1 Introduction

There are many real-life situations in which people or businesses are uncertain about their coalitional payoffs. Situations with uncertain payoffs in which the agents cannot await the realizations of their coalition payoffs cannot be modeled according to classical game theory. Several models that are useful to handle uncertain payoffs exist in the game theory literature. We refer here to chance-constrained games [?], cooperative games with stochastic payoffs [?], cooperative games with random payoffs [?]. In all these models stochastics plays an important role.

Interval cooperative games are models of cooperation where only bounds for payoffs of coalitions are known with certainty. Such games are called cooperative interval games. Let $I(\mathbb{R})$ be the set of all compact intervals of the real line $\mathbb{R}$. Formally, a *cooperative interval game* in coalitional form (Alparslan Gök, Miquel and Tijs [?] is an ordered pair $\langle N, w\rangle$ where $N = \{1, 2, \ldots, n\}$ is the set of players, and $w : 2^N \to I(\mathbb{R})$ is the characteristic function such that $w(\emptyset) = [0, 0]$, where $I(\mathbb{R})$ is the set of all nonempty, compact intervals in $\mathbb{R}$. For each $S \in 2^N$, the worth set (or worth interval) $w(S)$ of the coalition S in the interval game $\langle N, w\rangle$ is of the form $[\underline{w}(S), \overline{w}(S)]$. We denote by IG_N the family of all interval games with player set N. Note that if all the worth intervals are degenerate intervals, i.e. $\underline{w}(S) = \overline{w}(S)$ for each $S \in 2^N$, then the interval game $\langle N, w\rangle$ corresponds in a natural way to the classical cooperative game $\langle N, v\rangle$ where $v(S) = \underline{w}(S)$ for all $S \in 2^N$. Some classical TU-games associated with an interval game $w \in IG_N$ will play a key role, namely the *border* games $\langle N, \underline{w}\rangle$, $\langle N, \overline{w}\rangle$ and the *length* game $\langle N, |w|\rangle$, where $|w|(S) = \overline{w}(S) - \underline{w}(S)$ for each $S \in 2^N$. Note that $\overline{w} = \underline{w} + |w|$. An interval solution concept $\mathcal{F}$ on IG_N is a map assigning to each interval game $\langle N, w\rangle \in IG_N$ a subset of $I(\mathbb{R})^N$.

Cooperative interval games are very suitable to describe real-life situations in which people or firms that consider cooperation have to sign a contract when they cannot pin down the attainable coalition payoffs, knowing with certainty only their lower and upper bounds. The contract should specify how the players' payoff shares will be obtained when the uncertainty of the worth of the grand coalition is resolved at an ex post stage.

Note that the agreement on a particular interval allocation $(I_1, I_2, \ldots, I_n)$ based on an interval solution concept merely says that the payoff x_i that player i will receive in the interim or ex post stage is in the interval I_i. This is a very weak contract to settle cooperation within the grand coalition. It can be considered as a first step of cooperation, where the following step should transform an interval allocation into a classical payoff vector. Such procedures are described in Branzei, Tijs and Alparslan Gök [?].

The study of interval game solutions begins with extentions of classical theory of cooperative game solutions to interval games. For example, we can apply some single-valued solution concept to both border games, and in the case when the solution of the upper game weakly dominates that of the lower game, the corresponding interval vector could be admitted as the interval solution, *generated* by a classical cooperative game solution. Just in this manner the interval Shapley value for convex interval games was defined in Alparslan Gök, Branzei and Tijs [?]. The same approach can be applied to the extension of set-valued solutions as well (Alparslan Gök, Branzei and Tijs [?, ?]).

Naturally, the problem of existence of such interval solution arises. In fact if for some interval game $\langle N, w\rangle$ the characteristic function values of the lower and upper games on the grand coalition coincide, i.e., $\underline{w}(N) = \overline{w}(N)$, then for any single-valued classical solution φ the (vector) inequality $\varphi(N, \underline{w}) \leq \varphi(N, \overline{w})$ is impossible, and this approach cannot be applied to the extension of the solution φ to the interval game $\langle N, w\rangle$.

It is clear that the possibility of the extension of a classical cooperative game solution to interval games depends both on the class of interval games into consideration and on monotonicity properties of the classical cooperative game solution itself. Thus, in the paper by Alparslan Gök, Branzei and Tijs [?] the class of convex interval games was introduced. It turned out that interval games was introduced. It turned out that the most known cooperative game solutions such as the core, the Shapley value, the Weber set, and are extendable to the class of convex interval games. At last< in [?] it was shown that the Dutta–Ray solution also can be extended to the interavl games.

