Russian Academy of Sciences
St. Petersburg Department
of Steklov Mathematical Institute
Euler International Mathematical Institute

International Conference

“Painlevé Equations and Related Topics”

International Conference “Painlevé Equations and Related Topics”
St. Petersburg, June, 2011

International Conference
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Euler International Mathematical Institute
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CONFERENCE TOPICS

- General ODE equations
- Painlevé equations and their generalizations
- Painlevé property
- Discrete Painlevé equations
- Properties of solutions of all mentioned above equations:
  - Asymptotic forms and asymptotic expansions
  - Connections of asymptotic forms of a solution near different points
  - Convergency and asymptotic character of a formal solution
  - New types of asymptotic forms and asymptotic expansions
  - Riemann-Hilbert problems
  - Isomonodromic deformations of linear systems
  - Symmetries and transformations of solutions
  - Algebraic solutions
- Reductions of PDE to Painlevé equations and their generalizations
- ODE systems equivalent to Painlevé equations and their generalizations
- Applications of the equations and the solutions

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CONFERENCE SCHEDULE

June 17
Arrival day.
Registration in Euler Institute 11:00 – 17:00

June 18
9:00 – 10:30 Registration, coffee
10:30 – 11:00 Opening the Conference
11:00 – 13:30 Morning session
13:30 – 15:00 Lunch
15:00 – 18:00 Afternoon session
Welcome party 18:00

June 19
10:00 – 13:30 Morning session
13:30 – 15:00 Lunch
15:00 – 18:00 Afternoon session

June 20
Petrodvorets excursion
Banquette (with dances)

June 21
10:00 – 13:30 Morning session
13:30 – 15:00 Lunch
15:00 – 18:00 Afternoon session

June 22
10:00 – 13:30 Morning session
13:30 – 15:00 Lunch
15:00 – 17:30 Afternoon session
18:00 Boat tour

June 23
10:00 – 13:30 Morning session
14:00 Closing of the Conference
Departure

The Schlesinger system and isomonodromic deformations of bundles with connections on Riemann surfaces

Dmitry V. Artamonov

Abstract. Some representation of a pair "a bundle with a connection" on a Riemann surface, based on a representation of a surface as a factor of an exterior of a unit disk, is introduced. In this representation isomonodromic deformations of bundles with logarithmic connections are described by some modification of the Schlesinger system on a Riemann sphere (typically this modification is just the ordinary Schlesinger system) plus some linear system.

Mathematics Subject Classification (2000). Primary 34G56; Secondary 32G08, 32G34.

Keywords. The Schlesinger system, isomonodromic deformations, Riemann surfaces.

1. Introduction

Consider a linear fuchsian system on a Riemann sphere: $\frac{dy}{dz} = \sum_{i=1}^{n} \frac{B_i}{z-a_i} y$. Let us change positions of singularities a_i in such a way that the monodromy is preserved and singularities do not confluence. Then the residues B_i become functions of a_i. If one considers only the so called Schlesinger deformations (for typical fuchsian systems all their deformations are Schlesinger), these function are exactly solutions of the Schlesinger system $dB_i = -\sum_{j=1,\ i\neq j}^{n} \frac{[B_i, B_j]}{a_i - a_j} d(a_i - a_j)$.

Take a Riemann surfaces of positive genus instead of the Riemann sphere. In this case it is natural to consider the deformations of bundles with connections. Also it is natural to allow to change the complex structure on the surface.

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The isomonodromic deformations on Riemann surfaces were considered for example in [1],[2],[3]. The general case was considered by Krichever [2]. His approach is based on the fact that all holomorphic vector bundles are meromorphically trivial. But the equations of isomonodromic deformations from [2] differ much from the Schlesinger system..

In the case of genus 1 another approach is known: the elliptic Schlesinger system (see [1],[3]). This system of equations describes isomonodromic deformations on a torus and it looks like an ordinary Schlesinger system.

The problem of generalizing of the Schlesinger system to higher genus surfaces was posed in [1]. It will be proved that in the case of Riemann surfaces the isomonodromic deformations can be described by the Schlesinger system plus some collection of linear equations.

2. The space of parameters $\widetilde{T}$.

Let M be a Riemann surface of genus $g > 1$. Let us fix some initial point x_0 in M.

Definition 1. The space $\widetilde{T}$ of parameters of deformations is defined as follows. Let T be the Teichmuller space with n marked points $a_1, ..., a_n$, where $a_i \neq a_j$ for $i \neq j$. Then $\widetilde{T}$ is the universal covering of T.

Definition 2. The space $\widetilde{T_1}$, on which the isomonodromic families of pairs "a bundle with a connection" are defined, is constructed as follows. Let T_1 be the Teichmuller space with $(n+1)$ marked points $z, a_1, ..., a_n$, where $a_i \neq a_j$ for $i \neq j$. Then $\widetilde{T_1}$ is the universal covering of T_1 taken by variables a_i.

There exists a mapping $\widetilde{T_1} \to \widetilde{T}$, which forgets the marked point z. Let $\tau \in \widetilde{T}$. Denote as $\widetilde{T_1} \mid_\tau$ the preimage of τ under the mapping $\widetilde{T_1} \to \widetilde{T}$.

Only the local equations of isomonodromic deformations will be written. That is why we shall sometimes omit taking if the universal covering.

The deformed objects are pairs "a bundle with a connection" (not a form = a system of linear equations = a connection in a trivial bundle as in the case of genus 0).

3. The description of bundles with connections on a Riemann surface.

The surface is represented as a factor of an exterior of a unit disk by an action of a fuchsian group. A fundamental polygon U with $4g$ vertices (g is a genus of a surface) is chosen in such a canonical way that one of its vertices is identically ∞. The singularities become points in this polygon.

The deformed objects, bundles with connections (E, ∇) on the Riemann surface, are described by the following data:

1. A form ω on a Riemann Sphere. This form is of type $\omega = (\frac{C_k}{z^k} + ...\frac{C_1}{z} + \sum_i \frac{B_i}{z-a_i})dz$. All its singularities, except 0, belong to the fundamental polygon U. The 0 is a regular singularity.
2. A collection of nondegenerate matrices $S_{x_0^1,x_0^i}$, $i = 2, ..., 4g$, where the symbols x_0^i denote the vertices of the fundamental polygon.

The form ω is constructed as follows. Take an inverse image of a pair (E, ∇) on a surface under factorization $U \to M$ and get some pair (E_U, ∇_U) on a fundamental polygon U. Continue this pair to a bundle with a connection on the Riemann sphere. Take a meromorphic trivialization of this bundle, which is holomorphic in $\overline{\mathbb{C}} \setminus \{0\}$. The connection in this bundle is defined by a form with a regular singularity in zero. This is the form ω. For typical monodromy and typical positions of singularities, the trivialization can be chosen in such a way that the form is written just as $\omega = (\frac{C_1}{z} + \sum_{i=1}^n \frac{B_i}{z-a_i})dz$.

Define matrices $S_{x_0^1 x_0^i}$. All vertices $x_0^1, ..., x_0^{4g}$ of the polygon are glued to x_0 under factorization $U \to M$. A bundle with a connection (E_U, ∇_U) is an inverse image of the bundle with connection (E, ∇). There exist operators, that identify stalks $E_{Ux_0^1}$, $E_{Ux_0^i}$ of the bundle E_U over the vertices x_0^1 and x_0^i. The matrix $S_{x_0^1,x_0^i}$ is the matrix of this operator.

The procedure of construction of the form and the matrices is noncanonical. Different forms and matrices can give equivalent bundles with connections on a surface. We shall in fact consider not bundles with connections but the data, described above.

4. The isomonodromic deformations.

In this section (E, ∇) is a logarithmic connection. Let (E^1, ∇^1) be a pair on $\widetilde{T}_1$. For $\tau \in \widetilde{T}$ denote as $(E^1, \nabla^1) |_\tau$ the restriction of a bundle and a connection (E^1, ∇^1) to the subspace $\widetilde{T}_1 |_\tau$.

Definition 3. An isomonodromic family is a pair (E^1, ∇^1) on $\widetilde{T}_1$, such that the following holds

1. the pair (E^1, ∇^1) has singularities on hypersurfaces $z = a_i$ (i.e. hypersurfaces in $\widetilde{T}_1$, which are preimages of hypersurfaces $z = a_i$ in T_1).
2. for all $\tau \in \widetilde{T}$ the pairs $(E^1, \nabla^1) |_\tau$ have the same monodromy.

Such a family describes a deformation of a pair (E, ∇) on the marked Riemann surface corresponding to τ_0, with singularities corresponding to marked points of τ_0, if $(E^1, \nabla^1) |_{\tau_0} = (E, \nabla)$.

Definition 4. A family (E^1, ∇^1) is a Schlesinger family if the following holds. When the point in the Teichmuller space is fixed in some neighborhood of a hypersurface $z = a_i$ the connection ∇^1 is given in local coordinates by a form of type $\frac{B_i}{\zeta - a_i} d(\zeta - a_i) + h(\zeta, a_i)$, here $h(\zeta, a_i)$ is some holomorphic form, B_i are holomorphic functions of a_i.

Proposition 5. *For every logarithmic initial pair* (E, ∇) *at* $t_0 \in \widetilde{T}$ *there exists a unique its continuation to a Schlesinger isomonodromic family* (E^1, ∇^1) *on* $\widetilde{T}_1$.

Now let us write the equations of Schlelinger isomonodromic deformations. For each $\tau \in T$ close to τ_0 take a pair $(E^1, \nabla^1)\mid_\tau$ and consider the corresponding form and matrices. Let us write the equations describing the resulting family of forms and matrices.

Proposition 6. *The equations of the isomonodromic deformations of* ω *are the following:*

$$dB_i = -\sum_{j=1, i\neq j}^n \frac{[B_i, B_j]}{a_i - a_j} d(a_i - a_j) + \frac{\partial C_1}{\partial a_i} da_i, \quad i = 1, ..., n$$

$$\frac{\partial C_2}{\partial a_i} - \frac{\partial C_1}{\partial a_i} a_i = -[B_i, C_1], ..., \frac{\partial C_{l+1}}{\partial a_i} - \frac{\partial C_l}{\partial a_i} a_i = -[B_i, C_l], ..., -\frac{\partial C_k}{\partial a_i} a_i = -[B_i, C_k]$$

In the typical case, when $C_l = 0$ *for* $l > 1$, *in the notations* $a_0 = 0$, $B_0 = C_1 = -\sum_{i=1}^n B_i$ *this system turns into an ordinary Schlesinger system* $dB_i = -\sum_{j=0,\ i\neq j}^n \frac{[B_i, B_j]}{a_i - a_j} d(a_i - a_j)$.

Adding equation, that describe how the matrices $S_{x_0^1, x_0^i}$ are changing, we get the main result.

Theorem 7. *In terms of the data presented in the section* 3, *the Schlesinger isomodromic deformation locally are described as follows:*

1. *The vertices* x_0^i *of the fundamental polygon are changing according to the change of a point in the Teichmuller space.*
2. *The coefficients of the form* $\omega = (\frac{C_k}{z^k} + ... + \frac{C_1}{z} + \sum_i \frac{B_i}{z - a_i})dz$ *satisfy the equations listed in the Proposition* 6. *In the typical case* $C_l = 0$, $l > 0$ *in notations* $a_0 = 0$, $B_0 = -\sum_{i=1}^n B_i$ *this is just the Schlesinger system* $dB_i = -\sum_{j=0,\ i\neq j}^n \frac{[B_i, B_j]}{a_i - a_j} d(a_i - a_j)$.
3. *The matrices* $S_{x_0^1, x_0^i}$ *are isomonodromic families of solutions of the equation* $dS_{x_0^1 z} = \omega S_{x_0^1 z}$ *at the point* $z = x_0^i$.

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Asymptotic Classification of Solutions to 3rd and 4th Order Emden-Fowler Type Differential Equations

Irina Astashova

Abstract. For $n = 3$ and 4, $k > 1$, asymptotic classification of all solutions to the equation

$$y^{(n)} + p(x)\,|y|^{k-1}y = 0$$

is obtained. For $n = 3$ existence of a solution with any prescribed domain is proved.

Mathematics Subject Classification (2000). 34C15, 34E10.

Keywords. asymptotic classification, nonlinear differential equations.

1. Introduction

For $n = 3$ and 4, $k > 1$, by using topological methods, asymptotic classification of all solutions to the equation

$$y^{(n)} + p(x)\,|y|^{k-1}y = 0 \tag{1.1}$$

is obtained. For $n = 3$ existence of a solution with any prescribed domain is proved.

Remark 1.1. A similar result for the equation (1.1) with $n = 2$ was described in [1] and [3].

Remark 1.2. Asymptotic behavior of solutions to (1.1) near the bounds of domain is obtained in [2].

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2. Results

2.1. Asymptotic classification

Theorem 2.1. *(See figure 1.) Suppose $n = 3,\ k > 1$, and $p(x)$ is a globally defined positive continuous function with positive limits p_* and p^* as $x \to \pm\infty$. Then any non-trivial non-extensible solution to (1.1) is either*

1)-2) a Kneser solution on a semi-axis $(b, +\infty)$ satisfying

$$y(x) = \pm C_{3k}(p(b))\,(x-b)^{-\frac{3}{k-1}}\,(1+o(1)) \quad \text{as } x \to b+0,$$
$$y(x) = \pm C_{3k}(p^*)\,x^{-\frac{3}{k-1}}\,(1+o(1)) \quad \text{as } x \to +\infty,$$

where

$$C_{3k}(p) = \left(\frac{3(k+2)(2k+1)}{p\,(k-1)^3}\right)^{\frac{1}{k-1}};$$

or 3) an oscillating, in both directions, solution on a semi-axis $(-\infty, b)$ satisfying, at its local extremum points,

$$|y(x')| = |x'|^{-\frac{3}{k-1}+o(1)} \quad \text{as } x' \to -\infty,$$
$$|y(x')| = |b - x'|^{-\frac{3}{k-1}+o(1)} \quad \text{as } x' \to b+0;$$

or 4)-5) an oscillating near the right boundary and non-vanishing near the left one solution on a bounded interval (b', b'') satisfying

$$y(x) = \pm C_{3k}(p(b'))\,(x-b')^{-\frac{3}{k-1}}\,(1+o(1)) \quad \text{as } x \to b'+0,$$

and, at its local extremum points, $|y(x')| = |b'' - x'|^{-\frac{3}{k-1}+o(1)}$ as $x' \to b'' - 0$.

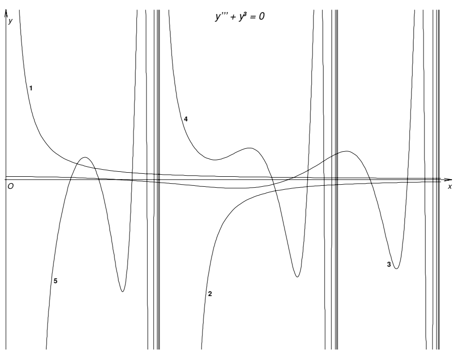

Figure 1

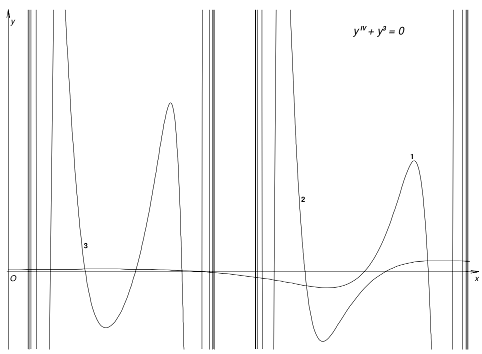

Figure 2

Theorem 2.2. *(See figure 2.) For $n = 4$, $k > 1$, and $p(x) \equiv p_0 > 0$, any non-trivial non-extensible solution to (1.1) is either*

1)-2) an oscillating, in both directions, solution on a semi-axis $(-\infty, b)$ or $(b, +\infty)$ satisfying, at its local extremum points, $|y(x)| \asymp |x-b|^{-\frac{4}{k-1}}$;

3) or an oscillating, in both directions, solution on a bounded interval (b', b'') satisfying, at its local extremum points, $|y(x)| \asymp \min\{x-b', b''-x\}^{-\frac{4}{k-1}}$.

Theorem 2.3. *(See figure 3.) For $n = 4$, $k > 1$, and $p(x) \equiv p_0 < 0$, any non-trivial non-extensible solution to (1.1) is either*

1)-4) a Kneser solution on a semi-axis $(-\infty, b)$ or $(b, +\infty)$ satisfying

$$y(x) = \pm C_{4k}(p_0)\,(x-b)^{-\frac{4}{k-1}}\,(1+o(1)) \qquad as\ \ x \to b,$$
$$y(x) = \pm C_{4k}(p_0)\,x^{-\frac{4}{k-1}}\,(1+o(1)) \qquad as\ \ x \to \pm\infty,$$

where

$$C_{4k}(p) = \left(\frac{4(k+3)(2k+2)(3k+1)}{p\,(k-1)^4}\right)^{\frac{1}{k-1}} ;$$

5) or a globally defined oscillating solution with arbitrary period;

6)-9) or a solution on a bounded interval (b', b'') satisfying, with independent signs $\pm$,

$$y(x) = \pm C_{4k}(p_0)\,(x-b')^{-\frac{4}{k-1}}\,(1+o(1)) \qquad as\ \ x \to b'+0,$$
$$y(x) = \pm C_{4k}(p_0)\,(b''-x)^{-\frac{4}{k-1}}\,(1+o(1)) \qquad as\ \ x \to b''-0;$$

10)-13) or a solution on a semi-axis $(-\infty, b)$ or $(b, +\infty)$ satisfying

$$y(x) = \pm C_{4k}(p_0))\,|x-b|^{-\frac{4}{k-1}}\,(1+o(1)) \ \ as\ x \to b$$

and oscillating at $\pm\infty$ with non-zero finite upper and lower limits.

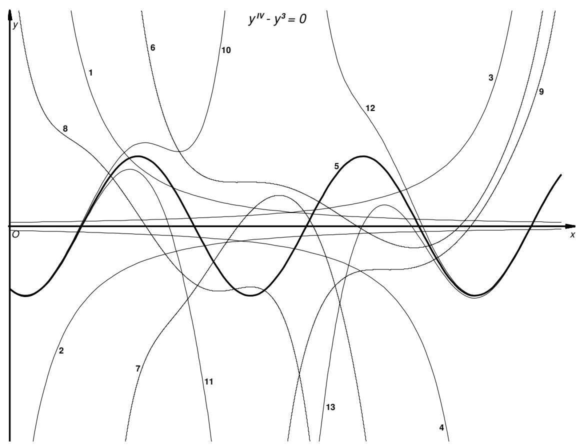

FIGURE 3

2.2. Existence of solution with prescribed domain

Definition. A solution $y(x)$ has a *resonance asymptote* $x = x^*$ if

$$\overline{\lim_{x\to x^*}}\, y(x) = +\infty, \qquad \underline{\lim_{x\to x^*}}\, y(x) = -\infty.$$

Theorem 2.4. *Suppose $n = 3,\ k > 1$, the function $p(x)$ is continuous and*

$$0 < p_* \le p(x) \le p^*. \tag{2.1}$$

Let $y(x)$ be a solution of (1.1) defined on $[x_0, x^*)$ with the resonance asymptote $x = x^*$. Then the position of the asymptote $x = x^*$ depends continuously on $y(x_0)$, $y'(x_0)$, $y''(x_0)$.

Theorem 2.5. *Suppose $n = 3,\ k > 1$, the function $p(x)$ is continuous and condition (2.1) holds. Then for any finite $x_* < x^*$ there exists a non-extensible solution $y(x)$ of (1.1) defined on (x_*, x^*) with the vertical asymptote $x = x_*$ and the resonance asymptote $x = x^*$.*

Corollary 2.6. *Suppose $n = 3,\ k > 1$, the function $p(x)$ is continuous and condition (2.1) holds. Then for any $x_* \in \mathbb{R}$ there exists a Kneser solution of (1.1) having the vertical asymptote $x = x_*$ and tending to 0 as $x \to +\infty$.*

Corollary 2.7. *Suppose $n = 3,\ k > 1$, the function $p(x)$ is continuous and condition (2.1) holds. For any $x^* \in \mathbb{R}$ there exists a non-extensible solution $y(x)$ of (1.1) having the resonance asymptote $x = x^*$ and tending to 0 as $x \to -\infty$.*

Theorem 2.8. *Suppose $n = 3,\ k > 1$, the function $p(x)$ is continuous and condition (2.1) holds. Then for any finite or infinite $x_* < x^*$ there exists a non-extensible solution $y(x)$ of (1.1) with domain (x_*, x^*).*

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On rational canonical parametrization of isomonodromic deformation equations phase space

Mikhail V. Babich

Abstract. I propose the transparent geometric model that makes possible to present the set of the rational Darboux coordinates on the phase space of the Isomonodromic Deformation equation.

In the foundation of the construction lie an observation that for the matrix from the orbit the projections of the kernel and image to the corresponding complementary coordinate subspaces are conjugated each other with respect to the canonical structure of the orbit.

Keywords. Standard Jordan form, momentum map, Lie-Poisson-Kirillov-Kostant form, rational symplectic coordinates.

Let us consider the deformation of the Fuchs equation

$$\frac{d}{dz}\Psi = \sum_{k=1}^{M} \frac{A^k}{z - z^k}\Psi; \;\; A^k \in \mathrm{sl}(N, \mathbb{C}); \;\; z, z^k \in \mathbb{C}. \tag{1}$$

It is known that the isomonodromic deformation of this equation may be associated with some Hamiltonian system defined on the space that we denote by $\mathcal{O}_{J^1} \times \mathcal{O}_{J^2} \times \cdots \times \mathcal{O}_{J^M} /\!\!/ \mathrm{SL}(N, \mathbb{C})$. This space is the quotient of the product of the several coadjoint orbits $\mathcal{O}_{J^k} := \cup_{g \in SL(N,\mathbb{C})} g J^k g^{-1} \ni A^k$ over the diagonal action of $\mathrm{SL}(N, \mathbb{C})$ intersected by the subspace $\sum_{k=1}^{M} A^k = 0$.

It is known that both the orbit $\mathcal{O}_J$ and $\mathcal{O}_{J^1} \times \mathcal{O}_{J^2} \times \cdots \times \mathcal{O}_{J^M} /\!\!/ \mathrm{SL}(N, \mathbb{C})$ are the symplectic manifolds equipped with the canonical symplectic forms ω_J and $\omega = \sum_k \omega_{J^k}$. First of all we will introduce the birational Darboux parametrization of $\mathcal{O}_J$. The birationality means that the transformations (in both directions) between the set of the matrix elements of $A \in \mathcal{O}_J$ and the set of the canonical coordinate functions in question are given by the rational functions.

Let us built a set of the canonical coordinates on the orbits $\mathcal{O}_{J^k}$. It is the heart of the construction.

We get the coordinates as the result of the following iterative process. Let us put in any order the eigenvalues of J: $\lambda_1, \lambda_2, \dots, \lambda_{n_{max}}$, where the eigenvalues corresponding to Jordan blocks are written several times, in accordance with the maximal size of the blocks corresponding to the eigenvalue. Let us define the iteration $A_{j-1} \to A_j$:

$$A_{j-1} = \begin{pmatrix} I & 0 \\ p_j & I \end{pmatrix} \begin{pmatrix} \lambda_j & q_j \\ 0 & A_j \end{pmatrix} \begin{pmatrix} I & 0 \\ p_j & I \end{pmatrix}^{-1}, \quad A_0 := A,$$

where the columns of $(Ip_j)^T$ form the basis of the eigenspace $\ker(A_{j-1} - \lambda_j I)$. It can be proved that

the process is correctly defined on the Zariski-open subset of the orbit and define the rectangular matrices p_j, q_j, which matrix elements $(p_j)_{st}, (q_j)_{ts}$ are canonically conjugated with respect to the Lie-Poisson-Kirillov-Kostant form on the orbit $\mathcal{O}_J$.

The coordinates I have presented are similar to the cylindrical coordinates introduced by Archimedes on a sphere (it is an orbit of O(3)) for the calculation of its area. Namely. The family of p-coordinates has *a group nature,* they parameterize some Abelian subgroups of triangular matrices, it is an analog of the longitude. The conjugated q-coordinates form another family. These coordinates are invariant with respect to the action of the corresponding Abelian subgroups on $\mathcal{O}_J$. They are an analog of the projection of the sphere on the diameter connecting the North and the South poles. It is *important for us* that the matrix elements depend on the coordinates from the q-family *linearly.*

Let us consider the isomonodromic deformations of the Schlesinger type:

$$dA^k + [A^k, dRR^{-1} + \sum_{i \neq k} A^i d\log(z^k - z^i)] = 0. \tag{2}$$

Matrix-function $R = R(z^1, \dots, z^M)$ is *the normalization matrix.* It is the value of the isomonodromic fundamental system of the solutions (1) in the infinity $R := \Psi(z|_{z=\infty}, z^1, \dots z^M)$.

The fixation of the normalization R is equivalent to the fixation of the section of $\mathcal{O}_{J^1} \times \cdots \times \mathcal{O}_{J^M} \to \mathcal{O}_{J^1} \times \cdots \times \mathcal{O}_{J^M} /\!\!/ \mathrm{SL}(N, \mathbb{C})$. We will fix this section explicitly in terms of the constructed coordinates on $\mathcal{O}_{J^1} \times \cdots \times \mathcal{O}_{J^M}$.

If the dimension of $\mathcal{O}_{J^1} \times \mathcal{O}_{J^2} \times \cdots \times \mathcal{O}_{J^M} /\!\!/ \mathrm{SL}(N, \mathbb{C})$ is enough high we can normalize (2) by setting equal to constant values the coordinates from the special subset $p_1^{norm}, p_2^{norm}, \dots, p_{N^2-1}^{norm}$ from the set of all p_i^k, and the special choice of the values of the coordinates q_i^{Σ} conjugated p_i^{norm} provides the necessary restriction $\sum_{k=1}^M A^k = 0$.

The dependence A^k on the coordinates q_i^k is linear, consequently *the solution of the linear equations* $\sum_{k=1}^M A^k = 0$ *is given by the rational functions* $q_i^\Sigma = q_i^\Sigma(\hat{p}, \hat{q})$, where by $\hat{p}, \hat{q}$ we denote the sets of all the coordinates without p_i^{norm} and q_i^Σ. They are removed from the sets.

The coordinates p_i^{norm} conjugated to q_i^Σ are equal to constants, consequently the symplectic form ω is equal to $\sum_i d\hat{p}_i \wedge d\hat{q}_i$.

Functions $\hat{p}, \hat{q}$ *form the set of the birational canonical coordinates.*

Example. *Let two matrices-residues in (1), say* A^M *and* A^{M-1} *have one-dimensional eigenspaces only, and at least one eigenspace of* A^{M-2}*, say* $\ker(A^{M-2} - \lambda_N^{M-2} I)$ *is one-dimensional too.* We present the matrices A^M, A^{M-1}, A^{M-2} for this case:

$$A^M = \begin{pmatrix} \lambda_1^M & q_1^\Sigma & \dots & q_{N-1}^\Sigma \\ 0 & \lambda_2^M & \dots & q_{N-1+N-2}^\Sigma \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_N^M \end{pmatrix},$$

$$A^{M-1} = \begin{pmatrix} \lambda_1^{M-1} & 0 & \dots & 0 \\ q_{N(N-1)/2+1}^\Sigma & \lambda_2^{M-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ q_{N(N-1)/2+N-1}^\Sigma & q_{N(N-1)/2+N-1+N-2}^\Sigma & \dots & \lambda_N^{M-1} \end{pmatrix},$$

$$A^{M-2} = \begin{pmatrix} ({A'}^{N-2})_{11} + q_{N(N-1)+1}^\Sigma & (A^{N-2})_{12} & \dots & (A^{N-2})_{1N} \\ (A^{N-2})_{21} & ({A'}^{N-2})_{22} + q_{N(N-1)+2}^\Sigma & \dots & (A^{N-2})_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ (A^{N-2})_{N1} & (A^{N-2})_{N2} & \dots & (A^{N-2})_{NN} \end{pmatrix}$$

It is important that the values $({A'}^{N-2})_{ii}, i = 1, \dots, N-1$ and $(A^{N-2})_{NN}$ from the diagonal *do not* depend on the variables q_j^Σ, and all the matrix elements of A^{N-2} *do not* depend on $q_j^\Sigma, j \le N(N-1)$.

We can see that the equation $\sum A^k = 0$ just fix the values of all q_i^Σ: to get $(\sum A^k)_{jj} = 0, j < N$ we fix $q_i^\Sigma, i > N(N-1)$; after that we fix $q_i^\Sigma, i \le N(N-1)$ to vanish off-diagonal entries of the equality $\sum A^k = 0$ using the freedom to set upper- and lower- triangular parts of A^M and A^{M-1}. The last term $(\sum A^k)_{NN}$ is equal to zero automatically because all the matrices are traceless.

We note that it is a case of a general position for $J^M, J^{M-1} J^{M-2}$, and we do not put any restrictions on the structure of all other matrices $J^k, k < M-2$.

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Exact simple solutions to PVI

Alexander B. Batkhin and Natalia V. Batkhina

Abstract. We use the method for computing exact solutions of the sixth Painlevé equation (PVI) in the form of finite sums of power functions with rational power exponents. This method is essentially uses algorithms of Power Geometry for computing power expansions of solutions to an ordinary differential equation and computer algebra. New exact solutions were obtained.

1. Description of the method

Knowledge of the exact solutions of the equation PVI for definite values of equation's parameters α, β, γ, δ is valuable for applications. Some elementary exact solutions in the form of algebraic functions were obtained with the help of Bäcklund transformations (see [1, § 48]). The modification of method for computing exact solutions, proposed in [2] for finding of exact solutions to N. Kovalewski equations, is used by authors. This method is based on fitting of two power series expansions near the origin and at infinity and getting conditions on the coefficients of expansions in the form of system of algebraic equations. It is possible to get the exact solution in the form of finite sum of power functions with rational degrees by solving this system of equations. The modification of the method consists in the fact that, using the form of asymptotic expansions of solutions to the equation PVI at the origin and at infinity, the general form of exact solution is composed. After substituting such solution into the equation PVI one can obtain the system of algebraic equations for unknown coefficients of exact solution and parameters of the equation. The obtained system is solved with the help of computer algebra system using Gröbner basis [3].

The choice of the form of the exact solution is determined with the help of Power Geometry algorithms (see [4, Ch. 1]). One can write the equation PVI as sum of differential monomials then compute the support and the polygon of this sum (see Fig. 1 taken from [4]). There are only four pairs of power expansions at

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the origin and at infinity that suitable for matching procedure:

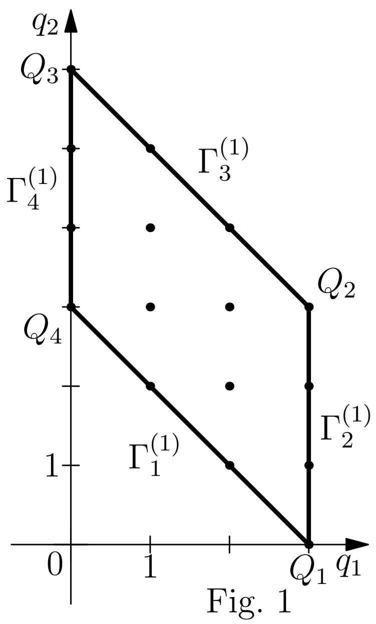

Fig. 1

1. the edge $\Gamma_4^{(1)}$ and the edge $\Gamma_3^{(1)}$;
2. the vertex Q_0 and the vertex Q_2;
3. the vertex Q_0 and the edge $\Gamma_3^{(1)}$;
4. the edge $\Gamma_4^{(1)}$ and the vertex Q_2.

The power expansion of a solution at the origin is a power series of increasing power exponents and power expansion of a solution at infinity is a power series of decreasing power exponents. Therefore the matching condition may be fulfilled if the initial expansion term of the expansion at the origin will be less then initial expansion term of the expansion at infinity. The initial expansion terms of expansions corresponding to the edges $\Gamma_4^{(1)}$ and $\Gamma_3^{(1)}$ have power exponents equal to 0 and 1. The asymptotic expansions corresponding to the edges Q_4 and Q_2 were computed in Theorem 2.2.1 and in § 5.1 [4] and are of the form

$$\mathcal{A}_0 : c_r x^r + \sum c_s x^s,\ 0 < \operatorname{Re} r < 1,\ s \in \{r + lr + m(1-r)\},$$
$$\mathcal{A}_\infty : c_r x^r + \sum c_s x^s,\ 0 < \operatorname{Re} r < 1,\ s \in \{r - lr - m(1-r)\},$$

where $l, m \geqslant 0$, $l + m > 0$, $l, m \in \mathbb{Z}$. Then the exact solutions should be sought in the form

$$y(x) = \sum_{k=0}^{l} a_k x^{k/l}, \quad l \in \mathbb{N}. \tag{1}$$

2. Results

If all $a_k - 0$ for $k = 1, \ldots, l-1$ then (1) gives the linear solution of the equation PVI. The following table summaries the families of linear solutions for different values of parameters of the equation PVI.

Sol.	Parameters				Notes
	α	β	γ	δ	
$ax+1$	$1/2$	$-\frac{(a+1)^2}{2}$	0	$a - a^2/2$	a – parameter
$ax+1-a$	γ/a^2	$-1/2$	γ	$1/2-\gamma(1-1/a)^2$	a, γ – parameters
$x+b$	$1/2$	$-\frac{(b+1)^2}{2}$	$\frac{(b-1)^2}{2}$	$1/2$	b – parameter

Here we present the list of solutions of the form (1) for $l = 2, \dots, 6$. There is an elementary solution $y = ax^n$, $n \in \mathbb{C}$ to the equation PVI for $\alpha = \beta = 0$, $\gamma = n^2/2$, $\delta = n(2-n)/2$, therefore we list only those solutions of the form (1) for which at least more then one coefficient $a_k \neq 0$.

$l = 2$	Parameters				Coefficients		
Sol. No	α	β	γ	δ	a_0	a_1	a_2
2-1	$1/2$	0	$1/8$	$-5/8$	$1/2$	$\pm 1/2$	0
2-2	$1/8$	$-9/8$	$1/2$	0	1	± 1	1
2-3	$1/2$	0	$3/8$	$9/8$	0	$\pm 1/2$	$1/2$
2-4	2	0	$3/8$	$9/8$	$1/4$	$\pm 1/2$	$1/4$

$l = 3$	Parameters				Coefficients			
Sol. No	α	β	γ	δ	a_0	a_1	a_2	a_3
3-1	$1/18$	$-2/9$	0	$1/2$	0	-1	-1	0
3-2	$2/9$	$-1/18$	$1/2$	0	0	$1/2$	$1/2$	0
3-3	$2/9$	$-1/18$	$1/2$	0	0	$-1/2(1+z_0)$	$z_0/2$	0
3-4	$1/2$	0	$2/9$	$-7/18$	$1/3$	$1/3$	$1/3$	0
3-5	$1/2$	0	$2/9$	$-7/18$	$1/3$	$-1/3(1+z_0)$	$z_0/3$	0
3-6	$1/2$	0	$8/9$	$5/18$	0	$1/3$	$1/3$	$1/3$
3-7	$1/2$	0	$8/9$	$5/18$	0	$-1/3(1+z_0)$	$z_0/3$	$1/3$
3-8	$1/18$	$-8/9$	0	$1/2$	1	1	1	1
3-9	$1/18$	$-8/9$	0	$1/2$	1	$-1-z_0$	z_0	1

where z_0, is the root of the equation $z^2 + z + 1 = 0$. Solutions 3-4 and 3-6 are connected by the Bäcklund transformation $T_4 : y_4(x; \alpha, \beta, -\delta + 1/2, -\gamma + 1/2) = xy(1/x; \alpha, \beta, \gamma, \delta)$.

$l = 4$	Parameters				Coefficients				
Sol. No	α	β	γ	δ	a_0	a_1	a_2	a_3	a_4
4-1	$9/32$	$-1/32$	$1/2$	0	0	$1/3$	$1/3$	$1/3$	0
4-2	$9/32$	$-1/32$	$1/2$	0	0	$\pm 1/3$	0	$\pm 1/3$	0
4-3	$1/32$	$-9/32$	0	$1/2$	0	$\mp i$	1	$\pm i$	0
4-4	$1/2$	0	$9/32$	$-9/32$	$1/4$	$\pm 1/4$	$1/4$	$\pm 1/4$	0
4-5	$1/2$	0	$9/32$	$-9/32$	$1/4$	$\mp i/4$	$-1/4$	$\pm i/4$	0
4-6	$1/2$	0	$25/32$	$7/32$	0	$\pm 1/4$	$1/4$	$\pm 1/4$	$1/4$

Solutions 4-4 and 4-6 are connected by the Bäcklund transformation T_4 as mentioned above.

