calculus using *webMathematica* is based on twelve interactive electronic tutorials. The tutorials are available on the web site <http://wm.iedu.ru>. Here are their titles: Real numbers, Numeric sequences, Continuity, Derivatives, Differential and the Taylor formula, Function graphs, Antiderivatives, Integration, Numeric and Power series, Fourier series, Critical points, and Double integral.

Among the tutorials we mentioned, there are two that can be called trainers. We have in mind the tutorials Derivatives and Antiderivatives. We will describe them in the next section.

Trainers

Every student who studied Calculus should master the skills of calculating the derivatives of functions. Our experience suggests that the most difficult are the first or the first few steps of differentiation. The tutorial Derivatives helps the student to make these steps. As is well known, the art of differentiation is based on a finite number of rules such as the rule of differentiation of constants, the Leibnitz rule, and the rules of differentiation of the basic elementary functions: power functions, the exponents, the trigonometric functions and so on. The tutorial contains six examples of the derivatives calculation done from beginning to end (see Figure 1).

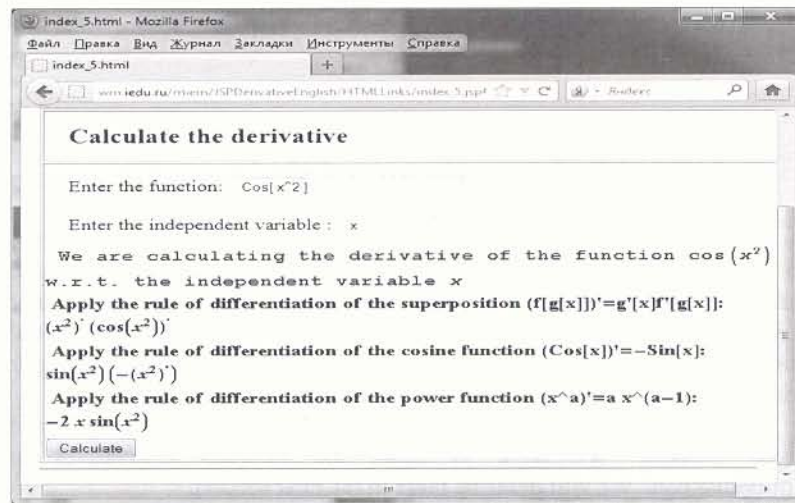


Figure 1. Calculation of the derivative of the function $\cos(x^2)$

Figure 1 shows the three steps of calculation of the first derivative of $\cos(x^2)$ which is the superposition of the cosine function and the power function. But if the student tries to solve his own problem there appear only half of the necessary steps and the student has to finish the calculation of the derivative by himself.

The tutorial Antiderivatives helps students to acquire the skills of calculating the Antiderivatives (indefinite integrals). Namely, to learn the substitution method, the integration by parts method, to become experienced in integrating the rational functions and the simplest irrational functions. The tutorial automates calculations associated with the methods so that students could better understand the essence of the methods, not wasting time on calculating. Figure 2 shows how it works.

