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OPTIMAL CONTROL PROBLEMS FOR WAVE EQUATIONS

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Abstract. The problem on minimizing a quadratic functional on trajectories of the wave equation is considered. We assume that the density of external forces is a control function. A control problem for a partial differential equation is reduced to a control problem for a countable system of ordinary differential equations by use of the Fourier method. The controllability problem for this countable system is considered. Conditions of the noncontrollability for some wave equations were obtained.

CONTENTS

Control problems for wave equations have wide applications in robotics, control of space constructions, etc. In particular, problems of the controllability and construction of optimal control are of important significance [8]. For many applications, it is natural to consider external effects that are pointwise or are distributed along a boundary. The problem of optimal control of an oscillating homogeneous rectangular membrane was considered in [1]. It was assumed that the force-type controls were distributed along its boundary. With the use of the Fourier method, an initial-boundary-value problem was solved and a problem of moments was formulated. For the problem of steering the system from a given initial state to a given terminal state in finite time, an effective approximate solution was proposed. In [3, 4], optimal conditions in the form of a variational inequality for the problem of pointwise control of the wave equation were obtained. In [3], the controllability conditions for the problem of minimization of the mean square deviation of the string from the equilibrium position at a fixed time moment were obtained by use of the smoothness of the solution of the corresponding boundary problem.

In the present paper, we consider the problem on minimizing the mean square deviation of the wave equations from the equilibrium position. We assume that the density of external forces is a control function. For control problems for rectangular and round membranes with δ-like density of external forces, the noncontrollability conditions are obtained. For the control problem for string oscillations, we prove that the optimal control has an infinite number of switchings on a finite time interval for the initial data from a small neighborhood of the equilibrium position.

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1. Control Problem for the Wave Equation. Fourier Method. Control Problem for Countable System of Ordinary Differential Equations

Consider the control problem for the wave equation

$$y_{tt} - a^2 \Delta y = u(t) f(x) \tag{1.1}$$

with the initial conditions

$$y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), \quad x \in \overline{\Omega}, \tag{1.2}$$

(Ω is a bounded domain in $\mathbb{R}^n$ with a piecewise smooth boundary S) and the boundary condition

$$y\Big|_S = 0, \quad t \geq 0. \tag{1.3}$$

Here $q(x,t) = u(t)f(x)$ is the density of external forces. We assume that the force profile function $f(x)$ is given and the control function $u(t)$ is bounded:

$$-1 \leq u(t) \leq 1. \tag{1.4}$$

We consider the problem of minimizing the mean-quadratic deviation from the equilibrium state

$$\int_0^\infty \int_\Omega y^2(x,t)dxdt \to \inf. \tag{1.5}$$

Using the Fourier method, we reduce the control problem for a partial differential equation to a control problem for a countable system of ordinary differential equations. We seek a solution $y(x,t)$ of problem (1.1)–(1.5) in the form of a formal series in the eigenfunctions of the operator $L = -\Delta$. The operator L with the domain

$$D_L = \left\{ v \in \overset{\circ}{H}{}^1(\Omega), \Delta v \in L_2(\Omega) \right\}$$

is a positive, self-adjoint operator (see [6]). The system of eigenfunctions $\{h_j(x)\}_{j=1}^\infty$, $h_j \in \overset{\circ}{H}{}^1(\Omega)$, of the operator L forms an orthonormal basis in $L_2(\Omega)$. Let $\{\lambda_j\}_{j=1}^\infty$ be the corresponding sequence of eigenvalues of the operator L. Each eigenvalue has a finite multiplicity, so we have

$$0 < \lambda_1 \leq \lambda_2 \leq \dots, \quad \lambda_j \to \infty, \quad j \to \infty.$$

We expand the solution $y(x,t)$ in the series with respect to the system $\{h_j(x)\}_{j=1}^\infty$:

$$y(x,t) = \sum_{j=1}^\infty s_j(t) h_j(x), \quad s_j(t) = (y, h_j)_{L_2(\Omega)} \tag{1.6}$$

where the Fourier coefficients s_j depend on time. Assume that $y_0(x)$, $y_1(x)$, $f(x) \in L_2(\Omega)$. Expand these functions in series with respect to the system $\{h_j(x)\}_{j=1}^{\infty}$:

$$f(x) = \sum_{j=1}^{\infty} c_j h_j(x), \quad c_j = (f, h_j)_{L_2(\Omega)}, \tag{1.7}$$

$$y_0(x) = \sum_{j=1}^{\infty} \alpha_j h_j(x), \quad \alpha_j = (y_0, h_j)_{L_2(\Omega)}, \tag{1.8}$$

$$y_1(x) = \sum_{j=1}^{\infty} \beta_j h_j(x), \quad \beta_j = (y_1, h_j)_{L_2(\Omega)}. \tag{1.9}$$