$l = 5$	Parameters				Coefficients					
Sol. No	α	β	γ	δ	a_0	a_1	a_2	a_3	a_4	a_5
5-1	$8/25$	$-1/50$	$1/2$	0	0	$1/4$	$1/4$	$1/4$	$1/4$	0
5-2	$8/25$	$-1/50$	$1/2$	0	0	A_1	$z_0^3/4$	$z_0^2/4$	$z_0/4$	0
5-3	$1/50$	$-8/25$	0	$1/2$	0	-1	-1	-1	-1	0
5-4	$1/50$	$-8/25$	0	$1/2$	0	B_1	z_1^3	z_1^2	z_1	0
5-5	$1/2$	0	$8/25$	$-11/50$	$1/5$	$1/5$	$1/5$	$1/5$	$1/5$	0
5-6	$1/2$	0	$8/25$	$-11/50$	$1/5$	C_1	$z_0^3/5$	$z_0^2/5$	$z_0/5$	0

where z_0 is the root of the equation $z^4+z^3+z^2+z+1=0$, $A_1 = -\frac{1}{4} - a_2 - a_3 - a_4$, $C_1 = -1/5 - a_2 - a_3 - a_4$ and z_1 is the root of the equation $z^4 - z^3 + z^2 - z + 1 = 0$, $B_1 = 1 - a_2 - a_3 - a_4$.

$l = 6$	Parameters				Coefficients						
Sol. No	α	β	γ	δ	a_0	a_1	a_2	a_3	a_4	a_5	a_6
6-1	$\frac{1}{72}$	$-\frac{25}{72}$	0	$\frac{1}{2}$	0	± 1	1	± 1	1	± 1	0
6-2	$\frac{1}{72}$	$-\frac{25}{72}$	0	$\frac{1}{2}$	0	$-(1+z_0)$	$-z_0$	1	$1+z_0$	z_0	0
6-3	$\frac{1}{72}$	$-\frac{25}{72}$	0	$\frac{1}{2}$	0	$1-z_1$	z_1	-1	$1-z_1$	z_1	0
6-4	$\frac{1}{2}$	0	$\frac{25}{72}$	$-\frac{13}{72}$	$\frac{1}{6}$	$\pm 1/6$	$\frac{1}{6}$	$\pm\frac{1}{6}$	$1/6$	$\pm 1/6$	0
6-5	$\frac{1}{2}$	0	$\frac{25}{72}$	$-\frac{13}{72}$	$\frac{1}{6}$	$-\frac{1+z_0}{6}$	$\frac{z_0}{6}$	$\frac{1}{6}$	$-\frac{1+z_0}{6}$	$z_0/6$	0
6-6	$\frac{1}{2}$	0	$\frac{25}{72}$	$-\frac{13}{72}$	$\frac{1}{6}$	$\frac{1-z_1}{6}$	$-\frac{z_1}{6}$	$-\frac{1}{6}$	$\frac{z_1-1}{6}$	$z_1/6$	0

where z_0 is the root of the equation $z^2 + z + 1$, z_1 is the root of the equation $z^2 - z + 1$.

Solution 3-1 can be obtained from elementary solutions $y = x^{4/3}$ or $y = x^{1/3}$ by the Bäcklund transformations $T_8 : y_8(x; -\delta + 1/2, -\gamma, -\beta, -\alpha + 1/2) = y(x; \alpha, \beta, \gamma, \delta)$ and $T_9 : y_9(x; \gamma, \delta - 1/2, \alpha, \beta + 1/2) = y(x; \alpha, \beta, \gamma, \delta)$, correspondingly. The same transformations connect the solution 5-3 with elementary solutions $y = x^{6/5}$ and $y = x^{1/5}$, correspondingly. It is possible that some other solutions among mentioned above may be obtained from the known elementary solutions of the equation PVI by the Bäcklund transformations but the authors does not known anything about it.

3. Final remarks

Solutions 3-4, 3-8, 4-4 and 5-5 can be written as the sum of finite geometrical progression and then can be generalized for the case of any power exponents. The direct substitution shows that the function $y = b(x-1)/(x^b - 1)$ is the exact solution of the equation PVI for $\alpha = 1/2$, $\beta = 0$, $\gamma = (1-b)^2/2$, $\delta = -(2+b)^2/2$.

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On the Malgrange isomonodromic deformations of non-resonant meromorphic connections

Yuliya P. Bibilo and Renat R. Gontsov

Abstract. Movable singularities of the equations governing the Malgrange isomonodromic deformation of a non-resonant rank 2 meromorphic connection are studied: we describe the theta-divisor of the deformation and estimate orders of movable poles of the equations in the case of irreducible monodromy.

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Keywords. Holomorphic vector bundle, meromorphic connection, irregular singularity, isomonodromic deformation, theta-divisor, Painlevé property.

1. Introduction

Consider a *meromorphic linear system* on the Riemann sphere $\overline{\mathbb{C}}$, i. e., a system of p linear ordinary differential equations with singularities $a_1^0, \ldots, a_n^0 \in \mathbb{C}$ and possibly ∞. By a conformal mapping one can always arrange that all the singularities are in the complex plane only. This means that one can reduce the system to the form

$$\frac{dy}{dz} = B(z)\, y, \qquad B(z) = \sum_{i=1}^{n} \sum_{j=1}^{r_i+1} \frac{B_{ij}^0}{(z - a_i^0)^j}, \tag{1.1}$$

where $y(z) \in \mathbb{C}^p$, B_{ij}^0 are $(p \times p)$-matrices and $\sum_{i=1}^n B_{i1}^0 = 0$, to ensure that ∞ is not a singular point.

The non-negative integers $r_1, \ldots, r_n$ are called the *Poincaré ranks* of the singularities $a_1^0, \ldots, a_n^0$ respectively. One can assume that the Poincaré ranks $r_1, \ldots, r_m$ are positive and $r_{m+1} = \ldots = r_n = 0$ (that is, the singular points $a_{m+1}^0, \ldots, a_n^0$ are *Fuchsian*) for some $0 \leqslant m \leqslant n$.

We consider the *non-resonant* case. This means that the leading term B_{i,r_i+1}^0 of each non-Fuchsian singularity a_i^0, $i = 1, \ldots, m$, has p distinct eigenvalues.

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The system (1.1) can be thought of as a meromorphic connection ∇^0 (more precisely, as an equation for horizontal sections with respect to this connection) on a holomorphically trivial vector bundle E^0 of rank p over $\overline{\mathbb{C}}$. In a neighbourhood of each (non-resonant) irregular singularity a_i^0 the local connection form $\omega^0 = -B(z)dz$ of ∇^0 is formally equivalent to the 1-form

$$\omega_{\Lambda_i^0} = \sum_{j=1}^{r_i+1} \frac{\Lambda_{ij}^0}{(z-a_i^0)^j}\, dz,$$

where $\Lambda_{i1}^0, \dots, \Lambda_{i,r_i+1}^0$ are diagonal matrices and the leading term Λ_{i,r_i+1}^0 are conjugated to B_{i,r_i+1}^0.

One should note that formally equivalent systems in a neighbourhood $O_{a_i^0}$ of an irregular singularity a_i^0 are not necessary holomorphically or meromorphically equivalent. The system (1.1) has in $O_{a_i^0}$ a formal fundamental matrix of the form

$$\widehat{Y}(z) = \widehat{F}(z)(z-a_i^0)^{-\Lambda_{i1}^0} e^{Q(z)}, \qquad Q(z) = \sum_{j=1}^{r_i} \frac{\Lambda_{ij}^0}{j}\,(z-a_i^0)^{-j},$$

where $\widehat{F}(z)$ is an invertible formal Taylor series in $(z-a_i^0)$. One can cover $O_{a_i^0}$ by a set of sufficiently small sectors $S_1, \dots, S_N$ such that in each S_k there exists a unique fundamental matrix $Y_k(z) = F_k(z)(z-a_i^0)^{-\Lambda_{i1}^0} e^{Q(z)}$ of the system with $F_k(z)$ having $\widehat{F}(z)$ as asymptotic series in S_k. In every intersection $S_k \cap S_{k+1}$ the fundamental matrices $Y_k(z)$, $Y_{k+1}(z)$ are connected by a constant matrix C_k: $Y_{k+1}(z) = Y_k(z)C_k$, which is called a *Stokes' matrix.* If a_i^0 is a non-resonant singularity, then two formally equivalent systems are holomorphically equivalent in $O_{a_i^0}$ if and only if they have the same sets of Stokes' matrices.

Further we will focus on the deformations of the system (1.1) (of the pair (E^0, ∇^0)) that allow the local formal equivalence class

$$\omega_{\Lambda_i} = \sum_{j=2}^{r_i+1} \frac{\Lambda_{ij}}{(z-a_i)^j}\, dz + \frac{\Lambda_{i1}^0}{z-a_i}\, dz, \qquad i = 1, \dots, m,$$

to vary in the sense that the diagonal matrices $\Lambda_{i2}, \dots, \Lambda_{i,r_i+1}$ vary in a neighbourhood of $\Lambda_{i2}^0, \dots, \Lambda_{i,r_i+1}^0$ with Λ_{i1}^0 held fixed. Thus for the set $\Lambda_i = \{\Lambda_{i2}, \dots, \Lambda_{i,r_i+1}\}$ of r_i diagonal matrices we denote by ∇_{Λ_i} the meromorphic connection on the holomorphically trivial vector bundle of rank p over O_{a_i} whose 1-form is ω_{Λ_i}.

2. The Malgrange isomonodromic deformation of the pair (E^0, ∇^0)

First we describe in more details the deformation space. For $k \in \mathbb{N}$ let us denote by Z^k the subset of the space $\mathbb{C}^k$ whose points have pairwise distinct coordinates. Then Z^n will be the space of pole locations and

$$\mathcal{C}_i = \underbrace{\mathbb{C}^p \times \ldots \times \mathbb{C}^p}_{r_i-1} \times Z^p, \qquad i = 1, \dots, m,$$

will be the space of local formal equivalence classes at the pole a_i. Define the deformation space $\mathcal{D}$ as the universal cover

$$\mathcal{D} = \widetilde{Z^n} \times \widetilde{\mathcal{C}}_1 \times \ldots \times \widetilde{\mathcal{C}}_m$$

of the Cartesian product $Z^n \times \mathcal{C}_1 \times \ldots \times \mathcal{C}_m$.

One has the standard projections

$$\begin{aligned} a = (a_1, \ldots, a_n) : \mathcal{D} &\to Z^n, \\ \Lambda_i = (\Lambda_{i2}, \ldots, \Lambda_{i,r_i+1}) : \mathcal{D} &\to \mathcal{C}_i, \qquad i = 1, \ldots, m. \end{aligned}$$

For $t \in \mathcal{D}$ we denote by $a_i(t)$ the i-th coordinate of the image of t under the first projection and by $\Lambda_i(t)$ the image of t under the second one. Denote then by t^0 the base point of the deformation space $\mathcal{D}$ corresponding to the system (1.1) (to the initial connection ∇^0), i. e., $a(t^0) = (a_1^0, \ldots, a_n^0)$, $\Lambda_i(t^0) = (\Lambda_{i2}^0, \ldots, \Lambda_{i,r_i+1}^0)$. Consider also the singular hypersurfaces

$$Y_i = \{(z,t) \in \overline{\mathbb{C}} \times \mathcal{D} \mid z = a_i(t)\} \subset \overline{\mathbb{C}} \times \mathcal{D}, \qquad i = 1, \ldots, n.$$

Now consider the fibre bundle $\mathcal{M}_i \to \mathcal{C}_i$, whose fiber over each point $\Lambda_i \in \mathcal{C}_i$ is the moduli space of local holomorphic equivalence classes of connections that are all formally equivalent to the connection ∇_{Λ_i}. A point of this fiber (a holomorphic equivalence class of connections) is determined by a corresponding set of Stokes' matrices. Let $\sigma_i^0 \in \mathcal{M}_i$ denote the holomorphic equivalence class of the connection $\nabla^0|_{O_{a_i^0}} \sim \nabla_{\Lambda_i^0}$ and let σ_i denote the unique (local) horizontal section of the fibre bundle $\mathcal{M}_i \to \mathcal{C}_i$ such that $\sigma_i(\Lambda_i^0) = \sigma_i^0$.

There exists [1, Th. 3.1] (see also [2, Th. 2.9]) the Malgrange isomonodromic deformation (E, ∇) of the pair (E^0, ∇^0), that is, the rank p holomorphic vector bundle E over $\overline{\mathbb{C}} \times \mathcal{D}$ and integrable meromorphic connection ∇ on E with a simple type r_i singularity over Y_i, $i = 1, \ldots, n$, satisfying the following properties:

1) the restriction of (E, ∇) to $\overline{\mathbb{C}} \times \{t^0\}$ is equivalent to (E^0, ∇^0);

2) the restriction of ∇ to $\overline{\mathbb{C}} \times \{t\}$ is formally equivalent to the local connection $\nabla_{\Lambda_i(t)}$ near $z = a_i(t)$;

3) the latter restriction belongs to the local holomorphic equivalence class $\sigma_i(\Lambda_i(t)) \in \mathcal{M}_i$.

According to the Malgrange-Helminck-Palmer theorem (see [2, §3]) the set

$$\Theta = \{t \in \mathcal{D} \mid E|_{\overline{\mathbb{C}} \times \{t\}} \text{ is non-trivial } \}$$

is either empty or $\Theta \subset \mathcal{D}$ is an analytic subset of codimension one. Thus the Malgrange isomonodromic deformation of the pair (E^0, ∇^0) determines an isomonodromic deformation

$$\frac{dy}{dz} = \Big(\sum_{i=1}^{n} \sum_{j=1}^{r_i+1} \frac{B_{ij}(t)}{(z - a_i(t))^j} \Big) y, \qquad B_{ij}(t^0) = B_{ij}^0,$$

of the system (1.1) for $t \in D(t^0)$, where $D(t^0)$ is a neighbourhood of the point t^0 in the space $\mathcal{D}$. The matrix functions $B_{ij}(t)$, holomorphic in $D(t^0)$, can be extended meromorphically to the whole space $\mathcal{D}$ having Θ as a polar locus.

3. Specificity of meromorphic 2×2-connections

The polar locus Θ (which is called the *Malgrange* Θ*-divisor*) possesses a local description analogous to that of the Θ-divisor for the Schlesinger equation governing isomonodromic deformations of Fuchsian systems (see [3]).

Now let us consider the two-dimensional case ($p=2$). Suppose that the initial system has at most $m=2$ irregular (non-resonant) singularities and they are of Poincaré rank 1. Thus we consider the system of the form

$$\frac{dy}{dz}=\left(\frac{B^0_{12}}{(z-a^0_1)^2}+\frac{B^0_{22}}{(z-a^0_2)^2}+\sum_{i=1}^{n}\frac{B^0_{i1}}{z-a^0_i}\right)y. \tag{3.1}$$

Theorem 3.1. *Let the monodromy of the system* (3.1) *be* irreducible *and let* (E,∇) *be the Malgrange isomonodromic deformation of the system* (3.1)*. Consider any point* $t^*\in\Theta$ *such that* $E|_{\overline{\mathbb{C}}\times\{t^*\}}\cong\mathcal{O}(1)\oplus\mathcal{O}(-1)$.

Then in a neighbourhood $D(t^*)$ *of* t^* *the* Θ*-divisor is given as a zero set of an* irreducible *holomorphic function* τ*, and the matrix functions* $B_{ij}(t)$ *have poles of at most* second *order along* $D(t^*)\cap\Theta$.

The latter means that $\tau^2(t)B_{ij}(t)$ are holomorphic matrix functions in $D(t^*)$.

Remark 3.2. For example, the Painlevé III and V equations can be described in terms of isomonodromic deformations satisfying the above theorem (see details in [4, Ch. 5, §§4,5]): for P_{III} one has $m=n=2$ and for P_V one has $m=1$, $n=3$. If $t^*\in\Theta$ and $E|_{\overline{\mathbb{C}}\times\{t^*\}}\cong\mathcal{O}(k)\oplus\mathcal{O}(-k)$, then the estimate $2k\leqslant m+n-2$ holds [5] when the monodromy of a connection is irreducible. Thus $2k\leqslant 2$ and hence $k=1$ in both cases.

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The sixth Painlevé transcendent as a generator of uniformizable orbifolds

Yurii V. Brezhnev

1. Algebraic solutions of $\mathcal{P}_6$ and uniformization theory

The sixth Painlevé transcendent

$$\mathcal{P}_6: \quad y_{xx} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right) y_x^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right) y_x$$
$$+ \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left\{\alpha - \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} - \left(\delta - \tfrac{1}{2}\right)\frac{x(x-1)}{(y-x)^2}\right\}$$

is known to be a rich source of nontrivial algebraic solutions $y = f(x)$ and genera of these solutions, as genera of corresponding algebraic curves $F(x, y) = 0$, may be made as great as is wished. The relation of such solutions to the uniformization theory is based on the $\wp$-representation of the $\mathcal{P}_6$:

$$-\frac{\pi^2}{4}\frac{d^2 z}{d\tau^2} = \alpha\,\wp'(z|\tau) + \beta\,\wp'(z-1|\tau) + \gamma\,\wp'(z-\tau|\tau) + \delta\,\wp'(z-1-\tau|\tau) \qquad (1)$$

obtainable via the transcendental change $(x, y) \mapsto (z, \tau)$ (Painlevé (1906), Manin–Babich–Bordag (1996)):

$$x = \frac{\vartheta_4^4(\tau)}{\vartheta_3^4(\tau)}, \qquad y = \frac{1}{3} + \frac{1}{3}\frac{\vartheta_4^4(\tau)}{\vartheta_3^4(\tau)} - \frac{4}{\pi^2}\frac{\wp(z|\tau)}{\vartheta_3^4(\tau)}. \qquad (2)$$

Thus, knowledge of $z(\tau)$-dependence leads to a parametric representation for solution $y = f(x)$ and, in particular, to parametric representation of algebraic solutions. In their full generality these dependencies are known for the Picard–Hitchin class of solutions. For example, Picard's case $\alpha = \beta = \gamma = \delta = 0$ corresponds to $z = A\tau + B$. In Hitchin's case $\alpha = \beta = \gamma = \delta = \frac{1}{8}$ the dependence $z(\tau)$ is more complicated (obtainable through Okamoto's transformations) but parametric form of solution is, however, found to be very compact

$$y_{\mathrm{Pic}} = -\frac{\vartheta_4^2(\tau)}{\vartheta_3^2(\tau)}\frac{\theta_2^2}{\theta_1^2}, \qquad y_{\mathrm{Hit}} = \frac{\vartheta_4^2(\tau)}{\vartheta_3^2(\tau)}\left\{\pi\frac{\vartheta_2^2(\tau)\cdot\theta_2\theta_3\theta_4}{\theta_1' + 2\pi A\theta_1} - \theta_2^2\right\}\frac{1}{\theta_1^2}, \qquad (3)$$

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where θ's are understood to be equal to $\theta_k(A\tau{+}B|\tau)$ with arbitrary constants A, B and $\theta_1' := \theta_1'(A\tau{+}B|\tau)$. Purely algebraic solutions correspond to $A\tau{+}B = \frac{\nu}{N}\tau + \frac{\mu}{N}$ with integral ν, μ, and N.

Uniformizing functions are known to be determined in terms of the auxiliary 2nd order linear Fuchsian ODEs $\Psi_{yy} = \frac{1}{2}\,\mathcal{Q}(x,y)\,\Psi$, where $\mathcal{Q}$, as a rational function of x and y, contains all the information about corresponding Riemann surface $\mathcal{R}$ (or orbifold $\mathfrak{T}$). Since the function $x = \chi(\tau)$ in (2) is the very well-known one and its Fuchsian $\mathbf{\Gamma}(2)$-equation $\Psi'' = -\frac{1}{4}\,\frac{x^2-x+1}{x^2(x-1)^2}\,\Psi$ is also known, we obtain nontrivial (solvable) Fuchsian equations for the second uniformizing function $y(\tau)$. Manipulations with Fuchsian equations themselves are not convenient because we constantly handle the multivalued functions-inversions; the ratios like $\tau = \Psi_1(x)/\Psi_2(x)$. For this reason we invert the standard Schwarz derivative $\{\tau, x\}$ into the 'reverse' object $[x,\tau] = -\{\tau, x\}$ and work with the autonomous ODEs

$$[y,\tau] = \mathcal{Q}(x,y)\,, \qquad \text{where} \;\; [y,\tau] := \frac{\dddot{y}}{\dot{y}^3} - \frac{3}{2}\,\frac{\ddot{y}^2}{\dot{y}^4}\,, \tag{4}$$

defining uniformizing single-valued functions and other single-valued objects.

2. On the general solution to equation (1)

Complete structure of the analytic continuations (a connection problem) of arbitrary solutions to $\mathcal{P}_6$ is the subject matter of the series works by D. Guzzetti (see, e. g., [1]). Analyzing these results, it would appear reasonable that the ramification structure of all (not necessarily algebraic) solutions to the $\mathcal{P}_6$-equation in the vicinity of critical points is described by a function series of the kind

$$y = A + R\big[(x-e)^a \ln^n(x-e)\big] + \cdots\,,$$

where $e = \{0,1\}$, $a \in \mathbb{C}$, $n \in \mathbb{Z}$, and $R[\ldots]$ is a rational function of its argument. In the language of uniformizing Painlevé substitution (2) this point is self-suggested: in the upper (τ)-half-plain $\mathbb{H}^+$ the x-function has an exponential behavior in the neighborhood of the points $x = \{0,1,\infty\}$:

$$x \overset{\tau\to 0}{=} 0 + 16\exp\Big(\frac{\pi}{\mathrm{i}\tau}\Big) + \cdots\,, \quad x \overset{\tau\to\infty}{=} 1 - 16\mathrm{e}^{\pi\mathrm{i}\tau} + \cdots\,, \quad x \overset{\tau\to 1}{=} \frac{1}{16}\exp\Big(\frac{\pi\mathrm{i}}{\tau-1}\Big) + \cdots$$

(the uniformizing τ-parameter itself is defined up to a fraction-linear transformation). It follows (the conjecture) that the y-function has also the single-valued character about each of the branch-point pre-images:

$$y(\tau) = A + B\,(\tau-\tau_{\mathrm{o}})^n \exp\Big(\frac{-a\pi\mathrm{i}}{\tau-\tau_{\mathrm{o}}}\Big) + \cdots\,, \qquad y(\tau) = A + B\tau^n \mathrm{e}^{a\pi\mathrm{i}\tau} + \cdots\,. \tag{5}$$

as $\tau \to \tau_{\mathrm{o}} \in \mathbb{R}$ or, respectively, $\tau \to +\mathrm{i}\infty$. For example, all asymptotics appearing in [1] fit this behavior. We can therefore rewrite Eqs. $\mathcal{P}_6$ and (1) in form of modification of purely 'algebraic' uniformizing Schwarz–Fuchs 3rd order ODE (4):

$$[y,\tau] = A y_x^{-4} + B y_x^{-3} + C y_x^{-2} + D y_x^{-1} + E\,, \tag{6}$$

where (A, B, C, D, E) are certain rational functions of x, y and quadratic polynomials in parameters $(\alpha, \beta, \gamma, \delta)$ (explicit expressions are too cumbersome to display here). Because of outstanding character of $\mathcal{P}_6$, this equation may be treated as a generator of 'infinite genus curves'. In the case of algebraic solutions the right hand side of Eq. (6) becomes a rational function $\mathcal{Q}(x, y)$, that is (4). We conjecture that all the Painlevé solutions to Eq. (6) are the globally single-valued analytic functions with the structure (5) and the domain of their existence is a half-plain (under suitable normalization of τ). It is known that solutions to the lower Painlevé equations $\mathcal{P}_{1\ldots5}$ (under an appropriate modification [2]) are the single-valued functions on $\overline{\mathbb{C}}$. In this respect, the pass from $\mathcal{P}_6$-equation over $\overline{\mathbb{C}}\backslash\{0, 1, \infty\}$ to the $\mathbb{H}^+$ and uniformization theory related to the coverings of a three punctured $\mathbf{\Gamma}(2)$-orbifold becomes very natural.

3. Calculus: Abelian integrals and affine (analytic) connections

Insomuch as we have not only τ-representations for the scalar (i. e. automorphic) functions on $\mathcal{R}$'s but rules for differential computations with theta-functions of arbitrary arguments [3] we can close the differential apparatus on orbifolds $\mathfrak{T}$ whose compactifications are corresponding Painlevé $\mathcal{R}$'s. This includes the additively automorphic functions (Abelian integrals), differentials, and covariant differentiation, say, of 1-differentials $\nabla = \partial_\tau - \mathbf{\Gamma}(\tau)$. The latter leads to necessity to introduce the geometric connection object $\mathbf{\Gamma}(\tau)$, which transforms according to the standard rule $\widetilde{\mathbf{\Gamma}}(\tilde{\tau})\, d\tilde{\tau} = \mathbf{\Gamma}(\tau)\, d\tau - d \ln \frac{d\tilde{\tau}}{d\tau}$ under $\mathrm{SL}_2(\mathbb{R})$-transformations and respects the factor topology of $\mathbb{H}^+\!/\pi_1(\mathfrak{T})$. The characteristic feature of the (complex) 1-dimensional case (orbifolds and Riemann surfaces) is that it is completely described by the invariant 3rd order ODE (4). Therefore closed collection of data for the theory is given by the set $\big\{y(\tau), \dot{y}(\tau), \ddot{y}(\tau)\big\}$ if, however, the automorphism group of the generator $y(\tau)$ coincides with $\pi_1(\mathfrak{T})$. In general, automorphisms of the field generators are not bound to coincide with $\pi_1(\mathfrak{T})$ since the choice of the pair (x, y) is not unique. It is found however that the set of Painlevé orbifolds coming from Picard–Hitchin's curves (3) is not the case: $\mathrm{Aut}\, y(\tau) \cong \pi_1(\mathfrak{T})$. In this regard the many Painlevé curves (we suggest that all) stand out majority of classical modular equations originated from purely group-algebraic considerations related to the group $\mathrm{PSL}_2(\mathbb{Z})$ or some its subgroups. By this means the expression

$$\mathbf{\Gamma}(\tau) = \frac{d}{d\tau} \ln \dot{y}(\tau) + \text{arbitrary (Abelian) 1-differential}$$

provides a general form of the sought-for connection on Painlevé $\mathfrak{T}$. We can normalize this $\mathbf{\Gamma}(\tau)$ to have only first order poles (residues) and, integrating the transformation law above, one can see that the sum of such residues is invariant

$$\int_{\partial\mathcal{R}} \widetilde{\mathbf{\Gamma}}(\tilde{\tau})\, d\tilde{\tau} = \int_{\partial\mathcal{R}} \mathbf{\Gamma}(\tau)\, d\tau = (2g-2)\cdot 2\pi\mathrm{i};$$

it depends only on genus and, in effect, is equal to the number of zeroes of a holomorphic differential $\dot{\mathfrak{u}}(\tau)$. Varying the holomorphic differentials $\dot{\mathfrak{u}}_k(\tau)$ we can impart the simpler from to the connection

$$\mathbf{\Gamma}(\tau)=\frac{d}{d\tau}\ln\dot{\mathfrak{u}}(\tau)+\sum_{k=1}^{g}\dot{\mathfrak{u}}_k(\tau)$$

and build the elementary $\mathbf{\Gamma}$ with a single pole (if genus $g>1$ then the analytic connection does always have a singularity). So we have the set of invariant objects $\big\{y(\tau),\dot{y}(\tau),\mathbf{\Gamma}(\tau)\big\}$ since functions $x(\tau)$, $y(\tau)$ are completely at hand. The remarkable fact is that *affine* (*analyitc*) *connection on an arbitrary* $\mathfrak{T}$ *satisfies an autonomous ODE* $\Xi(\dddot{\Gamma},\ddot{\Gamma},\dot{\Gamma},\Gamma)=0$ and there is an algorithm how to derive it.

For completeness we should involve into analysis the integrals of closed 1-forms on our $\mathfrak{T}$'s and $\mathcal{R}$'s, if only because there are exact 1-forms whose integrals lead to the scalar objects. On the other hand, uniformization of any higher genera curves is reduced to uniformization of *zero* genus orbifolds and the latter form towers and hierarchies. In the Painlevé uniformizing theory, in one way or another, many classical and nonclassical zero genus known orbifolds appear [3]. In turn they are related to nonzero genus curves which may cover elliptic ones, i. e. tori. We thus obtain a possibility to construct explicitly Abelian integrals if they come from an elliptic cover. Here is a good example along these lines.

The Chudnovsky orbifold defined by the Fuchsian equation $(z^3-z)\,\Psi''+(3\,z^2-1)\,\Psi'+z\,\Psi=0$ is related, through the Halphen transformation (zero genus elliptic cover) $z=\wp(\mathfrak{u})$, to the Fuchsian equation on the lemniscatic torus $\wp'^2=4\wp^3-4\wp$. Correlating these facts we derive the nice τ-representation for the everywhere finite object $\mathfrak{u}$ and analog of (4)—the uniformizing Schwarz equation:

$$[\mathfrak{u},\tau]=-2\,\wp(2\mathfrak{u})\,,\qquad \mathfrak{u}(\tau)=\frac{1}{2}\frac{\vartheta_3(\tau)}{\vartheta_2(\tau)}\cdot{}_2F_1\!\left(\frac{1}{2},\frac{1}{4};\frac{5}{4}\middle|\frac{\vartheta_3^4(\tau)}{\vartheta_2^4(\tau)}\right)$$

(the check is a good exercise). This is a first *explicit and analytic* τ-representation for an additively automorphic function (Abelian integral $\mathfrak{u}=\wp^{-1}(z)$) on an orbifold (Riemann surface) of a *negative* curvature -1. Under suitable cover this $u(\tau)$ may produce the τ-representation for $\mathfrak{u}$-integrals on higher genus curves; examples of the analogous ODEs and their solutions can also be obtained. All of them can be related to the Painlevé curves.

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Plane Power Geometry for single ODE and Painlevé equations

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Power Geometry (PG) gives algorithms for calculation of asymptotic forms and asymptotic expansions of solutions to one equation (algebraic, ordinary differential and partial differential) and to system of such equations. Here we consider such asymptotic expansions of solutions to single ordinary differential equation (ODE) which can be calculated by algorithms of planar Power Geometry, and we point out some applications to Painlevé equations.

1. General ODE

1. 1. Statement of the problem. Let x be independent and y be dependent variables, $x, y \in \mathbb{C}$. A *differential monomial* $a(x, y)$ is a product of an ordinary monomial $cx^{r_1}y^{r_2}$, where $c = \text{const} \in \mathbb{C}$, $(r_1, r_2) \in \mathbb{R}^2$, and a finite number of derivatives of the form $d^l y/dx^l$, $l \in \mathbb{N}$. A sum of differential monomials $f(x, y) = \sum a_i(x, y)$ is called the *differential sum*.

The main problem. Let a differential equation be given

$$f(x, y) = 0, \tag{1}$$

where $f(x, y)$ is a differential sum. As $x \to 0$, or as $x \to \infty$, for solutions $y = \varphi(x)$ to the equation (1), find all expansions of the form

$$y = c_r x^r + \sum c_s x^s, \quad c_r = \text{const} \in \mathbb{C}, \quad c_r \neq 0, \tag{2}$$

where c_s are polynomials in $\ln x$, and power exponents $r, s \in \mathbb{C}$, $\omega \text{Re}\, r > \omega \text{Re} s$. Here and below $\omega = -1, \text{if} x \to 0$, $\omega = 1, \text{if} x \to \infty$.

The procedure to compute expansions (2) consists of two steps: computation of the first approximations

$$y = c_r x^r, \quad c_r \neq 0 \tag{3}$$

and computation of further expansion terms in (2).

1. 2. Computation of truncated equations. To each differential monomial $a(x, y)$, we assign its (vector) *power exponent* $Q(a) = (q_1, q_2) \in \mathbb{R}^2$ by the following rules: $Q(cx^{r_1}y^{r_2}) = (r_1, \ r_2)$; $Q(d^l y/dx^l) = (-l, 1)$; when differential monomials are multiplied, their power exponents must be added as vectors $Q(a_1a_2) = Q(a_1) + Q(a_2)$.

The set $\mathbf{S}(f)$ of power exponents $Q(a_i)$ of all differential monomials $a_i(x, y)$ present in the differential sum $f(x, y)$ is called the *support of the sum* $f(x, y)$. Obviously, $\mathbf{S}(f) \in \mathbb{R}^2$. The convex hull $\Gamma(f)$ of the support $\mathbf{S}(f)$ is called the *polygon of the sum* $f(x, y)$. The boundary $\partial\Gamma(f)$ of the polygon $\Gamma(f)$ consists of the vertices $\Gamma_j^{(0)}$ and the edges $\Gamma_j^{(1)}$. They are called (generalized) *faces* $\Gamma_j^{(d)}$, where the upper index indicates the dimension of the face, and the lower one is its number. Each face $\Gamma_j^{(d)}$ corresponds to the *truncated sum*

$$\hat{f}_j^{(d)}(x, y) = \sum a_i(x, y) \ \text{ over } \ \mathrm{Q(a_i)} \in \Gamma_{\mathrm{j}}^{(\mathrm{d})} \cap \mathbf{S}(\mathrm{f}).$$

Example. Consider the third Painlevé equation

$$f(x, y) \overset{def}{=} -xyy'' + xy'^2 - yy' + ay^3 + by + cxy^4 + dx = 0, \tag{4}$$

assuming the complex parameters $a, b, c,\ d \neq 0$. Here the first three differential monomials have the same power exponent $Q_1 = (-1, 2)$, then $Q_2 = (0, 3)$, $Q_3 = (0, 1)$, $Q_4 = (1, 4)$, $Q_5 = (1, 0)$. They are shown in Fig. 1 in coordinates q_1, q_2. Their convex hull $\Gamma(f)$ is the triangle with three vertices $\Gamma_1^{(0)} = Q_1$, $\Gamma_2^{(0)} = Q_4$, $\Gamma_3^{(0)} = Q_5$, and with three edges $\Gamma_1^{(1)}$, $\Gamma_2^{(1)}$, $\Gamma_3^{(1)}$. The vertex $\Gamma_1^{(0)} = Q_1$ corresponds to the truncation $\hat{f}_1^{(0)}(x, y) = -xyy'' + xy'^2 - yy'$, and the edge $\Gamma_1^{(1)}$ corresponds to the truncation $\hat{f}_1^{(1)}(x, y) = \hat{f}_1^{(0)}(x, y) + by + dx$. ■

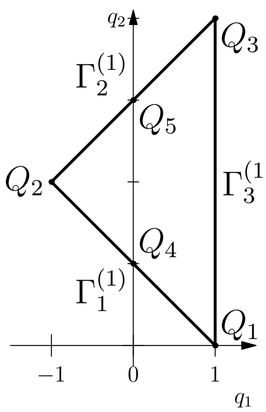

FIG. 1

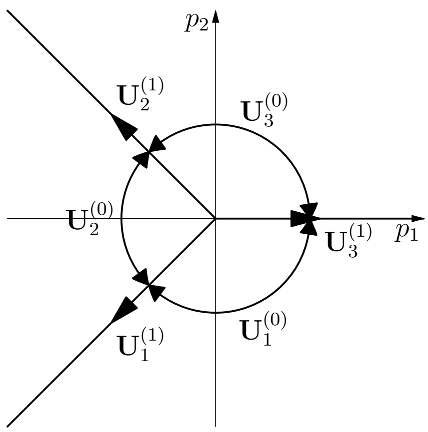

FIG. 2

Let the plane $\mathbb{R}_*^2$ be dual to the plane $\mathbb{R}^2$ such that for $P = (p_1, p_2) \in \mathbb{R}_*^2$ and $Q = (q_1, q_2) \in \mathbb{R}^2$, the scalar product $\langle P, Q\rangle \overset{def}{=} p_1q_1 + p_2q_2$ is defined. Each face $\Gamma_j^{(d)}$ in $\mathbb{R}^2_*$ corresponds to its own *normal cone* $\mathbf{U}_j^{(d)}$ formed by the outward normal vectors P to the face $\Gamma_j^{(d)}$. For the edge $\Gamma_j^{(1)}$, the normal cone $\mathbf{U}_j^{(1)}$ is the ray orthogonal to the edge $\Gamma_j^{(1)}$ and directed outward the polygon $\Gamma(f)$. For the vertex $\Gamma_j^{(0)}$, the normal cone $\mathbf{U}_j^{(0)}$ is the open sector (angle) in the plane $\mathbb{R}^2_*$ with the vertex at the origin $P = 0$ and limited by the rays which are the normal cones of the edges adjacent to the vertex $\Gamma_j^{(0)}$.

Example. For the the equation (4), the normal cones $\mathbf{U}_j^{(d)}$ of the faces $\Gamma_j^{(d)}$ are shown in Fig. 2. ■

Thus, each face $\Gamma_j^{(d)}$ corresponds to the normal cone $\mathbf{U}_j^{(d)}$ in the plane $\mathbb{R}^2_*$ and to the truncated equation

$$\hat{f}_j^{(d)}(x, y) = 0. \tag{5}$$

Theorem 1. *If the expansion* (2) *satisfies the equation* (1), *and* $\omega(1, \mathrm{Re}\ r) \in \mathbf{U}_j^{(d)}$, *then the truncation* $y = c_r x^r$ *of the solution* (2) *is the solution to the truncated equation* $\hat{f}_j^{(d)}(x, y) = 0$.

Hence, to find all truncated solutions $y = c_r x^r$ to the equation (1), we need to compute: the support $\mathbf{S}(f)$, the polygon $\Gamma(f)$, all its faces $\Gamma_j^{(d)}$, and their normal cones $\mathbf{U}_j^{(d)}$. Then for each truncated equation $\hat{f}_j^{(d)}(x, y) = 0$, we need to find all its solutions $y = c_r x^r$ which have one of the vectors $\pm(1, \mathrm{Re}\ r)$ lying in the normal cone $\mathbf{U}_j^{(d)}$.

