Geometric mitosis and Newton–Okounkov polytopes

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In [K], a convex-geometric algorithm was introduced for building new analogs of Gelfand–Zetlin polytopes for arbitrary reductive groups. Conjecturally, these polytopes coincide with the Newton–Okounkov polytopes of flag varieties for a geometric valuation. I outline an algorithm (*geometric mitosis*) for finding collection of faces in these polytopes that represent a given Schubert cycle. For GL_n and Gelfand–Zetlin polytopes, this algorithm reduces to a geometric version of Knutson–Miller mitosis introduced in [KST].

First, recall the *mitosis on parallelepipeds* from [KST, Section 6]. Let $\Pi(\mu,\nu)\subset\mathbb{R}^n$ be a parallelepiped given by inequalities $\mu_i\le x_i\le\nu_i$ for $i=1,\ldots,n$. For every face $\Gamma\subset\Pi(\mu,\nu)$, we now define a collection of faces $M(\Gamma)$ called the *mitosis* of Γ. Let k be the minimal number such that $\Gamma\subseteq\{x_i=\mu_i\}$ for all $i>k$ (in particular, $\Gamma\not\subseteq\{x_k=\mu_k\}$) and $\nu_i\ne\mu_i$ for at least one $i>k$. If no such k exists then $M(\Gamma)=\emptyset$. Under the isomorphism $\mathbb{R}^n\simeq\mathbb{R}^k\times\mathbb{R}^{n-k}$; $(x_1,\ldots,x_n)\mapsto(x_1,\ldots,x_k)\times(x_{k+1},\ldots,x_n)$ the face Γ gets mapped to $\Gamma'\times v$ where $v=(\mu_{k+1},\ldots,\mu_n)$ is a point and $\Gamma'\subset\mathbb{R}^k$ is a parallelepiped in $\mathbb{R}^k$. Let $E_i\subset\mathbb{R}^{n-k}$ for $i=k+1,\ldots,n$ be the segment with vertices $(\mu_{k+1},\ldots,\mu_{i-1},\mu_i,\nu_{i+1}\ldots,\nu_n)$ and $(\mu_{k+1},\ldots,\mu_{i-1},\nu_i,\ldots,\nu_n)$ (that is, the union $\bigcup_{i=k+1}^n E_i$ is a broken line that connects points $(\mu_{k+1},\ldots,\mu_n)$ and $(\nu_{k+1},\ldots,\nu_n)$). Then $M(\Gamma)$ consists of all faces $\Gamma'\times E_i$ for $k+1\le i\le n$ such that E_i is not a single point (in particular, $\dim\Delta=\dim\Gamma+1$ for any $\Delta\in M(\Gamma)$). Definition of $M(\Gamma)$ is motivated by the identity [KST, Proposition 6.8] for a Demazure-type operator applied to an exponential sum over Γ.

This geometric version of mitosis reduces easily to the combinatorial mitosis of [KnM] as follows. Every face of $\Pi(\mu,\nu)$ can be represented by a $2\times n$ table $(a_{ij})_{i=1,2,\ 1\le j\le n}$ whose cells are either filled with $+$ or empty. Namely, the face satisfies the equality $x_i=\mu_i$ or $x_i=\nu_i$ if and only if $a_{1i}=+$ or $a_{2i}=+$, respectively (in particular, if $\mu_i=\nu_i$ then the i-th column has two $+$). On the level of tables, operation M coincides the mitosis of [KnM] after reflecting our tables in a vertical line.

