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Asymptotic Solutions of the Navier–Stokes Equations and Systems of Stretched Vortices Filling a Three-Dimensional Volume*

V. P. Maslov ** **and A. I. Shafarevich*****

Moscow State University

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Abstract—We construct asymptotic solutions of the Navier–Stokes equations describing periodic systems of vortex filaments entirely filling a three-dimensional volume. Such solutions are related to certain topological invariants of divergence-free vector fields on the two-dimensional torus. The equations describing the evolution of of such a structure are defined on a graph which is the set of trajectories of a divergence-free field.

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1. ASYMPTOTIC SOLUTIONS OF THE NAVIER–STOKES EQUATIONS AND THE TOPOLOGICAL INVARIANTS OF HAMILTONIAN FIELDS ON THE TWO-DIMENSIONAL TORUS

1.1. The Structure of the Asymptotic Solution

The Navier–Stokes equations

$$\frac{\partial u}{\partial t} + (u, \nabla)u + \nabla P = \nu \Delta u, \qquad (\nabla, u) = 0. \tag{1}$$

describe the evolution of the velocity $u(x,t)$ and pressure fields $P(x,t)$ of an incompressible viscous liquid (u is a time-dependent vector field in $\mathbf{R}^3$ and P is a scalar function, ν is the the viscosity coefficient). The periodic systems of Taylor-scale stretched vortices filling a three-dimensional volume can be expressed as asymptotic solutions of these equations of the form

$$u(x,t) = U\biggl(\frac{S(x,t)}{h}, x, t\biggr) + hU_1 + \dots, \qquad P = \Pi\biggl(\frac{S(x,t)}{h}, x, t\biggr) + h\Pi_1 + \dots, \tag{2}$$

where $h \to 0$, $h^2 = \nu$, and $S(x,t) = (S_1, S_2)$ is the two-dimensional vector function, dS_1 and dS_2 are linearly independent everywhere in the domain under consideration, and $U(z,x,t)$ is 2π-periodic in the "rapid" variables $z = S/h$. The axis of the vortex passing at time t by the point x is directed along the vector $m(x,t)$ perpendicular to ∇S_1 and ∇S_2. Substituting (2) into (1) and equating to zero the coefficient of h^{-1} on the left-hand side of the equality leads to the following statement.

Statement 1.1. *If Eqs. (1) admit asymptotic solutions of the form (2), then the function U satisfies the equations*

$$\begin{cases} (v, \nabla_z)v + \nabla_z \Pi = 0, \\ (\nabla_z, v) = 0, \end{cases} \tag{3}$$

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**E-mail: v.p.maslov@mail.ru

***E-mail: shafarev@yahoo.com

and

$$(v, \nabla_z)w = 0, \tag{4}$$

where ∇_z *denotes differentiation with respect to the rapid variables* z *in the Euclidean metric* $g^{ij} = (\nabla S_i, \nabla S_j)$ *on the torus* $\mathbf{T}^2$, $v_j = \partial S_j/\partial t + dS_j(U)$, *and* $w = (U, m)$, *where* m *is the unit vector orthogonal to* ∇S_1 *and* ∇S_2.

Remark. The construction of asymptotic solutions of the form (2) leads to the replacement of the derivatives (a, ∇) (a is a vector field) by the derivatives $(a, \nabla) + h^{-1} dS_j(a)\partial/\partial z_j$, i.e., to the appearance of a connection in the Cartesian product $\mathbf{R}^3_x \times \mathbf{T}^2_z$, whose horizontal space is defined by the vectors $a_i\partial/\partial x_i + b_j\partial/\partial z_j$ for which $b_j = dS_j(a)$.

Remark. The equations for the two-dimensional vector field v are stationary Euler equations on the torus with the given metric; the last equation for the function w implies that this function is constant on the trajectories of the vector field v.

1.2. The Parametrization of the Solutions of the Euler Equations

The following step of the asymptotic procedure is to find equations for the parameters on which the solutions of Eqs. (3)–(4) depend. The papers [40–42, 54] dealt with the conjecture according to which the topological invariants of divergence-free vector fields with respect to area-preserving (volume-preserving, in the three-dimensional case) diffeomorphisms of the flow domain are parameters on which the solutions of the Euler equations depend. Let us relate this conjecture to our case.