Substituting (1.6)–(1.7) in (1.1) yields

$$\sum_{j=1}^{\infty} \left(\ddot{s_j}(t) + a^2 \lambda_j s_j(t) - c_j u(t) \right) h_j(x) = 0.$$

By virtue of the orthogonality of the system of eigenfunctions, we obtain a countable system of ordinary differential equations

$$\ddot{s_j}(t) + a^2 \lambda_j s_j(t) = c_j u(t), \quad j = 1, 2, \dots. \tag{1.10}$$

Using (1.8)–(1.9), we obtain the initial conditions for (1.10):

$$s_j(0) = \alpha_j, \quad \dot{s}_j(0) = \beta_j. \tag{1.11}$$

The cost functional (1.5) takes the form

$$\int_0^{\infty} \sum_{j=1}^{\infty} s_j^2(t) dt \to \inf. \tag{1.12}$$

Thus, we have the problem of the minimization of (1.12) on trajectories of the control system

$$\dot{s}_j = \tau_j, \quad \dot{\tau}_j = -\omega_j^2 s_j + c_j u, \tag{1.13}$$

$$s_j(0) = \alpha_j, \quad \tau_j(0) = \beta_j, \quad j = 1, 2, \dots, \tag{1.14}$$

$$-1 \le u(t) \le 1. \tag{1.15}$$

Here $\tau_j(t) = \dot{s}_j(t)$ and $\omega_j^2 = a^2 \lambda_j$, $j = 1, \dots$. Note that (1.12) implies

$$\lim_{T \to \infty} s(t) = \lim_{T \to \infty} \tau(t) = 0.$$

Introduce the notation

$$s(t) = (s_1(t), s_2(t), \dots), \quad \tau(t) = (\tau_1(t), \tau_2(t), \dots),$$
$$\alpha = (\alpha_1, \alpha_2, \dots), \quad \beta = (\beta_1, \beta_2, \dots), \quad c = (c_1, c_2, \dots).$$

Since $f, y_0, y_1 \in L_2(\Omega)$, we have $\alpha, \beta, c \in l_2$.

Assume that $c_j \neq 0$, $j = 1, 2, \dots$, and

$$|\omega_{j+1}| - |\omega_j| \ge \delta, \quad |\omega_j| \le K \cdot j, \quad j = 1, 2, \dots,$$

where K and δ are some positive constants. It is known (see [2]) that the set of trajectories of system (1.13), for which integral (1.12) converges, is nonempty. This implies the existence of solutions of problem (1.12)–(1.15) for arbitrary initial conditions (α, β) from certain open neighborhood of the origin in the space $l_2 \times l_2$. The uniqueness of an optimal trajectory follows from the strict convexity of functional (1.12) on solutions of system (1.13).

We apply a formal generalization of the Pontryagin maximum principle to problem (1.12)–(1.15). Let us define the Pontryagin function

$$H(\psi_1, \psi_2, s, \tau, q) = \sum_{j=1}^{\infty} \left(\psi_{1j}\tau_j - \psi_{2j}\omega_j^2 s_j + \psi_{2j}c_j u - \frac{1}{2}s_j^2 \right).$$

Here we denote by ψ_i, $i = 1, 2$, the vector $(\psi_{i1}, \psi_{i2}, \ldots)$. In the space $l_2 \times l_2 \times l_2 \times l_2$, let us consider the Hamiltonian system

$$\begin{aligned} \dot{\psi}_{1j} &= \psi_{2j}\omega_j^2 + s_j, & \dot{s}_j &= \tau_j, \\ \dot{\psi}_{2j} &= -\psi_{1j}, & \dot{\tau}_j &= -\omega_j^2 s_j + c_j\hat{u}(t), \end{aligned} \qquad j = 1, 2, \ldots, \tag{1.16}$$

where

$$\hat{u}(t) = \arg\max_{u \in [-1,1]} H = \operatorname{sgn} H_1(t), \quad H_1(t) = \sum_{j=1}^{\infty} \psi_{2j}(t)c_j. \tag{1.17}$$

It was proved in [2] that the conditions of the Pontryagin maximum principle are the necessary and sufficient optimal conditions in problem (1.12)–(1.15). Namely, the following assertions hold.

Lemma 1.1 (sufficient optimal condition)**.** *Let* $(\psi_1(t), \psi_2(t), s(t), \tau(t))$ *be an arbitrary solution of system* (1.16)–(1.17) *with the boundary conditions* $s(0) = \alpha$, $\tau(0) = \beta$, $s(t) = 0$, *and* $\tau(t) = 0$. *Then* $(s(t), \tau(t))$ *is a solution of the problem* (1.12)–(1.15).