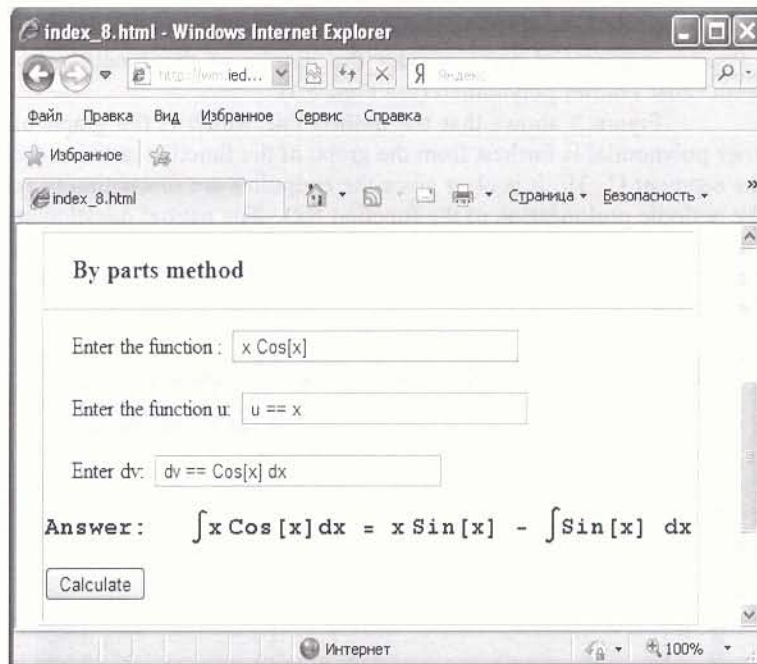


Figure 2. Integration by parts method

The integration by parts method is applied to calculation of the antiderivative of the function $x \cos(x)$. As is well known the method requires the presentation of the integrand as a product of the form $u(x) dv(x)$. If both of the factors are correctly chosen the integration is successful. Figure 2 shows this case since the integration of the function $x \cos(x)$ reduces to calculation of the antiderivative of the sine function if $u(x) = x$ and $dv(x) = \cos(x)dx$. But if the factors are chosen improperly the problem is not simplified.

Tutorials that deepen the knowledge

Automation of computations can significantly expand the content of the discipline. We will try to justify this assertion using the Fourier series as an example. When discussing this Calculus topic, the lecturer usually limits himself with the basic definitions, Fourier coefficients formulae, and the convergence issues. Calculation time savings enrich this topic with the new colors allowing students to achieve the deeper understanding of it. Not to be baseless, turn to the example.

Consider the function $f(x) = x^2 - x$ determined at the segment $[1, 2]$. Study the quality of the approximation of this function with its seventh-

order Fourier polynomial. Charge *webMathematica* with the computation of the Fourier polynomial and consider the graphs of the function and its seventh-order Fourier polynomial (see Figure 3).

Figure 3 shows that the dashed line which is the graph of the Fourier polynomial is furthest from the graph of the function at the endpoints of the segment $[1, 3]$. It is clear since the endpoints are discontinuity points of the periodic prolongation of the function $f(x)$. The natural question arises. Is it possible to find the seventh-order Fourier polynomial which approximates the function equally well at all the points of the segment? The answer is yes.

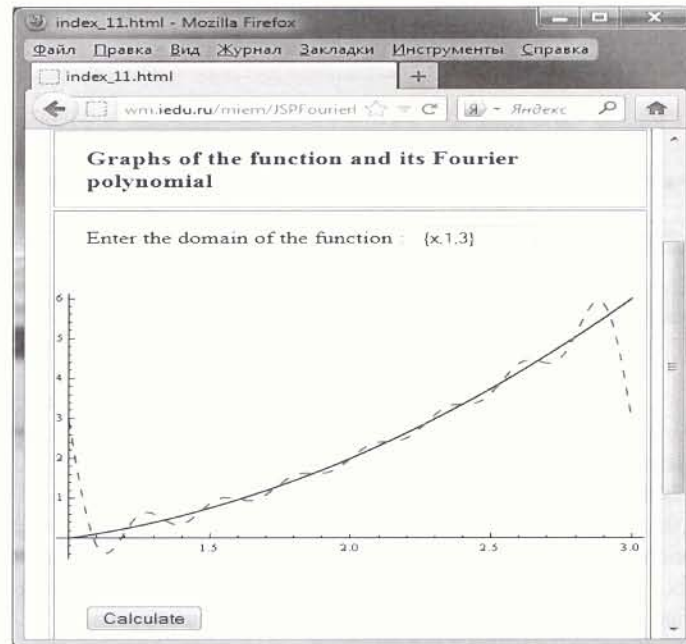


Figure 3. Graphs of the function and its Fourier polynomial

To do this, we make the even prolongation of the function $f(x)$ onto the segment $[-1, 1]$. The prolongation is given by the formula $f(x) = f(2 - x)$ for $x \in [-1, 1]$. Look at Figure 4 where the graph of the prolonged function is presented. Figure 4 shows that the periodic prolongation of the evenly prolonged function is continuous.