Consider a divergence-free vector field $v(z)$ on the two-dimensional torus; we assume that all the equilibrium positions of this field are nondegenerate and, in addition, almost all the trajectories are closed. Denote by Γ the quotient of the torus by the trajectories of v; obviously, Γ is a graph whose vertices correspond to equilibrium positions and edges to domains smoothly stratified into closed trajectories of v. There is a natural parametrization on each edge: to a periodic trajectory γ we assign the number

$$I = \frac{1}{2\pi}\int_\gamma z_1\, dz_2$$

(the action variable). The parametrized graph Γ is an invariant of the field v with respect to area-preserving diffeomorphisms of the torus; in addition, an (also invariant) function assigning to each periodic trajectory of the field v its frequency $\omega(I)$ is defined on this graph; obviously this function is continuous on Γ and smooth on each edge. If the field v satisfies the Euler equations, then the function of the frequency is related to the Bernoulli integral $B = \frac{1}{2}v^2 + \Pi$; for our present purposes, it is convenient to use the function B instead of ω.

Conjecture. *There exists an open* (*in a suitable sense*) *subset in the set of pairs* Γ, B, *where* Γ *is a parametrized graph and* B *is a continuous function on* Γ, *smooth on the edges, such that, for each pair from this open subset, there is a smooth solution* v, Π *of the Euler equations* (3) *for which the graph* Γ *is the set of trajectories of the field* v, *and* B *is the Bernoulli integral.*

Remark. Any divergence-free field v almost all of whose trajectories on the torus are closed is Hamiltonian; i.e., it can be expressed as the skew gradient of a scalar function ψ (the current function). In the coordinates z, we have

$$v_1 = -\frac{\partial\psi}{\partial z_2}, \qquad v_2 = \frac{\partial\psi}{\partial z_1},$$

where ψ is a periodic function of z. Obviously, Γ is a Reeb graph (the set of level lines) of the function ψ.

Remark. Equation (4) means that w is a well-defined function on Γ. Thus, the complete set of "parameters" on which the solutions of Eqs. (3)–(4) depend consists of triples Γ, B, w, where Γ is a parametrized graph and B, w are continuous functions on Γ which are smooth on the edges. The functions B, w also depend on the "slow" variables x, t. Below we obtain equations for these functions determining the evolution of the vortex system.

2. THE CO-KERNEL OF THE LINEARIZED EULER EQUATIONS

Equations for the parameters arise in the analysis of the next approximation in the asymptotic procedure. Namely, after substituting (2) into (1), we equate the coefficients of h^0 on both sides of (1), obtaining equations for the field U_1, which are the linearized equations (3)–(4) with right-hand side. The required equations arise from solvability conditions for this problem, i.e., conditions for the orthogonality of the right-hand side to the co-kernel of the linearized operator (3)–(4). First, let us describe the co-kernel of the linearized Euler operator on the two-dimensional torus; it consists of divergence-free vector fields ξ satisfying the equations

$$(v, \nabla_z)\xi - \frac{\partial v^*}{\partial z}\,\xi + \nabla_z\chi = 0. \tag{5}$$

Statement 2.1. *The co-kernel of the linearized Euler operator is infinite-dimensional; namely, it contains any divergence-free field commuting with v.*

Proof. Suppose that ξ is a field commuting with v; we substitute it into (5) and apply the operator curl_z. Replacing $(v, \nabla_z)\xi$ by $(\xi, \nabla_z)v = (\partial v/\partial z)\xi$, we obtain

$$\operatorname{curl}_z\biggl((v, \nabla_z)\xi - \frac{\partial v^*}{\partial z}\xi\biggr) = \operatorname{curl}_z\biggl(\biggl(\frac{\partial v}{\partial z} - \frac{\partial v^*}{\partial z}\biggr)\xi\biggr) = (\xi, \nabla_z)\operatorname{curl}_z v + (\nabla_z, \xi)\operatorname{curl}_z v.$$

Both summands on the right-hand side are zero, i.e., the field ξ is divergence-free and the scalar function $\operatorname{curl}_z v = \Delta\psi$ is constant on the trajectories v, and hence on the trajectories of the field ξ (commuting with v) as well. □