Let $u^*(t)$ be an optimal control of problem (1.12)–(1.15) and $(s^*(t), \tau^*(t))$ be an optimal trajectory.

Lemma 1.2 (necessary optimal condition)**.** *There exists a nontrivial function* $(\psi_1(\cdot), \psi_2(\cdot))$ *with values in* $l_2 \times l_2$ *satisfying the adjoint system of equations*

$$\dot{\psi}_{1j} = \psi_{2j}\omega_j^2 + s_j^*, \quad \dot{\psi}_{2j} = -\psi_{1j}, \quad j = 1, 2, \ldots,$$

such that the following maximum condition holds:

$$\max_{-1 \le u(t) \le 1} \left(\sum_{j=1}^{\infty} \psi_{2j}(t)c_j u(t) \right) = \sum_{j=1}^{\infty} \psi_{2j}(t)c_j u^*(t).$$

If $H_1(t) \neq 0$ along the trajectory, the optimal control is uniquely defined as a function of time from the maximum condition. Assume that there exists an interval (t_1, t_2) such that

$$H_1(t) \equiv 0 \quad \forall t \in (t_1, t_2).$$

We will differentiate the identity $H_1(t) \equiv 0$ by virtue of system (1.16) until a control with a nonzero coefficient occurs in the resulting expression. Assume that the series in (1.18) are convergent. We have

$$
\begin{aligned}
\frac{d}{dt}H_1(t)\Big|_{(1.16)} &= \frac{d}{dt}\left(\sum_{j=1}^{\infty}\psi_{2j}(t)c_j\right) = \left(-\sum_{j=1}^{\infty}c_j\psi_{1j}\right),\\
\frac{d^2}{dt^2}H_1(t)\Big|_{(1.16)} &= \frac{d}{dt}\left(-\sum_{j=1}^{\infty}c_j\psi_{1j}\right) = -\sum_{j=1}^{\infty}c_j\left(\psi_{2j}\omega_j^2+s_j\right),\\
\frac{d^3}{dt^3}H_1(t)\Big|_{(1.16)} &= -\frac{d}{dt}\sum_{j=1}^{\infty}c_j\left(\psi_{2j}\omega_j^2+s_j\right) = -\sum_{j=1}^{\infty}c_j\left(-\omega_j^2\psi_{1j}+\tau_j\right),\\
\frac{d^4}{dt^4}H_1(t)\Big|_{(1.16)} &= \sum_{j=1}^{\infty}c_j\omega_j^2\left(\psi_{2j}\omega_j^2+2s_j\right) - u\sum_{j=1}^{\infty}c_j^2.
\end{aligned}
\tag{1.18}
$$

Introduce the notation

$$
H_2 = -\sum_{j=1}^{\infty}c_j\psi_{1j},\quad H_3 = -\sum_{j=1}^{\infty}c_j\left(\psi_{2j}\omega_j^2+s_j\right),\quad H_4 = -\sum_{j=1}^{\infty}c_j\left(-\psi_{1j}\omega_j^2+\tau_j\right).
$$

From (1.18) it follows that

$$
H_1(t) = H_2(t) = H_3(t) = H_4(t) = 0,\quad t\in(t_1,t_2).
$$

Let us define singular solutions of (1.16)–(1.17) lying on the surface $H_1 = H_2 = H_3 = H_4 = 0$. Then the singular control is defined from (1.18):

$$
u^0(t) = \frac{\sum_{j=1}^{\infty}c_j\omega_j^2\left(\psi_{2j}\omega_j^2+2s_j\right)}{\sum_{j=1}^{\infty}c_j^2}. \tag{1.19}
$$

For problem (1.12)–(1.15), the following theorem holds.

Theorem 1.3 (see [2]). *Let $c_j \neq 0$ for all j and $\left(c_1\omega_1^3, c_2\omega_2^3,\dots\right)\in l_2$. Assume that there exist positive constants δ and K such that*

$$
|\omega_{j+1}| - |\omega_j| \geq \delta,\quad |\omega_j| \leq K\cdot j,\quad j=1,2,\dots.
$$

Then there exists an open neighborhood of the origin of the space (s,τ) such that the following statements hold for all initial conditions of problem (1.12)–(1.15) from this neighborhood.