1. 3. Types of expansions of solutions Any truncated equation (5) is quasihomogeneous, and it is not difficult to find all its power solutions of the form (3). For each such solution, we can compute its characteristic polynomial $\nu(k)$, roots k_j of which with $\omega\mathrm{Re}\ k_j < \omega\mathrm{Re}\ r$ are *critical numbers* [1]. If the truncated solution (3) has no critical numbers then the initial equation (1) has a solution of the form (2) where c_s are constants. Such solutions belong to

Type 1. *Power expansions.* For them $|\mathrm{Im}\ s/\mathrm{Re}\ s| <$ const. If the order κ of the characteristic polynomial $\nu(k)$ equals to the maximal order n of the derivatives in equation (1) then the expansion (2) converges [1–4]. If $\kappa < n$ then the power expansion (2) can be continued by an exponential expansion of type 6 (see below) as a solution to equation (1). There are other following types of expansions of form (2) for solutions to equation (1).

Type 2. c_r is a constant, but c_s are polynomials in $\log x$ *(power-logarithmic expansions).*

Type 3. c_r and c_s are power series in decreasing powers of $\log x$ *(complicated expansions).* The truncated solution (3) has its characteristic polynomial $\lambda(k)$. Absence of critical numbers is sufficient for existence of the expansion [5].

Type 4. r and s are real, but c_r are series in powers of x^i, and c_r contains finite number of such terms *(half-exotic expansions).*

Type 5. r, s and c_s are as in type 4, but c_r is a sum of infinitely many powers of x^i and they are bounded from one side *(exotic expansions).*

Types 4 and 5 differ from types 1 and 2 by the form of $c_r(x)$ and absence of restriction $|\mathrm{Im}\ s/\mathrm{Re}\ s| <$ const. In all expansions of types 1–5 the truncated solution $y = c_r x^r$ is a solution of the truncated equation (5) and can be easily found. Expansions of types 1–5 and algorithms of their computation were described in [1, 2]. Now we introduce new type.

Type 6. *Exponential expansions*

$$y = b_0(x) + Ce^{\varphi(x)} + \sum_{k=2}^{\infty} b_k(x) C^k e^{k\varphi(x)}, \tag{6}$$

where $b_0(x), b_k(x)$ and $\varphi'(x)$ are power expansions, C is an arbitrary constant. To the initial part $b_0(x)Ce^{\varphi(x)}$ there corresponds its characteristic polynomial $\mu(k)$. Absence of critical numbers k_j is enough for existence of the expansion (6).

2. Painlevé equations P_l

Supports of Painlevé equations $P_1,\ P_2,\ P_3,\ P_4,\ P_5, P_6$ are shown in Fig. 3 for generic case. Expansions of solutions of type 1 exist for all P_1–P_6; of types 2-5 exist only for $P_3,\ P_5,\ P_6$; and of type 6 exist only for P_1–P_5 and only near infinity [2, 5–11]. Some other applications of Power Geometry see in [12–16].

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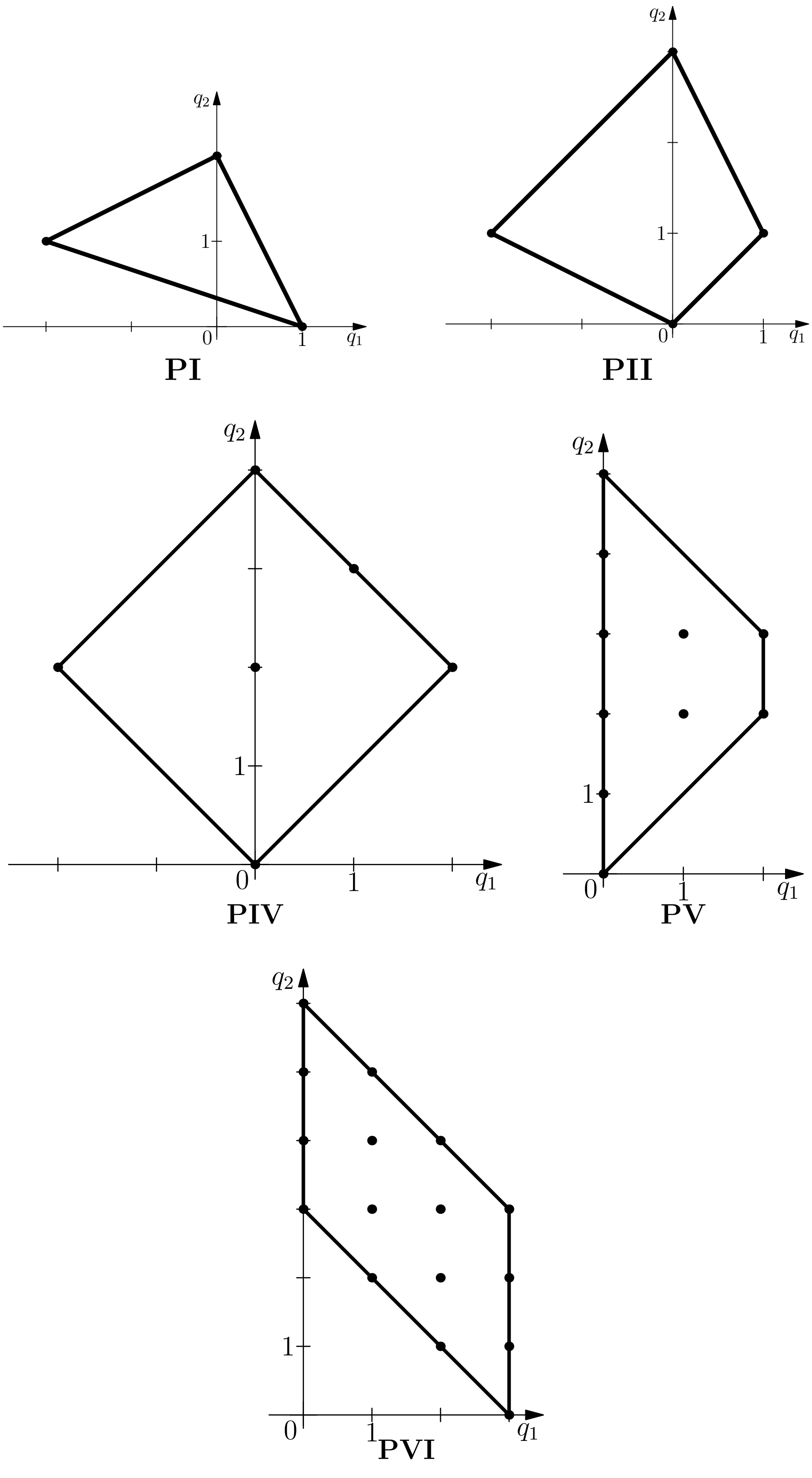

Fig. 3

Space Power Geometry for an ODE and Painlevé equations

Alexander D. Bruno

Abstract. Here we explain algorithms of the space Power geometry (PG) for one ordinary differential equation (ODE). The algorithms allow to calculate such asymptotic forms of solutions of the ODE, which cannot be calculated by algorithms of the plane PG [1]. We apply that approach to Painlevé equations P_l. It appears that P_1 – P_5 have elliptic asymptotic forms of solutions and P_2, P_4, P_5 have also periodic asymptotic forms and all their families are two-parameter. Similar (but more complicated) approach and results see in [2–4].

1. Space Power Geometry

Let x be independent and y be dependent variables, $x, y \in \mathbb{C}$. A *differential monomial* $a(x, y)$ is a product of an ordinary monomial $cx^{r_1}y^{r_2}$, where $c = \text{const} \in \mathbb{C}$, $(r_1, r_2) \in \mathbb{R}^2$, and a finite number of derivatives of the form $d^l y/dx^l$, $l \in \mathbb{N}$. The sum of differential monomials $f(x, y) = \sum a_i(x, y)$ is called the *differential sum.*

As $x \to \infty$ we consider the ODE

$$f(x,\, y) = 0, \tag{1.1}$$

where $f(x, y)$ is a differential sum. To each differential monomial $a(x, y)$, we assign its (vector) *power exponent* $\mathbf{Q}(a) = (q_1, q_2, q_3) \in \mathbb{R}^3$ by the following rules: $\mathbf{Q}(cx^{r_1}y^{r_2}) = (r_1,\ r_2,\ 0)$; $\mathbf{Q}(d^l y/dx^l) = (0,\, 1,\, l)$; power exponent of the product of differential monomials is the sum of power exponents of factors $Q(a_1a_2) = Q(a_1) + Q(a_2)$. The set $\tilde{\mathbf{S}}(f)$ of power exponents $\mathbf{Q}(a_i)$ of all differential monomials $a_i(x, y)$ present in the differential sum $f(x, y)$ is called the *support of the sum* $f(x, y)$. Obviously, $\tilde{\mathbf{S}}(f) \subset \mathbb{R}^3$. The convex hull $\mathbf{\Gamma}(f)$ of the support $\tilde{\mathbf{S}}(f)$ is called the *polyhedron of the sum* $f(x, y)$. The boundary $\partial\mathbf{\Gamma}(f)$ of the polyhedron $\mathbf{\Gamma}(f)$ consists of the vertices $\mathbf{\Gamma}_j^{(0)}$, the edges $\mathbf{\Gamma}_j^{(1)}$ and the faces $\mathbf{\Gamma}_j^{(2)}$. They are called (generalized) *faces* $\mathbf{\Gamma}_j^{(d)}$, where the upper index indicates the dimension of the face, and the lower one is its number. Each face $\mathbf{\Gamma}_j^{(d)}$ corresponds to the *space truncated sum* $\check{f}_j^{(d)}(x,y) = \sum a_i(x,y)$ over $\mathbf{Q}(a_i) \in \mathbf{\Gamma}_j^{(d)} \cap \tilde{\mathbf{S}}(f)$.

Example. Consider the third Painlevé equation

$$f(x,y) \overset{def}{=} -xyy'' + xy'^2 - yy' + ay^3 + by + cxy^4 + dx = 0, \tag{1.2}$$

assuming the complex parameters a, b, c, $d \neq 0$. Here the first two differential monomials have the same power exponent $\mathbf{Q}_1 = (1,2,2)$, then $\mathbf{Q}_2 = (0,2,1)$, $\mathbf{Q}_3 = (0,3,0)$, $\mathbf{Q}_4 = (0,1,0)$, $\mathbf{Q}_5 = (1,4,0)$, $\mathbf{Q}_6 = (1,0,0)$. They are shown in Fig. 1 in coordinates q_1, q_2, q_3. Their convex hull $\mathbf{\Gamma}(f)$ has 5 two-dimensional faces $\mathbf{\Gamma}_j^{(2)}$. The far face $\mathbf{\Gamma}_1^{(2)}$, spanned by $\mathbf{Q}_1$, $\mathbf{Q}_5$, $\mathbf{Q}_6$, corresponds to the truncation

$$\check{f}_1^{(2)} = -xyy'' + xy'^2 + cxy^4 + dx = x(-yy'' + y'^2 + cy^4 + d). \tag{1.3}$$

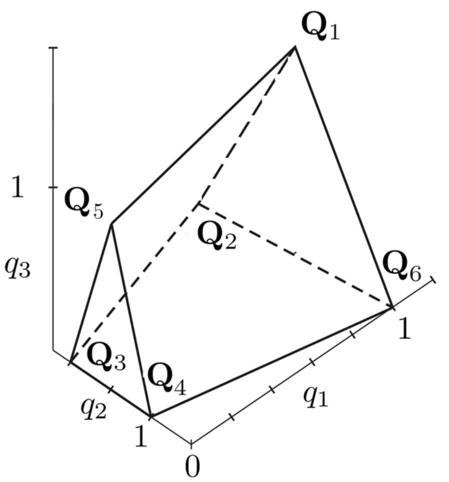

Fig. 1

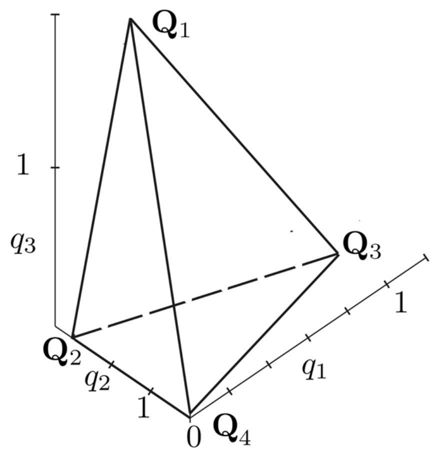

Fig. 2

Thus, each face $\mathbf{\Gamma}_j^{(d)}$ corresponds to the truncated equation

$$\check{f}_j^{(d)}(x,y) = 0. \tag{1.4}$$

Let $\mathbf{N}_j = (n_1, n_2, n_3)$ be the external normal to two-dimensional face $\mathbf{\Gamma}_j^{(2)}$. We will consider only normals with $n_1 > 0$, so we can assume that $n_1 = 1$.

If the face $\mathbf{\Gamma}_j^{(2)}$ has the normal $\mathbf{N}_j = (1,0,0)$ then the corresponding truncation $\check{f}_j^{(2)} = x^{q_1} g(y)$, where $g(y)$ contains y and its derivatives but does not contain

x. In that case the full sum $f(x,y)$ can be written as

$$f(x,y) = x^{q_1} g(y) + x^{q_1-\gamma} h(x,y),$$

where $\gamma > 0$ and $h(x,y)$ is a differential sum (see (1.3)).

Remark 1.1. If $y(x)$ is a solution to the equation $g(y) = 0$ with the property

$$0 < \varepsilon < |y(x)|, |y'(x)|, \ldots, |y^{(n)}| < \varepsilon^{-1}, \tag{1.5}$$

then $y(x)$ is the asymptotic form of the solutions to the full equation (1.1). Here ε is a small number.

Let the power transformation of variables $x, y \to u, v$:

$$y = x^{\alpha} v, \ u = x^{\beta}, \tag{1.6}$$

transforms $f(x,y)$ into $f^*(u,v)$.

Theorem 1.2. *Let the face* $\mathbf{\Gamma}_1^{(2)}$ *of* $\mathbf{\Gamma}(f)$ *have the normal* $\mathbf{N}_j = (1, n_2, n_3)$ *with* $n_3+1 > 0$, *then the power transformation* (1.6) *with* $\alpha = n_2$, $\beta = n_3+1$ *transforms the truncation* $\check{f}_j^{(2)}(x,y)$ *of* $f(x,y)$ *into the truncation*

$$\check{f}_j^{*(2)}(u,v) = u^{q_1} g(v) \tag{1.7}$$

of $f^*(u,v)$, *corresponding to the face* $\mathbf{\Gamma^*}_j^{(2)}$ *of* $\mathbf{\Gamma}(f^*)$ *with the normal* $\mathbf{N}_j^* = (1,0,0)$. *Here* $\check{f}_j^{*(2)}(u,v)$ *equals to* $\check{f}_j^{(2)}(x,y)$ *after substitution*

$$\beta^l \, u^{[\alpha + l(\beta-1)]/\beta} \, d^l v / du^l \tag{1.8}$$

instead of $y^{(l)} = d^l y / dx^l$.

So, if $v = \varphi(u)$ is a solution to the equation $g(v) = 0$ and $|\varphi(u)|$ is bounded from zero and infinity as y in (1.5), then the initial equation $f(x,y) = 0$ has a solution with asymptotic form

$$y \sim x^{\alpha} \varphi(x^{\beta}), \ x \to \infty. \tag{1.9}$$

2. Painlevé equations P_l

2.1. Equation P_1

$$f(x,y) \stackrel{def}{=} -y'' + 3y^2 + x = 0. \tag{2.1}$$

The support $\tilde{\mathbf{S}}(f)$ consists of three points $\mathbf{Q}_1 = (0,1,2)$, $\mathbf{Q}_2 = (0,2,0)$, $\mathbf{Q}_3 = (1,0,0)$. They lie in a plane with normal $\mathbf{N} = (1,\ 1/2,\ 1/4)$. Here $f = \check{f}_1^{(2)}$, i.e. the truncated equation (1.4) coincides with the initial equation (1.1). We make the power transformation (1.6) with $\alpha = 1/2$, $\beta = 5/4$. According to Theorem 1.2 and (1.8), equation (2.1) gives an equation with the truncation

$$\begin{aligned}\check{f}_1^{*(2)}(u,v) &= -\left(5/4\right)^2 u^{(1/2+2/4)(4/5)} \ddot{v} + 3u^{4/5} v^2 + u^{4/5} = \\ &= u^{4/5} \left[-(5/4)^2 \ddot{v} + 3v^2 + 1\right] \stackrel{def}{=} u^{4/5} g(v).\end{aligned}$$

Here and below $\dot{v} = dv/du$. Equation $g(v) = 0$ has the first integral

$$\dot{v}^2 = 2(4/5)^2(v^3 + v + C_0), \tag{2.2}$$

where here and below C_0 is an arbitrary constant. If the discriminant $-4 - 27C_0$ of the polynomial $v^3 + v + C_0$ is not zero then solutions to the equation (2.2) are elliptic functions $v = \varphi(u)$. Thus, P_1 has solutions with asymptotic forms (1.9), i.e. $y = x^{1/2}\varphi(x^{5/4})$. These asymptotic forms were found by Boutroux [5] and were studied in a lot of publications.

2.2. Equation P_2

$$f(x, y) \stackrel{def}{=} -y'' + 2y^3 + xy + a = 0, \tag{2.3}$$

where a is the complex parameter. The support $\tilde{\mathbf{S}}(f)$ consists of four points $\mathbf{Q}_1 = (0, 1, 2)$, $\mathbf{Q}_2 = (0, 3, 0)$, $\mathbf{Q}_3 = (1, 1, 0)$, $\mathbf{Q}_4 = 0$; see Fig. 2. Their convex hull $\mathbf{\Gamma}(f)$ is a tetrahedron. It has 4 faces with external normals $\mathbf{N}_1 = (0, 0, -1)$, $\mathbf{N}_2 = (-1, 0, 0)$, $\mathbf{N}_3 = (1, 1/2, 1/2)$, $\mathbf{N}_4 = (1, -1, 1/2)$. Two of them $\mathbf{N}_3$ and $\mathbf{N}_4$ have $n_1 > 0$.

First we consider $\mathbf{\Gamma}_3^{(2)}$ with the truncated equation

$$\check{f}_3^{(2)}(x, y) \stackrel{def}{=} -y'' + 2y^3 + xy = 0. \tag{2.4}$$

After transformation (1.6) with $\alpha = 1/2$, $\beta = 3/2$, using (1.8), we obtain the truncated equation

$$\begin{aligned} \check{f}_3^{*(2)}(u, v) &= -(3/2)^2 u^{(1/2+2\cdot 1/2)\cdot 2/3}\ddot{v} + 2uv^3 + uv = \\ &= u\left[-(3/2)^2\ddot{v} + 2v^3 + v\right] \stackrel{def}{=} u\,g(v) = 0. \end{aligned}$$

Equation $g(v) = 0$ has the first integral

$$\dot{v}^2 = \frac{4}{9}\left(v^4 + v^2 + C_0\right). \tag{2.5}$$

If the discriminant of the right hand part is different from zero, the equation (2.5) has elliptic solution $v = \varphi(u)$. So the equation P_2 has solution with asymptotic forms (1.9), i.e. $y \sim x^{1/2}\varphi(x^{3/2})$.

Now we consider $\mathbf{\Gamma}_4^{(2)}$ with truncated equation

$$\check{f}_4^{(2)}(x, y) \stackrel{def}{=} -y'' + xy + a = 0. \tag{2.6}$$

After transformation (1.6) with $\alpha = -1$, $\beta = 3/2$ using (1.8), we obtain the truncated equation

$$\check{f}_4^{*(2)}(u, v) \stackrel{def}{=} -(3/2)^2\, u^{(-1+2\cdot 1/2)\cdot 2/3}\ddot{v} + v + a \stackrel{def}{=} g(v) = 0.$$

It has the first integral

$$\dot{v}^2 = (4/9)(v^2 + 2av + C_0). \tag{2.7}$$

if the discriminant $a^2 - C_0 \neq 0$ then equation (2.7) has periodic solution (with a complex period) $v = \varphi(u)$, which gives asymptotic forms (1.9), i.e. $y \sim \varphi(x^{3/2})/x$, for solutions to equation P_2.

2.3. Equation P_3

See (1.2) and Fig. 1. Among faces $\mathbf{\Gamma}_j^{(2)}$ only one face $\mathbf{\Gamma}_1^{(2)}$ with truncated equation (1.3) has external normal $\mathbf{N} = (n_1, n_2, n_3)$ with $n_1 > 0$. It is $N_1 = (1, 0, 0)$, so $\alpha = 0$, $\beta = 1$ and we do not need to make the transformation (1.6). Equation (1.3) has the first integral

$$y'^2 = cy^4 + C_0 y^2 - d.$$

If $C_0^2 + 4cd \neq 0$, then solutions to the equation are elliptic functions $y = \varphi(x)$, which gives asymptotic forms $y \sim \varphi(x)$ for solutions of equation (1.2). Compare with [6].

2.4. Equation P_4

$$f(x, y) \stackrel{def}{=} -2yy'' + y'^2 + 3y^4 + 8xy^3 + 4(x^2 - a)y^2 + 2b = 0. \tag{2.8}$$

The support $\tilde{\mathbf{S}}$ has 6 points and the polyhedron $\mathbf{\Gamma}(f)$ is a tetrahedron about similar to Fig. 2. It has 4 faces $\mathbf{\Gamma}_j^{(2)}$ with normals $\mathbf{N}_1 = (0, 0, -1)$, $\mathbf{N}_2 = (-1, 0, 0)$, $\mathbf{N}_3 = (1, 1, 1)$, $\mathbf{N}_4 = (1, -1, 1)$. Only two of them have external normal vector $\mathbf{N} = (n_1, n_2, n_3)$ with $n_1 > 0$. Namely, $\mathbf{N}_3 = (1, 1, 1)$ and $\mathbf{N}_4 = (1, -1, 1)$.

First consider $\mathbf{\Gamma}_3^{(2)}$. It corresponds to the truncated equation

$$\check{f}_3^{(2)}(x, y) \stackrel{def}{=} -2yy'' + y'^2 + 3y^4 + 8xy^3 + 4x^2y^2 = 0. \tag{2.9}$$

After power transformation (1.6) with $\alpha = 1$, $\beta = 2$, we obtain the truncated equation

$$\begin{aligned} \check{f}_3^{*(2)}(u, v) &\stackrel{def}{=} -2 \cdot 4u^{(1+2+1)/2}\ddot{v}v + 4u^2\dot{v}^2 + 3u^2v^4 + 8u^2v^3 + 4u^2v^2 \\ &= 4u^2[-2\ddot{v} + \dot{v}^2 + (3/4)v^4 + 2v^3 + v^2] \stackrel{def}{=} 4u^2 g(v) = 0. \end{aligned} \tag{2.10}$$

Equation $g = 0$ has the first integral

$$\dot{v}^2 = v^4/4 + v^3 + v^2 + C_0 v.$$

If the discriminant differs from zero, i.e. $C_0 \neq 0$, $C_0 \neq 8/27$, then the last equation has elliptic solutions $v = \varphi(u)$ and initial equation P_4 has solutions with asymptotic forms $y \sim x\varphi(x^2)$.

Now consider $\mathbf{\Gamma}_4^{(2)}$ with truncated equation

$$\check{f}_4^{(2)}(x, y) \stackrel{def}{=} -2yy'' + y'^2 + 4x^2y^2 + 2b = 0. \tag{2.11}$$

After power transformation (1.6) with $\alpha = -1$, $\beta = 2$, we obtain the truncated equation

$$\check{f}_4^{*(2)}(u, v) \stackrel{def}{=} -2 \cdot 4v\ddot{v} + 4\dot{v}^2 + 4v^2 + 2b \stackrel{def}{=} g(v) = 0.$$

It has the first integral

$$\dot{v}^2 = v^2 + C_0 v - b/2. \tag{2.12}$$

If the discriminant $C_0^2 + 2b \neq 0$, then the equation (2.12) has periodic solutions $v = \varphi(u)$ and P_4 has solutions with asymptotic forms $y = \varphi(x^2)/x$.

2.5. Equation P_6

The polyhedron $\mathbf{\Gamma}$ for P_6 has not an appropriate face $\mathbf{\Gamma}_j^{(2)}$ having external normal vector $\mathbf{N}_j = (n_1, n_2, n_3)$ with $n_1 > 0$ and $n_1 + n_3 > 0$. So P_6 has not periodic and elliptic asymptotic forms of solutions [7].

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Lotka–Volterra equations in three and four dimensions satisfying the Kowalevski-Painlevé property

Pantelis A. Damianou

Abstract. We examine a class of Lotka-Volterra equations in three and four dimensions which satisfy the Kowalevski-Painlevé property. We restrict our attention to Lotka-Volterra systems defined by a skew symmetric matrix. We obtain a complete classification of such systems. The classification is obtained using Painlevé analysis and more specifically by the use of Kowalevski exponents. The imposition of certain integrality conditions on the Kowalevski exponents gives necessary conditions. We also show that the conditions are sufficient. In the four dimensional case we also examine the Liouville integrability of the systems.

Mathematics Subject Classification (2000). 34G20, 34M55, 37J35.

Keywords. Lotka-Volterra Equations, Kowalevski Exponents, Painleve Analysis.

The Lotka-Volterra model is a basic model of predator-prey interactions. The model was developed independently by Alfred Lotka (1925), and Vito Volterra (1926). It forms the basis for many models used today in the analysis of population dynamics. In three dimensions it describes the dynamics of a biological system where three species interact.

The most general form of Lotka-Volterra equations is

$$\dot{x}_i = \varepsilon_i x_i + \sum_{j=1}^{n} a_{ij} x_i x_j, \quad i = 1, 2, \ldots, n \ . \tag{1}$$

We consider Lotka-Volterra equations without linear terms ($\varepsilon_i = 0$), and where the matrix of interaction coefficients $A = (a_{ij})$ is skew-symmetric. The special case of Kac-van Moerbeke system (KM-system) was used to describe population evolution in a hierarchical system of competing individuals. The KM-system has close connection with the Toda lattice. The Lotka-Volterra equations were studied by many authors in its various aspects, e.g. complete integrability [7] Poisson and bi-Hamiltonian formulation ([5], [10], [12], [16]), stability of solutions and Darboux polynomials ([9], [18]). In [11] we examined such Lotka-Volterra equations in three dimensions satisfying the Kowalevski-Painlevé property. The basic tools for the required classification are, the use of Painlevé analysis, the examination of the eigenvalues of the Kowalevski matrix and other standard Lax pair and Poisson techniques. The Kowalevski exponents are useful in establishing integrability or non-integrability of Hamiltonian systems; see [1], [2], [6], [14], [15], [17], [19], [21]. The first step is to impose certain conditions on the exponents, i.e., we require that all the Kowalevski exponents be integers for every solution of the indicial equation. This gives a finite list of values of the parameters satisfying such conditions. This step requires some elementary number theoretic techniques as is usual with such type of classification.

The second step is to check that the leading behavior of the Laurent series solutions agrees with the weights of the corresponding homogeneous vector field defining the dynamical system. In our case the weights are all equal to one and therefore we must exclude the possibility that some of the Laurent series have leading terms with poles of order greater than one. To accomplish this step we use old-fashioned Painlevé Analysis, i.e., Laurent series. The application of Painlevé analysis and especially of the ARS algorithm (see [3], [4], [7], [8], [15]) is useful in calculating the Laurent solution of a system and check if there are $(n-1)$ free parameters.

In this classification of Lotka-Volterra systems we discover, as expected, some well known integrable systems like the open and periodic Kac-van Moerbeke systems and systems associated with simple Lie algebras.

We also have to point out that our classification is up to isomorphism. In other words, if one system is obtained from another by an invertible linear change of variables, we do not consider them as different. Modulo this identification we obtain only six classes of solutions.

The Lotka-Volterra system can be expressed in hamiltonian form as follows: Define a quadratic Poisson bracket by the formula

$$\{x_i, x_j\} = a_{ij} x_i x_j, \quad i,j = 1,2,\dots,n \ . \tag{2}$$

Then the system can be written in the form $\dot{x}_i = \{x_i, H\}$, where $H = \sum_{i=1}^{n} x_i$. The Louville integrability in the three-dimensional case can be easily established.

In dimension three the system is defined by a matrix of the form

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} ,$$

where a, b, c are constants. We use the notation (a, b, c) to denote this system. It turns out that the Lotka-Volterra systems which posses the Kowalevski-Painlevé property fall either into two infinite families or four exceptional cases:

Theorem 1. *The Lotka-Volterra equations in three dimensions satisfy the Kowalevski–Painlevé property if and only if* (a, b, c) *is in the class of*

(l_2) $(1, 0, 1)$

(l_3) $(1, -1, 1)$

(l_4) $(1, -1, 2)$

(l_6) $(1, -2, 3)$

(l_λ) $(1, 1, \lambda)$ $\lambda \in \mathbf{Z} \setminus 0.$

(l_0) $(1, 1 + \mu, \mu)$ $\mu \in \mathbf{R}.$

In the four dimensional case, the situation is much more complicated than in the three dimensional case. The system is not automatically Liouville integrable. We distinguish two different cases. First, the case where the Poisson bracket is of rank 2. Then we have two Casimirs and therefore in this case the system is integrable (since the Hamiltonian is a constant of motion). The interesting case is when the rank is full, i.e. four. In the classification of the systems which satisfy the Painleve-Kowalevski condition we obtained over 100 such cases. For example, there are 117 cases which satisfy this condition in the case of full rank. The analysis of Kowalevski exponents indicate that these systems should be integrable. We analyze some of these examples and show their integrability. Special cases include the KM system (Volterra lattice) and some systems associated with simple and affine Lie algebras.

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About Parametric Weakly Nonlinear ODE with Time-reversal Symmetries

Nataliya Dilna and Michal Fečkan

Abstract. We show the existence of periodic and symmetric solutions of parametric weakly nonlinear ODE possessing time-reversal symmetries. Local asymptotic behaviours of these solutions are established as well. Concrete examples are presented to illustrate the general theory.

Mathematics Subject Classification (2000). Primary 34C14, 34D20; Secondary 34C15.

Keywords. Periodic solution, symmetric systems, stability of solution, parametric equations.

1. Introduction

We consider in [3] the systems of differential equations under symmetric assumptions. More concretely, we consider a weakly nonlinear ordinary differential equation of the form

$$\dot{x} = \varepsilon f(x, \mu, t), \quad x \in \mathbb{R}^n, \, t \in \mathbb{R} \tag{1.1}$$

with parameters $\epsilon \in \mathbb{R}$, $\mu \in \mathbb{R}^k$, where ϵ is small, and with a C^∞-smooth function $f : \mathbb{R}^{n+k+1} \to \mathbb{R}^n$ symmetric in x, i.e. it holds

$$Af(x, \mu, t) = -f(Ax, \mu, -t - \tau)\,, \tag{1.2}$$

where $A : \mathbb{R}^n \to \mathbb{R}^n$ is a regular linear map, $\tau \in \mathbb{R}$ is fixed and, moreover, function f is T-periodic on t, i.e. it holds

$$f(x, \mu, t) = f(x, \mu, t + T). \tag{1.3}$$

Note condition (1.2) represents such a kind of symmetry for (1.1).

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On the other hand, there are several papers [6, 9, 11] studying ODE with symmetries when (1.2) is replaced with the following assumption

$$Af(Ax, \mu, t) = -f(x, \mu, -t-\tau)\,. \tag{1.4}$$

Moreover, most of these papers suppose additional condition $A^2 = \mathbb{I}$, and then (1.4) is called as property E. Furthermore, clearly property E is our assumption (1.2) with $A^2 = \mathbb{I}$. Consequently, our results are generalizations of some earlier results for weakly nonlinear ordinary differential equations with property E.

Without loss of generality, we suppose

$$Af(x, \mu, t) = -f(Ax, \mu, -t) \tag{1.5}$$

instead of (1.2). We introduce a vector space

$$X := \left\{x \in C^1(\mathbb{R}, \mathbb{R}^{n+1}) \mid x(t) = Ax(-t)\, \forall t \in \mathbb{R}\right\}. \tag{1.6}$$

Definition 1.1. By a *symmetric solution* x of equation (1.1) we mean $x \in X$ satisfying this equation.

The main goal of this paper is to find symmetric and periodic solutions (see Section 2) for equation (1.1) and to study their asymptotic properties (see Section 3). The results presented in this note are also generalizations of achievements for anti-periodic problems with $A = -\mathbb{I}$ [1], and continuations of [5]. Doubly symmetric solutions of reversible systems are studied in [8]. Symmetric properties of periodic solutions of nonlinear nonautonomous ordinary differential equations are studied also in [2]. We can also apply numerical methods from [10] for computation of symmetric solutions of (1.1). More results on periodic solutions in dynamical systems and ordinary differential equations are presented in [4, 7].

Furthermore, when in addition, f is odd in x, i.e. it holds $f(-x, \mu, t) = -f(x, \mu, t)$, then we extend our result to the study of *antisymmetric* and periodic solutions of (1.1), i.e. satisfying $-x(-t) = Ax(t)\, \forall t \in \mathbb{R}$ instead of $x \in X$.

2. Existence of symmetric and periodic solutions

If $x(t)$ is T-periodic and satisfying condition $Ax(t) = x(-t)$ then we get $x(T/2) = x(-T/2) = Ax(T/2)$, so

$$x(T/2) \in \ker(\mathbb{I} - A)\,. \tag{2.1}$$

On the other hand, if $x(\eta, \epsilon, \mu, T/2) \in \ker(\mathbb{I}-A)$ then $x(\eta, \epsilon, \mu, -T/2) = x(\eta, \epsilon, \mu, T/2)$, so $x(\eta, \epsilon, \mu, t)$ is T-periodic. Consequently, in order to find symmetric and periodic solutions of (1.1), we have to study the following equation

$$F(\eta, \mu, \epsilon) := Sx(\eta, \epsilon, \mu, T/2) = 0\,, \tag{2.2}$$

where $\mathbb{I} - S : \mathbb{R}^n \to \ker(\mathbb{I} - A)$ is a A-invariant projection, i.e. $AS = SA$. Let

$$V := \ker(\mathbb{I} - S); \quad p := \dim V = n - \dim\ker(\mathbb{I} - A)\,.$$

Since $F(\eta,\mu,0) = S\eta = 0$, we solve equation $\frac{1}{\epsilon}F(\eta,\mu,\epsilon) = 0$, $\epsilon \neq 0$. Now we suppose that

$$m := \dim\ker(\mathbb{I} - A) + k \geq p. \tag{2.3}$$

2.1. The case $\ker(\mathbb{I} - A) = \{0\}$

Then $S = \mathbb{I}$ and by (2.3), $m = k \geq p = n$. Now we can prove the following result.

Theorem 2.1. *If there exists* $\mu_0 \in \mathbb{R}^k$ *such that*

$$\int_0^{T/2} f(0,\mu_0,s)\,ds = 0 \quad \text{and} \quad \int_0^{T/2} D_\mu f(0,\mu_0,s)\,ds : \mathbb{R}^k \to \mathbb{R}^n \quad \text{is onto}. \tag{2.4}$$

Then there is a decomposition $\mathbb{R}^k = X_1 \oplus X_2$ *with* $\dim X_1 = n$ *and constants* $\epsilon_0 > 0$, $\delta_1^0 > 0$, $\delta_2^0 > 0$ *along with a unique* C^∞*-smooth function* $\mu_1(\mu_2,\epsilon) \in X_1$, $\epsilon \in (-\epsilon_0,\epsilon_0)$, $|\mu_2 - \mu_2^0| < \delta_2^0$ *such that* $\mu_1(\mu_2^0,0) = \mu_1^0$ *for* $\mu_0 = (\mu_1^0,\mu_2^0) \in X_1 \times X_2$ *with the following properties: For any* $|\mu_1 - \mu_1^0| < \delta_1^0$, $|\mu_2 - \mu_2^0| < \delta_2^0$ *and* $0 < |\epsilon| < \epsilon_0$, *equation* (1.1) *has a* T*-periodic and symmetric solution if and only if* $\mu_1 = \mu_1(\mu_2,\epsilon)$, *moreover this solution is unique, so that it is given by* $x(0,\epsilon,\mu_1(\mu_2,\epsilon),\mu_2,t)$ *and thus it is located near* 0 *in* $\mathbb{R}^n$.

Next we have the following result.

Theorem 2.2. *Assume* $\ker(\mathbb{I} - A) = \ker(\mathbb{I} - A^2) = \{0\}$. *Then* $x(t) = 0$ *is the only symmetric solution of* (1.1) *for any* $\epsilon \neq 0$ *small.*

Proof. By (1.5) we obtain $A^2 f(0,\mu,t) = f(0,\mu,t)$ and so $f(0,\mu,t) \in \ker(\mathbb{I} - A^2)$. Hence $f(0,\mu,t) = 0$ and the proof is finished. □

Moreover, we got result on the case $\ker(\mathbb{I} - A) \neq \{0\}$.