Example 1: If $\Pi(\mu,\nu)\subset\mathbb{R}^4$, where $\mu=(1,1,1,1)$ and $\nu=(2,2,1,2)$ (that is, $\mu_3=\nu_3$), then the vertex $\Gamma=\{x_1=\nu_1,\ x_2=\mu_2,\ x_4=\mu_4\}$ is represented by the table

	+	+	+
+		+	

The set $M(\Gamma)$ consists of two edges represented by the tables

	+	+	
+		+	

&

		+	
+		+	+

We now briefly recall a construction from [K, Section 3.3]. Let G be a connected reductive group of semisimple rank r. Let $\alpha_1,\ldots,\alpha_r$ denote simple roots of G, and $s_1,\dots,\ s_r$ the corresponding simple reflections. Fix a reduced decomposition $w_0 = s_{i_1}s_{i_2}\cdots s_{i_d}$ where w_0 is the longest element of the Weyl group of G. Let d_i be the number of s_{i_j} in this decomposition such that $i_j = i$. Consider the space

$$\mathbb{R}^d = \mathbb{R}^{d_1} \oplus \ldots \oplus \mathbb{R}^{d_r}$$

and choose coordinates $x = (x^1_1,\dots,x^1_{d_1};\dots;x^r_1,\dots,x^r_{d_r})$ with respect to this decomposition. Put $\sigma_i(x) = \sum_{j=1}^{d_i} x^i_j$. Define the projection p of $\mathbb{R}^d$ to the real span $\mathbb{R}^r$ of the weight lattice of G by the formula $p(x) = \sigma_1(x)\alpha_1 + \ldots + \sigma_r(x)\alpha_r$. Let λ be a dominant weight of G. There is an elementary convex-geometric algorithm for constructing a polytope $P_\lambda(i_1,\dots,i_d) \subset \mathbb{R}^d$ that yields the Weyl character $\chi(V_\lambda)$ of the irreducible G-module V_λ, that is,

$$\chi(V_\lambda) = \sum_{x\in P_\lambda\cap\ \mathbb{Z}^d} e^{p(x)}$$

(see Theorem [K, Theorem 3.6] for more details). The polytope P_λ can be used to extend the results of [KST] from GL_n to G since its *polytope ring* is isomorphic to the cohomology ring of the complete flag variety G/B (with rational coefficients).

In particular, if $G = SL_n$ and $w_0 = (s_1)(s_2s_1)(s_3s_2s_1)\ldots(s_{n-1}\ldots s_1)$, then we get the classical Gelfand–Zetlin polytope [K, Theorem 3.4]. However, if $G = Sp_4$ the resulting polytopes seem to be different from *string polytopes* of Berenstein–Littelmann–Zelevinsky.

Example 2: Take $G = Sp(4)$ (that is, $d = 4$ and $r = 2$) and $w_0 = s_2s_1s_2s_1$ (here α_1 denotes the shorter root and α_2 denotes the longer one). Let $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$ be a strictly dominant weight of Sp_4. Choose a point $a_\lambda = (a, b, c, d)$ such that $p(a_\lambda) = w_0\lambda = -\lambda$. Label coordinates in $\mathbb{R}^4$ by $x := x^1_1 - a$, $y := x^1_2 - b$, $z := x^2_1 - c$ and $t := x^2_2 - d$. The polytope $P_\lambda(2, 1, 2, 1)$ is given by inequalities

$$0 \le x \le \lambda_1, \quad z \le x + \lambda_2, \quad y \le 2z,$$

$$y \le z + \lambda_2, \quad 0 \le t \le \lambda_2, \quad t \le \frac{y}{2}$$

(see [K, Example 3.4]). It has 11 vertices, hence, it is not combinatorially equivalent to string polytopes for Sp_4 defined in [L].

Remark: Let $X = Sp_4/B$ be the complete flag variety for Sp_4, and L_λ the line bundle on X corresponding to the weight λ. Recently, I checked that after a unimodular change of coordinates $P_\lambda(2, 1, 2, 1)$ coincides with the Newton–Okounkov polytope $\Delta_v(X, L_\lambda)$ for the lowest term valuation v corresponding to the flag of translated Schubert varieties: $w_0X_{id} \subset s_1s_2s_1X_{s_2} \subset s_1s_2X_{s_1s_2} \subset s_1X_{s_2s_1s_2} \subset X$ (cf. [Ka, Remark 2.3]).