(1) *The optimal trajectory exists and is unique in the problem (1.12)–(1.15).*

(2) *In the space (s,τ,ψ_1,ψ_2), there exists a surface Σ of codimension 4 given by the equations*

$$
\sum_{j=1}^{\infty}\psi_{2j}c_j = 0,\quad \sum_{j=1}^{\infty}c_j\psi_{1j} = 0,
$$

$$
\sum_{j=1}^{\infty}c_j\left(\psi_{2j}\omega_j^2+s_j\right) = 0,\quad \sum_{j=1}^{\infty}c_j\left(-\psi_{1j}\omega_j^2+\tau_j\right) = 0,
$$

which is filled in by singular extremals of problem (1.12)–(1.15), the control on singular extremals of which are defined by (1.19).

(3) *For all initial conditions that do not belong to the projection of the singular surface Σ on the space (s,τ), the optimal trajectories arrive at Σ in a finite time with countable many control switchings.*

2. One-Dimensional Wave Equation. Control of an Oscillating String

Consider the problem on minimizing the mean-quadratic deviation from the equilibrium position for a homogenous string with fixed ends:

$$\int_0^\infty \int_0^1 y^2(x,t)dxdt \to \inf, \tag{2.1}$$

$$y_{tt}(x,t) - a^2 y_{xx}(x,t) = u(t)f(x), \quad |u(t)| \le 1, \quad x \in (0,1), \quad t \ge 0, \tag{2.2}$$

$$y(0,t) = y(1,t) = 0, \quad t \ge 0, \tag{2.3}$$

$$y\Big|_{t=0} = y_0(x), \quad y_t\Big|_{t=0} = y_1(x), \quad x \in (0,1). \tag{2.4}$$

Let us consider the boundary-value problem for the elliptic operator $-\frac{d^2}{dx^2}$ on the interval $(0,1)$:

$$-\frac{d^2}{dx^2}v = \lambda v, \quad x \in (0,1), \quad v(0) = v(1) = 0. \tag{2.5}$$

Problem (2.5) has the eigenfunctions

$$h_j(x) = \sqrt{2}\sin(\pi j x), \quad j = 1,2,\dots, \tag{2.6}$$

and the eigenvalues

$$\lambda_j = (\pi j)^2, \quad j = 1,2,\dots \tag{2.7}$$

(see [6]).

For problem (2.1)–(2.4), the following statement was proved in [5].

Theorem 2.1. *Let the functions $f(x)$, $y_0(x)$, and $y_1(x)$ satisfy the conditions*

$$f \in KC^3(0,1), \quad f(0) = f_{xx}(0) = 0, \quad f(1) = f_{xx}(1) = 0,$$

$$y_0 \in \overset{\circ}{H}{}^2(0,1), \quad y_1 \in \overset{\circ}{H}{}^1(0,1),$$

and all Fourier coefficients of the function f with respect to system (2.6) are nonzero. Then there exist positive constants γ_0 and γ_1 such that if

$$\|y_0\|_{L_2(0,1)} < \gamma_0, \quad \|y_1\|_{L_2(0,1)} < \gamma_1,$$

then for problem (2.1)–(2.4), the following assertions hold:

(1) *there exists a unique optimal control $u^*(x,t)$:*

$$u^*(x,t) \in H^2\big((0,1)\times(0,T)\big) \quad \forall T > 0;$$

(2) *optimal trajectories are chattering trajectories; this means that an optimal control has an infinite number of switchings in a finite time interval.*

Assume that an external force is applied at the point $x_0 \in (0,1)$. Namely, let $g(x)$ be a positive, even, continuous, finite function:

$$g(-x) = g(x), \quad x \in (-\varepsilon,\varepsilon), \qquad g(-\varepsilon) = g(\varepsilon) = 0, \qquad g(x) = 0, \quad |x| > \varepsilon,$$

where $0 < \varepsilon < 1$ and

$$\int_{-\varepsilon}^{\varepsilon} g(x)dx = 1.$$

Take a point $x_0 \in (0,1)$ and a parameter ε, $0 < \varepsilon < 1$, such that

$$(x_0 - \varepsilon, x_0 + \varepsilon) \subset (0,1).$$

Consider the function $f_{x_0}(x) : [0,1] \to \mathbb{R}$ defined as follows:

$$f_{x_0}(x) = \begin{cases} g(x - x_0), & |x - x_0| \le \varepsilon, \\ 0, & |x - x_0| > \varepsilon. \end{cases} \tag{2.8}$$

Lemma 2.2. *If $x_0 \in \mathbb{Q}$, then there exists a countable sequence of trivial Fourier coefficients of the function $f_{x_0}(x)$ with respect to system* (2.6).