3. Asymptotic properties of symmetric and periodic solutions

3.1. The case $A = -\mathbb{I}$

Theorem 3.1. *Suppose* $n = 1$ *in Theorem* 2.1. *If in addition*

$$\int_0^T D_{xx} f(0,\mu_0,t)dt \neq 0\,,$$

then the T*-periodic and symmetric solution* $x(0,\epsilon,\mu_1(\mu_2,\epsilon),\mu_2,t)$ *is a saddle-node.*

The next theorem is on the case $n > 1$.

Theorem 3.2. *Suppose* $n > 1$. *Let the assumptions of Theorem* 2.1 *be satisfied. If in addition*

$$\mathcal{B} := \frac{1}{2}\int_0^T D_{xx} f(0,\mu_0,t)dt$$

has a negative eigenvalue with eigenvector x_0 *such that* $\Re\sigma(B) > 0$ *for* $B := 2Q\mathcal{B}x_0 \cdot |[x_0]^\perp$ *with the orthogonal projection* $Q : \mathbb{R}^n \to [x_0]^\perp$ *then the* T*-periodic* *and symmetric solution* $x(0, \epsilon, \mu_1(\mu_2, \epsilon), \mu_2, t)$ *has a local saddle-node dynamics. Hence it is unstable.*

Also we got results on the case $A \neq -\mathbb{I}$.

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On Dependence of the First Eigenvalue of the Sturm – Liouville Problem with Dirichlet Boundary Conditions on Parameter of Integral Condition

Svetlana Ezhak

Abstract. Estimates of the first eigenvalue λ_1 of the Sturm-Liouville problem with Dirichlet boundary conditions and integral condition to the potential are obtained.

Mathematics Subject Classification (2000). 34L15.

1. Introduction

Consider the Sturm — Liouville problem:

$$y''(x) - Q(x)y(x) + \lambda y(x) = 0, \tag{1.1}$$

$$y(0) = y(1) = 0, \tag{1.2}$$

where $Q(x)$ is a non-negative bounded function on $[0, 1]$ such that

$$\int_0^1 Q^\alpha(x)dx = 1, \quad \alpha \neq 0. \tag{1.3}$$

A function $y(x)$ is called a solution of problem (1.1) – (1.2) if it's defined on $[0, 1]$, it satisfies condition (1.2), its derivative $y'(x)$ is absolutely continuous, and equation (1.1) holds almost everywhere on $(0, 1)$.

We estimate the first eigenvalue λ_1 of this problem for different values of α.

Remark 1.1. The Dirichlet problem for the equation $y''(x) + \lambda q(x)y(x) = 0$, where $q(x)$ is a non-negative bounded summable function on $[0, 1]$ satisfying (1.3), was considered in [1].

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Consider the functional

$$R[Q,y] = \frac{\int_0^1 y'^2(x)dx + \int_0^1 Q(x)y^2(x)dx}{\int_0^1 y^2(x)dx}. \tag{1.4}$$

According to the variation principle

$$\lambda_1 = \inf_{y(x)\in H_0^1(0,1)} R[Q,y],$$

where $H_0^1(0,1)$ is a function space, defined on $(0,1)$, satisfying (1.2) and having the generalized derivative of the first order, with the norm

$$\|y(x)\|_{H_0^1(0,1)} = \left(\int_0^1 (y^2(x) + y'^2(x))dx\right)^{1/2}.$$

Put

$$m_\alpha = \inf_{Q(x)\in A_\alpha} \lambda_1, \quad M_\alpha = \sup_{Q(x)\in A_\alpha} \lambda_1,$$

where A_α is the set of the non-negative bounded on $[0,1]$ functions such that $\int_0^1 Q^\alpha(x)dx = 1$.

2. Results

Theorem 2.1. 1. *If $\alpha > 1$, then $m_\alpha = \pi^2$, $M_\alpha < \infty$, and there exist functions $u(x) \in H_0^1(0,1)$ and $Q(x) \in A_\alpha$ such that*

$$\inf_{y(x)\in H_0^1(0,1)} R[Q,y] = R[Q,u] = M_\alpha.$$

2. *If $\alpha = 1$, then $m_1 = \pi^2$, $M_1 = \frac{\pi^2}{2} + 1 + \frac{\pi}{2}\sqrt{\pi^2+4}$, and there exist functions $u(x) \in H_0^1(0,1)$ and $Q(x) \in A_\alpha$ such that*

$$\inf_{y(x)\in H_0^1(0,1)} R[Q,y] = R[Q,u] = M_1.$$

3. *If $0 < \alpha < 1$, then $m_\alpha = \pi^2$, $M_\alpha = \infty$.*
4. *If $\alpha < 0$, then $m_\alpha > \pi^2$, $M_\alpha = \infty$, and there exist functions $u(x) \in H_0^1(0,1)$ and $Q(x) \in A_\alpha$ such that*

$$\inf_{y(x)\in H_0^1(0,1)} R[Q,y] = R[Q,u] = m_\alpha.$$

3. Proofs of some results

Proof. 1) Note that $m_\alpha \geq \pi^2$ for any α, $\alpha \neq 0$.

2) Suppose $\alpha > 1$. Consider the functional

$$G[y] = \frac{\int_0^1 y'^2(x)dx + \left(\int_0^1 |y(x)|^p dx\right)^{2/p}}{\int_0^1 y^2(x)dx}, \; p = \frac{2\alpha}{\alpha-1} \tag{3.1}$$

Using Hölder inequality and (1.3), we obtain

$$\inf_{y(x)\in H_0^1(0,1)} R[Q,y] \le \inf_{y(x)\in H_0^1(0,1)} G[y]. \tag{3.2}$$

Denote $m = m(\alpha) = \inf\limits_{y(x)\in H_0^1(0,1)} G[y]$. From (3.2) we have $M_\alpha \le m$. To prove $M_\alpha = m$, we need the following lemma:

Lemma. *Suppose $\alpha > 1$ ($p = \frac{2\alpha}{\alpha-1} > 2$), and $m = \inf_{y(x)\in H_0^1(0,1)} G[y]$. Then there exists function $u(x) \in H_0^1(0,1)$, which is positive on $(0,1)$, satisfies the equation*

$$u''(x) - u^{p-1}(x) + mu(x) = 0, \tag{3.3}$$

and the conditions

$$u(0) = u(1) = 0, \tag{3.4}$$

$$\int_0^1 u^p(x)dx = 1, \tag{3.5}$$

such that $m = G[u]$.

Here m is the solution of system of the equations

$$\begin{cases} \int_0^H \frac{du}{\sqrt{mH^2 - mu^2 - \frac{2}{p}H^p + \frac{2}{p}u^p}} = \frac{1}{2}, \\ \int_0^H \frac{u^p(x)du}{\sqrt{mH^2 - mu^2 - \frac{2}{p}H^p + \frac{2}{p}u^p}} = \frac{1}{2}. \end{cases}$$

and $H = \max_{x\in[0,1]} u(x)$.

We prove that $M_\alpha = m$. We have $m = G[u] = \dfrac{\int_0^1 u'^2(x)dx + (\int_0^1 |u(x)|^p dx)^{2/p}}{\int_0^1 u^2(x)dx}$,

where $u(x)$ satisfies (3.3) and (3.4) – (3.5).

On the other hand,

$$M_\alpha = \sup_{Q(x)\in A_\alpha} \lambda_1 = \sup_{Q(x)\in A_\alpha} \inf_{y(x)\in H_0^1(0,1)} R[Q,y] \le m.$$

Since $u(x) \in H_0^1(0,1)$ and $u^{\frac{2}{\alpha-1}}(x) \in A_\alpha$, substituting these values for $y(x)$ and $Q(x)$ in $R[Q,y]$, we receive

$$R[u^{\frac{2}{\alpha-1}}, u] = \frac{\int_0^1 u'^2(x)dx + \int_0^1 u^{\frac{2}{\alpha-1}}(x)u^2(x)dx}{\int_0^1 u^2(x)dx} =$$

$$= \frac{\int_0^1 u'^2(x)dx + (\int_0^1 u^p(x)dx)^{2/p}}{\int_0^1 u^2(x)dx} = G[u] = m.$$

Thus we have the pair of functions $Q(x)$ and $y(x)$, so that the functional $R[Q,y]$ is equal to m. Hence $M_\alpha = m$.

3) Suppose $\alpha = 1$. Consider

$$L[y] = \frac{\int_0^1 y'^2(x)dx + \max_{x\in[0,1]} y^2(x)}{\int_0^1 y^2(x)dx}. \tag{3.6}$$

We have $\inf\limits_{y(x)\in H_0^1(0,1)} R[Q,y] \le \inf\limits_{y(x)\in H_0^1(0,1)} L[y]$.

Let's prove that $M_1 = \frac{\pi^2}{2} + 1 + \frac{\pi}{2}\sqrt{\pi^2+4}$.

Consider the functions

$$Q^*(x) = \begin{cases} 0, & 0<x<\tau, \\ \gamma, & \tau<x<1-\tau, \\ 0, & 1-\tau<x<1, \end{cases} \quad y^*(x) = \begin{cases} \sin\sqrt{\gamma}x, & 0<x<\tau, \\ \sin\sqrt{\gamma}\tau, & \tau<x<1-\tau, \\ \sin\sqrt{\gamma}(1-x), & 1-\tau<x<1, \end{cases}$$

where $\tau = \frac{\pi}{2\sqrt{m}}$, $\gamma = \frac{\pi^2}{2} + 1 + \frac{\pi}{2}\sqrt{\pi^2+4}$.

Note that $y^*(x)$, $(y^*(x))'$ are continuous on $[0,1]$, $y^*(0) = y^*(1) = 0$. The function $Q^*(x)$ satisfies (1.3). Thus $y^*(x)$ is the first eigenfunction for problem (1.1) – (1.2) – (1.3) with the potential $Q(x) = Q^*(x)$, and γ is the first eigenvalue. Then $\gamma \le M_1 = \sup_{Q(x)\in A_\alpha} \lambda_1$. Since $L[y^*] = \gamma$, we have $\inf_{y(x)\in H_0^1(0,1)} L[y] \le \gamma$.

Thus we have a sequence of inequalities:

$$\gamma \le M_1 = \sup_{Q(x)\in A_\alpha} \lambda_1 = \sup_{Q(x)\in A_\alpha} \inf_{y(x)\in H_0^1(0,1)} R[Q,y] \le$$
$$\le \sup_{Q(x)\in A_\alpha} \inf_{y(x)\in H_0^1(0,1)} L[y] = \inf_{y(x)\in H_0^1(0,1)} L[y] \le \gamma.$$

Hence $M_1 = \gamma$, and M_1 is attained at the function $Q^*(x)$. □

Remark 3.1. Note that we proved that the constant $M_1 = \frac{\pi^2}{2} + 1 + \frac{\pi}{2}\sqrt{\pi^2+4}$ is the accurate estimate of λ_1 from above. In [2] for M_1 only the result $M_1 \le \frac{\pi^2}{2} + 1 + \frac{\pi}{2}\sqrt{\pi^2+4}$ was formulated. The result $M_\alpha < \infty$ for $\alpha > 1$ is also obtained in [2].

Remark 3.2. The results 3)–4) were proved in [3].

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Phase shift for some special solution Korteweg–de Vries equation

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Abstract. The aim of this work is to find the phase shift of special solution of Kortevege-de Vries equation in Whitham zone. It is done without using any averaging method. We use a complimentary condition of Kortevege-de Vries equation and a fifth order ordinary equation.

Mathematics Subject Classification (2000). Primary 37K10; Secondary 76L05.

Keywords. nondissipative shock waves, phase shift, Korteweg-de Vries equation.

1. Introduction

In this work we investigate a special solution $u(t,x)$ of Kortevege-de Vries equation

$$u_t + uu_x + u_{xxx} = 0. \tag{1.1}$$

This solution simultaneously is a solution to the following ordinary differential equation (ODE)

$$\left(u_{xxxx} + \frac{5u_{xx}u}{3} + \frac{5u_x^2}{6} + \frac{5u^3}{18}\right)'_x - \frac{2u + xu_x - 3t(u_{xxx} + uu_x)}{6} = 0 \tag{1.2}$$

which is obtained as a combination of stationary parts of two symmetries of KdV equation. One of them is the fifth order higher (generalized) symmetry of the KdV equation,

$$u_{\tau_5} = \left(u_{xxxx} + \frac{5u_{xx}u}{3} + \frac{5u_x^2}{6} + \frac{5u^3}{18}\right)'_x \tag{1.3}$$

and the second one is classical dilation symmetry

$$u_{\tau_r} = 2u + xu_x - 3t(u_{xxx} + uu_x). \tag{1.4}$$

The equation (1.2) may be called as first high analog of Painleve I equation.

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This solution plays the main role in the problem of nondissipating shock waves [1, 2, 3]. In [1] the asymptotic behavior of these simultaneous solutions in the zone of undamped oscillations is given by quasisimple wave solutions to Whitham equations. But the phase shift was not found there. In this work we find the asymptotic behavior of these solutions including the phase shift by method suggested in [4].

2. Evaluation of phase shift

We are aimed at constructing asymptotics of the solution in the zone of undamped oscillations, when $t \to \infty$. Following familiar techniques, we change the variables by

$$u = t\,U(t,s), \quad s = \frac{x}{t^2}.$$

Then equations (1.1) and (1.2) take the form

$$\begin{aligned} t^{-5}U_{sss} + tU_t - 2sU_s + UU_s + U &= 0, \\ t^{-10}U_{sssss} + \frac{1}{6}t^{-5}(20U_sU_{ss} + (10U+3)U_{sss}) &+ \\ \tfrac{1}{6}(5U^2 - s + 3U)U_s - \tfrac{1}{3}U &= 0. \end{aligned} \tag{2.1}$$

We now look for a solution U of the system in the form of asymptotic series

$$U = U_0(\varphi,s) + t^{-7/4}U_1(\varphi,s) + t^{-7/2}U_2(\varphi,s) + \dots, \tag{2.2}$$

where U_0, U_1 and U_2 are 2π-periodic in the fast variable φ. This latter is assumed to be of the form

$$\varphi = t^{5/2}f(s) + n(s),$$

where by $n(s)$ is meant precisely the phase shift.

For the unknown function U_0 we get the nonlinear system

$$\begin{aligned} (f')^2\partial_\varphi^3 U_0 + \left(\tfrac{5f}{2f'} - 2s + U_0\right)\partial_\varphi U_0 &= 0, \\ (f')^4\partial_\varphi^5 U_0 + \frac{1}{6}\left(20\partial_\varphi U_0\partial_\varphi^2 U_0 + (10U_0+3)\partial_\varphi^3 U_0\right)(f')^2 &+ \\ \tfrac{1}{6}(5U_0^2 - s + 3U_0)\partial_\varphi U_0 &= 0, \end{aligned}$$

and nonhomogeneous linear systems for unknown functions U_1, U_2, which are omitted.

From the compatibility condition of the equations for U_0 we obtain a second order equation

$$(f')^2\partial_\varphi^2 U_0 + \frac{1}{2}U_0^2 - 2U_0 s + \frac{5}{2}s + 12s^2 + \frac{5}{4}(-24s + 2U_0 - 3)\frac{f}{f'} + \frac{75}{4}\frac{f^2}{(f')^2} = 0. \tag{2.3}$$

From this equation we can write:

$$(f'\partial_\varphi U_0)^2 = 2sU_0^2 - \frac{1}{3}U_0^3 - sU_0(24s+5) + (24s+3-U_0)\frac{5U_0 f}{2f'} - \frac{75U_0 f^2}{2(f')^2} + \frac{25}{4}b(s). \tag{2.4}$$

Here $b(s)$ is an arbitrary function (constant of integration).

From the compatibility condition of the equations for U_1 we obtain a system of the first order nonlinear equations for functions

$$a(s) = 5/2f/f', b(s),$$

which are omitted. From the compatibility condition of the equations for U_2 we obtain the equation of the next form:

$$\begin{array}{rcl} \partial_\varphi U_0(n''' + A_1 n'' + A_2 n') + \partial_s^3 U_0 + B_1 \partial_s^2 U_0 \partial_s U_0 + B_2 \partial_s^2 U_0 & + & \\ B_3(\partial_s U_0)^3 + B_4(\partial_s U_0)^2 + B_5 \partial_s U_0 + B_6 & = & 0, \end{array} \qquad (2.5)$$

where

$$A_i = A_i(s, f, a, b), \quad B_i = B_i(U_0, s, f, a, b)$$

some functions.

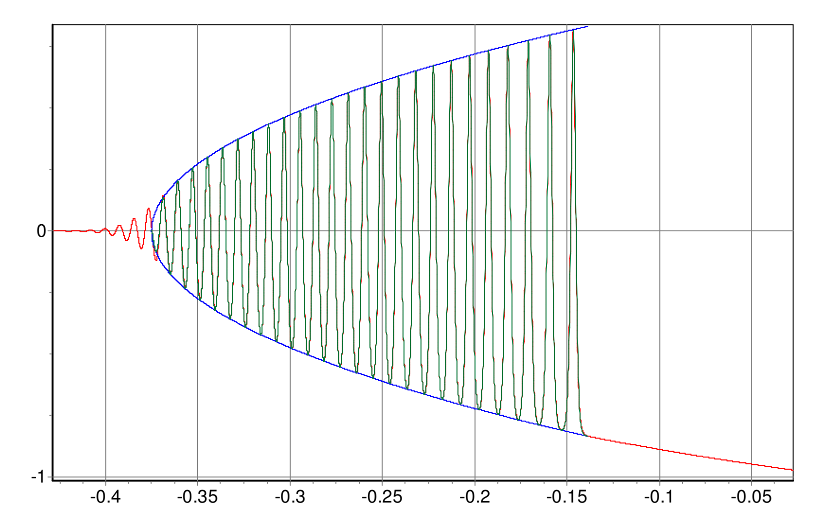

FIGURE 1. The numerical simulation for the function $U(t, z)$ corresponding to $t = 19$

From (2.5) we can immediately get:

$$n''' + A_1 n'' + A_2 n' = 0. \qquad (2.6)$$

The common solution of (2.6) have the form:

$$n(s) = C_1 + C_2 n_1(s) + C_2 n_2(s). \qquad (2.7)$$

By using numerical experiments we finally found

$$n(s) = \pi.$$

We observe the difference between numerical and asymptotic solutions and found that it decreases as $t^{-5/2}$ for this value of $n(s)$. On figure 1 one can observe numerical solutions for function $U(t,z)$ for $t=19$.

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On Fuchsian reduction of differential equations

Valentina A. Goloubeva

Abstract. The historical review and present results of the multidimensional Fuchsian theory and also exposition of some results concerning Fuchsian reduction of certain nonlinear equations are given.

Mathematics Subject Classification (2000). Primary 35Q40; Secondary 35Q15.

Keywords. Griffits method of reducing of the pole order, Knizhnik-Zamolodchikov equation, central elements of the algebra.

At the end of fifties of XX century I.M. Gelfand studied and delivered a course on the quantum electrodynamics and the theory of elementary particles and stated the problem generalizing the well known Riemann-Hilbert problem:

Construct some system of partial differential equations of hypergeometric type for the Feynman integral which is a term of some of finite order of the series for scattering matrix of quantum electrodynamics. The notion of the hypergeometric type partial differential equation was not known.

This statement was based on the ramification properties of such integrals on the Landau varieties investigated in some particular cases in the physical papers.

The first publication of this hypothesis was done by T. Regge in 1965 in the volume of Proceedings of the Conference "Battelle Rencontres" (1965). T. Regge was known with this hypothesis by O. Parasyuk who at this time actively worked in Moscow in the vicinity of N.N. Bogolyubov and I.M. Gel'fand,

However, from the beginning of the XX century there were known hypergeometric funcfions of two variables of Appell and Kampe de Feriet. R, Gérard try to study their properties which were very similar to the properties of the Gauss ordinary hypergeometric function and even to solve the multidimensional local Riemann-Hilbert problem, but this dream was not realized because he made the mistake,

These investigations were continued by V.A. Golubeva who obtained the systems of partial differential equations of Fuchsian type for hypergeometric functions of two variables of Appell and Kampe de Feriet and obtained the commutation

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relations between the coefficients of these equations. The definition of multidimensional Fuchsian type equations was given.

The partial differential equations for certain terms of the series for the S-matrix were obtained by V. Golubeva. These equations were of the Fuchsian type. The method of their derivation was based on Ph. Griffiths algebraic-geometric theorem on reduction of order of pole of rational differential form. This method often gives not Fuchsian system, but some Fuchsian reduction permits to obtain global Fuchsian system.

The next step in investigation of ramification properties of Feynman integrals was the V. Enolskii paper who obtained the local expansion of the Feynman integral in the vicinity of the ordinary point of its Landau variety,

After the construction the first series of the multidimensional Fuchsian systems the question was stated whether there are the other objects satisfying the partial differential equations of Fuchsian type. So the first multidimensional Fuchsian systems were obtained. The next problem was to find a sufficiently reach class of physical models which can be described by equations of Fuchsian type. Happiness, the physicists prepared such models in the conformal field theory. The well known Knizhnik-Zamolodchikov equation appeared the equation of Fuchsian type whose singular locus is the set of hyperplanes in $\mathbb{C}^n$ turned out to be the reflection hyperplanes of Weyl group of A_n type and with the coefficients belonging to the tensor power of the corresponding Lie algebra A_n. This equation was investigated in details (see, for ex. Chr. Kassel, 1995). It was natural to state the problem of construction and investigation of the the equation of Knizhnik-Zamolodchikov type with other symmetry algebras. For ex., such equation with symmetries corresponding to the Lie algebra B_n was presented in A. Leibman's paper (1994) who using the Casimir element of the second order obtained the integrability relations for this equations and gave the solutions of this relations. He considered only the one-parametric case of the model. However, for the characterization of the models with symmetries described by the Lie algebra of B_n it is necessary to use two parameters. At the present time the problem of construction two-parametric Knizhnik-Zamolodchikov equation of B_n type is not solved. Using the higher Casimir element it is possible to construct Knizhnik-Zamolodchikov equations of some other types but not two-parametric.

It is necessary to note that the coefficients of Knizhnik-Zamolodchikov equations can be not only elements of (tensor products)of algebras, but also elements of other nature, for ex. knots and links. The application of central elements such as Casimir and Capelli elements of higher order for generation of new Knizhnik-Zamolodchikov equations is very perspectively.

The contemporary statement of the multidimensional problem analogues to the Riemann-Hilbert-Gelfand-Regge problem looks in the following manner:

In $\mathbb{C}^n$ the arrangement of hyperplanes is given being the reflection hyperplanes of Weyl group of one of the types $B_n, C_n, D_n, G_2, F_4, E_6, etc.$. To construct the equations of the Knizhnik-Zamolodchikov type with singular locus on the given arrangement.

Note. In the statement of the problem above the monodromy is not prescribed (as in the one-dimensional case) because it depends on the symmetry algebra of given arrangement. The choice of the coefficients sets of the equations is possible in reach limits.

At present time in many sections of mathematical physics the method of Fuchsian reduction is applied as for linear so for non linear system equations. It consists in applying the series of local transformations as independent so dependent variables for deriving the local equations of Fuchsian type. Such a transformation permits to use a reach technique of local developments of the theory of Fuchsian and Painleve equations. The method of Fuchsian reduction is applied in astronomy, general relativity, nonlinear optics, soliton theory, differential geometry etc. In particular, in the theory of solitons the Fuchsian reduction permits to give the answer to the question whether the formal solutions of the completely integrable equations represent actual solutions.

In the talk as the historical review and present results of the multidimensional Fuchsian theory and also exposition of some results concerning Fuchsian reduction of certain nonlinear equations will be given.

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On convergence of a formal solution to an ODE

Irina V. Goryuchkina

Abstract. In 2004 Prof. Bruno had formulated a theorem on convergence of a power series solution to an ordinary differential equation. We proved this theorem in two cases: for rational power exponents in the expansion and for complex and irrational (but not rational) power exponents with one complex or irrational generatrix of the set of power exponents in the expansion. In the proofs we used Power Geometry, the majorant method and some new trick. These results are new.

Mathematics Subject Classification (2000). Primary 34M55; Secondary 34E05.

Keywords. Formal solutions, convergence of series, majorant method.

1. The general case

We consider an ordinary differential equation of the form

$$f(x, y, y', \dots, y^{(n)}) = 0, \tag{1.1}$$

where $f(x, y, y', \dots, y^{(n)})$ is a polynomial of its variables.

Let for $|x| \to 0$ equation (1.1) have formal solution of the form

$$y = \sum c_s x^s, \ \ s \in \mathbf{K} \subset \mathbb{C} \tag{1.2}$$

where power exponents s are complex, $\operatorname{Re} s$ increase, number of power exponents s with the same real parts $\operatorname{Re} s$ is finite, coefficients c_s are complex constants. We enumerate power exponents s in order of increasing real parts $\operatorname{Re} s : \operatorname{Re} s_0 \leq \operatorname{Re} s_1 \leq \operatorname{Re} s_2 \leq \dots$.

By the substitution

$$y = \sum_{s=s_0}^{s_m} c_s x^s + u, \tag{1.3}$$

where $m \in \mathbb{Z}$, $m \geq 0$, $\operatorname{Re} s_m \geq n$, power exponents s and coefficients c_s are from (1.2), we reduce equation (1.1) to the form

$$f_1(x,\, u) \stackrel{def}{=} \mathcal{L}(x) u + g(x,\, u,\, u', \dots, u^{(n)}) = 0, \tag{1.4}$$

where the linear differential operator $\mathcal{L}(x)$ has form

$$\mathcal{L}(x) = x^v \sum_{l=1}^{n} a_l x^l \frac{d^l}{dx^l}, \tag{1.5}$$

$\mathcal{L}(x) \not\equiv 0$, $v \in \mathbb{C}$, a_l are complex constants. The function g contains terms independent of u, $u', \dots, u^{(n)}$, linear terms in u, $u', \dots, u^{(n)}$ of the form $c\, x^{v_1+l}\, u^{(l)}$ with $\operatorname{Re} v_1 > \operatorname{Re} v$, $l \le n$, $c = \mathrm{const} \in \mathbb{C}$, and nonlinear terms in u, $u', \dots, u^{(n)}$.

The linear differential operator $\mathcal{L}(x)$ has eigenvalues $\lambda_1, \dots, \lambda_n$. We order these eigenvalues by increasing real parts $\operatorname{Re}\lambda_1 \le \dots \le \operatorname{Re}\lambda_n$. We suppose that $\operatorname{Re} s_m \ge \operatorname{Re}\lambda_n$ in substitution (1.3). Then equation (1.4) has unique solution of the form

$$u = \sum_{s=s_{m+1}}^{\infty} c_s x^s, \tag{1.6}$$

where power exponents $s \in \mathbb{C}$, $\operatorname{Re} s$ increase, complex coefficients c_s are uniquely determined constants.

Theorem 1.1. [1] *If in the equation* (1.4)*, which we obtain after substitution* (1.3) *in the equation* (1.1)*, the order of the highest derivative in* $\mathcal{L}(x)u$ *is equal to the order of the highest derivative in the sum* f_1*, then the series* (1.6) *converges for sufficiently small* $|x|$ *and* $\arg x \in (-\pi, \pi)$.

2. The case of rational power exponents

Let for $x \to 0$ equation (1.1) have the formal solution

$$y = \sum_{s=s_0}^{\infty} c_s x^s, \tag{2.1}$$

where $s \in \mathbb{Z}$, $s \ge s_0 > -\infty$, s increase, c_s are constant coefficients.

After substitution (1.3) with $s \in \mathbb{Z}$, $s_m \ge n$, $s_m \ge s_0$, $s_m \ge \operatorname{Re}\lambda_n$, the equation (1.1) takes form (1.4). Equation (1.4) contains integer power exponents of x and nonnegative integer powers of u, $u', \dots, u^{(n)}$.

For $x \to 0$ equation (1.4) has unique solution of the form

$$u = \sum_{s=s_m+1}^{\infty} c_s x^s, \tag{2.2}$$

where power exponents $s \in \mathbb{N}$, s increase, complex coefficients c_s are uniquely determined constants.

Theorem 2.1. *If in the equation* (1.4)*, which we obtain by means of substitution* (1.3) *with* $s \in \mathbb{Z}$*,* $s_m \ge n$*,* $s_m \ge s_0$*,* $s_m \ge \operatorname{Re}\lambda_n$ *from the equation* (1.1)*, the order of the highest derivative in* $\mathcal{L}(x)u$ *is equal to the order of the highest derivative in the sum* f_1*, then the series* (2.2) *converges for sufficiently small* $|x|$.

Here we consider case when expansion (2.2) contains only integer power exponents. The case of rational power exponents with a finite common denominator m is reduced to this case by the substitution $z = x^{1/m}$.

The constraint on $\arg x$ is necessary only for expansions (1.6) with irrational or complex power exponents and can be dropped in the case under study.

The proof of Theorem 2.1 was published in [2] and [3].

3. The case of complex power exponents

We consider formal solution of the form (1.2), where

$$\mathbf{K} = \{s_0 + m_1 r_1 + m_2 r_2,\ m_1, m_2 \in \mathbb{Z},\ m_1, m_2 \geq 0\}, \tag{3.1}$$

$$\begin{gathered} s_0 \in \mathbb{C} \setminus \mathbb{Q},\ \ r_1 = \langle R_1,\, (1, s_0)\rangle,\ r_2 = \langle R_2,\, (1, s_0)\rangle, \\ R_1 = (\alpha_1,\, \beta_1),\ R_2 = (\alpha_2,\, \beta_2),\ R_1,\, R_2 \in \mathbb{Z}^2. \end{gathered} \tag{3.2}$$

After substitution (1.3) with $s \in \mathbf{K}$, the equation (1.1) takes the form (1.4). The function f_1 contains complex power exponents of x and nonnegative integer power exponents $u, u', \dots, u^{(n)}$.

For $|x| \to 0$, $\arg(x) \in (-\pi, \pi)$, the equation (1.4) has unique solution of the form (1.6) with power exponents $s \in \mathbb{C}$.

Theorem 3.1. *If in the equation* (1.4)*, which we obtain after substitution* (1.3) *from the equation* (1.1)*, the order of the highest derivative in $\mathcal{L}(x)u$ is equal to the order of the highest derivative in the sum f_1, then the series* (1.6) *converges for sufficiently small $|x|$ and* $\arg(x) \in (-\pi, \pi)$.

The proof of Theorem 3.1 was published in [4]. The similar theorem is true for irrational s_0, r_1, r_2 in (3.1).

4. On solutions of the sixth Painlevé equation

According to Theorems 2.1 and 3.1 all power expansions of solutions to the sixth Painlevé equation near its three singular points are convergent. Near regular point all expansions are power and form 17 families. They converge according to theorem 2.1, and in some cases according to the Cauchy Theorem.

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Asymptotic expansions and forms of solutions to the sixth Painlevé equation

Irina V. Goryuchkina

Abstract. Using algorithms of Power Geometry, we have found asymptotic expansions and asymptotic forms of solutions to the sixth Painlevé equation for all values of its four complex parameters near its three singular points $x = 0, 1, \infty$ and near its regular points $x = x_0 \neq 0, 1, \infty$. There are five types of asymptotic expansions of solutions, namely, power, power-logarithmic, complicated, exotic and half-exotic. Near all singular points of the equation they form 117 families. Most of these expansions are new. Near regular points of the equation there are 17 families of power expansions. Among them 8 families are new. Besides we stated that near singular points of the sixth Painlevé equation the Boutroux type elliptic asymptotic forms are absent in contrary to other Painlevé equations.

Mathematics Subject Classification (2000). Primary 34M55; Secondary 34E05.

Keywords. Painlevé equations, asymptotic forms, asymptotic expansions.

1. Asymptotic expansions near singular points of the equation

The sixth Painlevé equation is

$$y'' = \frac{(y')^2}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right) - y'\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right) + \\ + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[a + b\frac{x}{y^2} + c\frac{x-1}{(y-1)^2} + d\frac{x(x-1)}{(y-x)^2}\right], \qquad (1.1)$$

where a, b, c, and d are complex parameters, x and y are complex variables, and $y' = dy/dx$. This equation has three singular points $x = 0$, $x = \infty$, and $x = 1$, and three main symmetries that allow to transfer expansions of solutions near one its singular point to expansions of solutions near the same or other its singular point. These symmetries are associated with three changes of variables

$$1)\ \ x = z,\ y = z/w;\quad 2)\ \ x = 1/z,\ y = 1/w;\quad 3)\ \ x = 1 - z,\ y = 1 - w. \qquad (1.2)$$

At first we obtained 21 families of such asymptotic expansions of solutions to the equation (1.1) near the singular point $x = 0$, for which the order of the first term is less than unity. We refer to these expansions as *basic expansions*. All other asymptotic expansions of solutions to the equation (1.1) near its three singular points $x = 0$, $x = 1$, and $x = \infty$ we obtained from the basic expansions using symmetries (1.2). Altogether the expansions form 117 families [1]. At $x = 0$ we seek asymptotic expansions of the form

$$y = c_r x^r + \sum_s c_s x^s, \tag{1.3}$$

where power exponents r and s are complex, $\operatorname{Re} s > \operatorname{Re} r$, $\operatorname{Re} s$ increase. They belongs to one of three types: power, power-logarithmic or complicated. Expansions of these types have a finite number of terms s with the same value $\operatorname{Re} s$. Coefficients c_r and c_s are varied according to types:
Type 1. c_r and c_s are constant (*power expansions*);
Type 2. c_r is constant, c_s are polynomials in $\log x$ (*power-logarithmic expansions*);
Type 3. c_r and c_s are series in decreasing powers of $\log x$ (*complicated expansions*).

At $x = 0$ we also consider expansions of the form

$$y = \sum_\rho c_\rho x^\rho + \sum_s c_s x^s, \tag{1.4}$$

where power exponents ρ and s are complex, all $\operatorname{Re}\rho$ are the same, $\operatorname{Re}\rho < \operatorname{Re} s$, $\operatorname{Re} s$ increase, the first sum contains more than one term, complex coefficients c_ρ and c_s are polynomials in $\log x$. We differ two types of the expansions (1.4).
Type 4. The first sum in (1.4) contains a finite number of terms. Coefficients c_ρ are constants for extreme values of $\operatorname{Im}\rho$. The number of power exponents s with the same real parts $\operatorname{Re} s$ are finite (*half-exotic expansions*);
Type 5. The first sum in (1.4) contains infinite number of terms, but $\operatorname{Im}\rho$ are bounded either below or above. Coefficient c_ρ is constant for extreme value of $\operatorname{Im}\rho$ (*exotic expansions*).
Besides we suppose that $\arg x$ is bounded from two sides. Here we give examples for each type of expansions.

Theorem 1.1. *For $x \to 0$ there exists the family of power expansions of solutions to the equation* (1.1) *with two parameters c_r and r, and constant coefficients, which has form*

$$\mathcal{A}_0 : y = c_r x^r + \sum_s c_s x^s, \tag{1.5}$$

where complex power exponent r is arbitrary with $\operatorname{Re} r \in (0,1)$, complex power exponents $s \in \{r + lr + m(1-r),\ l, m \geq 0;\ l + m > 0;\ l, m \in \mathbb{Z}\}$; complex coefficient c_r is arbitrary nonzero constant, other complex coefficients c_s are uniquely determined constants. Family $\mathcal{A}_0$ exists for all values of parameters of equation (1.1).

The family $\mathcal{A}_0$ was known.

Theorem 1.2. *For $x \to 0$ and $a = c \neq 0$, $2\sqrt{2a} \in \mathbb{Z}\backslash\{0\}$ there exists the family of power-logarithmic expansions of solutions to the equation* (1.1) *of the form*

$$\mathcal{B}_2: \quad y = 2 + \sum_{s=1}^{\infty} c_s(\log x)x^s, \tag{1.6}$$

where coefficients $c_s(\log x)$ are polynomials in $\log x$.

Theorem 1.3. *For $x \to 0$ and $a \neq c \neq 0$ there exists the family of complicated expansions of solutions to the equation* (1.1) *of the form*

$$\mathcal{B}_3: \quad y = \psi_0 + \sum_{\sigma=1}^{\infty} \psi_\sigma x^\sigma, \tag{1.7}$$

where

$$\psi_0 = \frac{2}{c-a}\frac{1}{\log^2 x} + \frac{c_{-3}}{\log^3 x} + \sum_{s=4}^{\infty}\frac{c_{-s}}{\log^s x} = \frac{2(c-a)}{(c-a)^2(\log x + C_0)^2 - 2a},$$

coefficients c_{-3} and C_0 are arbitrary constants, other coefficients c_{-s} are uniquely determined constants; ψ_σ are series in decreasing powers of $\log x$.

Theorem 1.4. *For $x \to 0$ there exists the family of half-exotic expansions of solutions to the equation* (1.1) *with two parameters c_ρ and ρ, and constant coefficients, which has form*

$$\mathcal{H}_0: \quad y = c_\rho x^\rho + c_1 x + c_{2-\rho}x^{2-\rho} + \sum_s c_s x^s + \ldots, \tag{1.8}$$

where $\rho - 1$ is pure imaginary arbitrary constant, s runs over the set $\{l + k(\rho - 1);\ l, k \in \mathbb{Z};\ l \geq 2, |k| \leq l\}$, complex coefficient c_ρ is an arbitrary constant, other complex coefficients c_1, $c_{2-\rho}$ and c_s are uniquely determined constants.