Remark. The space in the co-kernel of the linearized Euler operator is generated by the variation of an arbitrary function B and can be interpreted as the space of functions on the graph Γ. Indeed, let us introduce the variables "action–angle" I, φ ([2, 3]) in an arbitrary domain of the torus smoothly stratified into the trajectories v. In these coordinates, the fields v and ξ are of the form $\omega(I)\partial/\partial\varphi$ and $\lambda(I)\partial/\partial\varphi$, respectively, where $\omega(I)$ and $\lambda(I)$ are the frequencies of these fields on the trajectory corresponding to the parameter I. Thus, the field ξ is given by the function $\lambda(I)$ defined on the graph Γ.

Remark. The co-kernel of the operator corresponding to Eq. (4) obviously consists of all functions constant on the trajectories of the field v, i.e., of functions on Γ. Thus, the pairs of the functions on this graph lie in the co-kernel of the linearized operator of system (3)–(4).

3. SOLVABILITY CONDITIONS FOR THE CORRECTION EQUATION

The equations of the first approximation (i.e., the equalities obtained by equating the summands of order 1 arising after substituting (2) into (1)) have the following form:

$$\begin{cases} (v_1, \nabla_z)v + (v, \nabla_z)v_1 + \nabla_z\Pi_1 = -F, \\ (\nabla_z, v_1) = -G, \\ (v, \nabla)w_1 + (v_1, \nabla)w = -H, \end{cases} \tag{6}$$

where the vector function F and the scalar functions G, H are expressed in terms of v, w, Π, and by v_1, w we denote the functions

$$v_1^j = dS_j(U_1)|_{M_t}, \qquad w_1 = (U_1, m)|_{M_t}.$$

The conditions for the orthogonality to the space (described in the previous section) in the co-kernel of the linearized Euler operator imply a relation between F, G, and H; more precisely, the following statement is valid.

Theorem 3.1. *Suppose that there exist smooth solutions of Eqs.* (6)*. Then the functions F, G and H satisfy the equalities*

$$\int_\gamma (F, v)\,d\varphi + a(I)\frac{\partial B}{\partial I} = 0, \qquad \frac{\partial a}{\partial I} + \int_\gamma G\,d\varphi = 0, \qquad \int_\gamma H\,d\varphi + a\frac{\partial w}{\partial I} = 0. \tag{7}$$

Here $B = \Pi + \frac{1}{2}v^2$ is the Bernoulli function, γ is an arbitrary closed trajectory of the field v, φ is the angular coordinate on the trajectories, and a is an auxiliary function on the graph Γ.

Proof. Let us prove the first equality in (7). To this end, let us multiply the vector equation in (6) scalarly by v, obtaining

$$(F,v)+(v,(v,\nabla_z)v^1+(v^1,\nabla_z)v+\nabla_z\pi_1)=0.$$

Transforming the last three summands, we find that

$$(F,v)+(v,\nabla_z)[(v,v^1)+\Pi_1]+(v^1,\nabla_z)B=0.$$

After integrating along the trajectory γ, we see that the second summand vanishes; in addition, since the Bernoulli integral B is constant on the trajectories, it follows that the last summand after the integration will have the form

$$\int_\gamma (v^1,\nabla_z)B\,d\varphi=\frac{\partial B}{\partial I}\int_\gamma v_I^1\,d\varphi,$$

where $v^1=v_I^1\partial/\partial I+v_\varphi^1\partial/\partial\varphi$. This implies the first equality in (7) and the equality

$$a=\int_\gamma v_I^1\,d\varphi.$$

Similarly, the second and third equalities are obtained by integrating along the trajectory γ of the second and third equations in (6). □

Remark. The conditions obtained above follow from the conditions for the orthogonality of the right-hand side of (6) to the infinite-dimensional space (described above) in the co-kernel of the linearized Euler operator (see Statement 2.1). Namely, equalities (7) are orthogonality conditions written in a specially chosen "utility basis for this space," which, roughly speaking, consists of δ-shaped fields with supports on the trajectories of the field v. It is this particular choice of the "basis" that allows us to reduce the determination of the orthogonality conditions to averaging along the trajectories.