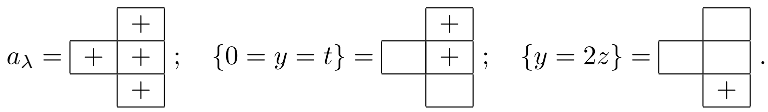
By construction, the intersection of the polytope $P_\lambda := P_\lambda(i_1,\dots,i_d)$ with $(c + \mathbb{R}^{d_i})$ is either a parallelepiped $\Pi(\mu(c), \nu(c))$ or is empty for any $c \in \mathbb{R}^d$. This property of P_λ gives r mitosis operations $M_1,\dots,\ M_r$ corresponding to parallelepipeds $P_\lambda\cap(c+\mathbb{R}^{d_1}),\dots,\ P_\lambda\cap(c+\mathbb{R}^{d_r})$, respectively. Mitosis on parallelepipeds allows us to produce collections of faces of P_λ that represent a given Schubert cycle in G/B (in the sense of [KST, Theorem 5.1]), that is, the exponential sum over the union of these faces yields the Demazure characters. The algorithm is as follows. For an element $w \in W$ of the Weyl group, denote by $[X_w] = [\overline{BwB/B}]$ the Schubert cycle corresponding to w. Let $s_{j_1} \dots s_{j_\ell}$ be a reduced decomposition of $w_0 w w_0^{-1}$ such that $(j_1, \dots, j_\ell)$ is a subword of $(i_1, \dots, i_d)$. Then $[X_w]$ is represented by the union of faces produced from a vertex of P_λ by applying successively the operations $M_{j_\ell}, \dots, M_{j_1}$. For $G = SL_n$ and $w_0 = (s_1)(s_2 s_1)(s_3 s_2 s_1) \dots (s_{n-1} \dots s_1)$, this algorithm can be described combinatorially using mitosis of Knutson–Miller on pipe-dreams (see [KST]).

For other reductive groups, one can also describe the mitosis algorithm combinatorially using suitable analogs of pipe-dreams.

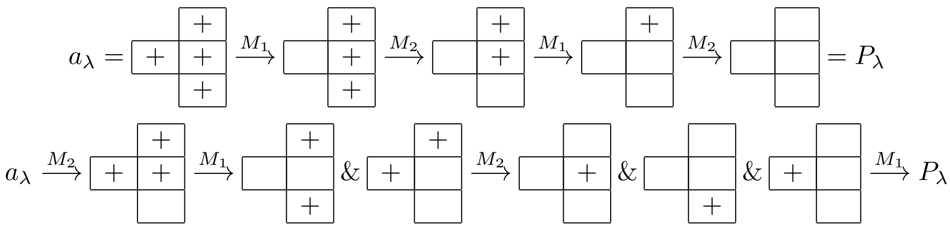
Example 3: We continue Example 2. The vertex a_λ is the intersection of 4 facets: $0 = x$, $y = 2z$, $0 = t$, $t = \frac{y}{2}$. Let us encode faces that contain a_λ by tables using the following rules:

	$+ \Longleftrightarrow 0 = t$
$+ \Longleftrightarrow 0 = x$	$+ \Longleftrightarrow t = \frac{y}{2}$
	$+ \Longleftrightarrow y = 2z$

Here are three examples:



Every face Γ defines two (possibly degenerate) rectangles $\Pi_1(\Gamma) = \Gamma \cap \{z = z_0,\ t = t_0\}$ and $\Pi_2(\Gamma) = \Gamma \cap \{x = x_0,\ y = y_0\}$ (we choose x_0, y_0, z_0 and t_0 so that the dimensions of $\Pi_1(\Gamma)$ and $\Pi_2(\Gamma)$ are maximal possible). For instance, the face $\Gamma = \{0 = y = t\}$ defines two segments. Note that $\Pi_i(\Gamma)$ is a face of the rectangle $\Pi_i(P_\lambda)$, and hence, there is a well-defined operation M_i of mitosis on parallelograms for $i = 1, 2$. It is not hard to check that in terms of tables, M_1 and M_2 do the following:



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