Theorem 1.5. *For $x \to 0$ there exists the family of exotic expansions of solutions to the equation* (1.1) *with two parameters C_1 and ρ, and constant coefficients, which has form*

$$\begin{aligned}\mathcal{B}_0^\tau: \quad y &= \frac{\rho^2}{\beta\cos^2[\log(C_1 x)^\gamma] + \alpha\sin^2[\log(C_1 x)^\gamma]} + \sum_{\operatorname{Re} s \geq 1} c_s x^s = \\ &= x^\rho\left(c_\rho + \sum_{k=1}^{\infty}\tilde{c}_k x^{k\rho}\right) + \sum_{\operatorname{Re} s\geq 1} c_s x^s,\end{aligned} \tag{1.9}$$

where ρ is a pure imaginary nonzero arbitrary constant, $s \in \{\rho + l\rho + m(1 - \rho);\ l, m \geq 0;\ l + m > 0;\ l, m \in \mathbb{Z}\}$, $\tau = \operatorname{sgn}(\operatorname{Im}\rho)$, $\alpha + \beta = (\rho^2 - 2c + 2a)/(2a)$, $\alpha\beta = \rho^2/(2a)$, $2\gamma = i\rho$, complex coefficients $\tilde{c}_k$ and c_s are uniquely determined constants.

2. Asymptotic expansions near a regular point of the equation

In neighborhood of a regular point $x = x_0 \neq 0,\ 1,\ \infty$ of the sixth Painlevé equation (1.1) there are 17 families of power expansions of its solutions for all values of its four parameters [2]. The expansions are Laurent or Taylor series. Among them 1 family of expansions has the pole of the second order, 2 families of expansions have poles of the first order, other ones are families of Taylor expansions. 8 families of expansions are new (compare with [3]).

3. Boutroux type elliptic asymptotic forms

Theorem 3.1. *Near singular points $x = 0,\ 1,\ \infty$ of the sixth Painlevé equation, the Boutroux type elliptic asymptotic forms of solutions are absent* [4] *in contrary to other Painlevé equations* [5]*,* [6]*,* [7]*.*

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Painlevé Formal Test and Briot-Bouquet Systems

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Hierarchies of ordinary differential equations with Painlevé property and properties their solutions are a topic of great current interest. For higher-order differential equations, the Painlevé classifcation problem proved to be very dicult, and the most complete results have only been obtained for equations of polynomial type.

In the present paper, for the differential equations

$$w^{(n)} = R(w^{(n-1)}, \dots, w, z), \tag{1}$$

where $R(w^{(n-1)}, \dots, w, z)$ is a rational function of the first n arguments whose coeffcients are analytic with respect to z in the domain $G \subset \mathbb{C}$, we clarify the relationship between the resonance method, which is also referred to as the Painlevé formal test, and the representation of these equations in the form of Briot-Bouquet systems.

The test itself is shortly the successive performing of the following steps.

We find the least possible power of the variable t in the formal expansion of a solution of equation (1) of the form

$$w = \sum_{j=1}^{\infty} c_j t^{j-k}, \ t = z - z_0. \tag{2}$$

If $c_0 \neq 0$ and the numbers k are positive integers, then the first step is complete, and the analysis of the equation can be continued. Note that the case of negative integers k can be reduced to the previous case by the replacement $w \to w^{-1}$, and the case $k = 0$ can be reduced to it by the replacement $w \to w + c_0$. The second step of the test is related to finding the Fuchs indices for each pair $(c_0; k)$. The Fuchs indices, or resonances, are defined as the indices j_r of the coefficients c_j in the expansion of the solution (2) for which the coefficients c_j remain arbitrary. The Fuchs indices are found from the polynomial equation $Q(r) = 0$, which is referred to as the resonance equation. If, for each pair $(c_0; k)$, all roots of the resonance equation are simple and roots other than -1 and possibly zero are positive integers, then the second step of the formal test is considered to be complete. If all coefficients c_j in the expansion (2) can be determined, and the number of arbitrary coefficients among them is equal to $n-1$, then we assume that the third step is complete for the differential equation (1), and, therefore, the whole Painlevé formal test is complete.

Therefore, the passage of the Painlevé formal test for the differential equation (1) eliminates the presence of movable algebraic singular points of solutions of that equation. In this connection, obviously, it is assumed that there can exist movable singular points of logarithmic type or movable essentially singular points.

The expansion (2) used in the above-described test is formal; the proof of its converges required additional investigations. It turns out that, to prove the formal expansion (2) in the general case, one can use the theory of Briot-Bouquet systems

$$tu_j' = f_j(t, u_1, \ldots, u_n), \ j = 1, \ldots, n \tag{3}$$

with functions f_j analytic in some neighborhood of the point $t = u_1, \ldots, u_n = 0$ and with the conditions $f_j(t, u_1, \ldots, u_n) = 0$.

First we prove that for all pairs $(c_0; k)$; $k \in \mathbb{N}$; $c_0 \neq 0$; one root of the resonance equation $Q(r) = 0$ is equal to -1. The main result is

Theorem 1. If equation (1) passes the Painlevé formal test, then, in a neighborhood of a singular point, it can be reduced to a Briot-Bouquet system (3), and expansions of the form (2) for equation (1) are convergent.

We also consider the invese result to theorem 1 and the case of rational k. In the last case we looking for the algebraic solutions of equation (1).

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Solutions of the Chazy System

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The six Painlevé equations, were discovered by Painlevé and Gambier through the classification problem for ordinary differential equations, whose solutions have no movable branch points. Today this property is referred to as the Painlevé property. For higher-order equations, the Painlevé classification problem proves to be very complicated, and the most complete results have so far been obtained only for higher-order polynomial equations. The Chazy paper [1] is one of the first papers on the classification of higher-order equations with respect to the Painleve property; it deals with the analysis of the Painlevé property of the equation

$$y''' = \sum_{k=1}^{6} \frac{(y' - a_k')(y'' - a_k'')}{y - a_k} + \sum_{k=1}^{6} \frac{A_k(y' - a_k')^3 + B_k(y' - a_k')^2 + C_k(y' - a_k')}{y - a_k} + \\ + Dy'' + Ey' + \prod_{k=1}^{6}(y - a_k) \sum_{k=1}^{6} \frac{F_k}{y - a_k}; \tag{1}$$

here the poles $a_k = a_k(z)$ are finite and distinct and, in general, are functions of the independent variable z. The paper [1] also presents the Chazy system of 31 algebraic and differential equations

$$\sum_{j=1}^{6} A_j = 0, \sum_{j=1}^{6} a_j A_j = -6, \sum_{j=1}^{6} a_j^2 A_j = -2\sum_{j=1}^{6} a_j, 2A_k^2 + \sum_{j=1}^{6} \frac{A_k - A_j}{a_k - a_j} = 0, \qquad (\mathcal{A})$$

where $k = 1, \ldots, 6$ $(j \neq k)$,

$$\sum_{j=1}^{6}(B_k - B_j)\Big(-\frac{A_k}{2} - \frac{1}{a_k - a_j}\Big) + A_k' - \sum_{j=1}^{6} \frac{a_k' - a_j'}{a_k - a_j}(A_k - 3A_j) - \frac{3}{2}A_k \sum_{i=1}^{6} a_i' A_i = 0, \quad (\mathcal{B})$$

$$\Big(-2A_k C_k - \sum_{j=1}^{6} \frac{C_k - C_j}{a_k - a_j}\Big) + \sum_{j=1}^{6} \frac{3A_j(a_k' - a_j')^2 + +(2B_j - B_k)(a_k' - a_j') + a_k'' - a_j''}{a_k - a_j} - \\ - B_k^2 + B_k' - B_k D + E = 0, \qquad (\mathcal{C})$$

$$2D+\sum_{j=1}^{6}(B_j-3a_j'A_j)=0, \qquad (\mathcal{D})$$

$$\sum_{j=1}^{6}F_j=0, \sum_{j=1}^{6}a_jF_j=0, \sum_{j=1}^{6}a_j^2F_j=0,$$

$$-a_k'''-B_kC_k+C_k'+D(a_k''-C_k)+Ea_k'+F_k\prod_{j=1}^{6}(a_k-a_j)+$$
$$+\sum_{j=1}^{6}\frac{A_j(a_k'-a_j')^3+B_j(a_k'-a_j')^2-(C_k-C_j)(a_k'-a_j')+(a_k'-a_j')(a_k''-a_j'')}{a_k-a_j}=0, \qquad (\mathcal{F})$$

for 26 unknown functions $A_k=A_k(z)$, $B_k=B_k(z)$, $C_k=C_k(z)$, $D=D(z)$, $E=E(z)$, and $F_k=F_k(z)$, whose solution, as Chazy claims, determines necessary and sufficient conditions for Painleve property of (1).

The problem of solving the Chazy system remains open. Moreover, in some papers the equations of the system are written out in a form different from the original, and Chazy himself did not present the derivation of the system in [1].

In the present paper, we give a derivation of system $(\mathcal{A})$–$(\mathcal{F})$, the solution of system $(\mathcal{A})$ expressed in expanded form via the parameters a_k [2], and the solution of system $(\mathcal{B})$–$(\mathcal{F})$ in the case of constant a_k as well as in the general case under some restrictions, which permit one to write out the solution in closed form. All computations have been carried out with the Mathematica computer algebra system.

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Solving PVI by Isomonodromy Deformations

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We consider the sixth Painlevé equation, with independent variable x and dependent variable $y(x)$, in standard form [8], with parameters α, β, γ and $\delta \in \mathbf{C}$. *Solving* PVI means that: **i)** We determine the critical behavior (or asymptotic expansion) of $y(x)$, by an explicit formula in terms of two integration constants. **ii)** We solve the *connection problem*, namely we find the *explicit* relations among couples of integration constants at different critical points. We review here the results of the isomonodromy deformation method, which answers i) and ii).

PVI is the isomonodromy deformation equation [9] of a 2×2 Fuchsian system

$$\frac{d\Psi}{d\lambda} = \left[\frac{A_0(x)}{\lambda} + \frac{A_x(x)}{\lambda - x} + \frac{A_1(x)}{\lambda - 1} \right] \Psi, \quad y(x) = \frac{x(A_0)_{12}}{x[(A_0)_{12} + (A_1)_{12}] - (A_1)_{12}}. \quad (1)$$

The eigenvalues of $A_i(x)$ are algebraic functions of α, β, γ and δ. Let M_0, M_x, M_1 be the x-independent monodromy matrices of Ψ w.r.t. a given basis of loops. There is a one-to-one correspondence between a point in the space of monodromy data and a branch $y(x)$ (except for one M_i or $M_1 M_x M_0 = I$. See [5], [6]). As a consequence, the integration constants of $y(x)$ can be parameterized in terms of the monodromy data. Jimbo [8] provided the critical behavior and its parameterization in terms of monodromy data for a wide class of solutions. We review here the matching procedure of [4], which gives the results of [8] plus the other critical behaviors and their parameterization.

i) Behavior of $y(x)$: We consider $x \to 0$ ($x \to 1$ and $x \to \infty$ can be obtained from the results at $x = 0$ by symmetries). Let us divide the λ-plane into two domains. The "out" domain is $|\lambda| \geq |x|^{\delta_{OUT}}$, $\delta_{OUT} > 0$. (1) can be approximated by:

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{A_0 + A_x}{\lambda} + \frac{x A_x}{\lambda^2} \sum_{n=0}^{N_{OUT}} \left(\frac{x}{\lambda} \right)^n + \frac{A_1}{\lambda - 1} \right] \Psi_{OUT}, \quad (2)$$

The "in" domain is $|\lambda| \leq |x|^{\delta_{IN}}$, $0 < \delta_{IN} \leq \delta_{OUT}$. (1) can be approximated by:

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} - A_1 \sum_{n=0}^{N_{IN}} \lambda^n \right] \Psi_{IN}, \quad (3)$$

where N_{IN}, N_{OUT} are suitable integers. We impose that

$$\Psi_{OUT}(\lambda,x)\sim\Psi_{IN}(\lambda,x),\ |x|^{\delta_{OUT}}\le|\lambda|\le|x|^{\delta_{IN}},\ x\to 0 \qquad (4)$$

This *matching* produces the leading terms of $A_0(x)$, $A_1(x)$ and $A_x(x)$ as $x\to 0$, therefore it produces the leading term(s) of $y(x)$ in (1). The results are below:

1) Two-complex parameter (c_{11}, $\sigma\in\mathbf{C}$, $0\le\Re\sigma<1$) solutions [8],[3],[4],[6]

$$y(x)=\sum_{n=1}^{\infty}x^n\sum_{m=-n}^{n}c_{nm}x^{m\sigma},\ c_{11}\neq 0,\ c_{nm}=c_{nm}(c_{11},\sigma),\quad \text{convergent}$$

In particular, for $\sigma=2i\nu$, $\nu\in\mathbf{R}$ the above becomes a three-real parameter solution

$$y(x)=x\Big[A\sin(\nu\ln x+\phi)+B+\sum_{n=1}^{\infty}x^n\sum_{m=-n-1}^{n+1}c_{nm}x^{m\sigma}\Big],\ \phi=\phi(c_{11}) \qquad (5)$$

2) Three-real parameters ($\nu\in\mathbf{R}$, $d\in\mathbf{C}$) solutions [6]

$$y(x)=\Big[\sum_{n=0}^{\infty}x^n\sum_{m=-n-1}^{n+1}c_{nm}\ e^{2imd}x^{2im\nu}\Big]^{-1}\quad\text{convergent}$$

$$=\Big[A\sin(\nu\ln x+\phi(d))+B+\sum_{n=1}^{\infty}x^n\sum_{m=-n-1}^{n+1}c_{nm}e^{2imd}x^{2im\nu}\Big]^{-1} \qquad (6)$$

All c_{nm}, A, B are determined by d and ν. (6) is obtained form (5) by the symmetry $y(x)\mapsto x/y(x)$. If $\arg x$ and $\arg(1-x)$ are bounded (namely, when we considered $y(x)$ as a branch), then (6) admits two infinite sequences of *movable poles* in the neighborhood of $x=0$ (see [7]).

3) If $\sqrt{-2\beta}\pm\sqrt{1-2\delta}\notin\mathbf{Z}$, there are one-complex parameter ($c_{11}\in\mathbf{C}$) solutions [4],[6]

$$y(x)=\sum_{n=1}^{\infty}x^n\sum_{m=0}^{n}c_{nm}x^{m\sigma}=\sum_{k=0}^{\infty}y_k(x)\ (c_{11}\ x^{\sigma})^k,\ \sigma\in\{\sigma^+_{\beta\delta},\sigma^-_{\beta\delta}\},\quad\text{convergent}$$

where $\sigma^{\pm}_{\beta\delta}:=(\sqrt{-2\beta}\pm\sqrt{1-2\delta})\text{sgn}(\Re(\sqrt{-2\beta}\pm\sqrt{1-2\delta}))$, and $y_k(x)$ are Taylor expansions of the form $y_1(x)=O(x)$, $y_k(x)=O(x^k)$. c_{nm} are determined by c_{11}. For $c_{11}=0$, the above becomes a Taylor series with no free parameter. If $|\sqrt{-2\beta}\pm\sqrt{1-2\delta}|<1$, the above solutions are defined also for $\sigma\in\{\pm(\sqrt{-2\beta}+\sqrt{1-2\delta}),\pm(\sqrt{-2\beta}-\sqrt{1-2\delta})\}$.

4) If $\sqrt{-2\beta}+\sqrt{1-2\delta}=N$ or $\sqrt{-2\beta}-\sqrt{1-2\delta}=N$, $N\in\mathbf{Z}$, there are one-complex parameter ($a\in\mathbf{C}$) solutions [4],[5]

$$y(x)=\sum_{n=1}^{|N|}b_nx^n+\Big(a+b_{|N|+1}\ln x\Big)x^{|N|+1}+\sum_{n=|N|+2}^{\infty}P_n(\ln x;a)x^n,\ a\in\mathbf{C} \qquad (7)$$

where b_n are functions of α,β,γ and δ, a is a free parameter, $P_n(\ln x;a)$ are polynomials of $\ln x$ of degree $n-|N|$, with coefficients determined by a, α,β,γ

and δ. In particular, the above reduces to a Taylor series [4],[10]

$$y(x) = \sum_{n=1}^{|N|} b_n x^n + a x^{|N|+1} + \sum_{n=|N|+2}^{\infty} b_n(a) x^n \text{ convergent}$$

if either $\{\sqrt{2\alpha}+\sqrt{2\gamma}, \sqrt{2\alpha}-\sqrt{2\gamma}\} \cap \{|N|-1, 1-|N|\} \neq \emptyset$ or if $2\beta = -N^2$.

5) If $2\beta \neq 2\delta - 1$, besides log-solutions (7), there are one-complex parameter ($a \in \mathbf{C}$) logarithmic solutions [8], [4], [5]

$$y(x) = \left[\frac{2\beta+1-2\delta}{4}(a+\ln x)^2 + \frac{2\beta}{2\beta+1-2\delta}\right] x + \sum_{n\geq 2}^{\infty} P_n(\ln x; a) x^n, \ a \in \mathbf{C}.$$

3.1) The symmetry $y(x) \mapsto x/y(x)$ applied to case 3) gives one-parameter ($a \in \mathbf{C}$) solutions

$$y(x) = \sum_{k=0}^{\infty} y_k(x)(ax^{\omega})^k, \ \omega \in \{\omega^+_{\alpha\gamma}, \omega^-_{\alpha\gamma}\}, \ y_0(x) = O(1), \ y_k(x) = O(x^{k-1}),$$

where $\omega^{\pm}_{\alpha\gamma} := (\sqrt{2\alpha} \pm \sqrt{2\gamma})\mathrm{sgn}(\Re(\sqrt{2\alpha} \pm \sqrt{2\gamma})) \notin \mathbf{Z}$, and $y_k(x)$ are Taylor series.

4.1) The symmetry $y(x) \mapsto x/y(x)$ applied to case 4) with $N \neq 0$, gives solutions existing when $\sqrt{2\alpha}+\sqrt{2\gamma} = N$ or $\sqrt{2\alpha}-\sqrt{2\gamma} = N$:

$$y(x) = \sum_{n=0}^{|N|-1} b_n x^n + \Big(a + b_N \ln x\Big) x^{|N|} + \sum_{n=|N|+1}^{\infty} P_n(\ln x; a) x^n, \ N \neq 0$$

where b_n are functions of α, β, γ and δ, a is a free parameter, $P_n(\ln x; a)$ are polynomials of $\ln x$ of degree $n - |N| + 1$, with coefficients determined by a, α, β, γ and δ. If $\{\sqrt{-2\beta}+\sqrt{1-2\delta}, \sqrt{-2\beta}-\sqrt{1-2\delta}\} \cap \{|N|-1, 1-|N|\} \neq \emptyset$ or if $2\alpha = N^2$, the solutions become convergent Taylor expansions. The symmetry $y(x) \mapsto x/y(x)$ applied to case 4) for $N = 0$ gives the solutions

$$y(x) = \frac{1}{(a \pm \sqrt{2\alpha}\ln x) + \sum_{n=1}^{\infty} P_n(\ln x; a)} \sim \pm \frac{1}{\sqrt{2\alpha}\ln x}, \ \alpha = \gamma$$

5.1) Symmetry $y(x) \mapsto x/y(x)$ applied to 6) gives

$$y(x) = \frac{2}{(\gamma - \alpha)\ln^2 x}\left[1 + \frac{1}{\ln x} + O\left(\frac{1}{\ln^2 x}\right)\right], \ \alpha \neq \gamma$$

$\diamond$ Cases 1), 3) for $|\Re\sigma| < 1$, 4) for $N = 0$, and 5) are obtained by matching solutions of OUT and IN-systems with $N_{OUT} = N_{IN} = 0$. Case 2) is obtained from (5) via symmetry $y(x) \mapsto x/y(x)$. Cases 3) and 4) with $N \neq 0$ are obtained using OUT and IN-systems with at least one N_{IN} or N_{OUT} greater than zero. For the convergence of 1), 2) and 3) see method of [11] and [3]. For convergence of Taylor expansions see [10].

The set of the above behaviors obtained in [8], [3], [4], [5] and [6], coincides with the set of expansions obtained by power geometry, summarized in [2].

ii) Parameterization in terms of monodromy data: Every solution $y(x)$ is associated to a point in the space of monodromy data of (1). The power of the method of isomonodromic deformations is that it allows to parameterize the free parameters (integration constants) of $y(x)$ in terms of monodromy data. For a given point in the space of monodromy data, namely for a given $y(x)$, the parameterization can be done at each of the critical points $x = 0, 1, \infty$. In this way the connection problem is solved. In order to achieve the goal, we need to compute the monodromy data of (1). This is done as follows. *Once the matching* $\Psi_{OUT} \leftrightarrow \Psi_{IN}$ *in (4) has been completed*, we have to match Ψ_{OUT} with a fundamental solution Ψ of (1) at $\lambda = \infty$, and we have to match Ψ_{IN} with *the same* Ψ in another region of the λ-plane, typically around $\lambda = 0$ or x. If this is done, then M_1 of Ψ coincides with M_1^{OUT} of Ψ_{OUT}, while M_0 and M_x coincide with M_0^{IN} and M_x^{IN} of Ψ_{IN}. The crucial point is that we are able to compute the monodromy matrices M_1^{OUT}, M_0^{IN} and M_x^{IN} exactly, namely that we are able to solve the systems in terms of linear special functions. We cannot write here the rather long parameterization formulas. The reader may see [8], [1] [4], [5] and [6]

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Parametric Stokes phenomenon for the second Painlevé equation

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Abstract. The second Painlevé equation with a large parameter $(P_{\rm II})$ is analyzed by using the exact WKB analysis. The purpose of this study is to investigate the problem of the degeneration of P-Stokes geometry of $(P_{\rm II})$, which is related to a kind of Stokes phenomena for asymptotic (formal) solutions of $(P_{\rm II})$. We formulate the connection formula for this Stokes phenomenon, and confirm it in two ways: the first one is by computing the "Voros coefficient" of $(P_{\rm II})$, and the second one is by using the isomonodromic deformation theory. Our main claim is that the connection formulas derived by these two completely different methods coincide.

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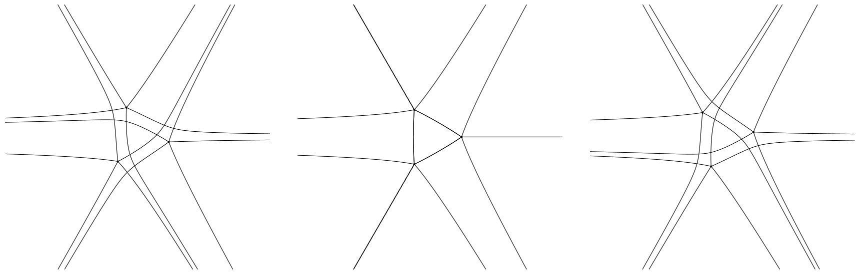
1. Introduction

We analyze the second Painlevé equation with a large parameter η

$$(P_{\rm II}) : \frac{d^2\lambda}{dt^2} = \eta^2(2\lambda^3 + t\lambda + c)$$

by using the exact WKB analysis. (The general theory of the exact WKB analysis for Painlevé equations is presented in [1].) The purpose of this study is to investigate the problem of the degeneration of P-Stokes geometry (existence of P-Stokes curves connecting two turning points) of $(P_{\rm II})$. (See Definition 2.1 for the definition of P-Stokes curve.) Here the following figures describe the P-Stokes curves of $(P_{\rm II})$ near $\arg c = \frac{\pi}{2}$. The degeneration of P-Stokes curves observed when $\arg\, c = \frac{\pi}{2}$ suggests that a kind of Stokes phenomena occurs when c varies near $\arg\, c = \frac{\pi}{2}$, that is, the correspondence between asymptotic (formal) solutions and true solutions of $(P_{\rm II})$ changes discontinuously before and after the degeneration. We call this phenomenon "parametric Stokes phenomenon" because this Stokes phenomenon (or the degeneration of Stokes geometry) occurs when the parameter



c contained in $(P_{\rm II})$ varies. In this case, asymptotic solutions mean the following 1-parameter family of transseries solutions (1-parameter solutions) of $(P_{\rm II})$:

$$\lambda(t,c,\eta;\alpha) = \lambda^{(0)}(t,c,\eta) + \alpha\eta^{-\frac{1}{2}}\lambda^{(1)}(t,c,\eta)e^{\eta\phi_{\rm II}} + (\alpha\eta^{-\frac{1}{2}})^2\lambda^{(2)}(t,c,\eta)e^{2\eta\phi_{\rm II}} + \cdots .$$

Here α is a free parameter, $\lambda^{(k)}(t,c,\eta) = \lambda_0^{(k)}(t,c) + \eta^{-1}\lambda_1^{(k)}(t,c) + \eta^{-2}\lambda_2^{(k)}(t,c) + \cdots$ $(k \ge 0)$ are formal power series of η^{-1} and $\phi_{\rm II} = \phi_{\rm II}(t,c)$ is some function. (Note that $\lambda^{(0)}(t,c,\eta)$ itself is a formal power series solution of $(P_{\rm II})$, called 0-parameter solution.) We will formulate the connection formula for the parametric Stokes phenomenon for 1-parameter solutions.

2. Connection formula for the parametric Stokes phenomenon

Before formulating the connection formula, we discuss normalizations of 1-parameter solutions. By letting $\tilde{\lambda}^{(1)}(t,c,\eta;\alpha) = \alpha\eta^{-\frac{1}{2}}\lambda^{(1)}(t,c,\eta)e^{\eta\phi_{\rm II}}$, we know that $\tilde{\lambda}^{(1)}$ satisfies the following second order linear ordinary differential equation

$$\frac{d^2\tilde{\lambda}^{(1)}}{dt^2} = \eta^2(6\lambda^{(0)}(t,c,\eta)^2 + t)\,\tilde{\lambda}^{(1)}, \tag{2.1}$$

that is, the Fréchet derivative of $(P_{\rm II})$ at $\lambda = \lambda^{(0)}$. Thus $\tilde{\lambda}^{(1)}$ can be taken as a WKB solution ([1]) of (2.1) of the form

$$\begin{aligned}\tilde{\lambda}^{(1)}(t,c,\eta;\alpha) &= \alpha\frac{1}{\sqrt{R_{\rm odd}(t,c,\eta)}}\exp\Big(\int^t R_{\rm odd}(t,c,\eta)dt\Big) \\ &= \alpha\eta^{-\frac{1}{2}}(\lambda_0^{(1)}(t,c) + \eta^{-1}\lambda_1^{(1)}(t,c) + \eta^{-2}\lambda_2^{(1)}(t,c) + \cdots)e^{\eta\phi_{\rm II}},\end{aligned} \tag{2.2}$$

where $R_{\rm odd}$ is the odd part of a formal power series solution $R = \eta R_{-1} + R_0 + \eta^{-1}R_1 + \cdots$ of the Riccati equation

$$R^2 + \frac{dR}{dt} = \eta^2\big(6\lambda^{(0)}(t,c,\eta)^2 + t\big) \tag{2.3}$$

associated with (2.1). (Thus we have $\phi_{\rm II} = \int^t R_{-1}dt$.) We note that $\lambda^{(k)}$ $(k \ge 2)$ are determined uniquely once the normalization of $\tilde{\lambda}^{(1)}$ (i.e. the normalization of the integral $\int^t R_{\rm odd}(t,c,\eta)dt$ in (2.2)) is fixed.

Definition 2.1 ([1]). (i) A point t is called a *P-turning point of (P_{II})* if t satisfies $6\lambda_0^{(0)\,2} + t = 0$.
(ii) For a P-turning point $t = \tau$, a real one-dimensional curve defined by

$$\mathrm{Im} \int_\tau^t \sqrt{6\lambda_0^{(0)\,2} + t}\, dt = 0$$

is said to be a *P-Stokes curve of (P_{II})*.

Here we introduce two normalizations of $\tilde{\lambda}^{(1)}$. The first normalization of $\tilde{\lambda}^{(1)}$ is "the normalization at a P-turning point":

$$\tilde{\lambda}_\tau^{(1)}(t, c, \eta; \alpha) = \alpha \frac{1}{\sqrt{R_{\mathrm{odd}}}} \exp\left(\int_\tau^t R_{\mathrm{odd}}\, dt \right), \tag{2.4}$$

where τ is a P-turning point of (P_{II}). The second one is "the normalization at ∞":

$$\tilde{\lambda}_\infty^{(1)}(t, c, \eta; \alpha) = \alpha \frac{1}{\sqrt{R_{\mathrm{odd}}}} \exp\left(\eta \int_\tau^t R_{-1}\, dt + \int_\infty^t \left(R_{\mathrm{odd}} - \eta R_{-1} \right) dt \right). \tag{2.5}$$

We define the 1-parameter solution $\lambda_\tau(t, c, \eta; \alpha)$(resp. $\lambda_\infty(t, c, \eta; \alpha)$) by using $\tilde{\lambda}_\tau^{(1)}$(resp. $\tilde{\lambda}_\infty^{(1)}$) for the normalization of $\tilde{\lambda}^{(1)}$. Then the connection formula for the parametric Stokes phenomena for 1-parameter solutions can be described as follows:

Connection formula for the 1-parameter solutions of (P_{II}). *Let ε be a sufficiently small positive number.*

(i) If the true solutions represented by $\lambda_\infty(t, c, \eta; \alpha)$ for arg $c = \frac{\pi}{2} - \varepsilon$ and by $\lambda_\infty(t, c, \eta; \tilde{\alpha})$ for arg $c = \frac{\pi}{2} + \varepsilon$ coincide, then the following holds:

$$\tilde{\alpha} = \alpha. \tag{2.6}$$

(ii) If the true solutions represented by $\lambda_\tau(t, c, \eta; \alpha)$ for arg $c = \frac{\pi}{2} - \varepsilon$ and by $\lambda_\tau(t, c, \eta; \tilde{\alpha})$ for arg $c = \frac{\pi}{2} + \varepsilon$ coincide, then the following holds:

$$\tilde{\alpha} = \left(1 + e^{2\pi i c \eta}\right) \alpha. \tag{2.7}$$

Assuming the Borel summability of the 1-parameter solutions, we derive the above connection formulas in two ways:

A: derivation through the analysis of "the Voros coefficient of (P_{II})",
B: derivation by using the isomonodromic deformation.

Our main claim is that the connection formulas derived by these two completely different methods coincide. In this abstract, due to the lack of space, we explain only an outline of the derivation of the connection formula by **A**. The derivation by **B** will be discussed in the talk.

3. Derivation of the connection formulas through the analysis of the Voros coefficient of (P_{II})

We define the Voros coefficient of (P_{II}). It plays an important role in the analysis of the parametric Stokes phenomenon for the 1-parameter solutions of (P_{II}).

Definition 3.1. The Voros coefficient $W(c,\eta)$ of (P_{II}) is defined as follows:

$$W(c,\eta) \quad = \quad \int_{\tau}^{\infty} \big(R_{\mathrm{odd}}(t,c,\eta) - \eta R_{-1}(t,c)\big) dt. \tag{3.1}$$

$W(c,\eta)$ appears as a difference of the above two normalizations of $\tilde{\lambda}^{(1)}$:

$$\tilde{\lambda}_{\tau}^{(1)}(t,c,\eta;\alpha) = e^{W(c,\eta)}\,\tilde{\lambda}_{\infty}^{(1)}(t,c,\eta;\alpha). \tag{3.2}$$

Then we obtain the following theorem.

Theorem 3.2. *The Voros coefficient $W(c,\eta)$ of (P_{II}) is represented explicitly as follows:*

$$W(c,\eta) = -\sum_{n=1}^{\infty} \frac{2^{1-2n}-1}{2n(2n-1)} B_{2n}(c\eta)^{1-2n}. \tag{3.3}$$

Here B_{2n} is the 2n-th Bernoulli number defined by

$$\frac{w}{e^{w}-1} = 1 - \frac{w}{2} + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}\, w^{2n}. \tag{3.4}$$

A key to the proof of Theorem 3.2 is the calculation of the difference $W(c,\eta) - W(c-\eta^{-1},\eta)$, which is done by making use of the Bäcklund transformation that induces the translation of the parameter $c \mapsto c-\eta^{-1}$ of (P_{II}). Using the expression (3.3), we can analyze the parametric Stokes phenomenon for $W(c,\eta)$.

Corollary 3.3 ([2])**.** *By denoting the Borel resummation operator by $\mathcal{S}$, we obtain the following:*

$$\mathcal{S}\big[e^{W(c,\eta)}\big|_{\arg c = \frac{\pi}{2}-\varepsilon}\big] = (1+e^{2\pi i c\eta})\,\mathcal{S}\big[e^{W(c,\eta)}\big|_{\arg c = \frac{\pi}{2}+\varepsilon}\big], \tag{3.5}$$

where ε is a sufficiently small positive number.

Assume that the connection formula (2.6) for λ_{∞} is true. Then, combining Corollary 3.3, (2.6) and (3.2), we can derive the connection formula (2.7) for the 1-parameter solution $\lambda_{\tau}(t,c,\eta;\alpha)$.

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Solution of the equivalence problem for the second order ODE's with the degenerate Cartan's invariants

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Abstract. A class of second order ordinary differential equations that possesses the constant (degenerate) Cartan's invariants is investigated. Four basic types of these equations were found. For any type of equations the equivalence problem is solved. As the examples, it solved for the Painleve II equation, Painleve III equation with three zero parameters, Emden equations and for some other equations from the handbook Kamke.

Mathematics Subject Classification (2000). 53A55, 34A26, 34A34, 34C14, 34C20, 34C41.

Keywords. Differential invariant, Problem of equivalence, Point transformation, Painleve equation, Emden equation.

1. Four types of the equations

Let us consider the following class of the second order ODE's:

$$y'' = P(x, y) + 3\,Q(x, y)y' + 3\,R(x, y)y'^2 + S(x, y)y'^3. \tag{1.1}$$

It is well-known fact that it closed under the general point transformations $\tilde{x} = \tilde{x}(x, y)$, $\tilde{y} = \tilde{y}(x, y)$. Let we have two arbitrary equations (1.1). The problem of existence of the point transformation that connects these equations is called *the equivalence problem.* The main approach that allows to solve the equivalence problem is based on the theory of invariant.

The invariant theory of equation (1.1) goes back to the classical works of the end of XIXth - beginning of the XXth centuries by R.Liouville, S.Lie, A.Tresse, E.Cartan; later it continues in the works of the end of XX century by C. Grissom,

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G. Thompson and G. Wilkens; N. Kamran, K.G. Lamb & W.F. Shadwick; J. Hietarinta, V. Dryuma, L.A. Bordag, N. Kh. Ibragimov, R.A.Sharipov. It remains an active research topic in the XXI century, see works by C. Bandle and L.A. Bordag, V.V.Kartak, N.H. Ibragimov, S.V. Meleshko.

In the present paper we use notations from works R.A.Sharipov to calculation the Cartan's invariants. The explicit formulas are very huge and can't be written at the present paper, see [1].

Here we investigate equations (1.1) with conditions

$$\alpha \neq 0, \quad F = 0, \quad M \neq 0, \quad I_1 = const \neq 0, \quad I_2 = 0. \tag{1.2}$$

Theorem 1.1. *Each equation (1.1) with conditions (1.2) can be transformed by point transformations into the form:*

$$y'' = P^*(y) + t(x)y + s(x),$$

where

$$P^*(y) = \begin{cases} e^y, & \text{if } \; I_1 = \dfrac{3}{5}; \\ -\ln y, & \text{if } \; I_1 = -\dfrac{9}{10}; \\ y(\ln y - 1), & \text{if } \; I_1 = -\dfrac{12}{5}; \\ \dfrac{y^{C+2}}{(C+1)(C+2)}, & \text{if } \; I_1 = \dfrac{3(C+5)}{5C}, \; C = const \neq -5, \, -2, \, -1, \, 0. \end{cases}$$

Definition 1.2. Let us say that equation (1.1) has Type I if hold the conditions (1.2) where $I_1 = 3/5$.

Theorem 1.3. *A complete list of cases which can be distinguished for equations of Type I. Here J_3, J_6, J_1 and K are invariants. Formulas are into the paper [2].*

Type	J_3	J_6	J_1	K	*Canonical form*
I.1	0	0	–	0	$y'' = e^y$
I.2	$\neq const$	J_3	–	0	$y'' = e^y + 1$
I.3	$\neq const$	$\neq J_3$, $\neq const$	*const=a*	0	$y'' = e^y + y + a$
I.4	$\neq const$	J_3	–	$const = k \neq 0$	$y'' = e^y + \frac{4}{kx^2}$
I.5	$\neq const$	J_3	–	$\neq const$	$y'' = e^y + s(x)$, $s(x) \neq const$
I.6	$\neq const$	$\neq J_3$, $\neq const$	$\neq const$	$\neq const$	$y'' = e^y + t(x)y + s(x)$, $t(x) \neq 0$

Definition 1.4. Let us say that equation (1.1) has Type II if hold the conditions (1.2) where $I_1 = -9/10$.

Theorem 1.5. *A complete list of cases which can be distinguished for equations of Type II. Here J_6, J_9 and J are invariants. Formulas are into the paper [2].*

Type	J_6	J_9	J	*Canonical form*
II.1	0	0	$-$	$y'' = -\ln y$
II.2	$\neq$ *const*	*0*	$a =$*const*$\neq 0$	$y'' = -\ln y + y + a$
II.3	*0*	$\neq$ *const,*	-	$y'' = -\ln y + s(x)$, $s(x) \neq const$
II.4	$\neq$ *const*	$\neq$*const*	$\neq$*const*	$y'' = -\ln y + t(x)y + s(x)$, $t(x) \neq 0$

Definition 1.6. Let us say that equation (1.1) has Type III if hold the conditions (1.2) where $I_1 = -12/5$.