However. both the prenucleolus and the tau-value are not aggregate monotonic on the class of convex TU games [?], [?]. Therefore, interval analogues of these solutions either should be defined by another manner, or perhaps they exist in some other class of interval games. Both approaches are used in the paper: the prenucleolus of a convex interval game is defined by lexicographical minimization of the lexmin relation on the set of joint excess vectors of lower and upper games. On the other hand, the τ-value is shown to satisfy extendability condition on a subclass of convex games – on the class of totally positive convex games.

The interval prenucleolus is determined in Section 2, and the proof of existence of the interval τ-value on the class of interval totally positive games is given in Section 3.

2 The Nucleolus for interval games

An *interval game* is an ordered triple $\langle N, (\underline{w}, \overline{w})\rangle$, where N is a finite set of players, $\underline{w}, \overline{w} : 2^N \to \mathbb{R}$ are the *lower* and *upper* characteristic functions satisfying inequalities $\underline{w}(S) \leq \overline{w}(S)$ for each coalition $S \subset N$. Cooperative game with transferable utilities (TU) $\langle N, \underline{w}\rangle, \langle N, \overline{w}\rangle$ are called, respectively,the *lower* and the *upper* games of the interval game $\langle N, (\underline{w}, \overline{w})\rangle$.

Denote be G_N an arbitrary class of TU games with the players' set N, and by IG_N denote the class of interval games with the players' set N such that for every interval game $\langle N, (\underline{w}, \overline{w})\rangle \in IG_N$ both the lower and the upper games $\langle N, \underline{w}\rangle, \langle N, \overline{w}\rangle$ belong to the class G_N.

Denote by $X(N, \underline{w}), X(N, \overline{w})$ the sets of feasible payoff vectors of the lower and the upper games, and by $Y(N, \underline{w}), Y(N, \overline{w})$ – the set of *efficient* payoff vectors:

$$X(N, \underline{w}) = \{x \in \mathbb{R}^N \mid \textstyle\sum_{i \in N} x_i \leq \underline{w}(N)\},$$
$$X(N, \overline{w}) = \{x \in \mathbb{R}^N \mid \textstyle\sum_{i \in N} x_i \leq \overline{w}(N)\},$$

$$Y(N, \underline{w}) = \{x \in X(N, \underline{w}) \mid \textstyle\sum_{i \in N} x_i = \underline{w}(N)\},$$
$$Y(N, \overline{w}) = \{x \in X(N, \overline{w}) \mid \textstyle\sum_{i \in N} x_i = \overline{w}(N)\}.$$

Definition 1 A *single-valued solution (value)* ϕ for the class IG_N of interval games is a mapping assigning to every interval game $\langle N, (\underline{w}, \overline{w})\rangle \in IG_N$ a pair of payoff vectors $\phi(N, (\underline{w}, \overline{w})) = (x, y) \in \mathbb{R}^N \times \mathbb{R}^N$, satisfying the conditions $x \in X(N, \underline{w}), y \in X(N, \overline{w})$ and $x \leq y$.

Let $\mathcal{N}$ be an arbitrary *universe* set of players. Denote by $G_{\mathcal{N}}$ the class of cooperative games

$$\mathcal{G}_{\mathcal{N}} = \bigcup_{N \subset N} \mathcal{G}_N,$$

and by $IG_{\mathcal{N}}$ – the class of interval games

$$IG_{\mathcal{N}} = \bigcup_{N \subset N} IG_N.$$

A value φ for the class $G_{\mathcal{N}}$ *generates* the interval value ϕ for the class $G_{\mathcal{N}}$, if for every interval game $\langle N, (\underline{w}, \overline{w})\rangle \in IG_{\mathcal{N}}$

$$\phi(N, (\underline{w}, \overline{w})) = (\varphi(N, \underline{w}), \varphi(N, \overline{w}). \tag{1}$$

Now consider the class G^c_N of convex TU games with the players' set N.