Proof. We denote by $A_j(x_0)$ the Fourier coefficient of the function $f_{x_0}(x)$ with respect to the system (2.6):

$$A_j(x_0) = \Big(f_{x_0}(x),\, h_j(x)\Big)_{L_2(0,1)}.$$

Then

$$A_j(x_0) = \int\limits_{x_0-\varepsilon}^{x_0+\varepsilon} \sqrt{2}\sin(\pi j x) g(x - x_0)dx = \int\limits_{-\varepsilon}^{\varepsilon} \sqrt{2}\sin(\pi j(x + x_0))g(x)dx$$

$$= \sqrt{2}\int\limits_{-\varepsilon}^{\varepsilon} \Big(\sin(\pi j x)\cos(\pi j x_0) + \cos(\pi j x)\sin(\pi j x_0)\Big)g(x)dx = I_j^{(1)}(x_0) + I_j^{(2)}(x_0),$$

where

$$I_j^{(1)}(x_0) = \sqrt{2}\cos(\pi j x_0)\int\limits_{-\varepsilon}^{\varepsilon} \sin(\pi j x)g(x)dx,$$

$$I_j^{(2)}(x_0) = \sqrt{2}\sin(\pi j x_0)\int\limits_{-\varepsilon}^{\varepsilon} \cos(\pi j x)g(x)dx.$$

Since the function $g(x)$ is even, we obtain

$$I_j^{(1)}(x_0) = 0.$$

If $x_0 \in \mathbb{Q}(x_0 \in (0,1))$, then there exists $j_0 \ge 1$, $j_0 \in \mathbb{N}$, such that $j_0 x_0 \in \mathbb{N}$. Hence $\sin(\pi j_0 x_0) = 0$ and $I_{j_0}^{(2)}(x_0) = 0$. Therefore,

$$A_{j_0}(x_0) = 0.$$

We obviously have $A_{m j_0} = 0$, $m \in \mathbb{N}$. The lemma is proved. $\square$

For $m \in \mathbb{N}$, $x_0 \in (0,1)$, $0 < \varepsilon < 1$ ($(x_0 - \varepsilon, x_0 + \varepsilon) \subset (0,1)$), we consider the function

$$f_{m,x_0}(x) = \begin{cases} E_{m,x_0}\big(\cos\frac{\pi}{2\varepsilon}(x - x_0)\big)^{2m}, & |x - x_0| \le \varepsilon, \\ 0, & |x - x_0| > \varepsilon, \end{cases} \tag{2.9}$$

where E_{m,x_0} is a positive constant such that

$$\int\limits_{x_0-\varepsilon}^{x_0+\varepsilon} f_{m,x_0}(x)dx = 1.$$

If $x_0 \in \mathbb{Q}$, then Lemma 2.2 holds for the function (2.9); therefore, system (1.13) is uncontrollable. But for the function $f_{m,x_0}(x)$, we prove the following improved result.

Lemma 2.3. *All Fourier coefficients of the function* (2.9) *with respect to system* (2.6) *are nonzero if and only if* $x_0 \notin \mathbb{Q}$ *and* $\varepsilon \notin \mathbb{Q}$.

Proof. Let $x_0 \in (0,1)$. We denote by $c_j(m,x_0)$, $j = 1,2\ldots$, the Fourier coefficient of the function $f_{m,x_0}(x)$ with respect to the system $\{\sqrt{2}\sin(\pi jx)\}_{j=1}^{\infty}$:

$$c_j(m,x_0) = \Big(f_{m,x_0}(x), \sqrt{2}\sin(\pi jx)\Big)_{L_2(0,1)} = \sqrt{2}E_{m,x_0}p_j^{(m)}(x_0),$$

where

$$p_j^{(m)}(x_0) = \int\limits_{x_0-\varepsilon}^{x_0+\varepsilon} \Big(\cos\frac{\pi}{2\varepsilon}(x-x_0)\Big)^{2m}\sin(\pi jx)dx.$$

Find E_{m,x_0} and $p_j^{(m)}(x_0)$:

$$p_j^{(m)} = \frac{A_j^{(m)}}{B_j^{(m)}}\sin(\pi j\varepsilon)\sin(\pi jx_0), \quad A_j^{(m)} = (2m)!\cdot 2^{(1-2m)},$$

$$B_j^{(m)} = \pi j\prod_{\substack{k=-m\\k\neq 0}}^{m}(k+j\varepsilon), \quad E_{m,x_0} = \frac{\sqrt{\pi}\Gamma(1+m)}{2\varepsilon\Gamma\left(\frac{1}{2}+m\right)} = \frac{(2m)!!}{2\varepsilon(2m-1)!!}.$$

Thus,

$$c_j(m,x_0) = \frac{((2m)!!)^2\sin(\pi j\varepsilon)\sin(\pi jx_0)}{\varepsilon 2^{2m}\pi j\prod\limits_{\substack{k=-m\\k\neq 0}}^{m}(k+j\varepsilon)}. \tag{2.10}$$