Theorem 1.7. *A complete list of cases which can be distinguished for the equations (*1.1*) of Type III, J_6, J_9 and J are invariants. Formulas are into the paper* [2].

Type	J_6	J_9	J	*Canonical form*
III.1	0	0	0	$y'' = y(\ln y - 1)$
III.2	0	$b^2 =$*const*$\neq 0$	*0*	$y'' = y(\ln y - 1) \pm bxy$
III.3	$\neq$ *const*	*0*	$a =$*const*$\neq 0$	$y'' = y(\ln y - 1) + a$
III.4	$\neq$ *const*	$b^2 =$*const*$\neq 0$	$\neq$ *const*	$y'' = y(\ln y - 1) \pm bxy + 1$
III.5	$\neq$ *const*	$\neq$ *const*	$\neq$ *const*	$y'' = y(\ln y - 1) + t(x)y + s(x)$, $s(x) \neq 0$

Definition 1.8. Let us say that equation (1.1) has Type IV if hold the conditions (1.2) where $I_1 = 3(C+5)/5C$, $C = const$, $C \neq 0, -1, -2, -5$.

Theorem 1.9. *A complete list of cases which can be distinguished for the equations (*1.1*) of Type IV. Here J_1, J_2, K, J, J_9 and K_1 are invariants. Explicite formulas are into the paper* [2].

Type	J_1	J_2	K	J	J_9	K_1	*Canonical form*
IV.1	0	0	-	-	0	0	$y'' = \frac{y^{C+2}}{(C+1)(C+2)}$
IV.2	0	$\neq 0$	-	-	0	0	$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + 1$
IV.3	0	$\neq 0$	-	-	$\neq 0$	0	$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + x$
IV.4	0	$\neq 0$	-	-	$\neq 0$	$\neq 0$	$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + s(x)$
IV.5	$\neq 0$	0	$\neq$*const*	*0*	$\neq 0$	0	$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + xy$
IV.6	$\neq 0$	0	$k =$*const*$\neq 0$	*0*	$\neq 0$	$\neq 0$	$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + \frac{4y}{kx^2}$
IV.7	$\neq 0$	$\neq 0$	$k =$*const*$\neq 0$	$\neq$*const*	$\neq 0$	$\neq 0$	$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + \frac{4y}{kx^2} + 1$
IV.8	$\neq 0$	$\forall$	*0*	$a =$*const*	0	0	$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + y + a$
IV.9	$\neq 0$	$\forall$	$\neq$*const*	$a =$*const*	$\neq 0$	$\forall$	$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + t(x)y + at(x)^{\frac{C+2}{C+1}}$
IV.10	$\neq 0$	$\neq 0$	$\neq$*const*	$\neq$*const*	$\neq 0$	0	$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + xy + cx + d$
IV.11	$\neq 0$	$\neq 0$	$\neq$*const*	$\neq$*const*	$\neq 0$	$\neq 0$	$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + t(x)y + s(x)$

Example. Equation Painleve III depends on 4 parameters (a, b, c, d)

$$PIII(a,b,c,d): \qquad y'' = \frac{y'^2}{y} - \frac{y'}{x} + \frac{(ay^2+b)}{x} + cy^3 + \frac{d}{y}.$$

It is known that all equations Painleve III with 3 zero parameters are equivalent. In this way suppose that $a \neq 0$, $b = c = d = 0$. For the equation $PIII(a, 0, 0, 0)$ conditions (1.2) are hold and invariants are equal to:

$$I_1 = \frac{3}{5}, \quad I_2 = 0, \quad I_3 = \frac{1}{15}.$$

As $J_3 = 0$, then according to Theorem 1.3, equation $PIII(a, 0, 0, 0)$ has Type I.1 and can be reduced:

$$y'' = \frac{y'^2}{y} - \frac{y'}{x} + \frac{ay^2}{x} \quad \to \quad \tilde{y}'' = e^{\tilde{y}}.$$

Example. Equation Painleve II depends on 1 parameter a

$$PII(a): \qquad y'' = 2y^3 + xy + a.$$

It satisfies to the conditions (1.2) with invariant $I_1 = 18/5$. According to Theorem 1.9 it has Type IV, case $C = 1$. Let us calculate the additional invariants

$$J_1 = \frac{x}{12y^2}, \quad J_2 = \frac{a}{12y^3}, \quad J = \frac{2a\sqrt{3}}{x\sqrt{x}}, \quad K = \frac{1}{x^3}, \quad J_9 = \frac{1}{1728y^6}, \quad K_1 = 0.$$

We see that the equation PII(a) has Type IV.10 if $a \neq 0$ and Type IV.5 if $a = 0$.

Let us calculate x, y and a via invariants: $x = 1/\sqrt[3]{K}$, $y = 1/2\sqrt{3}\sqrt[6]{J_9}$, $a = J_2/2\sqrt{3}\sqrt{J_9}$.

Theorem 1.10. *Equation* (1.1) *of Type IV is equivalent to Painleve II equation with parameter* $\pm a$ *if and only if*

$$C = 1, \quad \frac{J_2}{2\sqrt{3}\sqrt{J_9}} = a = const, \quad K_1 = 0. \tag{1.3}$$

The explicit point transformation is $\tilde{x} = 1/\sqrt[3]{K(x,y)}$, $\tilde{y} = 1/(2\sqrt{3}\sqrt[6]{J_9(x,y)})$.

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On some estimates of the minimal eigenvalue for the Sturm — Liouville problem with third-type boundary conditions and integral condition

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Abstract. We consider the Sturm–Liouville problem with symmetric boundary conditions and an integral condition. We estimate the first eigenvalue λ_1 of this problem for different values of the parameters.

Mathematics Subject Classification (2000). 34L15.

1. Introduction

Consider the Sturm — Liouville problem:

$$y''(x) - q(x)y(x) + \lambda y(x) = 0, \tag{1.1}$$

$$\begin{cases} y'(0) - k^2 y(0) = 0, \\ y'(1) + k^2 y(1) = 0, \end{cases} \tag{1.2}$$

where $q(x)$ belongs to the set A_γ ($\gamma \neq 0$) of non-negative bounded summable functions on $[0,1]$ such that $\int_0^1 q^\gamma(x)dx = 1$.

We estimate the first eigenvalue $\lambda_1(q)$ of this problem for different values of γ and k.

According to the variation principle $\lambda_1(q) = \inf\limits_{y \in H_1(0,1)\setminus\{0\}} R(q, y)$, where

$$R(q, y) = \frac{\int\limits_0^1 {y'}^2(x)dx + \int\limits_0^1 q(x)y^2(x)dx + k^2\left(y^2(0) + y^2(1)\right)}{\int\limits_0^1 y^2(x)dx}. \tag{1.3}$$

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Put $m_\gamma = \inf\limits_{q\in A_\gamma} \lambda_1(q)$, $M_\gamma = \sup\limits_{q\in A_\gamma} \lambda_1(q)$.

Remark 1.1. Dirichlet problem for the equation (1.1), $q(x) \in A_\gamma$ was considered in [3], [4]. Different problems for the equation $y'' + \lambda q(x)y = 0$, $q(x) \in A_\gamma$ was considered in [1], [2].

2. Results

Theorem 2.1. *The following assertions are valid:*

1. *If* $\gamma \in (-\infty, 0) \cup (0, 1)$, *then* $M_\gamma = +\infty$, *and there exists the minimizing sequence* $q_\varepsilon(x) \in A_\gamma$ *such that* $M_\gamma = \lim\limits_{\varepsilon\to 0} \lambda_1(q_\varepsilon)$.
2. *If* $\gamma \geq 1$ *and* $k = 0$, *then* $M_\gamma = 1$, *and this estimate is attained at* $q(x) \equiv 1$.
3. *If* $\gamma = 1$ *and* $k \neq 0$, *then* $M_1 = \xi_*$, *where* ξ_* *is the solution to the equation* $\arctan \frac{k^2}{\sqrt{\xi}} = \frac{\xi - 1}{2\sqrt{\xi}}$, *and this estimate is attained at*
$$q(x) = q_*(x) = \begin{cases} 0, & 0 \leq x < \tau, \\ \xi_*, & \tau \leq x < 1 - \tau, \\ 0, & 1 - \tau \leq x < 1, \end{cases} \qquad \text{where } \tau = \frac{1}{\sqrt{\xi_*}} \arctan \frac{k^2}{\sqrt{\xi_*}}.$$
4. *If* $\gamma > 1$, *then for* $k = 0$ *we have* $m_\gamma = 0$, *and for* $k \neq 0$ *we have* $m_\gamma = \lambda_1^0$, *where* λ_1^0 *is the minimal positive solution to the equation*
$$(k^4 - \lambda)\sin\sqrt{\lambda} + 2k^2\sqrt{\lambda}\cos\sqrt{\lambda} = 0; \tag{2.1}$$
and there exists the minimizing sequence $q_\varepsilon(x) \in A_\gamma$ *such that* $m_\gamma = \lim\limits_{\varepsilon\to 0} \lambda_1(q_\varepsilon)$.
5. *If* $\gamma > 0$, *then* $m_\gamma \to \pi^2$ *as* $k \to \infty$.

3. Proofs of some results

Results 1)–3) were proved in [5], [6]. Here we prove results 4)–5).

Suppose $\gamma > 1$ and $k \neq 0$. Let us prove that $m_\gamma = \lambda_1^0$, where λ_1^0 is the minimal positive solution to equation (2.1).

Proof. By definition,
$$m_\gamma = \inf_{q\in A_\gamma} \Big(\inf_{y\in H_1(0,1)\setminus\{0\}} R(q, y) \Big) = \lambda_1^0 + \inf_{q\in A_\gamma} \inf_{y} \frac{\int\limits_0^1 q(x)y^2dx}{\int\limits_0^1 y^2dx}, \tag{3.1}$$
where λ_1^0 is the first eigenvalue of the problem for the equation
$$y''(x) + \lambda y(x) = 0 \tag{3.2}$$
with conditions (1.2).

Let y_1^0 be the first eigenfunction of problem (3.2), (1.2), then

$$y_1^0 = C_1 \cos\sqrt{\lambda_1^0}x + C_2 \sin\sqrt{\lambda_1^0}x.$$

Put

$$q_\varepsilon(x) = \begin{cases} \varepsilon^{-1/\gamma}, & 0 < x < \varepsilon, \\ 0, & \varepsilon < x < 1. \end{cases}$$

Suppose $M_1 \geq \left(y_1^0(x)\right)^2 \geq M_2 > 0$ at $x \in [0,1]$, where M_1, M_2 are constants (such constants exist since $y_1^0(x)$ is bounded and non-trivial function). Hence,

$$\inf_{q\in A_\gamma} \inf_{y} \frac{\int\limits_0^1 q(x)y^2dx}{\int\limits_0^1 y^2dx} \leq \frac{\int\limits_0^1 q_\varepsilon(x)\left(y_1^0\right)^2 dx}{\int\limits_0^1 \left(y_1^0\right)^2 dx} \leq \frac{M_1}{M_2}\cdot \varepsilon^{1-1/\gamma} \to 0 \quad \text{as } \varepsilon \to 0.$$

Therefore, $m_\gamma = \lambda_1^0$. □

Suppose $\gamma > 0$. Let us prove that $m_\gamma \to \pi^2$ as $k \to \infty$.
First let us prove that if $\gamma > 0$, then $m_\gamma \leq \pi^2$.

Proof. Put

$$y_\delta(x) = \begin{cases} \sin\frac{\pi x}{\delta}, & 0 < x < \delta, \\ 0, & \delta < x < 1 \end{cases} \quad \text{and} \quad q_\delta(x) = \begin{cases} 0, & 0 < x < \delta, \\ (1-\delta)^{-\frac{1}{\gamma}}, & \delta < x < 1, \end{cases}$$

where $\delta \to 1-0$.

Then we have

$$R(q_\delta,\, y_\delta) = \frac{\frac{\pi^2}{2\delta} + 0 + k^2\sin^2\frac{\pi}{\delta}}{\delta/2}.$$

Therefore we obtain

$$m_\gamma = \inf_{q\in A_\gamma}\left[\inf_{y\in H_1(0,1)\setminus\{0\}} R(q,\, y)\right] \leq R(q_\delta,\, y_\delta) \to \pi^2 \text{ as } \delta \to 1-0.$$

□

Now let us prove that if $\gamma > 0$, then $m_\gamma \to \pi^2$ as $k \to \infty$.

Proof. It follows from (3.1) that $m_\gamma \geq \lambda_1^0$, where λ_1^0 is the minimal positive solution to equation (2.1). For $k^2 > \pi/2$ we have $\cos\sqrt{\lambda_1^0} \neq 0$, so we may seek λ_1^0 as the minimal positive solution to the equation

$$\tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}k^2}{(\lambda - k^4)}. \tag{3.3}$$

Using the properties of the functions $\tan(\sqrt{\lambda})$ and $\frac{2\sqrt{\lambda}k^2}{(\lambda-k^4)}$ (see fig. 1), we get:

- if $k^2 > \pi$, then $\lambda_1^0 \in \left(\pi^2/4;\, \pi^2\right)$;
- if $k \to \infty$, then $\lambda_1^0 \to \pi^2$.

Therefore, $m_\gamma \to \pi^2$ as $k \to \infty$. □

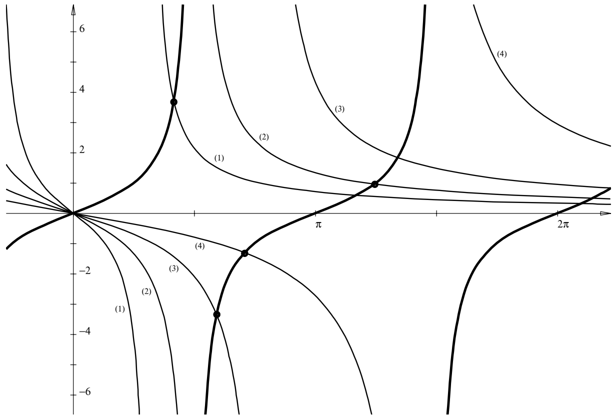

FIGURE 1. The solution of the equation $\tan t = \frac{2tk^2}{(t^2-k^4)}$ for different values of k.

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Euler integral symmetry and deformed hypergeometric equation

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Abstract. Deformed hypergeometric equation, linear differential equation of the second order with three regular singularity and one apparent singularity, is under consideration. Euler integral symmetry for this equation is described. Analytic continuation of the corresponding contour integral gives the possibility to calculate the monodromy group for this equation in explicit terms. Solutions of the deformed hypergeometric equation in integral form are obtained too.

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Linear special functions (like hypergeometric functions, Heun class functions etc.) connected with nonlinear special functions (solutions of the Painleve equations) by two ways. Firstly, Painleve equations govern isomonodromy deformations of the linear equations [1], [2]. Secondly, explicit information about monodromy groups can be exploited at the study of solutions of Painleve equations (A.Its and colleagues). Here the monodromy group of the deformed hypergeometric equation

$$w''(z) + \left[\frac{\alpha}{z} + \frac{\beta}{z-1} - \frac{1}{z-\lambda}\right] w'(z) + \frac{1}{z(z-1)}\left[\frac{\chi}{z-\lambda} + \frac{\sigma}{\lambda(\lambda-1)}\right] w(z) = 0 \quad (1)$$

is under consideration. This equation has like the hypergeometric one 3 regular singular points $z = 0, z = 1, z = \infty$, and, moreover, apparent (or "false") regular singular point of the first order at $z = \lambda$. It has 4 free parameters $\alpha, \beta, \lambda, \chi$, the value of the parameter σ is determined by the condition: solutions have not logarithmic behavior in the neighbourhood of the point $z = \lambda$. It can be shown, that $\sigma = -(\beta\lambda - 2\lambda + \alpha\lambda + 1 + \chi - \alpha)\chi$. The order of the apparent singularity is equal to 1 in accordance with residue at $z = \lambda$ in the coefficient at $w'(z)$. This equation can be considered as a particular case of the Heun equation [3] when one of regular singularity is apparent one. Our aim is to calculate the monodromy group for this equation, we apply here Euler integral symmetry and procedure of analytic continuation. This approach was discussed before in [4], [5].

Initial point is the linear differential system

$$(Az+B)W'(z) = CW(z), \tag{2}$$

where A, B, C are constant $m\times m$ matrices. If $m = 2$, this system can be reduced to the deformed hypergeometric equation. The singular regular points of this system are the roots of the $det(Az + B)$ and, may be, the infinity.

Theorem 1. *Let $V(t)$ be the solution of the system*

$$(At+B)V'(t) = [C-(\mu+1)A]\,V(t), \tag{3}$$

branching in the neighbourhood of some regular singularity $t = t_*$ and contour L is a double loop on complex plane around the point $t = z$ and $t = t_*$. Then $W(z)$ defined as

$$W(z) = \int_L (z-t)^\mu V(t)dt, \tag{4}$$

is the solution of the system (2) branching at the same point $z = t_*$.

Note, that integral relation (4) is a scalar one, namely, it links corresponding components of the vector-functions $V(t)$ and $W(z)$. So, if $m = 2$ reduction of systems (2) and (3) to the differential equations leads to the integral relation between solutions of the deformed hypergeometric equations. In order to describe this link in explicit terms it is necessary to express parameters of (1) in terms of parameters of (2). Then it is necessary to calculate parameters of deformed hypergeometric equation, which is a reduction of the system (3). These cumbersome calculations were realized with help of Maple, we omit here theirs details. Results of this calculations we summarize in the Proposition 2. Here we need in some convenient notions.

Let denote by $w_h^{(0)}(z)$ normalized at $z = 0$ holomorphic solution of equation (1), $w_b^{(0)}(z)$ normalized branching at $z = 0$ solution of this equation, $w_h^{(0)}(z) \sim 1$, $w_b^{(0)}(z) \sim 1\cdot z^{1-\alpha}$ at $z \sim 0$, $w_h^{(1)}(z), w_b^{(1)}(z)$ normalized holomorphic and branching at $z = 1$ solutions, $w_h^{(1)}(z) \sim 1$, $w_b^{(1)}(z) \sim 1\cdot(z-1)^{1-\beta}$ at $z-1 \sim 0$.

Theorem 2. *Let $v_b^{(0,1)}(t)$ are normalized branching solutions of equation*

$$v''(t) + \left[\frac{\widetilde{\alpha}}{t} + \frac{\widetilde{\beta}}{t-1} - \frac{1}{t-\widetilde{\lambda}}\right]v'(t) + \frac{1}{t(t-1)}\left[\frac{\widetilde{\chi}}{t-\widetilde{\lambda}} + \frac{\widetilde{\sigma}}{\widetilde{\lambda}(\widetilde{\lambda}-1)}\right]v(t) = 0, \tag{5}$$

fixed by theirs behavior in the regular singular points $t = 0$ or $t = 1$ respectively, the contour $L = L_{0,1}$ be double loop including either $t = 0$ (for $L^{(0)}$), or $t = 1$ (for $L^{(1)}$), and point $t = z$, and

$$\widetilde{\alpha} = \alpha+\mu+1, \widetilde{\beta} = \beta+\mu+1, \tag{6}$$

$$\widetilde{\lambda} = \frac{\lambda(\alpha\lambda-\lambda-\alpha+\mu\lambda-\mu+\beta\lambda+\chi)}{-2\lambda+\beta\lambda+\alpha\lambda+1-\alpha+\chi} \tag{7}$$

$$\widetilde{\chi} = -\frac{1}{-2\lambda+\beta\lambda+\alpha\lambda+1-\alpha+\chi}(-\chi^2-\alpha\mu\lambda-\beta\mu\lambda-\alpha\chi\lambda+ \tag{8}$$
$$\alpha\lambda^2+\beta\lambda^2-\lambda^2-\beta\lambda+\lambda+\beta\mu\lambda^2-\chi+\alpha\lambda^2\mu+\alpha\chi-\alpha\lambda+\mu^2\lambda^2-\mu^2\lambda-\beta\lambda\chi+2\chi\lambda).$$

Then function $w(z)$, defined by relation

$$w(z) = (\nu_{0,1})^{-1}\int_{L_{0,1}}(z-t)^{\mu}v_b^{(0,1)}(t)dt, \tag{9}$$

coincides with $w_b^{(0,1)}(z)$. *Here*

$$\nu_{0,1} = [1-exp(2\pi i\mu)]\,[1-exp(2\pi i(\kappa_{0,1}))]\,B(\mu+1,1+\kappa_{0,1}),$$

$\kappa_{0,1}$ *are characteristic exponents for the solutions* $v_b^{(0,1)}(t)$ *at the corresponding singular point.*

Let $W^{(0,1)}(z) = (w_b^{(0,1)}(z), w_h^{(0,1)}(z))^T$. Then, in accordance with analytic theory of linear differential equations [1], [2], $W^{(0)}(z) = RW^{(1)}(z)$, where R is the corresponding connection matrix for the equation (1). Considering the analytic continuation with help of integral relation (9), one obtains:

$$R_{11} = \frac{\exp(2\pi i\kappa_1)-1}{\exp(-2\pi i\beta)-1}\frac{\Gamma(\kappa_1+1)\Gamma(\kappa_0+\mu+2)}{\Gamma(\kappa_0+1)\Gamma(\kappa_1+\mu+2)}T_{11}, \tag{10}$$

where T is the corresponding connection matrix for the equation (5). Note, that integral transform (9) depends on the free parameter μ. Changing this parameter one can change the parameters of equation (5) in accordance with (6)-(8). Let us fix parameter μ by condition: apparent singularity of equation (5) $t = \widetilde{\lambda}$ coincides with regular singularity $t = 0$. It means, that $\mu = -(\alpha\lambda-\alpha+\beta\lambda-\lambda+\chi)/(\lambda-1)$, and equation (5) reads:

$$v''(t)+\left[\frac{-\alpha\lambda-1+\alpha-\beta+2\lambda-\chi}{(t-1)(\lambda-1)}+\frac{-\beta\lambda+\lambda-\chi}{t(\lambda-1)}\right]v'(t)+ \tag{11}$$
$$\frac{(\beta\lambda-\lambda+\chi)(-\lambda^2+t\lambda\alpha+t\beta\lambda+\lambda-2t\lambda+t-t\alpha+t\chi)}{(\lambda-1)^2\lambda t^2(t-1)}v(t) = 0.$$

Coefficient at $v(t)$ in the equation (11) has pole of the second order at $t = 0$, so this equation can be transformed into hypergeometric one by simple substitution. Connection matrices for hypergeometric equation are described by Kummer relations [6]. So, one can calculate the value of T_{11} and, consequently, R_{11}. Let

$$\Delta = \sqrt{\frac{(\alpha-\beta)^2}{4}+\frac{(\beta\lambda-\lambda+\chi)(-\lambda+\alpha\lambda+\chi+1-\alpha)}{\lambda(\lambda-1)}}, \tag{12}$$

then

$$R_{11} = -\frac{(\lambda-1)\exp(i\pi\beta)\Gamma(\beta-1)\Gamma(2-\alpha)}{(\beta\lambda+\chi-\lambda)\Gamma(\frac{\beta-\alpha}{2}+\Delta)\Gamma(\frac{\beta-\alpha}{2}-\Delta)}. \tag{13}$$

Relation (13) describes one entry of connection matrix for the equation (1), this connection matrix corresponds to the pair of singular points $z = 0, z = 1$. Other entries of this matrix can be evaluated with help of elementary symmetries of the deformed hypergeometric equation. Namely, substitutions $w(z) = z^{1-\alpha}w_1(z)$, $w(z) = (z-1)^{1-\beta}w_2(z)$, and change of variable $z = 1-s$ produce the same equation for functions $w_1(z), w_2(z), w(s)$ with different sets of parameters. In more details, the first substitution transforms equation with parameters $[\alpha, \beta, \lambda, \chi]$ into equation with parameters $[2-\alpha, \beta, \lambda, 1+\chi+\lambda\alpha-\lambda-\alpha]$, and exchanges branching and holomorphic at $z = 0$ solutions, solutions fixed by behavior at $z = 1$ does not change its character. So,

$$R_{21}(\alpha, \beta, \lambda, \chi) = R_{11}(2-\alpha, \beta, \lambda, 1+\chi+\lambda\alpha-\lambda-\alpha). \tag{14}$$

Analogously,

$$R_{12}(\alpha, \beta, \lambda, \chi) = R_{11}(\alpha, 2-\beta, \lambda, \chi-\lambda+\beta\lambda). \tag{15}$$

Combining,

$$R_{22}(\alpha, \beta, \lambda, \chi) = R_{11}(2-\alpha, 2-\beta, \lambda\alpha+\beta\lambda-2\lambda+1+\chi-\alpha). \tag{16}$$

These relations give us connection matrix R, which corresponds to the pairs of singularities $z = 0, 1$. Equation (1) has once more regular singularity $z = \infty$. But change of independent variable $z = s^{-1}$ (which exchanges points 0 and ∞)transforms equation (1) into equation, which differs from equation (1) by elementary factor only (with different set of parameters). This symmetry gives the possibility to calculate the connection matrix which corresponds to the pair of singularities $z = \infty, z = 1$. Solutions of the deformed hypergeometric equation in integral form are obtained too.

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Integral Euler symmetries for confluent Heun equation and symmetries of Painlevé equation P^5

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Confluent Heun equation is a second order equation with two regular singularities which are assumed to be $z_1 = 0$, $z_2 = 1$ and one irregular singularity at infinity of s-rank $R(\infty) = 2$ [1]. If the fourth singularity $z = q$, namely the apparent one, is added to this equation we term the generated equation as the deformed Confluent Heun equation *CHE1*.

$$z(z-1)w''(z)+(-tz(z-1)+c(z-1)+dz-\frac{1}{z-q})w'(z)+(-taz+t\lambda+\frac{pq}{z-q})w(z) = 0. \quad (1)$$

It is known that the isomonodromy condition for equation (1) is equivalent to the Painleve equation P^5.

The Euler integral transforms for Heun equation has been studied before. They lead to symmetries for Heun equations and as a result symmetries of the corresponding P^6 [2]. First idea that integral transforms lead to inner symmetries appeared in the paper by Fock [3]

The problem of transforms for *CHE1* is rather sophisticated and needs computer algebra systems for particular calculations. Here we give an implicit solution of this problem using the ideas proposed in [2] based on study of an auxiliary linear 2×2 system for which the Euler transform and subsequent symmetries are much simpler. As a result the symmetries of *CHE1* and resulting symmetries of P^5 are found. They generate a subset of Okamoto transforms for P^5.

The starting linear system is assumed to be of the form

$$(z^2A + zB + C)\frac{dW}{dz} = (-\alpha zA + E)W. \quad (2)$$

Here A, B, C, E are 2×2 matrices which do not depend on z, $W(z)$ is a 2-vector function. Particularly

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & \tau \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} c\tau + c + \tau & -c\tau - c - \tau \\ c & -c \end{pmatrix}, E = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}. \quad (3)$$

Here c and τ are parameters determining characteristic exponents at regular singularities. The determinant of $Az^2 + Bz + C$ can be easily computed

$$\det(Az^2 + Bz + C) = -(c + \tau)z(z - 1) \tag{4}$$

The choice of matrices A, B, C was the crucial point in our studies. It is based on (4) but has much freedom which can either simplify intermediate computations or make them more complicated. The roots of the polynomial $\det(z^2A + zB + C)$ are regular singularities of CHE1.

Reduction of (2) to the second order equation leads to the CHE1 and the corresponding relations between parameters of system (2) and equation (1) can be obtained. However, in several cases these relations are complicated (include hundred of terms) and are presented only in the report.

Further the solution of the auxiliary system CHE1 $\Phi(s)$ is introduced:

$$\left(s^2A + sB + C\right)\frac{d\Phi(s)}{ds} = ((\alpha + 2)sA + E - (-\alpha + 1)B)\Phi(s) = 0. \tag{5}$$

It can be shown by integration by parts, that if $\Phi(s)$ is appropriate solution of (**??**) and L is appropriate contour on complex plane, then

$$W(z) = \int_L (z - s)^{\kappa}\Phi(s)ds, \tag{6}$$

is a solution of initial system (2). Reduction of the system (5) leads to CHE1 too, but with the transformed coefficients. Due to the scalar nature of integral relation (6), the corresponding integral relation can be written for solutions of initial CHE1 and transformed CHE1. Analytic continuation with help of this integral give the possibility to link the monodromy matrices of the initial and transformed CHE1.

The rather boring computations where computer algebra systems can help allow to find the explicit relations between parameters of initial CHE1 and transformed CHE1.

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Third order differential equation with Painlevé property

A. Kessi and Y. Adjabi

Introduction

Painleve, Chazy and Garnier have obtained partial results on the Painlevé classification of rational third-order differential equations of the form

$$u''' = A_2(u', u, x)\, u''^2 + A_1(u', u, x)\, u'' + A_0(u', u, x)\,. \tag{1}$$

In this work we study the case where $A_2 = \dfrac{1-\frac{1}{\eta}}{u'+c_1}$;

$$u''' = \tfrac{1-\frac{1}{n}}{u'+c_1} u''^2 + A_1(u', u, x)\, u'' + A_0(u', u, x)\,. \tag{2}$$

Third order equation (2), such that A_1 is analytic in x and rational in its other arguments, was considered in Chazy and Martynov and others. I.P. Martynov starts with the following simplified equation. i.e. equation which contains the leading terms with leading order $p=-1$ as $x \longrightarrow x_0$ only

$$u''' = \beta \frac{(u''-2uu')^2}{u'-u^2} + buu'' + cu'^2 + du^2u' + fu^4 + a_1\frac{u''u'}{u} + b_1\frac{u'^3}{u^2}, \tag{3}$$

and investigates the values of $b,\ c,\ d,\ e,\ f,\ a_1,\ b_1$ and β such that the equation is of Painlevé type.

We consider the following third order differential equation:

$$u'''\left(\delta u' + \alpha u^2\right) = \beta u''^2 + au''u'u + bu''u^3 + cu'^3 + du'^2u^2 + eu'u^4 + fu^6 + a_1\frac{u''u'^2}{u} + b_1\frac{u'^4}{u^2}. \tag{4}$$

The main aim of this work is to find necessary conditions for equation (4) to be a Painlevé-type equation, and determine all the values of all the coefficients of the equation (4), so that it will be with fixed critical points, moreover we study some obtained equations. The simplified equation corresponding is

$$u''' = \beta\frac{u''^2}{u'} + a_1\frac{u''u'}{u} + b_1\frac{u'^3}{u^2}. \tag{5}$$

Firstly, by taking into account these requirements, we find that the equation (5) is with fixed critical points only if the couple (a_1, b_1) takes one of the following values

$$(a_1, b_1) \in \left\{ \begin{array}{ll} \eta = -2 & : \left(0,\ \frac{1-m^2}{2m^2}\right), \\ \eta = 2 & : \left(\frac{2m-1}{m},\ -\frac{(m-1)(3m+1)}{2m^2}\right);\ \left(2,\ -\frac{3(m-1)(m+1)}{2m^2}\right), \\ \eta \geqslant 2 & : \left(\frac{(m-1)(2+\eta)}{\eta m},\ -\frac{(m-1)(\eta m+m-1)}{\eta m^2}\right), \\ \eta = 3 & : \left(\frac{5m-2}{3m},\ -\frac{2(m-1)(2m+1)}{3m^2}\right), \\ \eta = 5 & : \left(\frac{7m-2}{5m},\ -\frac{2(m-1)(3m+2)}{5m^2}\right), \\ \eta = \infty & : \left(\frac{m-1}{m},\ \frac{1-m}{m}\right);\ \left(1,\ \frac{1}{m^2}-1\right), \\ \eta \text{ integer} & : \left(\frac{2}{\eta}+1,\ -\frac{1}{\eta}-1\right), \end{array} \right\},$$

where m is an integer number different from 0.

Secondly, equation (4) satisfies the Painleve test if all its solutions can be expanded in the formal Laurent series:

$$u = (x - x_0)^r \sum_{i=0}^{\infty} u_i \, (x - x_0)^i , \tag{6}$$

with u_0 non zero and r is the leading power that needs to be found, we see that $r = -1$, and the indicial equation of the equation (4) is

$$(\delta - \alpha u_0)\, r^3 + \left(b u_0^2 + (6\alpha - a)\, u_0 + a_1 + 4\beta - 6\delta\right) r^2 + Q\,(u_0)\, r + P\,(u_0) = 0, \tag{7}$$

where

$$\left\{ \begin{array}{l} Q\,(u_0) = e u_0^3 - (3b + 2d)\, u_0^2 + (3c + 5a - 11\alpha)\, u_0 - 12\beta + 17\delta - 7a_1 - 4b_1, \\ P\,(u_0) = 6 f u_0^4 - 5 e u_0^3 + 4\,(d + 2b)\, u_0^2 + 3\,(6\alpha - 2a - c)\, u_0 + 4a_1 + 2b_1 + 2\,(4\beta - 6\delta)\,, \end{array} \right.$$

and u_0 is root of the following equation :

$$f u_0^4 - e u_0^3 + (d + 2b)\, u_0^2 + (6\alpha - 2a - c)\, u_0 + 2a_1 + b_1 + (4\beta - 6\delta) = 0,\ u_0 \neq 0. \tag{8}$$

So that the equation (4) will be with fixed critical points, it is necessary that, for each values of u_0 solution of the equation(8), the solutions of the equation (7) will be distinct integer numbers.

There are two cases to consider, according to whether α is or not zero.

I. Equation with $\alpha = 0$

By applying Painlevé test method, we find that the list of equation with integer Fuchs's indices. For example

$$\begin{cases} u'''u' = \frac{3}{2}u''^2 - 4u'^3 - 8u'^2u^2 + 2\dfrac{u''u'^2}{u}, \\ u'''u' = u''^2 + \frac{10}{3}u''u'u + \frac{16}{3}u''u^3 - \frac{26}{3}u'^3 - \frac{44}{3}u'^2u^2 + 2\dfrac{u''u'^2}{u}, \\ u'''u' = \frac{3}{2}u''^2 + \frac{31}{6}u''u'u + \frac{1}{3}u''u^3 - \frac{89}{12}u'^3 - \frac{3}{2}u'^2u^2 - \frac{1}{12}u'u^4 + 2\dfrac{u''u'^2}{u}, \end{cases}$$

$$\begin{cases} u'''u' = \frac{2}{3}u''^2 + \frac{1}{3}u''u^3 + u'^2u^2 - \frac{1}{3}u^6 + \dfrac{u''u'^2}{u}, \\ u'''u' = \frac{4}{3}u''^2 + \frac{5}{3}u''u^3 - 5u'^2u^2 + \frac{1}{3}u^6 + \dfrac{u''u'^2}{u}, \\ u'''u' = \frac{1}{2}u''^2 + \frac{5}{2}u'^2u^2 - \frac{1}{2}u^6 + \dfrac{u''u'^2}{u}. \end{cases}$$

II. Equation with $\alpha \neq 0$

Without loss of generality one can see that one of the roots of (8) may be taken equal to -1 , and the corresponds indicial equation of the equation (7) can be writen as

$$(r+1)\big((\alpha+1)\,r^2 + Mr + N\big) = 0, \text{ and } u_0 = -1,$$

where

$$\begin{aligned} M &= b + a - 7 - 7\alpha + a_1 + 4\beta, \\ N &= 24 + 18\alpha - (e + 2d + 3c + 4b + 6a) - 16\beta - 8a_1 - 4b_1. \end{aligned}$$

We will distinguish two cases according to that $(N,\ M) = (0,\ 0)$, or $(N,\ M) \neq (0,\ 0)$.

II.1. The case where $\alpha = -1$ *and* $M = N = 0$, equation (4) takes the form (3), which is already studied by Martynov [4]. We obtained a list of equations with integer Fuchs's indices. For example

$$\begin{cases} u''' = \dfrac{1}{2}\dfrac{(u'' - 2uu')^2}{u' - u^2} + 2uu'' + 8u^2u' - 8u^4 + \dfrac{u''u'}{u}, \\ u''' = \dfrac{1}{2}\dfrac{(u'' - 2uu')^2}{u' - u^2} - \frac{5}{2}uu'' - \frac{37}{24}u'^2 + \frac{125}{24}u^2u' + \frac{125}{24}u^4 + \frac{3}{2}\dfrac{u''u'}{u} - \frac{7}{8}\dfrac{u'^3}{u^2}, \end{cases}$$

$$\begin{cases} u''' = \dfrac{1}{2}\dfrac{(u'' - 2uu')^2}{u' - u^2} + 7uu'' + \frac{265}{24}u'^2 - \frac{833}{24}u^2u' + \frac{343}{24}u^4 + \dfrac{u''u'}{u} - \frac{5}{8}\dfrac{u'^3}{u^2}, \\ u''' = \dfrac{1}{2}\dfrac{(u'' - 2uu')^2}{u' - u^2} + \frac{20}{3}uu'' + \frac{118}{9}u'^2 - \frac{200}{9}u^2u' + \frac{4}{3}\dfrac{u''u'}{u} - \frac{8}{9}\dfrac{u'^3}{u^2}, \\ u''' = \dfrac{1}{2}\dfrac{(u'' - 2uu')^2}{u' - u^2} - \frac{8}{3}uu'' - \frac{46}{9}u'^2 + \frac{128}{9}u^2u' + \frac{5}{3}\dfrac{u''u'}{u} - \frac{10}{9}\dfrac{u'^3}{u^2}, \\ u''' = \dfrac{4}{5}\dfrac{(u'' - 2uu')^2}{u' - u^2} + \frac{6}{5}uu'' - \frac{\left(125n^2 - 153\right)}{108}a_6u'^2 - 2a_6u^2u' + a_6u^4 + \frac{6}{5}\dfrac{u''u'}{u} - \frac{4}{5}\dfrac{u'^3}{u^2}. \end{cases}$$

equtions are not considered in [4].