The class $IG^c_{\mathcal{N}}$ of *convex* interval games with the universe set of players $\mathcal{N}$ is defined as follows:

$$\langle N, (\underline{w}, \overline{w})\rangle \in IG^c_{\mathcal{N}} \iff N \subset \mathcal{N} \text{ and the games } \langle N, \overline{w}\rangle, \langle N, \underline{w}\rangle, \langle N, \overline{w} - \underline{w}\rangle \text{ are convex .}$$

For any fixed vector $x \in \mathbb{R}^N$ and coalition $S \subset N$ we denote by x_S the projection of x on the subspace $\mathbb{R}^S$, and by $x(S)$ we denote the sum $x(S) = \sum_{i \in S} x_i$.

The existence of interval values generated by a TU game value φ, i.e. satisfying inequalities (1),is equivalent to the following monotonicity property of the value φ on the class of convex TU games:

Convex monotonicity (CvM). If $\langle N, v\rangle$, $\langle N, v'\rangle$, $\langle N, v' - v\rangle$ are convex TU games such that $v'(S) \geq v(S)$ for all $S \subset N$, then $\varphi(N, v') \geq \varphi(N, v)$.

Let us compare this property with the other monotonicity cooperative game solutions: [1]:

Aggregate monotonicity (AM). If $v'(N) > v(N)$ and $v'(S) = v(S)$ for all $S \subsetneqq N$, then $\varphi(N, v') \geq \varphi(N, v)$.

Contribution monotonicity (CM). For each $i \in N$ inequalities
$v'(S \cup \{i\}) - v'(S) \geq v(S \cup \{i\}) - v(S)$ for all $S \not\ni i$ imply $\varphi_i(N, v') \geq \varphi_i(N, v)$.

Weak contribution monotonicity (WCM) (Hokari, van Gellekom [**?**]). If for all $i \in N$ and all coalitions $S \not\ni i$ the inequalities $v'(S \cup \{i\}) - v'(S) \geq v(S \cup \{i\}) - v(S)$ hold, then $\varphi(N, v') \geq \varphi(N, v)$.

Note that all these properties were defined for games with the same sets of players. It is clear that

$$CM \implies WCM \implies AM. \tag{2}$$

Let us check where convex monotonicity is placed in relations (2).

Proposition 1 *On the class of convex games G^c_N*

$$WCM \implies CvM \implies AM.$$

Proof. Let $\langle N, v\rangle, \langle N, v'\rangle, \langle N, v' - v\rangle$ be convex games such that $v'(S) \geq v(S)$ for all $S \subset N$. Then for all $i \in N$ and $S \not\ni i$

$$v'(S \cup \{i\}) - v'(S) \geq v(S \cup \{i\}) - v(S). \tag{3}$$

[1]The definitions of the properties are given for an arbitrary class of TU games, so this class is not indicated.

If a value φ on G_N^c satisfies weak contribution monotonicity, then $\varphi(N,v') \geq \varphi(N,v)$, and φ satisfies convex monotonicity.

From (2) and Proposition 1 it follows that the values satisfying contribution monotonicity generate the corresponding interval values, and the values not satisfying aggregate monotonicity do not.

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Let $\langle N,(\underline{w},\overline{w})\rangle$ be a convex interval game, $x=\mathcal{N}(N,\underline{w}), y=\mathcal{N}(N,\overline{w})$– the nucleoli of the lower and upper games respectively (recall that for convex games the nucleolus coincides with the prenucleolus). If $x \leq y$, then the pair (x,y) is called the *nucleolus* of the interval game $\langle N,(\underline{w},\overline{w})\rangle$. However the inequality $x \leq y$, in general does not hold. In fact, with each convex TU game $\langle N,v\rangle$ we can associate an interval game $\langle N,(\underline{w},\overline{w})\rangle$ such that $\underline{w}=v, \overline{w}=v'$ where $v'(N)=v(N)+a, a>0, v'(S)=v(S)$ for other $S \subset N$. The nucleolus over the class of convex games does not satisfy aggregate monotonicity (Hokari, 2000) implying that $x \not\leq y$ may happen, and the nucleolus for interval games in the definition above does not exist.

Therefore, the problem is to define the interval nucleolus $\mathcal{IN}$ for all convex interval games such that

$$\mathcal{IN}(N,(\underline{w},\overline{w}))=(x,y), x=\mathcal{N}(N,\{w\}), y=\mathcal{N}(N,\overline{w}), \text{ if } x \leq y.$$

If the last inequality does not hold, define the nucleolus by the following way.