Therefore,

$$c_j(m,x_0) = 0 \iff \sin(\pi j\varepsilon)\sin(\pi jx_0) = 0 \iff j\varepsilon \in \mathbb{Z} \ \text{ or } \ jx_0 \in \mathbb{Z}.$$

Hence the assertion holds. □

Corollary 2.4. *Assume that* $y_0 \in \overset{\circ}{H}{}^2(0,1)$, $y_1 \in \overset{\circ}{H}{}^1(0,1)$, *and the function* $f_{m,x_0}(x)$ *is defined by* (2.9), *where* $x_0 \notin \mathbb{Q}$, $\varepsilon \notin \mathbb{Q}$, *and* $m \geq 2$. *Then for problem* (2.1)–(2.4), *Theorem* 2.1 *holds.*

3. Two-Dimensional Wave Equation. Control of Membrane Oscillations

Consider the problem of minimizing the mean-quadratic deviation from the equilibrium position for a homogenous membrane with a fixed boundary:

$$\int\limits_0^\infty \iint\limits_\Omega y^2(x,t)dxdt \to \inf,$$

$$y_{tt}(x,t) - a^2\Delta y_{tt}(x,t) = u(t)f(x), \quad |u(t)| \leq 1,$$

$$x = (x_1,x_2) \in \Omega \subset \mathbb{R}^2, \quad t \geq 0,$$

$$y\big|_{t=0} = y_0(x), \quad y_t\big|_{t=0} = y_1(x), \quad x \in \Omega,$$

$$y\big|_S = 0, \quad t \geq 0.$$

3.1. Oscillations of a rectangular membrane. Consider oscillations of a rectangle membrane with a fixed boundary:

$$y\big|_{x_1=0} = y\big|_{x_1=a} = y\big|_{x_2=0} = y\big|_{x_2=b} = 0, \quad t \geq 0.$$

The initial conditions are as follows:

$$y\big|_{t=0} = y_0(x_1, x_2), \quad y_t\big|_{t=0} = y_1(x_1, x_2), \quad x_1 \in (0, d), \quad x_2 \in (0, b).$$

Eigenfunctions and the corresponding eigenvalues of the boundary-value problem for the operator $-\Delta$ have the form

$$h_{jk}(x_1, x_2) = \frac{2}{\sqrt{db}} \sin \frac{\pi j}{d} x_1 \sin \frac{\pi k}{b} x_2, \tag{3.1}$$

$$\lambda_{jk} = \left(\frac{\pi j}{d}\right)^2 + \left(\frac{\pi k}{b}\right)^2 \tag{3.2}$$

(see [7]).

Let $\varepsilon_1, \varepsilon_2 > 0$. Consider a positive, even, continuous, finite function $g(x_1, x_2)$:

$$g(-x_1, x_2) = g(x_1, x_2), \quad g(x_1, -x_2) = g(x_1, x_2), \quad |x_1| < \varepsilon_1, \quad |x_2| < \varepsilon_2,$$

$$\int_{-\varepsilon_2}^{\varepsilon_2} \int_{-\varepsilon_1}^{\varepsilon_1} g(x_1, x_2) dx_1 dx_2 = 1,$$

and $g(x_1, x_2) = 0$ if $|x_1| \geq \varepsilon_1$ or $|x_2| \geq \varepsilon_2$. Let $\alpha \in (0, d)$, $\beta \in (0, b)$, and

$$(\alpha - \varepsilon_1, \alpha + \varepsilon_1) \subset (0, d), \quad (\beta - \varepsilon_2, \beta + \varepsilon_2) \subset (0, b).$$

Consider the following function on $(0, d) \times (0, b)$:

$$f_{\alpha,\beta}(x_1, x_2) = \begin{cases} g(x_1 - \alpha, x_2 - \beta), & |x_1 - \alpha| \leq \varepsilon_1, \quad |x_2 - \beta| \leq \varepsilon_2, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 3.1. *Let $\alpha/d \in \mathbb{Q}$ or $\beta/b \in \mathbb{Q}$. Then the sequence of the Fourier coefficients of the function $f_{\alpha,\beta}(x_1, x_2)$ with respect to system* (3.1) *contains infinitely many zeros.*