4. EVOLUTION OF THE VORTEX SYSTEM

Equalities (7) are equations determining the evolution of the "parameters" of the vortex structure, i.e., of the functions B,w given on the graph Γ and depending on the "slow" variables x, t. These equations must be considered simultaneously with the Euler equations (3)–(4); the following statement demonstrates their structure more fully.

Theorem 4.1. *Equalities* (7) *are equivalent to the system of equations*

$$\begin{cases}\dfrac{\partial B}{\partial t}+(\langle U\rangle,\nabla)B+a\dfrac{\partial B}{\partial I}+Q(B,w)=\dfrac{\partial}{\partial I}D^2\dfrac{\partial}{\partial I}B,\\ \dfrac{\partial a}{\partial I}+\langle(\nabla,U)\rangle+E(B,w)=0,\\ \dfrac{\partial w}{\partial t}+(\langle U\rangle,\nabla)w+a\dfrac{\partial w}{\partial I}+K(B,w)=\dfrac{\partial}{\partial I}D^2\dfrac{\partial}{\partial I}w.\end{cases}\tag{8}$$

Here the angular brackets denote averaging along the trajectories of the field v,

$$D^2=g\left\langle\left(\frac{\partial z}{\partial\varphi}\right)^2\right\rangle,$$

where $g=\det(\nabla S_i,\nabla S_j)$ and the functions Q,E,K depend on B,w,I,x,t.

Proof. Consider the first of the equalities (7). The function F is the orthogonal (to the vortex axis) component of the vector field

$$\hat F=\frac{\partial U}{\partial t}+(U,\nabla U)+\nabla\Pi+\nabla S_j\frac{\partial\Pi}{\partial z_j}-(\nabla S_j,\nabla S_k)\frac{\partial^2 U}{\partial z_j\partial z_k}.$$

Let us multiply this vector scalarly by ∇S_k, obtaining

$$(\nabla S_k, \hat{F}) = \left(\frac{\partial}{\partial t} + (U, \nabla)\right)(U, \nabla S_k) - \left(U, \frac{\partial}{\partial t}\nabla S_k + \frac{\partial^2 S_k}{\partial x^2} U\right) + (\nabla S_k, \nabla \Pi) - (\nabla S_j, \nabla S_m)\frac{\partial^2}{\partial z_j \partial z_m}(U, \nabla S_k).$$

Let us replace $(U, \nabla S_k)$ by $v_k - \partial S_k/\partial t$, multiply the obtained equality by $g_{kj} v_j$, sum over j, k, and integrate over the periodic trajectory of the field v. Taking into account the equality $v^2/2 = B - \Pi$, where the Bernoulli function B is constant on the trajectories and, therefore, is factored out of the integral, and performing cumbersome calculations similar to those in [40], we obtain the first of the equalities (8). The other two equations are obtained in a similar way (by projecting the vector $\hat{F}$ onto the vortex axis m and writing out the second equality in (7) in explicit form); note that the function w, just as B, is constant on the trajectories of the Eulerian field v. □

Remark. The summands

$$\frac{\partial}{\partial I} D^2 \frac{\partial}{\partial I} B, \qquad \frac{\partial}{\partial I} D^2 \frac{\partial}{\partial I} w$$

in Eqs. (8) describe the effect of viscosity of the liquid on the vortex system under consideration. Note that the "coefficient of viscosity" D^2 depends on the unknown functions B, w. A similar phenomenon arises in the description of well-developed turbulence: dynamical equations contain the so-called turbulent viscosity depending on an unknown velocity field. Note that the expression for turbulent viscosity it is not known beforehand; it is chosen from physical considerations or from those of the maximal simplicity of the model. In our case, D^2 is some definite (although intricately defined) function of B, w.

Remark. Equations (8) with respect to the variable I are given on the graph Γ. Below we discuss conditions at the vertices of the graph that are satisfied by the functions a, B, w.

5. THE KIRCHHOFF CONDITIONS

Obviously, the functions w, B are continuous on the graph Γ. The derivatives of these functions, as well as the function a, satisfy Kirchhoff's equalities of electric network theory at each vertex. Namely, the following assertion is valid.