For each payoff vectors $x \in X(N,\underline{w}), y \in X(N,\overline{w})$ define the sets

$$X^y=\{x \in X(N,\underline{w}) \,|\, x \leq y\}, \quad Y^x=\{y \in X(N,\overline{w} \,|\, y \geq x\}.$$

The nucleoli of games $\langle N,\underline{w}\rangle, \langle N,\overline{w}\rangle$ on the sets X^y, Y^x, i.e. the maximums of the lexmin relations of excess vectors

$$-e(x,\underline{w})=\{x(S)-\underline{w}(S)\}_{S\subset N}, \quad -e(y,\overline{w})=\{y(S)-\overline{w}\}_{S\subset N}$$

over the domains X^y, Y^x respectively, denote by $x^y=\mathcal{N}(N,\underline{w},X^y), y^x=\mathcal{N}(N,\overline{w},Y^x)$ respectively. They exist by the Schmeidler theorem (Schmeidler 1969). Thus, we have mappings $\underline{N}: X(N,\underline{w}) \to X(N,\underline{w}), \overline{N}: X(N,\overline{w}) \to X(N,\overline{w})$ defined by

$$\underline{N}(x)=x^{y^x}, \quad \overline{N}(y)=y^{x^y}.$$

Both mappings $\underline{N}, \overline{N}$ are continuous and, though their domains are not compact, have fixed points, The proof of their existence are similar to the proof of the existence of the prenucleolus for TU games [4].

Let x^* be a fixed point of the mapping $\underline{N}$, Then by the definition of the mappings $\underline{N}, \overline{N}$

$$x^*=\mathcal{N}(N,\underline{w},X^{y^{x^*}}), \quad y^{x^*}=\mathcal{N}(N,\overline{w},Y^{x^*}). \tag{4}$$

that means that $y^*=y^{x^*}$ is a fixed point of the mapping $\overline{N}$.

Definition 2 Given an interval game $\langle N,(\underline{w},\overline{w})\rangle$, the set of fixed points $\{(x^*,y^*)\}$ of the mappings $\underline{N}, \overline{N}$ such that $y^*=y^{x^*}$ is called the *Interval Nucleolus Set* ($\mathcal{INS}$) of this interval game.

It is clear that if $\mathcal{N}(N,\underline{w}) \le \mathcal{N}(N,\overline{w})$, then $(\mathcal{N}(N,\underline{w}),\mathcal{N}(N,\overline{w})) \in INS(N,(\underline{w},\overline{w}))$, and this definition is well-defined.

However the Interval Nucleolus Set, in general, is not single-valued. For example, $\left(\mathcal{N}(N,\underline{w}),\mathcal{N}(N,\overline{w},Y^{\mathcal{N}(N,\underline{w})}\right)$, $\left(\mathcal{N}(N,\underline{w},X^{\mathcal{N}(N,\overline{w})}),\mathcal{N}(M,\overline{w})\right) \in \mathcal{INS}(N,(\underline{w},\overline{w}))$. Thus, a more precise definition of the interval nucleoli is necessary to obtain its uniqueness.

Note that in Definition 2 for each $(x^*,y^*) \in INS(N,(\underline{w},\overline{w}))$ the determining of $\mathcal{N}(N,\underline{w},X^{y^*}),\mathcal{N}(N,\overline{w},$ was fulfiled independently, in fact, both prenucleoli were calculated under given y^*,x^* respectively. Since nucleoli express the idea of minimization of relative dissatisfaction of players and coalitions, i.e. excess vectors, in TU games, the interval nucleolous should minimize dissatisfactions at once both in lower and upper games. Thus, we can try to minimize lexicographically the vector of excesses of both games. For each pair of payoff vectors $(x,y), x \le y, x \in X(N,\underline{w}), y \in X(N,\overline{w})$ denote by $\mathcal{E}(x,y) \in \mathbb{R}^{2^{n+1}-4}$ the vector of excesses $\underline{w}(S)-x(S),\overline{w}(T)-y(T), S,T \subset N, S,T \ne N,\emptyset$, arranged in a weakly decreasing manner.

Then we come to the following

Definition 3 The *interval nucleolus ($\mathcal{IN}$)* of an interval game $\langle N,(\underline{w},\overline{w})\rangle$ is a pair (x^*,y^*) of payoff vectors $x^* \in X(N,\underline{w}), y^* \in X(N,\overline{w})$ such that $x^* \le y^*$) and

$$-\mathcal{E}(x^*,y^*) \succ_{lexmin} -\mathcal{E}(x,y) \quad \textit{for all} \quad x \in X(N,\underline{w}), y \in X(N,\overline{w}), x \le y. \tag{5}$$

Theorem 1 *There exists the unique interval nucleolus on the set of convex interval games.*

Proof. The proof of the existence of the interval nucleolus is similar to that of Schmeidler of the existence of the nucleolus and the prenucleolus for TU games.