Proof. We denote by

$$A_{jk}(\alpha, \beta) = \left(f_{\alpha,\beta}(x_1, x_2), \frac{2}{\sqrt{db}} \sin \frac{\pi j}{d} x_1 \sin \frac{\pi k}{b} x_2 \right)_{L_2(0,d)\times(0,b)}$$

the Fourier coefficient of the function $f_{\alpha,\beta}(x_1, x_2)$ with respect to the system (3.1). We have

$$A_{jk}(\alpha, \beta) = \int_{\beta-\varepsilon_1}^{\beta+\varepsilon_1} \int_{\alpha-\varepsilon_1}^{\alpha+\varepsilon_1} g\left(x_1 - \alpha, x_2 - \beta\right) \frac{2}{\sqrt{db}} \sin \frac{\pi j}{d} x_1 \sin \frac{\pi k}{b} x_2 dx_1 dx_2$$

$$= \int_{-\varepsilon_2}^{+\varepsilon_2} \int_{-\varepsilon_1}^{+\varepsilon_1} g(x_1, x_2) \frac{2}{\sqrt{db}} \sin \left(\frac{\pi j}{d}(x_1 + \alpha)\right) \sin \left(\frac{\pi k}{b}(x_2 + \beta)\right) dx_1 dx_2 = J_{jk}^{(1)} + J_{jk}^{(2)},$$

where

$$J_{jk}^{(1)} = \frac{2}{\sqrt{db}} \cos\left(\frac{\pi j \alpha}{d}\right) \int\limits_{-\varepsilon_2}^{+\varepsilon_2} \int\limits_{-\varepsilon_1}^{+\varepsilon_1} g(x_1, x_2) \sin\left(\frac{\pi j}{d} x_1\right) \sin\left(\frac{\pi k}{b}(x_2 + \beta)\right) dx_1 dx_2,$$

$$J_{jk}^{(2)} = \frac{2}{\sqrt{db}} \sin\left(\frac{\pi j \alpha}{d}\right) \int\limits_{-\varepsilon_2}^{+\varepsilon_2} \int\limits_{-\varepsilon_1}^{+\varepsilon_1} g(x_1, x_2) \cos\left(\frac{\pi j}{d} x_1\right) \sin\left(\frac{\pi k}{b}(x_2 + \beta)\right) dx_1 dx_2.$$

Since the function $g(x_1, x_2)$ is symmetric, we have

$$\int\limits_{-\varepsilon_1}^{+\varepsilon_1} g(x_1, x_2) \sin\left(\frac{\pi j}{d} x_1\right) dx_1 = 0.$$

Hence $J_{jk}^{(1)} = 0$ for all $j, k = 1, 2 \ldots$.

If $\alpha/d \in \mathbb{Q}$, then there exists $j_0 \in \mathbb{N}$ such that $j_0 \frac{\alpha}{d} \in \mathbb{N}$ implies $\sin\left(\frac{\pi j_0 \alpha}{d}\right) = 0$, hence $J_{j_0 k}^{(2)} = 0$ for all $k = 1, 2 \ldots$. The case $\beta/b \in \mathbb{Q}$ is considered similarly. The lemma is proved. □

3.2. Oscillations of s round membrane. Consider the control problem of oscillations of a round membrane. The equation of oscillations in the polar coordinate system (r, φ) with the origin at the center of the membrane has the form

$$y_{tt}(r, \varphi, t) - a^2 \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial y\,(r, \varphi, t)}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 y\,(r, \varphi, t)}{\partial \varphi^2}\right) = u(t) f(r, \varphi),$$
$$0 \le r < \rho, \quad 0 \le \varphi < 2\pi, \quad t \ge 0,$$

where ρ is the radius of the membrane. We assume that the boundary edge is fixed:

$$y\big|_{r=\rho} = 0.$$

The initial conditions are as follows:

$$y\big|_{t=0} = y_0(r, \varphi), \quad y_t\big|_{t=0} = y_1(r, \varphi), \quad 0 \le r < \rho, \quad 0 \le \varphi < 2\pi.$$

The eigenfunctions of the boundary-value problem for the operator

$$-\Delta = -\left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial y}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 y}{\partial \varphi^2}\right)$$

with the domain

$$\left\{h \in C^2(K_\rho) : h(r, \varphi + 2\pi) = h(r, \varphi),\ 0 \le \varphi < 2\pi,\ 0 < r < \rho,\ h\big|_{r=\rho} = 0,\ |h(0, \varphi)| < \infty\right\}$$

have the form

$$\begin{matrix} d_{jk} J_k\left(\sqrt{\lambda_j^{(k)}}\, r\right) \sin k\varphi, \\ d_{jk} J_k\left(\sqrt{\lambda_j^{(k)}}\, r\right) \cos k\varphi, \end{matrix} \qquad k = 0, 1, \ldots, \quad j = 1, 2, \ldots, \tag{3.3}$$