Statement 5.1. *In each interior vertex* (*i.e., vertex of degree greater than* 1) *of the graph* Γ*, the functions* a*,* $D^2\frac{\partial B}{\partial I}$*, and* $D^2\frac{\partial w}{\partial I}$ *satisfy the Kirchhoff conditions*

$$a_{\text{out}} = a_{\text{in}}, \qquad \left(D^2\frac{\partial B}{\partial I}\right)_{\text{out}} = \left(D^2\frac{\partial B}{\partial I}\right)_{\text{in}}, \qquad \left(D^2\frac{\partial w}{\partial I}\right)_{\text{out}} = \left(D^2\frac{\partial w}{\partial I}\right)_{\text{in}},$$

where the index "out" *denotes the sum of the limits of the corresponding function at the given vertex along the outgoing edges and the index* "in" *denotes the sum of the limits of the function along the incoming edges.*

Proof. A vertex of the graph corresponds to a separatrix of the vector field v. Consider a smooth finite vector field v_0, coinciding with v^1 in a neighborhood Q of the separatrix bounded by closed trajectories, and let us integrate its divergence over Q. We obtain

$$\int_Q (\nabla_z, v_0)\, dz = \int_{\gamma^+} (v_0, ds) - \int_{\gamma^-} (v_0, ds);$$

here the summands on the right-hand side of the equality are the flows of the field v_0 across the external (γ^+) and internal (γ^-) boundaries in the neighborhood Q. On the other hand,

$$\int_{\gamma^+} (v_0, ds) = 4\pi a_{\text{out}}, \quad \int_{\gamma^-} (v_0, ds) = 4\pi a_{\text{in}}$$

(recall that $v^1 = v^1_I \partial/\partial I + v^1_\varphi \partial/\partial\varphi$, where $a = \langle v^1_I \rangle$). Since the neighborhood Q can be taken arbitrarily small, the formulas given above imply Kirchhoff's equalities for the function a. Similarly, we can obtain Kirchhoff's equalities for the other functions as well, but, instead of (∇_z, v^1), we must integrate $\Delta_z B$ and $\Delta_z w$, respectively. □

Remark. At the vertices of degree 1 of the graph Γ (they correspond to the elliptic equilibrium positions of the field v), the function a vanishes (the proof is similar).

Remark. In the general position, all the interior vertices Γ are vertices of degree 3; in this case, the Kirchhoff conditions take the form

$$a_1 = a_2 + a_3, \quad \left(D^2\frac{\partial B}{\partial I}\right)_1 = \left(D^2\frac{\partial B}{\partial I}\right)_2 + \left(D^2\frac{\partial B}{\partial I}\right)_3, \quad \left(D^2\frac{\partial w}{\partial I}\right)_1 = \left(D^2\frac{\partial w}{\partial I}\right)_2 + \left(D^2\frac{\partial w}{\partial I}\right)_3,$$

where the index "1" denotes the limit along the incoming edge, while the indices "2" and "3" are the limits along the outgoing edges.

6. REYNOLDS STRESSES

It is well known (see, for example, [6]) that, the averaging of hydrodynamic equations is accompanied by the appearance of summands describing the effect of fluctuations on the mean field (Reynolds stresses); in our case, the role of the mean field is played by the integral of the velocity field over the two-dimensional torus of the "fast" variables z. Namely, the following statement is valid.

Theorem 6.1. *Suppose that Eqs. (8) have a smooth solution. Then the following relations hold*:

$$\frac{\partial \bar U}{\partial t} + (\bar U, \nabla \bar U) + \nabla \bar\Pi + \overline{(\Theta, \nabla\Theta) + \Theta(\nabla, \Theta)} = 0, \qquad (\nabla, \bar U) = 0. \tag{9}$$