Let (x_1,y_1) be the solution of the problem (5) without the condition $x \le y$, i.e.

$$-\mathcal{E}(x_1,y_1) \succ_{lexmin} -\mathcal{E}(x,y) \quad \text{for all} \quad x \in X(N,\underline{w}), y \in X(N,\overline{w}). \tag{6}$$

Then $x_1 = \mathcal{N}(N,\underline{w}), y_1 = \mathcal{N}(N,\overline{w})$, and there exists a solution of the problem (5). The uniqueness of the solution follows from convexity of the domain $\{(x,y)\,|\,x \in X(N,\underline{w}), y \in X(N,\overline{w}), x \le y\}$. □

Corollary 1 *The interval nucleolus belongs to the interval core.*

Proof. For all $(x,y) \in \mathcal{C}(N,(\underline{w},\overline{w}))$ we have $x \le y$ and

$$\max_{S\subsetneq N}(\underline{w}(S)-x(S)) \le 0, \quad \max_{S\subsetneq N}(\overline{w}(S)-y(S)) \le 0.$$

Therefore, for each pair of vectors $(z,u) \notin \mathcal{C}(N,\underline{w},\overline{w}), z \le u$

$$-\mathcal{E}(x,y) \succ_{lexmin} -\mathcal{E}(z,u),$$

and the maximum of the lexmin relation cannot be out of the interval core.

For the interval nucleolus an analogue of Kohlberg's characterization [3] can be proved:

For each vectors $x \in X(N,\underline{w}), y \in X(\overline{w})$ and $\alpha \in \mathbb{R}$ denote by

$$B_0(x,y) = \{i \in N\,|\,x_i = y_i\}, \underline{B}_\alpha(x) = \{S \subset N\,|\,\underline{w}(S)-x(S) \ge \alpha\} \;\; \overline{B}_\alpha(y) = \{S \subset N\,|\,\overline{w}(S)-y(S) \ge \alpha\}.$$

Theorem 2 *A pair $(x^*,y^*) = \mathcal{IN}(N,\underline{w},\overline{w})$ if and only if the collections of coalitions $\underline{B}_\alpha(x^*) \cup \{\{i\} \subset N\,|\,i \notin B_0(x^*,y^*)\}, \overline{B}_\alpha(y^*) \cup \{\{i\}\,|\,i \in B_0(x^*,y^*)\}$ are empty or weakly balanced with positive weights for coalitions from $\underline{B}_\alpha(x^*), \overline{B}_\alpha(y^*)$ for each α.*

section*Example

Consider the example of convex game $\langle N, v\rangle$ from [2] showing not aggregate monotonicity of the nucleolus.

$$\begin{aligned}
&N=\{1,2,3,4\},\ \ v(\{i\})=0 \forall i\in N,\\
&v(\{1,3\})=0,\ \ v(S)=2 \text{ for other } S, |S|=2,\\
&v(\{1,2,3\}=4,\ \ v(S)=6 \text{ for other } S, |S|=3,\\
&v(N)=10.
\end{aligned}$$

Then the nucleolus $\mathcal{N}(N,v) = (2,2,2,4)$. Let $\langle N, v'\rangle$ be the game whose characteristic function v' differs from v only on the grand coalition:

$v'(N)=12, v'(S)=v(S)$ for other $S\subset N$.

Then the nucleolus $\mathcal{N}(N,v')=(3,3,3,3)$.

Consider the interval game $\langle N,(v,v')\rangle$. Denote $x=\mathcal{N}(N,v), y=\mathcal{N}(N,v')$. Then $x\not\leq y$, and we have

$$\mathcal{N}(N,v',Y^{\mathcal{N}(N,v)})=(8/3,8/3,8/3,4), \qquad \mathcal{N}(N,v,X^{\mathcal{N}(N,v')})=(7/3,7/3,7/3,3).$$

The interval nucleolus in the definition above is equal $\mathcal{IN}(N,(v,v'))=(2,2,2,4)$. In fact, it is not difficult to show that the interval nucleolus has a form $z_a=(a,a,a,10-3a)$, and the vector of excesses $\{e(S,x)\}_{S\subset N}$ lexicographically dominates all vectors $\{e(S,z_a)\}_S$.

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