(see [7]), where $J_k(z)$ is the Bessel function:

$$J_k(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+k+1)} \left(\frac{z}{2}\right)^{2n+k},$$

and

$$d_{jk}^2 = \frac{2}{\sqrt{\pi}}\rho^{-2}\left(J_k'\left(\mu_j^{(k)}\right)\right)^{-2};$$

here $\mu_j^{(k)}$ is the jth positive root of the Bessel function $J_k(z)$. The corresponding eigenvalues have the form

$$\lambda_j^{(k)} = \left(\frac{\mu_j^{(k)}}{\rho}\right)^2.$$

Consider the function

$$f(r,\varphi) = g(\varphi-\varphi_0)R(r), \tag{3.4}$$

where $0 < \varphi_0 < 2\pi$, $R(r)$ is a continuous function, and $g(\varphi)$ is a continuous finite function:

$$g(\varphi) = 0, \quad |\varphi| \geq \varepsilon, \quad \varepsilon > 0. \tag{3.5}$$

We consider two cases:

(1) $g(\varphi)$ is an even function:

$$g(-\varphi) = g(\varphi), \quad \varphi \in [-\varepsilon, \varepsilon]; \tag{3.6}$$

(2) $g(\varphi)$ is an odd function:

$$g(-\varphi) = -g(\varphi), \quad \varphi \in [-\varepsilon, \varepsilon]. \tag{3.7}$$

Let $(\varphi_0 - \varepsilon, \varphi_0 + \varepsilon) \subset (0, 2\pi)$.

Lemma 3.2. *Assume that the function f has the form* (3.4) *and the function $g(\varphi)$ satisfies* (3.5) *and* (3.6) *or* (3.5) *and* (3.7). *Then the sequence of the Fourier coefficients of f with respect to the system* (3.3) *contains infinitely many zeros.*

Proof. Consider the system of eigenfunctions of the boundary-value problem for the operator $-\Delta$:

$$\begin{cases} h_{jk}(r,\varphi) = d_{jk}J_k\left(\sqrt{\lambda_j^{(k)}}r\right)\sin k(\varphi-\varphi_0), \\ \tilde{h}_{jk}(r,\varphi) = d_{jk}J_k\left(\sqrt{\lambda_j^{(k)}}r\right)\cos k(\varphi-\varphi_0), \end{cases} \quad k = 0,1,\ldots, \quad j = 1,2,\ldots. \tag{3.8}$$

We denote by

$$C_{jk}(\varphi_0,\varepsilon) = \big(f(r,\varphi), h_{jk}(r,\varphi)\big), \quad \tilde{C}_{jk}(\varphi_0,\varepsilon) = \big(f(r,\varphi), \tilde{h}_{jk}(r,\varphi)\big)$$

the Fourier coefficients of the function f with respect to the system (3.8):

$$C_{jk}(\varphi_0,\varepsilon) = \int\limits_0^{\rho}\int\limits_{\varphi_0-\varepsilon}^{\varphi_0+\varepsilon} g(\varphi-\varphi_0)R(r)d_{jk}J_k\left(\sqrt{\lambda_j^{(k)}}r\right)\sin k(\varphi-\varphi_0)d\varphi dr$$

$$= d_{jk}\int\limits_0^{\rho}\psi_{jk}^{(1)}(\varepsilon)R(r)J_k\left(\sqrt{\lambda_j^{(k)}}r\right)dr,$$

$$\psi_{jk}^{(1)}(\varepsilon) = \int\limits_{-\varepsilon}^{+\varepsilon} g(\varphi)\sin k\varphi\, d\varphi,$$

$$\tilde{C}_{jk}(\varphi_0,\varepsilon) = \int\limits_0^{\rho}\int\limits_{\varphi_0-\varepsilon}^{\varphi_0+\varepsilon} g(\varphi-\varphi_0)R(r)d_{jk}J_k\left(\sqrt{\lambda_j^{(k)}}r\right)\cos k(\varphi-\varphi_0)\,d\varphi dr$$

$$= d_{jk}\int\limits_0^{\rho}\psi_{jk}^{(2)}(\varepsilon)R(r)J_k\left(\sqrt{\lambda_j^{(k)}}r\right)dr,$$

$$\psi_{jk}^{(2)}(\varepsilon) = \int\limits_{-\varepsilon}^{+\varepsilon} g(\varphi)\cos k\varphi\, d\varphi.$$

If the function g satisfies the conditions (3.5) and (3.6), then we have $\psi_{jk}^{(1)}(\varepsilon) = 0$ for all j and k. If the function g satisfies the conditions (3.5) and (3.7), then we have $\psi_{jk}^{(2)}(\varepsilon) = 0$ for all j and k. The lemma is proved. □

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