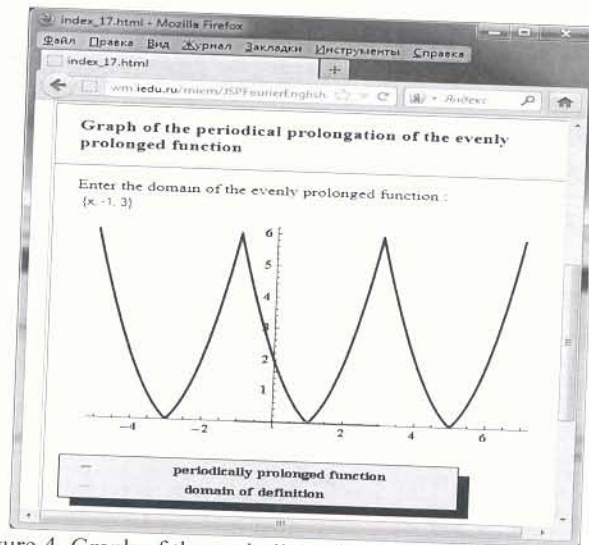


Figure 4. Graph of the periodic prolongation of the evenly prolonged function

We can therefore expect that the seventh-order Fourier polynomial of the evenly prolonged function will approximate the original function equally well at all the points of the segment $[1, 3]$. Figure 5 demonstrates it quite clearly. The three lines are depicted at this figure. The solid line is a graph of the function. The dashed line with the short segments is the graph of the Fourier polynomial of the function. We are already familiar with it. The other dashed line is a graph of the Fourier polynomial of the evenly prolonged function. We see that the last graph nearly coincides with the graph of the function at all the points of the segment.

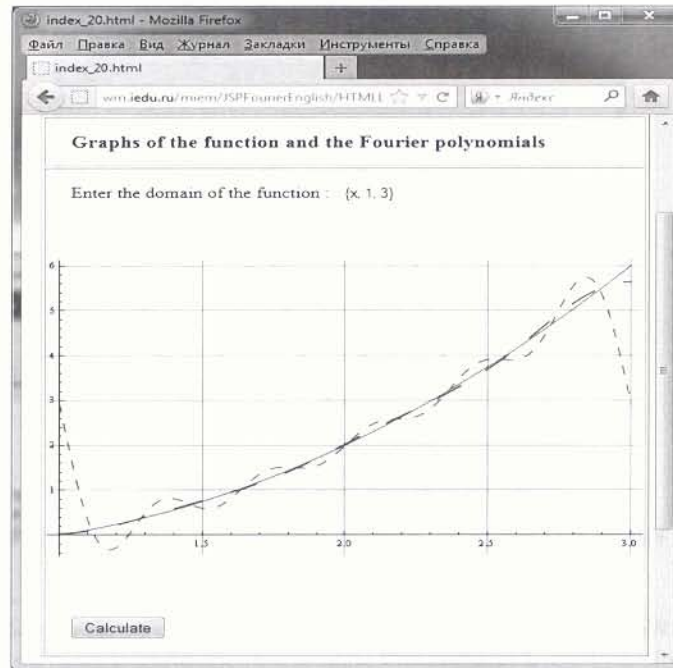


Figure 5. Graphs of the function and the Fourier polynomials

The Fourier series example demonstrates that relegating calculation to webMathematica we can not only consider the traditional problem of Fourier coefficients computation and the convergence issue, but examine how we can improve the approximation properties of the Fourier polynomials. The latter problem is of practical importance.

Conclusions

Technology always has had great influence on Calculus learning. Remember that the term Calculus takes its origin from pebbles on the sea shore where ancient Greek mathematicians were busy with counting and teaching. How far we have gone from this technology! The paper demonstrated that *webMathematica* can influence both on theory and practice of mathematics learning. It deepens the background and improves the drilling skills. But the potential of *webMathematica* is far from being exhausted. The new pedagogy of its implementation is very desirable.

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