II.2. The case when $\alpha \neq 0$ and $(N,\ M) \neq (0,\ 0)$

By applying Painlevé test method, we find that the list of equation with integer Fuchs's indices. For example

$$\begin{cases} u'''\left(u'-u^2\right)=u''^2-4u''u'u-u''u^3+4u'^3+u'^2u^2+\dfrac{u''u'^2}{u}-\dfrac{u'^4}{u^2}, \\ u'''\left(u'-u^2\right)=\frac{1}{2}u''^2-4u''u^3+2u'^3+12u'^2u^2-8u'u^4, \\ u'''\left(u'-u^2\right)=u''^2-3u''u'u-u''u^3+3u'^3+\frac{1}{2}u'^2u^2+2\dfrac{u''u'^2}{u}-\frac{3}{2}\dfrac{u'^4}{u^2}, \\ u'''\left(u'-u^2\right)=\frac{2}{3}u''^2-\frac{14}{3}u''u'u-\frac{1}{3}u''u^3+\frac{4}{3}u'^3+\frac{16}{3}u'^2u^2, \\ u'''\left(u'-u^2\right)=\frac{1}{2}u''^2-3u''u'u-u''u^3-3u'^3+\frac{19}{2}u'^2u^2+\frac{3}{2}u^6-6u'u^4+2\dfrac{u''u'^2}{u}. \end{cases}$$

equtions are not considered in [4].

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Dissipationless shocks and Painlevé equations

Christian Klein

The Cauchy problem for dissipationless equations as the Korteweg de Vries (KdV), Camassa-Holm (CH) and nonlinear Schrödinger (NLS) equation with small dispersion of order $\varepsilon \ll 1$, is characterized by the appearance of a zone of rapid modulated oscillations of wave-length of order ε. Near the gradient catastrophe of the dispersionless equation ($\varepsilon = 0$), a multi-scales expansion gives an asymptotic solution in terms of a fourth order generalization of Painlevé I for KdV and CH, and of a Painlevé I transcendent for NLS. At the leading edge of the oscillatory zone for KdV and CH, a corresponding multi-scales expansion yields an asymptotic description of the oscillations where the envelope is given by a solution to the Painlevé II equation. We study the applicability of these approximations for several PDEs and random matrix models numerically.

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From the tau function of Painlevé VI equation to the geometry of moduli spaces

Dmitry Korotkin and Peter Zograf

Abstract. The isomonodromic tau function of Hitchin's solution to the Painlevé VI equation can be naturally generalized for an arbitrary Riemann-Hilbert problem with quasi-permutation monodromies (the corresponding tau function also appears in the theory of Frobenius structures on Hurwitz spaces). Such a tau function can be interpreted as a holomorphic section of the Hodge line bundle on a Hurwitz space. Analysing the asymptotic behavior of the tau function near the boundary of the Hurwitz spaces we obtain new relations between divisor classes on these spaces. Similar results hold for the spaces of holomorphic 1-differentials on Riemann surfaces.

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Hitchin solved the Painlevé VI equation with coefficients $(1/8, -1/8, 1/8, 3/8)$ in [5], and his objective was to classify all $SU(2)$ invariant solutions of self-dual Einstein's equations. The isomonodromic tau function corresponding to his solution was found later in [7] and looks as follows:

$$\tau = \tau_B^{-1/2}\theta[p,q](0|\sigma); \tag{1}$$

here p, q are two complex valued constants, $\theta[p,q](u|\sigma)$ is the theta function with characteristics and σ is the period of elliptic curve $\nu^2 = z(z-1)(z-x)$. The function $\tau_B(x)$ (called the *Bergman tau function* by the reasons explained below) is given by

$$\tau_B = K(x(x-1))^{1/4}, \tag{2}$$

where $K(x)$ is the elliptic integral of the first kind. Formula (1) makes it possible to considerably simplify the original Hitchin's description of the corresponding Einstein manifolds [1].

On the other hand, a straightforward generalization of (1) immediately yields the isomonodromic tau function for an arbitrary Riemann-Hilbert problem with off-diagonal 2×2 monodromy matrices; in that case σ becomes the period matrix of the hyperelliptic curve of genus g given by $\nu^2 = \prod_{i=1}^{2g+2}(z - z_i)$; p and q are two arbitrary vectors from $\mathbf{C}^g$. The Bergman tau function τ_B has in this case the form

$$\tau_B = \det A \prod_{i<j}(z_i - z_j)^{1/4} , \tag{3}$$

where A is the matrix of a-periods of non-normalized differentials $z^{i-1}dz/\nu$, $i = 1, \dots, g$. The expression (3) is also well-known as the partition function of the Ashkin-Teller model [15].

Further generalization of the formulas (1) and (3) appears in the form of an isomonodromic tau function of an arbitrary Riemann-Hilbert problem with quasi-permutation monodromy matrices [8]. Each Riemann-Hilbert problem of this kind in matrix dimension d is naturally associated to a d-sheeted covering of the complex plane, which defines a Riemann surface C of some genus g. Denoting the period matrix of this Riemann surface by σ as before, we find that the corresponding tau function is given by the formula (1), where the vectors p and q are determined by the entries of monodromy matrices. To define the Bergman tau function τ_B is this general case we introduce the canonical bimeromorphic differential $B(x, y) = d_x d_y \log E(x, y)$, where $E(x, y)$ is the prime form on C. Near the diagonal $x = y$ the bidifferential B behaves as follows: $B(x, y) = ((\xi(x) - \xi(y)^{-2} + S_B(x) + ...)d\xi(x)d\xi(y)$, wher ξ is a local parameter. The term $S_B(x)$ transforms as a projective connection under the change of a local parameter; it is called the *Bergman projective connection* [14]. Another projective connection, denoted by S_{df}, is defined by the Schwarzian derivative of f: $S_{df} = \{f(x), \xi(x)\}$, where $f(x)$ is the meromorphic function on C, or the covering of the Riemann sphere. The space of pairs (C, f) for given genus g and degree d of f is called the Hurwitz space $\mathcal{H}_{g,d}$. We assume that all poles and critical values z_i of the function f are simple, C is considered up to an isomorphism, and f up to a linear fractional transformation $\tilde{f} = (af + b)(cf + d)^{-1}$; moreover, z_i serve as local coordinates on $\mathcal{H}_{g,d}$.

The difference of two projective connections $S_B - S_{df}$ is a meromorphic quadratic differential on the Riemann surface C. One can verify that the 1-form

$$\mathbf{q} = -\frac{1}{6}\sum_{i=1}^{n} \text{Res}|_{x=x_i} \frac{S_B - S_{df}}{df} dz_i$$

is closed on the Hurwitz space; here x_i are the critical points of f, $i = 1, \dots, n$ (i.e. $df(x_i) = 0$) and $z_i = f(x_i)$. Then the Bergman tau function on the Hurwitz space (called so after the Bergman projective connection S_B) is defined as a local potential for the 1-form $\mathbf{q}$:

$$d\log\tau_B = \mathbf{q} \tag{4}$$

For the hyperelliptic coverings an explicit integration of (4) gives (3); for an arbitrary covering τ_B is given by a more complicated expression, cf. [9]. The function τ_B is a universal object appearing in different problems - from the theory of Frobenius manifolds [6] to Hermitian matrix models [2] and the spectral theory of Riemann surfaces [13].

It was shown in [11] that the product $\eta = \tau_B^{24(n-1)} V^{-6}$, where $V = \prod_{i<j}(z_i - z_j)$ denotes the Vandermonde determinant of critical values, is invariant under Möbius transformations of the function f. On the other hand, under symplectic transformations of canonical basis on the Riemann surface given by a $2g\times 2g$ matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, it transforms as follows: $\eta \to \eta\,[\det(C\sigma + D)]^{24(n-1)}$. Therefore, η is a section of the $24(n-1)$th power of the Hodge line bundle on the Hurwitz space (moreover, it is holomorphic and non-vanishing as long as critical values of the function f remain finite and distinct). The Hurwitz space $\mathcal{H}_{g,d}$ can be compactified by means of admissible covers [4], and the boundary divisors of the compactification $\overline{\mathcal{H}}_{g,d}$ we denote by $\Delta_\mu^{(k)}$, where $k = 2, \ldots, g+d-1$ and $\mu = [m_1, \ldots, m_r]$ is a partition of d describing the ramification over the double point (see [4, 11] for details). The asymptotics of η near the boundary divisors can be explicitly computed, yielding the following relation [11]:

Theorem 1. *For the Hodge class $\lambda \in \mathrm{Pic}(\overline{\mathcal{H}}_{g,d}) \otimes \mathbf{Q}$ we have*

$$\lambda = \sum_{k=2}^{g+d-1} \sum_{\mu=[m_1,\ldots,m_r]} \prod_{i=1}^{r} m_i \left(\frac{k(n-k)}{8(n-1)} - \frac{1}{12}\left(d - \sum_{i=1}^{r} \frac{1}{m_i} \right) \right) \delta_\mu^{(k)} . \qquad (5)$$

(here $\delta_\mu^{(k)}$ are the classes of the divisors $\Delta_\mu^{(k)}$).

Recently this theorem got a different proof that is based on the Grothendieck-Riemann-Roch theorem, cf. [3].

A natural version of the Bergman tau function also exists on the moduli spaces of holomorphic differentials on Riemann surfaces [13]. These spaces were extensively studied in the theory of dynamical systems (see for example [10]). To each holomorphic 1-differential w on C one can naturally associate a flat metric with conical singularities $|w|^2$ that has trivial holonomy. As it was shown in [13], the regularized determinant of Laplace operator $\Delta^{|w|^2}$ for this metric is given by the formula:

$$\det \Delta^{|w|^2} = \mathrm{Area}(C, |w|^2)\,\{\det \Im\sigma\} |\tau_B(C, w)|^2 \qquad (6)$$

where the Bergman tau function τ_B is defined by a system of equations similar to (4), and can be computed in terms of theta functions as well.

Analysing the behavior of τ_B near the boundary divisor of the projectivized moduli space $\mathbb{P}(\overline{H}_g)$ of holomorphic 1-differentials, we derive a formula for the Hodge class λ similar to (5), cf. [12]:

Theorem 2. *In the rational Picard group $\mathrm{Pic}(\mathbb{P}(\overline{H}_g)) \otimes \mathbf{Q}$ of the space $\mathbb{P}(\overline{H}_g)$*

$$\lambda = \frac{g-1}{4}\psi + \frac{1}{24}\delta_{deg} + \frac{1}{12}\delta_0 + \frac{1}{8}\sum_{j=1}^{[g/2]} \delta_j . \qquad (7)$$

Here λ is the pullback to $\mathbb{P}(\overline{H}_g)$ of the Hodge class on the compactified Riemann moduli space $\overline{\mathcal{M}}_g$, ψ is the tautological class of the projectivization, δ_{deg} is the class *of the divisor of 1-differentials with multiple zeros, and* δ_j, $j = 0, \dots, [g/2]$, *are the pullbacks of the classes of the boundary divisors of* $\overline{\mathcal{M}}_g$.

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Isomonodromic deformations and Jordan-Pochhammer systems

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For a Fuchsian system

$$\frac{\mathrm{d}\, y(z)}{\mathrm{d}\, z} = \left(\sum_{i=1}^{n} \frac{B_i}{z - a_i} \right) y(z). \tag{1}$$

we consider the Schlesinger family of Fuchsian systems

$$\frac{\mathrm{d}\, y(z,\, a)}{\mathrm{d}\, z} = \left(\sum_{i=1}^{n} \frac{B_i(a)}{z - a_i} \right) y(z,\, a). \tag{2}$$

with initial conditions $B_i(a^0) = B_i,\ \ i = 1,\, 2,\, \dots\, n$, where $a^0 = (a_1^0,\, a_2^2,\, \dots,\, a_n^0) \in \mathbb{C}_*^n = \{(a_1,\, a_2^2,\, \dots,\, a_n)|\ a_i \neq a_j,\ i \neq j,\ i,j = 1,\, 2,\, \dots, n\}$. In small neighborhood of the $a^0 = (a_1^0,\, a_2^2,\, \dots,\, a_n^0)$ the sufficient condition on the family (2) that it was the isomonodromic family for the system (1) is the system of the Schlesinger equations on matrices $B_i(a)$, $i = 1,\, 2,\, \dots n$

$$\mathrm{d}\, B_i(a) = - \sum_{j=1,\, j\neq i}^{n} [B_i(a),\, B_j(a)] \frac{\mathrm{d}(a_i - a_j)}{a_i - a_j}. \tag{3}$$

Here $[B_i,\, B_j]$ denote the commutator of matrices. Non-linear Pfaff system (3) is integrable in Frobenius sense [2,4] and consequently in sufficient small neighborhood of the a^0 there exist the local solution $B(a) = (B_1(a),\, B_2(a),\, \dots,\, B_n(a))$ of the Pfaff system (3) with every initial value $B(a^0) = (B_1(a^0) = B_1,\, B_2(a^0) = B_2,\, \dots,\, B_n(a^0) = B_n)$. As well-known [2] that eigenvalues all matrices $B_i(a)$ are constants, that is, don't depend from a. Malgrange was proved [7] that the local solution $B(a)$ has meromorphic continuation on whole universal covering $\widetilde{C}_*^n$. In general, polar divisor Θ of the solution $B(a)$ on $\widetilde{C}_*^n$ is non-empty and it is defined by zero of the tau-function Miva $\tau(a)$ which is a solution of the equation $\mathrm{d} \ln \tau(a) = \kappa \sum_{i\neq j,\, i,j=1}^{n} tr(B_i(a)B_j(a)) \frac{d(a_i - a_j)}{a_i - a_j}$. Theta-divisor Malgrange Θ depends from initial data $B(a^0)$ and now it is called simple Malgrange divisor. The example, when Malgrange is non-empty, give us divisor movable poles of solutions

Painlevé VI equation. Last equation is rational reduction of the (3) for matrices $B_i(a)$ the size 2×3. For general solution $B(a)$ of the (3) with initial values $B(a^0) = (B_1, B_2, \ldots, B_n)$ when the monodromy representation of the (1) is irreducible there exist estimates of the order of poles of the $B(a)$ on the Malgrange divisor [2].

If the system (1) is a system of the order two with reducible monodromy representation then under some condition on eigenvalues of matrices B_i, $i = 1, 2, \ldots, n$, (that is, under additional demand on initial data for system (3)) we obtain that Malgrange divisor of corresponding solution is empty set. In this case we prove also that solution $B(a)$ has power growth in points of the divisor of fixed singularities $H = \cup_{1\le i<j\le n}\{a_i - a_j = 0\}$ and the ramification of the $B(a)$ around H has very special form.

Now we remind some information about special class of multidimensional integrable Fuchsian systems on the $\mathbb{C}^n$ that detailed are considered in works [5, 6] (scalar analog in class of ordinary differential equations of higher orders see in works [1] and [8]).

Jordan-Pochhammer system on the $\mathbb{C}^n$ is a meromorphic system with logarithmic poles of the order one on diagonal hyperplanes the following form

$$\mathrm{d}\, y(a) = \left(\sum_{1\le i<j\le n} J_{ij}(\lambda) \frac{\mathrm{d}(a_i - a_j)}{a_i - a_j} \right) y(a). \tag{4}$$

Here $J_{ij}(\lambda)$ are following matrices the size $n \times n$:

$$J_{ij}(\lambda) = \begin{pmatrix} 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \ldots & \lambda_j & \ldots & -\lambda_i & \ldots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \ldots & -\lambda_j & \ldots & \lambda_i & \ldots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \end{pmatrix} \begin{matrix} \\ \\ i \\ \vdots \\ j \\ \\ \\ \end{matrix}$$

Here λ is ordered collection of complex numbers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. The matrices $J_{ij}(\lambda)$ are satisfied the relations

$$[J_{ij}(\lambda),\, J_{ik}(\lambda) + J_{jk}(\lambda)] = 0, \;\; 1 \le i < j < k \le n; \tag{$*$}$$

$$[J_{ij}(\lambda),\, J_{kl}(\lambda)] = 0, \;\; \{i, j\} \cap \{k, l\} = \emptyset. \qquad (**).$$

The relations (*) and (**) are equivalent the Frobenius condition of the integrability of system (4) (that is, the system (4) is integrable). Then there exist fundamental matrix of solutions $Y(a)$ of the (4) that have holomorphic continuation on universal covering and deck transformations of elements g of the fundamental groups $\pi_1(\mathbb{C}^n_*, a^0)$ on universal covering $\widetilde{C}^n_*$ act on $Y(a)$ by rule $Y(g^* a) = Y(a)M(g)$, where $M(g)$ monodromy matrix corresponding $g \in \pi_1(\mathbb{C}^n_*, a^0)$. The entries y_{ij} of the fundamental matrix of solutions of the system (4) have the following integral representations [6]

$$y_{ij}(a_1,\dots,a_n) = \lambda_i \int_{\gamma_j} (t-a_1)^{\lambda_1}\cdots(t-a_n)^{\lambda_n}\frac{\mathrm{d}\,t}{t-a_i}, \qquad (5)$$

where $\gamma_j,\ j=1,\dots,n$ is a basis in homology group $H_1(\mathbb{C}P^1\setminus\{a_1,\dots,\,a_n,\infty\},\,\mathcal{L}_\chi)$ with local coefficients $\mathcal{L}_\chi$. The local system $\mathcal{L}_\chi$ is defined with a representation $\chi:\pi_1(\mathbb{C}P^1\setminus\{a_1,\dots,\,a_n,\infty\},\,t_0)=F_n\to\mathbb{C}^*$ of the fundamental group $\pi_1(\mathbb{C}P^1\setminus\{a_1,\dots,\,a_n,\infty\},\,t_0)=F_n$, that map generators $x_1,\dots,x_n$ this free groups F_n in non-zero complex numbers $q_1=e^{2\pi i\lambda_1},\dots,q_n=e^{2\pi i\lambda_n}$.

Now we suppose that coefficients of the system (1) have traces equal to zero (that is, $B_i\in sl_2(\mathbb{C}),\ i=1,\,\dots,\,n$) then $B_i(a)\in sl_2(\mathbb{C}),\ i=1,\,\dots,\,n$. Denote $\lambda_i,\ i=1,\dots,n$ eigenvalues of matrices B_i and λ_∞ eigenvalue of matrix $B_\infty = -\sum_{i=1}^n B_i$. We suppose also that B_∞ is the dianalizable matrix.

Theorem 1. *Let monodromy representation of Fuchsian system (1) be reducible. If for some choice of signs $\varepsilon_i=\pm1,\ i\in\{1,\,2,\,\dots,\,n,\,\infty\}$ the sum of eigenvalues $\sum_{i=1}^m \varepsilon_i\lambda_i+\varepsilon_\infty\lambda_\infty$ is equal to zero then the Schlesinger equation (3) under conjugation $CB_i(a)C^{-1}$ by a constant matrix C of all matrices $B_i(a)$, $i=1,2,\dots,n$ is reduced to integrable Jordan-Pochhammer system (4) on the $\mathbb{C}^n$.*

Proof. In [3] Gontsov is proved that under assumptions above the all matrices $B_i(a)$ can be reduced with constant matrix C to upper-triangular mitrices

$$\widetilde{B}_i(a) = \begin{pmatrix} \varepsilon_i\lambda_i & \widetilde{b}_i^{12}(a) \\ 0 & -\varepsilon_i\lambda_i \end{pmatrix}.$$

Conjugating with C the Schlesinger system (3)and compute commutators $[\widetilde{B}_i(a),\widetilde{B}_j(a)]$ we obtain system for $\widetilde{b}_i^{12}(a),\ i=1,\,\dots,\,n$

$$\mathrm{d}\,\widetilde{b}_i^{12}(a) = \left(\sum_{j\neq i,\,j=1}^{n}(\varepsilon_j\lambda_j\widetilde{b}_i^{12}-\varepsilon_i\lambda_i\widetilde{b}_j^{12})\frac{\mathrm{d}(a_i-a_j)}{a_i-a_j}\right). \qquad (6)$$

Last system is coincide with Jordan-Pochhammer system (4) for $\tilde{\lambda} = (\varepsilon_1\lambda_1,\dots,\varepsilon_n\lambda_n)$. Q.E.D.

Corollary 1. *Under conditions of the theorem 1 Malgrange divisor of Schlesinger system (3) is the empty set.*

Proof. The statement of corollary is the consequence of linearity and integrability of Jordan-Pochhammer system and properties of their solutions pointed above. Q.E.D.

The monodromy representations of Jordan-Pochhammer systems (4) have enough descriptions [5, 6, 8]. For case $\lambda_1=\lambda_2=\cdots=\lambda_n$ the monodromy representation of the (4) is equivalent to Burau representations [5]. In case does not equal all $\lambda_i, i=1,\dots,n$ is equivalent to some generalization of Gassner representation [6]. Explicit form monodromy matrices of these representations permit us to describe of ramification of solutions $B(a) = (B_1(a), \ldots, B_n(a)$ of Schlesinger systems.

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Asymptotics of Angelesco polynomials and double scaling limit at pushing point

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Abstract. Multiple orthogonal polynomials on the two disjoint intervals with Jacobi weights are studied. The strong asymptotics is derived by the method of matrix Riemann—Hilbert problem. The double scaling limit near the pushing point is considered.

Keywords. multiple orthogonal polynomials, Angelesco systems, Riemann—Hilbert problems, Painlevé equations.

Multiple orthogonal polynomials originally appear as the denominators of the Hermite—Padé rational approximants. In the last years they attracted a lot of attention because of various applications in random matrices, spectral theory of difference operators and diophantine approximations.

Consider two intervals on the real line $\Delta_1 = [a_1, b_1]$ and $\Delta_2 = [a_2, b_2]$ and two positive weight functions w_1 and w_2 on them. Given a multi-index $(n_1, n_2) \in \mathbb{Z}_+^2$ the multiple orthogonal polynomial P_{n_1,n_2} is the monic polynomial of degree at most $n_1 + n_2$ such that the following orthogonality conditions hold

$$\int_{\Delta_j} P_{n_1,n_2}(x)\, x^k\, w_j(x)\, dx = 0, \quad k = 0, \dots, n_j - 1, \quad j = 1, 2.$$

Angelesco systems introduced in [1] are the special case of multiple orthogonal polynomials with orthogonality on non-overlapping intervals: $\mathring{\Delta}_1 \cap \mathring{\Delta}_2 = \emptyset$. For the Angelesco system the polynomial P_{n_1,n_2} is unique and has exactly n_j zeros on $\mathring{\Delta}_j$.

Kalyagin [2] obtained the asymptotics of $P_{n,n}$ for the special case when the intervals have a common point $b_1 = a_2$ and the weights are the Jacobi weights. At this case the polynomial P_{n_1,n_2} preserves certain properties of classical Jacobi

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polynomials, namely there exists the Rodrigues formula and the linear differential equation.

The Angelesco system with general weights and intervals was considered by Gonchar and Rakhmanov in [3]. They introduced a new approach on the asymptotics analysis of Hermite—Padé approximants based on the equilibrium of the vector potential. In particular they describe the weak limit of the zero-counting measure for $P_{n,n}$:

$$\nu_n(x) := \frac{1}{n} \sum_{P_{n,n}(y)=0} \delta(x-y) \ \to\ \lambda_1 + \lambda_2, \quad n \to \infty.$$

The vector measure (λ_1, λ_2) is the extremal measure that minimizes the energy functional:

$$J(\mu_1, \mu_2) = I(\mu_1, \mu_1) + I(\mu_1, \mu_2) + I(\mu_2, \mu_2)$$

subject to $\{S(\mu_j) \subset \Delta_j,\ \int d\mu_j = 1,\ j = 1, 2\}$, where $S(\mu)$ is the support of the measure μ and $I(\mu, \nu)$ is the mutual logarithmic energy:

$$I(\mu, \nu) = \int\int \ln \frac{1}{|t-x|}\, d\mu(t)\, d\nu(x).$$

In [3] the so-called pushing effect was described. Assume that $|\Delta_1| \geq |\Delta_2|$ and $b_1 \leq a_2$, then there exists a critical value $b^* = b^*(a_1, a_2, b_2)$ such that

$$\begin{cases} S(\lambda_1) = \Delta_1 & \text{and} \quad S(\lambda_2) = \Delta_2, \quad \text{if} \quad b_1 \leq b^*, \\ S(\lambda_1) = \Delta_1^* := [a_1, b^*] & \text{and} \quad S(\lambda_2) = \Delta_2, \quad \text{if} \quad b_1 \geq b^*. \end{cases}$$

In the electrostatic interpretation one can say that the charges on the two conductors repel and for the certain locations the charges on the second (the smaller) conductor push the charges on the first conductor from the edge. This affects the behavior of the equilibrium measure at the end-point of the support:

$$\begin{cases} d\lambda_1(x) \asymp \frac{dx}{\sqrt{b_1 - x}}, & x \to b_1 - 0, \quad \text{if} \quad S(\lambda_1) = \Delta_1, \\ d\lambda_1(x) \asymp \sqrt{b^* - x}dx, & x \to b^* - 0, \quad \text{if} \quad S(\lambda_1) = \Delta_1^* \end{cases}$$

That is we have the transition from hard to soft edge.

We study the strong asymptotics of Angelesco multiple orthogonal polynomials in the case of non-intersecting intervals ($b_1 < a_2$) and modified Jacobi weights: $w_j(x) = (x - a_j)^{\alpha_j}(b_j - x)^{\beta_j} h_j(x)$, $1/h_j \in H(\Delta_j)$, $\alpha_j, \beta_j > -1$, $j = 1, 2$. General result on the strong asymptotics was obtained in [4]. Our analysis is based on the steepest descent method for the matrix Riemann—Hilbert problems [5],[6]. The case of symmetric intervals and Chebyshev weights was already considered in [7]. Generalized Angelesco—Nikishin systems were studied in [8]. The key feature of the current work is the explicit solution of the boundary problem for the Szegö function, where we use the result of [9].

Our particular interest is to describe the local behavior of the polynomials in the neighborhood of the end-point b_1 when the transition from hard to soft edge occurs. The transition is studied in a double scaling limit where $b_1 - b_1^{cr} \asymp n^{-2/3}$ and $n \to \infty$. We mention two papers here: [10] where the phase transition for the Angelesco system with touching intervals is investigated and [11] where the transition from hard to soft edge for random matrix ensembles is described. We expect that the result is related to the Painlevé II equation.

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Third Order Equation with an Irrational Right-Hand Side with the Painleve Property

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Abstract. In this paper the Painleve analysis of third order ordinary differential equations with irrational right-hand side is presented.

Mathematics Subject Classification (2000). 34M55.

Keywords. Painleve classification, differential equation, critical point.

Consider the differential equation

$$x''' = f_1(x, x', x'') + \frac{x'}{x^2} f_2(x, x', x'') + \frac{1}{x^2} \left(f_3(x, x', x'')\right)^{3/2}, \tag{1}$$

where $f_k(x, x', x'') = a_k x x'' + b_k x'^2 + c_k x^2 x' + d_k x^4$, a_k, b_k, c_k, d_k, $k = 1, 2, 3$ are constants; moreover, $f_3(x, x', x'')$ is not a complete square. We single out all classes of equations (1) with the Painleve property. We indicate which functions are integrated the resulting equations. The equation (1) determine one of components of the quadratic third-order system.

Require that the solutions of equation

$$f_3(x, x', x'') = 0 \tag{2}$$

are solutions of equation (1) [1]. If performed these requirement, then equation (1) replace the system

$$x'' + \frac{b_3}{a_3}\frac{x'^2}{x} + \frac{c_3}{a_3} x x' + \frac{d_3}{a_3} x^3 = \eta w^2 x, \quad w' = w^2 + \left(\alpha x + \beta \frac{x'}{x}\right) w, \tag{3}$$

where $\eta = \frac{4}{a_3^3}, \alpha = \frac{1}{2}(\frac{c_3}{a_3} + a_1), \beta = \frac{b_3}{a_3} + \frac{1}{2}(a_2 - 1)$. From (3) we have that $2w = f_3'(x, x', x'')/f_3(x, x', x'') - (\beta + 2)x'/x - \alpha x$. The following statement is valid.

Lemma 1. *Equation (1) has the Painleve property if and only if the system (3) has the Painleve property.*

Holds

Lemma 2. *If equation (1) has the Painleve property, then the equation (2) has the Painleve property.*

Proof. Setting in the system (3) $w = \varepsilon\omega$, we obtain the simplified system

$$x'' + \frac{b_3}{a_3}\frac{x'^2}{x} + \frac{c_3}{a_3}xx' + \frac{d_3}{a_3}x^3 = 0, \quad \omega' = \Big(\alpha x + \beta\frac{x'}{x}\Big)\omega,$$

as $\varepsilon = 0$ which should have the Painleve property. It follows that lemma 2 is valid.

Setting in the system (3) $z = \varepsilon\tau$, at $\varepsilon = 0$ we obtain the simplified system

$$\ddot{x} + \frac{b_3}{a_3}\frac{\dot{x}^2}{x} = 0, \quad \dot{w} = \beta\frac{\dot{x}}{x}w, \tag{4}$$

as $\varepsilon = 0$, where $\dot{x} = \frac{dx}{d\tau}, \ddot{x} = \frac{d^2x}{d\tau^2}, \dot{\omega} = \frac{d\omega}{d\tau}$. The first equation of the system (4) has the Painleve property if and only if

$$\frac{b_3}{a_3} = \frac{1}{n} - 1, \tag{5}$$

where $n \in \mathbb{Z}\setminus\{0\}$ or $n = \infty$. If $\beta \neq 0, a_3 + b_3 \neq 0$, then we find $w = C_1(\tau - \tau_0)^{\beta n}$, where C_1, τ_0 are arbitrary constants (in the following, τ_0 and $C_i, i = 1; 2$, stand for arbitrary constants). Consequently, $\tau = \tau_0$ is not a critical point, only for

$$\beta = \frac{\mu}{n}, \tag{6}$$

where $\mu, n \in \mathbb{Z}\setminus\{0\}$. □

The following statement is valid.

Lemma 3. *If system (3) has the Painleve property then the condition (5) is true and at $\beta \neq 0, n \in \mathbb{Z}\setminus\{0\}$ condition (6) is true.*

Introducing in the system (3) the parameter ε by the formulas $w = \varepsilon^{-1}\omega, z = \varepsilon\tau$, we obtain the simplified system

$$\ddot{x} = \Big(1 - \frac{1}{n}\Big)\frac{\dot{x}^2}{x} + \eta w^2 x, \quad \dot{\omega} = \omega^2 + \beta\frac{\dot{x}}{x}\omega, \tag{7}$$

as $\varepsilon = 0$, where $\dot{x} = \frac{dx}{d\tau}, \ddot{x} = \frac{d^2x}{d\tau^2}, \dot{\omega} = \frac{d\omega}{d\tau}$. Let $\beta = 0$. The second equation in system (7) has a solution $\omega = -1/(\tau - \tau_0)$, then the first equations in system (7) acquires the form

$$\ddot{x} = \Big(1 - \frac{1}{n}\Big)\frac{\dot{x}^2}{x} + \eta\frac{x}{(\tau - \tau_0)^2}. \tag{8}$$

The equation (8) has general solution $x = C_1 e^{C_2(\tau - \tau_0)}(\tau - \tau_0)^{-\eta}$, for $n = \infty$ and

$$x = (\tau - \tau_0)^{\frac{1}{2}\left(n - \sqrt{n(n+4\eta)}\right)}\Big(C_1 + C_2(\tau - \tau_0)^{\frac{1}{2}\sqrt{n(n+4\eta)}}\Big)^n,$$

for $n \in \mathbb{Z}\setminus\{0\}$. If $\beta = 0, n = \infty$, then system (7) has Painleve property only if

$$\eta \in \mathbb{Z}\setminus\{0\}. \tag{9}$$

If $\beta = 0, n \in \mathbb{Z} \setminus \{0\}$, then system (7) has Painleve property if and only if one of the following conditions:

$$n = 2m, \eta = m(s^2 - 1)/2, m \in \mathbb{Z} \setminus \{0\}; \tag{10}$$

$$n = 2m + 1, \eta = (2m + 1)(l^2 - l), m \in \mathbb{Z}; \tag{11}$$

where $s \in \mathbb{Z} \setminus \{-1; 1\}, l \in \mathbb{Z} \setminus \{0; 1\}$ is true. Consequently, the following statement is valid

Lemma 4. *Let $\beta = 0$. If system (3) has the Painleve property, then the condition (9) for $n = \infty$ or one of the conditions (10), (11) is true.*

Let $\beta \neq 0$. From system (7), for ω, we have the equation

$$\ddot{\omega} = \Big(1 - \frac{1}{\mu}\Big)\frac{\dot{\omega}^2}{\omega} + \Big(1 + \frac{2}{\mu}\Big)\omega\dot{\omega} + \Big(\mu\frac{\eta}{n} - \frac{1}{\mu}\Big)\omega^3, \tag{12}$$

for $n \in \mathbb{Z} \setminus \{0\}$ and the equation

$$\ddot{\omega} = \frac{\dot{\omega}^2}{\omega} + \omega\dot{\omega} + \eta\beta\omega^3, \tag{13}$$

for $n = \infty$. Equation (12) has the Painleve property only if one of the following conditions [2,3]:

$$\eta = 10n \text{ or } \eta = n \text{ at } \mu = 1; \eta = 5n/4 \text{ at } \mu = 2;$$

$$\eta = 3n/2 \text{ at } \mu = 3; \eta = 2n \text{ at } \mu = 5; \mu = -2$$

is true. Equation (13), where $\beta \neq 0$, has not the Painleve property [2,3]. From system (7) we find that

$$x = \omega^{\frac{1}{\beta}} e^{-\frac{1}{\beta}\int \omega dt}, \tag{14}$$

where ω is a solution to the of equation (12). The following statement is valid

Lemma 5. *If system (3) at $\beta \neq 0$ has the Painleve property, that one of the following conditions*

1) $n = (\mu + 2)p, \eta = (\mu + 2)(\mu + 3)p/4$, where $p \in \mathbb{Z} \setminus \{0\}, \mu \in \{1; 2; 3; 5\}$;

2) $n = 3p, \eta = 30p$, where $p \in \mathbb{Z} \setminus \{0\}$, for $\mu = 1$;

3) $\eta = pn(p+n)/(2p+n)^2$, where $p, n \in \mathbb{Z}\setminus\{0\}, p \neq -n, p \neq -\frac{n}{2}$, for $\mu = -2$;

4) $\eta = n/4$, where $n \in \mathbb{Z} \setminus \{0\}$, for $\mu = -2$

is true.