Here $\Theta = U - \bar U$ and the bar denotes averaging over the torus $\mathbf{T}^2$:

$$\bar f = \int_{\mathbf{T}^2} f(z)\, dz_1\, dz_2.$$

Proof. Substituting the functions (2) into the Navier–Stokes equation (1), we obtain summands containing the parameter h to the zeroth power. The requirement of the equality to zero of the sum of these summands takes the form

$$\begin{aligned}\left(\frac{\partial S_j}{\partial t} + (U, \nabla S_j)\right)\frac{\partial U_1}{\partial z_j} + (U_1, S_j)\frac{\partial U}{\partial z_j} + \nabla S_j \frac{\partial \Pi_1}{\partial z_j}&\\ + \frac{\partial U}{\partial t} + (U, \nabla U) + \nabla\Pi - (\nabla S_j, \nabla S_k)\frac{\partial^2 U}{\partial z_j \partial z_k} &= 0,\end{aligned} \tag{10}$$

$$\frac{\partial}{\partial z_j}(\nabla S_j, U_1) + (\nabla, U) = 0. \tag{11}$$

In the first equation, we put $U = \bar U$ and average this equality over the torus, obtaining

$$\frac{\partial \bar U}{\partial t} + (\bar U, \nabla)\bar U + \overline{(\Theta, \nabla)\Theta} + \nabla\bar\Pi + \overline{v^1_j \frac{\partial U}{\partial z_j}} = 0. \tag{12}$$

Averaging Eq. (11), we see that

$$(\nabla, \bar U) = 0.$$

Thus, we have proved the second of the equalities (9). Further, multiplying Eq. (11) by U and integrating over the torus, we obtain

$$\overline{U(\nabla_z, v^1)} + \overline{(\nabla, U)U} = 0,$$

whence, in view of the equality $(\nabla, \bar{U}) = 0$, we have

$$\overline{v_j^1 \frac{\partial U}{\partial z_j}} = \overline{(\nabla, \Theta)\Theta}$$

(here, in the first summand, the derivatives are moved from v^1 to U). Substituting this relation into (12), we obtain

$$\frac{\partial \bar{U}}{\partial t} + (U, \nabla)\bar{U} + \overline{(\Theta, \nabla)\Theta} + \overline{(\nabla, \Theta)\Theta} + \nabla\mathcal{P} \quad \square$$

Remark. The summands

$$\kappa = \overline{(\Theta, \nabla)\Theta + \Theta(\nabla, \Theta)}$$

in Eqs. (9) exactly coincide with the Reynolds stresses (see, for example, [6, 20, 36]); expressed in coordinates, they are

$$\kappa_i = \overline{(\nabla, \Theta\Theta_i)}.$$

Remark. Equations (9) are conditions for the orthogonality of the right-hand side of (6) to a constant vector field. Such a field, naturally, commutes with any field, in particular, with v and, therefore, lies in the co-kernel of the linearized Euler operator.

Remark. In the limit of vanishing viscosity ($h \to 0$), the vortex structure described above does not become the solution of the Euler equations. The weak limit of this structure satisfies the Reynolds equations; the "envelope" of rapid oscillations (which is, obviously, nonzero) is a function of the slow variables x, t, varying according to a complicated law; to find this limit, we must find the maximum with respect to the rapid variables of the solution of Eqs. (8).

7. SCENARIO FOR THE OCCURRENCE OF TURBULENCE

"Reynolds stresses" appearing in equality (9) do not vanish as $h \to 0$. As is well known, the presence of such summands leads to an increase in the internal energy and entropy of a liquid. In the papers of one of the authors (V. P. Maslov "Gibbs paradox, liquid phase as an alternative to the Bose-condensate, and homogeneous mixtures of new ideal gases", Math. Notes **89** (3), 2011, 366–373 and V. P. Maslov "Incompressible liquid in thermodynamics, new entropy and the scenario for the occurrence of turbulence for the Navier-Stokes equation", Math. Notes **90** (6), 2011, 859–866), a new theory was developed, according to which the molecules in a gas or a liquid combine into "clusters" as soon as the entropy reaches some critical value. By our scenario, it is these clusters (moving in an incompressible liquid) that generate turbulence.

Note that, as a rule, the transition to turbulence is related to the presence of "coherent structures"; the solutions of the Navier–Stokes equations constructed in the present paper describe the evolution of one of such structure types. The phenomenon of the occurrence of a coherent structure requires a special study; possibly, it is related to the small compressibility of a real liquid and the transformation of discontinuities as compressibility tends to zero.

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