Using lemmas 1–5, Painleve analysis of system (3) we obtain the following statement

Theorem 1. *Equation* (1) *has the Painleve property if and only if one of the following conditions:*

1) $a_1 = c_3 = d_1 = d_3 = 0, a_2 = 3 - 2/n, b_1 = -c_2, b_2 = 2/n - 2$, $b_3 = (1/n - 1)a_3, c_1 = -d_2, a_3^3 = 8/(m(s^2 - 1)), n = 2m$, $s \in \mathbb{Z} \setminus \{-1; 1\}, m \in \mathbb{Z} \setminus \{0\}$ or $a_3^3 = 4/((2m + 1)(l^2 - l))$, $n = 2m + 1, l \in \mathbb{Z} \setminus \{0; 1\}, m \in \mathbb{Z}$;

2) $a_1 = 1, a_2 = 3 - 2/n, a_3^3 = 16/(n(s^2 - 1)), b_1 = -c_2, b_2 = 2/n - 2$, $b_3 = (1/n - 1)a_3, c_3 = -a_3, d_1 = 0, d_2 = -c_1 - 2n/(n + 2)^2, d_3 = na_3/(n + 2)^2$, *where* $n \in \mathbb{Z} \setminus \{-2;-1;0\}$, $s \in \mathbb{Z} \setminus \{-1;1\}$, $(1-s)n/2 \in \mathbb{Z}$, $(1-s)n/2 + sk \neq -1$, *if* $n \in \mathbb{N}$, *then* $k = \overline{0,n}$; *if* $n \in \mathbb{Z}_- \setminus \{-2;-1\}$ *then* $k = 0,1,2,\ldots$;

3) $a_1 = c_3 = d_1 = d_3 = 0, a_2 = 3, b_1 = -c_2, b_2 = -2, b_3 = -a_3,\ c_1 = -d_2, 4/a_3^3 \in \mathbb{Z} \setminus \{0\}$;

4) $a_1 = -c_3/a_3, a_2 = 3, b_1 = -c_2, b_2 = -2, b_3 = -a_3, c_1 = -d_2, c_3 \neq 0, d_1 = d_3 = 0, 4/a_3^3 \in \mathbb{Z}_-$;

5) $a_1 = c_3 = d_1 = d_3 = 0, a_2 = 3 + 2(\mu-1)/n, b_1 = -c_2,\ b_2 = 2(1/n-1)(1+\mu/n), b_3 = (1/n-1)a_3, c_1 = -d_2$ *and one of the following conditions:*

a) $n = (\mu+2)p, a_3^3 = 16/((\mu+2)(\mu+3)p)$, *where* $\mu \in \{1;2;3;5\}, p \in \mathbb{Z} \setminus \{0\}$;

b) $n = 3p, a_3^3 = 2/(15p)$, *where* $p \in \mathbb{Z} \setminus \{0\}$, *for* $\mu = 1$;

c) $a_3^3 = 16/n$ *or* $a_3^3 = 4(2p+n)^2/(pn(p+n))$ *where* $p, n \in \mathbb{Z} \setminus \{0\}$, $p \neq -n, p \neq -n/2$, *for* $\mu = -2$

is true;

6) $a_1 = 3, a_2 = 1, a_3^3 = 1/5, b_1 = -c_2, b_2 = b_3 = 0, c_1 = 6 - d_2,\ c_3 = a_3, d_1 = -4, d_3 = -a_3$;

7) $a_1 = 1, a_2 = 2, a_3^3 = 8/15, b_1 = -1 - c_2, b_2 = -1, b_3 = -a_3/2, c_1 = 3 - d_2, c_3 = a_3, d_1 = -1, d_3 = -a_3/2$;

8) $a_1 = 0, a_2 = 1, a_3^3 = -2/3, b_1 = -4/3 - c_2, b_2 = -4/9, b_3 = d_3 = -c_3 = -2a_3/3,\ c_1 = 8/3 - d_2, d_1 = -4/9$;

9) $a_1 = 1 - 2\mu/(n+2), a_2 = 3 + 2(\mu-1)/n, b_1 = -c_2 - 6\mu/(n(n+2)), b_2 = 2(1-n)(n+\mu)/n^2, b_3 = (1/n-1)a_3,\ c_1 = -d_2 + 2(\mu(n+3)-n)/(n+2)^2, c_3 = -a_3, d_1 = -2n\mu/(n+2)^3, d_3 = a_3 n/(n+2)^2$ *and one of the following conditions:*

a) $n = (\mu+2)p, a_3^3 = 16/((\mu+2)(\mu+3)p)$, *where* $\mu \in \{1;2;3;5\}, p \in \mathbb{Z}_-$;

b) $n = (\mu+2)p, a_3^3 = 16/((\mu+2)(\mu+3)p)$, *where* $\mu \in \{1;2;3;5\}, p \in \mathbb{N} \setminus \{1\}$ *or* $n = 3p, a_3^3 = 2/(15p)$, *where* $p \in \mathbb{N}$, *for* $\mu = 1$, *and there has been a correlation* $\sum_{k=1}^{m-1}(m-k)A_k a_{m-k} = -m a_m$, *where* $m = -a_0 - 1, A_0 = 1, A_k = \sum_{l=0}^{k-1}(m-l)A_l a_{k-l}/(k(m-k)), k = \overline{1, m-1}$, *where* a_k *are expansion coefficients* $(\mu+2)p(\dot{\omega} - \omega^2)/(\mu\omega) = a_0/(z-z_0) + a_1 + a_2(z-z_0) + \ldots$, *where* ω *is a solution to the equation (12);*

c) $n = 3p, a_3^3 = 2/(15p)$, *where* $p \in \mathbb{Z}_-$, *for* $\mu = 1$;

d) $a_3^3 = 16/n$, *at* $n \in \mathbb{Z}_- \setminus \{-2;-1\}$ *or* $a_3^3 = 4(n-2m)^2/(nm(m-n))$, *where* $m \neq \dfrac{n}{2}, n - m > 0, m, n \in \mathbb{N}$, *for* $\mu = -2$

is true;

10) $a_1 = 2, a_2 = 6, a_3^3 = 8(p-1)^2/(p(2-p)), p \in \mathbb{Z} \setminus \{0;1;2\}, b_1 = -3 - c_2, b_2 = -6, b_3 = -3a_3/2,\ c_1 = -d_2, c_3 = 0, d_1 = -1, d_3 = -a_3/2$;

11) $a_1 = 2, a_2 = 6, a_3 = -2, b_1 = -3 - c_2, b_2 = -6, b_3 = 3, c_1 = -d_2, c_3 = 0, d_1 = -1, d_3 = 1$

is true.

The solutions of these equations can be expressed via either elementary functions, elliptic functions, or solutions of linear equations.

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Quasi-rational solutions to the focusing NLS equation and multiple rogue waves generation

Vladimir B. Matveev

Focusing NLS equation last years was considered as one of the most credible models for description of the rogue waves propagation in the optical fibers. The related solutions can be regarded as a special modulations of the plane wave solution, such that their amplitude attend one or several big maximums at a finite number of points of the (x, t) plane, in such a way that the amplitude of the solution at the point of the maximum is 3 or more times greater than the amplitude of the seed plane-wave solution, and it tends to the amplitude of the seed solution when $x^2 + t^2 \to \infty$. Surprisingly, only very isolated solutions of this kind which are known as Peregrine breather or higher Peregrine breathers were studied until very recently in connection with rogue waves. These solutions might be also called "extreme" rogue waves. Their amplitude attends only one biggest possible maximum in the (x, t) plane surrounded by the small maxima. In our recent works with Ph. Dubard it was shown that, in fact, the generic rogue waves solutions attend several maximums of the amplitude in (x, t) plane . They can be described by a compact formula involving Wronskian determinants of order 2n, which can be obtained by appropriate reduction of the general Darboux dressing formula for the ψ-function for the non-stationary **linear** Schrödinger equation.[1] For given n and the amplitude of the seed plane waves, solution equal 1 the related solutions depends on 2n real parameters. Peregrine breather corresponds to $n = 1$. It depends on two trivial translation parameters. In the case $n = 2$, the amplitude of solution corresponding to generic choice of the parameters has three big maximums of the height close to that of the Peregrine breather. We explain that a second and higher order Peregrine breathers can be obtained by appropriate specialization of the free parameters in this generic construction and that when we take the parameters to be close enough to the higher order Peregrine breather we obtain the solution which also can be considered as "extreme rogue wave". Therefore for $n \geq 2$ the "extreme" rogue waves solutions are not isolated any more, contrary to the case of Peregrine breather.

[1]This formula is equivalent to that obtained in more complicated way by Eleonski, Krichever and Kulagin in their almost forgotten article written 25 years ago.

The discussed solutions to the NLS equation are also used by us to construct a large family of smooth, real localized rational solutions of the KP-I equation quite different from the multi-lumps solutions. Some part of the related results can be found in our article: P.Dubard , V.B. Matveev ,"Multi-rogue wave solutions to the focusing NLS equation and the KOP-I equation " J. of Nat. Hazards earth System sci. , v.11, 667-672(2011), www.nat-hazards-earth-syst-sci-net/11/667/2011/

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A monodromy problem and some functions connected with Painlevé VI

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Abstract. We consider an integral equation connected with the problem of construction of a linear equation with given monodromy group.

Mathematics Subject Classification (2000). Primary 34M56; Secondary 34M50, 34M55, 45F05.

Keywords. isomonodromic deformations, Belavin - Polyakov - Zamolodchikov equations.

Preliminaries I. Let L be a non-selfintersecting closed curve passing through all ramification points (t_1, t_2, …, t_m, clockwise) of functions y_1, …, y_n, that form basis of solutions to some Fuchsian equation. We assume L smooth. Let D be the domain bounded by L and $y^+ = (y_1^+, \ldots, y_n^+)$ be branches of these functions in D. The condition that $y^- = y^+$ on the arc (t_m, t_1) of L uniquely defines the analytical continuation y^- of y^+ onto $D^- = \overline{\mathbb{C}} \backslash \overline{D}$. It follows that $y^+ = y^- M_1$ on (t_1, t_2), $y^+ = y^- M_1 M_2$ on (t_2, t_3), etc., where M_i are the monodromy matrices. Thus, we get $y^+ = y^- C$ on L, where $C(\zeta)$ is a piecewise constant matrix. This is a particular case of the Riemann - Hilbert (or linear conjugation) problem: given C, find analytic vector functions y^+ in D and y^- in D^- such that $y^+ = y^- C$ on L. The Sokhotskij - Plemelji formulas imply the following singular integral equation for the function $\varphi = (y^- + y^+)/2$:

$$\varphi(x) = \frac{1}{\pi i} \int_{\partial D} \varphi(\zeta) \frac{C(\zeta) - E}{C(\zeta) + E} \frac{d\zeta}{\zeta - x} + y^-(\infty). \tag{0.1}$$

In [1], Its and Harnad considered equation (0.1) with $C(\zeta) = E + f(\zeta) g(\zeta)$, $f(\zeta)$ and $g^T(\zeta)$ are $n \times l$ matrices ($l \leq n$) such that $g(\zeta) f(\zeta) = 0$. Suppose that C satisfies the following condition: for any nonzero $y^-(\infty)$ the equation (0.1) is soluble. Then the matrix problem $Y^+ = Y^- C$, $Y^-(\infty) = E$, reduces to the equations on the function $\Phi = (Y^- + Y^+)/2$:

$$\Phi(x) = E + \int \Phi(\zeta)\frac{f(\zeta)g(\zeta)}{2\pi i(\zeta - x)}\,d\zeta \;\Big[= E + \int \frac{F(\zeta)g(\zeta)\,d\zeta}{2\pi i(\zeta - x)}\Big],$$

$$\Phi^{-1}(x) = E - \int \frac{f(\zeta)g(\zeta)}{2\pi i(\zeta - x)}\Phi^{-1}(\zeta)\,d\zeta \;\Big[= E - \int \frac{f(\zeta)G(\zeta)\,d\zeta}{2\pi i(\zeta - x)}\Big].$$

Here $F = \Phi f$ and $G = g\Phi^{-1}$ are solutions of integral equations whose kernel has no singularity on the diagonal:

$$\begin{cases} F(x) - \int F(\zeta)K(\zeta,x)\,d\zeta = f(x), \\ G(x) - \int K(x,\zeta)G(\zeta)\,d\zeta = g(x), \end{cases} \qquad K(\zeta,x) = \frac{1}{2\pi i}\frac{g(\zeta)f(x)}{\zeta - x}. \tag{0.2}$$

The resolvent is defined by the equation $R(x,z) - \int K(x,\zeta)R(\zeta,z)\,d\zeta = K(x,z)$ and can be written explicitly ([1]):

$$R(x,z) = \frac{1}{2\pi i}\frac{G(x)F(z)}{x - z}. \tag{0.3}$$

Preliminaries II. The following case of the monodromy problem was considered in the papers [1], [2]. L is the system of nonintersected intervals, the number of ramification points is even ($m = 2k$), matrix $C = E + f_i g_i$ on the arcs (t_{2i-1}, t_{2i}), f_i and g_i^T are $n \times l$ constant matrices ($l \le n$), and $g_i f_i = 0$. On the remainder of L it is assumed that $C = E$. Then the problem reduces (as it was explained in Preliminaries I) to the integral equations (0.2)

$$F(x) - \int_{t_1}^{t_2} \frac{F(\zeta)\,d\zeta}{\zeta - x}\frac{g_1 f_j}{2\pi i} - \ldots - \int_{t_{2k-1}}^{t_{2k}} \frac{F(\zeta)\,d\zeta}{\zeta - x}\frac{g_k f_j}{2\pi i} = f_j, \tag{0.4}$$

where $x \in (t_{2j-1}, t_{2j})$, $j = \overline{1,k}$. Here, the kernel is continuous. Differentiating on variables t_j we get, using formulas from [3] and (0.3), the system

$$\frac{\partial F(x)}{\partial t_j} = -\frac{A_j}{x - t_j}F(x), \qquad A_j = \frac{(-1)^j}{2\pi i}F(t_j)G(t_j). \tag{0.5}$$

It follows from the integral equation that $F(x)$ is homogeneous on x, t_j and holomorphic on x near infinity:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial t_1} + \ldots + \frac{\partial F}{\partial t_m} = 0, \quad x\frac{\partial F}{\partial x} + t_1\frac{\partial F}{\partial t_1} + \ldots + t_m\frac{\partial F}{\partial t_m} = 0.$$

Thus, if the matrices M_i satisfy conditions $(M_i - E)^2 = 0$, $i = \overline{1,2k}$, and $M_{2i-1} = M_{2i}^{-1}$, $i = \overline{1,k}$, and are independent of t_j, then they are generators of the monodromy group of the Fuchsian system

$$\frac{\partial F}{\partial x} = AF, \qquad A = \sum_{i=1}^{m}\frac{A_i}{x - t_i}. \tag{0.6}$$

From (0.6), (0.5) follows the Schlesinger system

$$\frac{\partial A_j}{\partial t_i} = \frac{[A_i, A_j]}{t_i - t_j}, \quad i \ne j, \qquad \frac{\partial A_i}{\partial t_i} = -\sum_{j\ne i}\frac{[A_i, A_j]}{t_i - t_j}. \tag{0.7}$$

3. The τ-function was introduced in the paper [4] in connection with the system (0.6). Its construction involves the closed form $\omega = \sum H_i\, dt_i = d \ln \tau$, where

$$H_i = \frac{1}{2} \operatorname*{res}_{x=t_i} (\mathrm{tr} A^2) \equiv \sum_{j \neq i}^{m} \frac{\mathrm{tr} A_j A_i}{t_i - t_j}.$$

In what follows, we describe the results of [5]. We assume that A_i are 2×2 matrices and $\mathrm{tr} A_i = 0$, $\det A_i = 0$. According to (0.6), we have

$$\sum_{i=1}^{m} \frac{1}{x - t_i} \frac{\partial F}{\partial t_i} = \frac{\partial^2 F}{\partial x^2} - \sum_{i=1}^{m} \frac{H_i}{x - t_i} F.$$

For $\Phi = \tau F$ this read as

$$\sum_{i=1}^{m} \frac{1}{x - t_i} \frac{\partial \Phi}{\partial t_i} = \frac{\partial^2 \Phi}{\partial x^2}; \tag{0.8}$$

(0.8) is a special case of the equation on the correlator Φ in the conformal field theory

$$\sum_{i=1}^{m} \frac{1}{x - t_i} \frac{\partial \Phi}{\partial t_i} = \kappa \frac{\partial^2 \Phi}{\partial x^2} - \sum_{i=1}^{m} \frac{\Delta_i}{(x - t_i)^2} \Phi,$$

(the Virasoro algebra have central charge $c = -6\kappa + 13 - 6\kappa^{-1}$, formula (5.17) in [6]; we get (0.8) setting $c = 1$ and $\Delta_i = 0$).

The Neumann series for (0.4) gives an example of a solution to (0.8) (with polylogarithms). We apply linear superposition of solutions of (0.8) to construct solutions to systems (0.6), (0.7). This is only one aspect of the problem of complete integration of system (0.7) in special functions.

4. We demonstrate the reduction to P-VI in the particular case $m = 2k = 4$ in (0.4). The monodromy matrices satisfy the relations

$$M_1 = \begin{pmatrix} 1 & 2\pi i \gamma \\ 0 & 1 \end{pmatrix} = M_2^{-1}, \qquad M_3 = \begin{pmatrix} 1 & 0 \\ 2\pi i \gamma & 1 \end{pmatrix} = M_4^{-1}.$$

The equation for $F(x)$ follows from (0.4). Suppose that $t_1 = 1$, $t_2 = t$, $t_3 = 0$, and $t_4 \to \infty$. Then

$$F(x) - \lambda \int_1^t \frac{\ln x - \ln \zeta}{x - \zeta} F(\zeta)\, d\zeta = \begin{bmatrix} 1 \\ \ln x \end{bmatrix}, \tag{0.9}$$

where $\lambda = \gamma^2$. Due to (0.3), (0.5), we get

$$F = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \quad R(x, z) = \frac{\psi_1(x)\psi_2(z) - \psi_2(x)\psi_1(z)}{x - z}.$$

Set $\widetilde{a} = \psi_1(1)$, $a = \psi_1(t)$. It follows that the function $\sigma = t a^2 + \widetilde{a}^2$ is a solution to the σ-form (see [2]) of Painlev'e VI equation

$$[t(t-1)\sigma'']^2 = 4\sigma'(t\sigma' - \sigma)((1-t)\sigma' + \sigma). \tag{0.10}$$

Note that $\sigma = t(t-1)R(t,t)$. Let δ be the Fredholm determinant for (0.9). Using the identity $R(t,t) = (\ln\delta)'$, we get a formula for σ with an entire function δ on λ. One can check that we get solutions to P-VI by substitution to (0.10)

$$\sigma = -\lambda(t-1) - \lambda^2 t(t-1)\int_1^t \frac{(\ln x - \ln t)^2}{(x-t)^2}\,dx + \dots.$$

Moreover, for the function $\phi = \delta\psi_1$ we have the equation ([7])

$$\frac{t(t-1)}{x(x-1)(x-t)}\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2} + \left(\frac{1}{x} + \frac{1}{x-1}\right)\frac{\partial\phi}{\partial x}. \tag{0.11}$$

The Fredholm theory, for the integral equation (0.9), gives the expansion $\phi = 1 + \lambda\phi_1 + \lambda^2\phi_2 + \dots$ of solution to equation (0.11). Coefficients at the powers of λ satisfies to (0.11), for example,

$$\phi_1 = \int_1^t [K(x,z) - K(z,z)]\,dz = \int_1^t \Big(\frac{\ln x - \ln z}{x-z} - \frac{1}{z}\Big)\,dz.$$

The complexity of ϕ_i grows quickly as i increases. Noting that all branches of ϕ_i satisfy (0.11), one can get simpler solutions; in particular, ϕ_1 generate solutions $\ln(x-1) - \ln(x-t)$ and $\ln x - \ln(x-t)$.

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Particular solutions of q-Painlevé equations and q-hypergeometric equations

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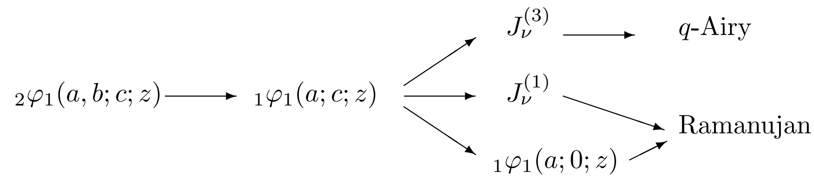
Abstract. We study a degeneration diagram of linear q-difference equations of hypergeometric type, which are second order q-difference equations whose coefficients are linear functions. We obtain seven q-hypergeometric equations, including two types of q-Bessel equations and two types of q-Airy functions. We explain how our degeneration scheme corresponds to a degeneration diagram of q-Painlevé equations.

Mathematics Subject Classification (2000). Primary 34M55; Secondary 33E17.

Keywords. q-Painlevé equation, basic hypergeometric series.

1. Introduction

We give a unified theory for q-special functions, which come from degeneration of the basic hypergeometric functions ${}_2\varphi_1(a,b,c;q;x)$. We obtain seven types of q-special functions. We have two different the q-Bessel functions. We also have two q-Airy equations, which are essentially equivalent.



We also show that a relation to hypergeometric type of q-Painlevé equations and our classification of q-special functions. See [6] for details.

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2. q-difference equation of the hypergeometric type

We call a q-difference equation of the second order with the linear coefficients

$$(a_0 + b_0 x)u(xq^2) + (a_1 + b_1 x)u(xq) + (a_2 + b_2 x)u(x) = 0 \qquad (2.1)$$

it the hypergeometric type. We denote the solution space of the above equation as

$$\Phi \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}; x$$

following [1]. We set $\mathrm{lq}\, x = \log x / \log q$.

Theorem 2.1. *A q-difference equation of the hypergeometric type has transformations which keep the hypergeometric type:*

(A) Change $x \to cx$:

$$\Phi \left[\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}; cx\right] = \Phi \left[\begin{matrix} a_1 & a_2 & a_3 \\ cb_1 & cb_2 & cb_3 \end{matrix}; x\right]$$

(B) Change $u \to x^\gamma u$ ($c = q^\gamma$)

$$x^\gamma\, \Phi \left[\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}; x\right] = \Phi \left[\begin{matrix} c^2 a_1 & ca_2 & a_3 \\ c^2 b_1 & cb_2 & b_3 \end{matrix}; x\right]$$

(C) Change $x \to 1/x$

$$\Phi \left[\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}; \frac{1}{x}\right] = \Phi \left[\begin{matrix} b_3 & b_2 & b_1 \\ q^2 a_3 & q^2 a_2 & a_1 \end{matrix}; x\right]$$

(D) Change $u \to (ax;q)_\infty/(bx;q)_\infty u$:

$$\Phi \left[\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}; x\right] = x^s \frac{(-b_3 x/a_3;q)_\infty}{(-b_1 x/a_1 q;q)_\infty}\, \Phi \left[\begin{matrix} a_3 & a_2 & a_1 \\ qb_3 & b_2 & q^{-1} b_1 \end{matrix}; x\right]$$

where $s = \mathrm{lq}(a_3/a_1)$.

We classify q-difference equations of the hypergeometric type up to the transformations in Theorem 1. Then we obtain seven classes of q-difference equations:

Theorem 2.2. *A q-difference equation* (2.1) *of the hypergeometric type reduces to one of the following equation by transforms in theorem* 2.1*.ip* $= \sqrt{q}$)

1) When $a_1 a_3 b_1 b_3 \neq 0$, Heine's hypergeometric ${}_2\varphi_1(a,b;c;q;x)$:

$$(c - abqx)u(xq^2) - [c + q - (a+b)qx]u(qx) + q(1-x)u(x) = 0.$$

2) When $b_3 = 0$, $a_1 a_3 b_1 b_2 \neq 0$, ${}_1\varphi_1(a;c;q;x)$:

$$(c - aqx)u(xq^2) - (c + q - qx)u(qx) + qu(x) = 0.$$

3-1) When $b_1 = b_2 = 0$, $a_3 \cdot a_2 a_1 b_3 \neq 0$, Jackson's Bessel functions $J_\nu^{(1)}(x;q)F$

$$u(xp^2) - (p^\nu + p^{-\nu})u(xp) + (1 + x/4)u(x) = 0.$$

3-2) When $b_1 = b_3 = 0$, $a_2 \cdot a_3 a_1 b_2 \neq 0$, Hahn-Exton's Bessel functions $J_\nu^{(3)}(x;q)$:

$$u(xp^2) + [-(p^\nu + p^{-\nu}) + p^{2-\nu} x]u(xp) + u(x) = 0.$$

3-3) When $b_3 = a_1 = 0$, $a_2b_2 \cdot a_3b_1 \neq 0$, *q-Hermite-Weber* ${}_1\varphi_0(a; -; q; x)$

$$axu(xq^2) + (1-x)u(xq) - u(x) = 0.$$

4-1) When $b_1 = a_2 = b_3 = 0$, *q-Airy* $Ai_q(x) = {}_1\varphi_1(0; -q; q; -x)$ *:*

$$u(xq^2) + xu(xq) - u(x) = 0.$$

4-2) When $a_1 = b_2 = b_3 = 0$, *the Ramanujan function* ${}_0\varphi_1(-; 0; q; -tq)$*:*

$$qxu(xq^2) - u(xq) + u(x) = 0.$$

In the study of differential equations, shearing transformations are useful to study irregular singular points whose Poincaré rank is non-integer.

Shearing transformations are also useful for q-differential equations when a slope of the Newton diagram is non-integer.

For a q-difference equation

$$a(x)u(xq^2) + b(x)u(xq) + c(x)u(x) = 0,$$

a *shearing transformation* is the following transformation:

$$x = t^2, \ p = \sqrt{q}, \ v(t) = u(x).$$

Then we have

$$a(t^2)v(tp^2) + b(t^2)v(tp) + c(t^2)v(t) = 0.$$

Lemma 2.3. *In Theorem* 2.2*, (4-2) is equivalent to (4.1) by shearing transformation.*

By connection formula of the q-Airy function by T. Morita, we obtain a relation between the q-Airy function and the Ramanujan function [5]:

$$\mathrm{A}_{q^2}(-q^3/x^2) = \frac{q^2}{(q, -1; q)_\infty}\left(-\theta(-x/q)\mathrm{Ai}_q(xq^2) + \theta(x/q)\mathrm{Ai}_q(-xq^2)\right).$$

3. Hypergeometric solutions of the q-Painlevé equations

As the same as the Painlevé differential equations have particular solutions represented by (confluent) hypergeometric functions, the q-Painlevé equations also have special solutions written by q-hypergeometric functions.

In [3], they has studies q-hypergeometric solutions of the q-Painlevé equations. The degeneration diagram of q-hypergeometric solutions of the q-Painlevé equations is as follows:

q-**P**	q-P_{VI}	$\to$	q-P_{V}	$\to$	q-P_{IV} q-P_{III}	$\to$	q-P_{II}	$\to$	q-P_{I}
HG	${}_2\varphi_1$	$\to$	${}_1\varphi_1$	$\to$	${}_1\varphi_1(a; 0; z)$ ${}_1\varphi_1(0; b; z)$	$\to$	${}_1\varphi_1(0; -q; z)$	$\to$	none
	(1)	$\to$	(2)	$\to$	(3-3) (3-2)	$\to$	(4-1)	$\to$	none

Comparing our list in Theorem 2.2, we do not have (3-1) and (4-2). The equation (4-2) is related to (4-1) by a shearing transformation. The equation (3-1) appears in another form of q-P_{III}.

It is know that there are several types of the q-Painlevé equations. For q-P_{III}, one is called q-$P_{\mathrm{III}}(A_5^{(1)})$ by Sakai [8]:

$$\frac{y\bar{y}}{a_3a_4} = -\frac{\bar{z}(\bar{z}-b_2t)}{\bar{z}-b_3}, \quad \frac{z\bar{z}}{b_3} = -\frac{y(y-a_1t)}{a_4(y-a_3)}.$$

Another one is known by Ramani, Grammaticos and Hietarinta [7]:

$$\frac{\overline{w}\underline{w}}{a_3a_4} = \frac{(w-a_1s)(w-sa_2)}{(w-a_1)(w-a_4)}, \tag{3.1}$$

which is a symmetric specialization of q-P_{VI} found by Jimbo and Sakai [2]. And $J_\nu^{(1)}(x;q)$, a solution of (3-1), is a special solution of (3.1) shown by [4].

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Asymptotic expansions of solutions to the fifth Painlevé equation

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Abstract. By means of Power Geometry we obtained all asymptotic expansions of solutions to the equation P_5 of the following five types: power, power-logarithmic, complicated, exotic and half-exotic for all values of 4 complex parameters of the equation. They form 16 and 30 families in the neighbourhood of singular points $z = \infty$ and $z = 0$ correspondingly. There exist 10 families in the neighbourhood of nonsingular point. Over 20 families are new.

1. Introduction

We consider the fifth Painlevé equation

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}, \quad (1.1)$$

where $\alpha, \beta, \gamma, \delta$ are complex parameters, z is an independent complex variable, w is a dependent one. The fifth Painlevé equation (1.1) has two singular points: $z = 0$ and $z = \infty$. The aim of the present work is to find all asymptotic expansions of solutions to the fifth Painlevé equation. We obtain these expansions using the methods of Power Geometry [1,2]. We are looking for the expansions of the following form near the singular points of the equation:

$$w = c_r(z)z^r + \sum_{s\in\mathbf{K}} c_s(z)z^s, \quad (1.2)$$

where $c_r(z), c_s(z), r, s \in \mathbb{C}$, $\mathbf{K} \subset \{\ s \mid \text{Re } s > \text{Re } r\}$ for the expansions in the neighbourhood of $z = 0$ and $\mathbf{K} \subset \{\ s \mid \text{Re } s < \text{Re } r\}$ for the expansions in the neibourhood of $z = \infty$; the set $\mathbf{K}$ is countable.

We obtain the expansions (1.2) of the following five types:

Type 1. $c_r(z)$ and $c_s(z)$ are constant (*power expansions*).

Type 2. $c_r(z)$ is constant, $c_s(z)$ are polynomials in $\log z$ (*power-logarithmic*).

Type 3. $c_r(z)$ and $c_s(z)$ are power series in $\log z$ (*complicated expansions*).

Type 4. r, $s \in \mathbb{R}$, $c_r(z)$ and $c_s(z)$ are series in z^i, and c_r is a sum of countable number of terms, the set of power exponents of z^i in c_r is bounded either from above, or from below (*exotic expansions*).
Type 5. r, $s \in \mathbb{R}$, $c_r(z)$ is a finite sum of powers of z^i with complex coefficients and $c_s(z)$ are power series over z^i (*half-exotic*).

The apparatus of the Power Geometry [1,2] permits to work with ordinary differential equations which have the form $f(z, w, w', w'', \ldots) = 0$, where f is a polynomial in its variables (we call the left part of the equation a differential sum). To transform the equation (1.1) to a differential sum we multiply it by $z^2 w(w-1)$. The next step of the equation's exploration is construction of its polygon $\Gamma(f)$. The polygon $\Gamma(f)$ for the equation (1.1) is shown in Fig. 1. The edges $\Gamma_j^{(1)}$, $j = 1, 2, 3$ give the expansions in the neighbourhood of $z = \infty$, the edge $\Gamma_1^{(1)}$ gives the expansions near $z = 0$.

The equation (1.1) has the symmetry

$$(z, w, \alpha, \beta, \gamma, \delta) = (\tilde{z}, \frac{1}{\tilde{w}}, -\tilde{\beta}, -\tilde{\alpha}, -\tilde{\gamma}, \tilde{\delta}). \tag{1.3}$$

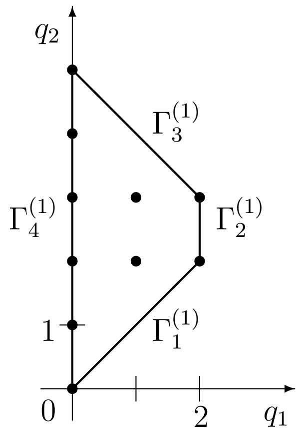

Fig. 1

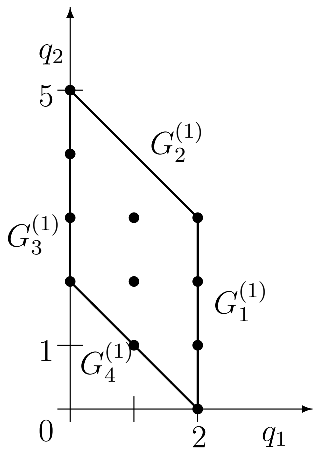

Fig. 2

2. Asymptotic expansions of solutions near infinity [3]

Theorem 2.1. *In the neighbourhood of $z = \infty$ there exist 10 asymptotic expansions of solutions to the equation* (1.1) *(power expansions):*

If $\alpha\beta\delta \neq 0$ there exist the following 5 *expansions of solutions*

$$\mathcal{D}_k : w = (-1)^k \sqrt{\frac{\beta}{\delta}} \frac{1}{z} + \left(-\frac{2\beta}{\delta} + (-1)^k \frac{\gamma}{2\delta} \sqrt{\frac{\beta}{\delta}} \right) \frac{1}{z^2} + \sum_{s=3}^{\infty} \frac{c_{sk}}{z^s}.$$

$$\mathcal{E}_1: \ w = -1 + \frac{2\gamma}{\delta z} + \sum_{s=2}^{\infty} \frac{c_s}{z^s}.$$

$$\mathcal{F}_k: \ w = (-1)^k \sqrt{-\frac{\delta}{\alpha}} z + 2 + (-1)^k \frac{1}{2} \frac{\gamma}{\sqrt{-\alpha\delta}} + \sum_{s=1}^{\infty} \frac{c_{s,k}}{z^s},$$

where c_s, c_{sk} are uniquely determined complex constants, $k = 1, 2$.

If $\alpha\beta\gamma \neq 0$, $\delta = 0$ there exist the following 5 expansions of solutions

$$\mathcal{D}_k: \ w = (-1)^k \sqrt{-\frac{\beta}{\gamma}} \frac{1}{\sqrt{z}} + \frac{\beta}{\gamma} \frac{1}{z} + \sum_{s=3}^{\infty} \frac{c_{s,k}}{z^{s/2}}.$$

$$\mathcal{E}_2: \ w = 1.$$

$$\mathcal{F}_k: \quad w = (-1)^k \sqrt{-\frac{\gamma}{\alpha}} \sqrt{z} + 1 + \sum_{s=1}^{\infty} \frac{c_{s,k}}{z^{s/2}},$$

where c_s, c_{sk} are uniquely determined complex constants, $k = 3, 4$. For every expansion except $\mathcal{E}_2$ there exist two corresponding exponential additions of the form $b(z)Ce^{\varphi(z)}$, where C is an arbitrary constant, $\varphi'(z)$, $b(z)$ are power series. Moreover, every exponential addition can be continued to the exponential expansion of the following type:

$$w = \sum_{k=0}^{\infty} b_k(z) C^k e^{k\varphi(z)}, \tag{2.1}$$

where C is an arbitrary constant, $\varphi'(z)$, $b_k(z)$ are power series.

If $\beta = 0$ there exist two families of expansions $\mathcal{U}_1$, $\mathcal{U}_2$ of the type (2.1) with $b_0 \equiv 0$. If $\alpha = 0$ there exist two families $\mathcal{V}_1$, $\mathcal{V}_2$, which can be obtained from $\mathcal{U}_1$, $\mathcal{U}_2$ by the symmetry (1.3). If $\gamma = \delta = 0$ there exist one family for $\alpha = 0$ and one more for $\beta = 0$

3. Asymptotic expansions of solutions near zero [4]

The following truncated equation corresponds to the edge $\Gamma_4^{(1)}$

$$-z^2 w(w-1)w'' + z^2 \left(\frac{3}{2}w - \frac{1}{2}\right)(w')^2 - zw(w-1)w' + (w-1)^3(\alpha w^2 + \beta) = 0. \tag{3.1}$$

As the differential operator (the first variation of (3.1)) vanishes only if $w = 1$ thus $w = 1$ is its special solution. We substitute $w = 1 + \tilde{w}$ into (1.1) and obtain an equation $g(z, \tilde{w}) = 0$ with polygon $\tilde{\Gamma} = \Gamma(g)$ having edges $G_j^{(1)}$, $j = 1, 2, 3, 4$ and 4 vertices (see Fig. 2)). As we consider the case $z \to 0$, the appropriate truncated solutions can be given by equations corresponding to vertex $(0, 2)$ and edges $G_3^{(1)}, G_4^{(1)}$.

Theorem 3.1. *In the neighbourhood of $z = 0$ there exist the following 8 families of expansions, corresponding to the edge $G_4^{(1)}$:*

$$\mathcal{H}_1: \ y = 1 - \frac{2\delta}{\gamma} x + \sum_{s \in \mathbf{K}} c_s x^s,$$

where $a = \left(\operatorname{sgn} \operatorname{Re} \frac{\gamma}{\sqrt{-2\delta}}\right) \frac{\gamma}{\sqrt{-2\delta}}$, $\mathbf{K} = \{s : s = l + m + ma, \ l, m \in \mathbb{Z}, \ l, m \geq 0, \ l + m > 0\}$, c_{a+1} *is an arbitrary constant, exists if* $\gamma\delta \neq 0$, $\gamma^2/2\delta = -p^2, p \in \mathbb{R} \setminus \mathbb{N}$. *If* $a \in \mathbb{Q}$ *the expansion converges according to Theorem* 1.7.2 [2]*;*

$$\mathcal{H}_2: \ y = 1 - \frac{2\delta}{\gamma} x + \sum_{s=1}^{\infty} c_s x^s,$$

where c_s, $1 \leq s \leq a$ *are constant and* c_s, $s \geq a + 2$ *are polynomials in* $\log z$ *with uniquely determined coefficients,* $c_{a+1} = A \log x + C$, *where* C *is an arbitrary constant, exists if* $\gamma\delta \neq 0$, $\gamma^2/2\delta = -n^2, n \in \mathbb{N}$.

$$\mathcal{H}_3: \ y = 1 - \frac{\delta}{\gamma} x - \frac{\gamma}{2} (\ln x + C)^2 x + \sum_{p=2}^{\infty} \varphi_p x^p,$$

where C *is an arbitrary constant,* φ_p *are series in decreasing powers of* $\log x$ *with uniquely determined coefficients;*

$$\mathcal{H}_1^\tau: \ y = 1 + \left(-\frac{2\delta}{\gamma} + C x^{i\tau\gamma/\sqrt{2\delta}}\right) x + \sum_{\operatorname{Re}\, s > 1} c_s x^s, \ \gamma^2/\delta \in \mathbb{R}_+, \tau = \pm 1, \ C \neq 0;$$

$$\mathcal{H}_4: \ y = 1 + \left(c_r x^{ir} - \frac{\gamma}{r^2} + \frac{\gamma^2 - 2\delta r^2}{4 c_r r^4} x^{-ir}\right) x + \sum_{\operatorname{Re}\, s > 1} c_s x^s, \ r \in \mathbb{R} \setminus \{0\}, \ c_r \in \mathbb{C},$$

r *and* c_r *are arbitrary constants.*

The families $\mathcal{H}_1$, $\mathcal{H}_1^\tau$, $\mathcal{H}_2$ *and* $\mathcal{H}_3$ *are one-parametric, the family* $\mathcal{H}_4$ *is two-parametric.*

If $\gamma \neq 0$, $\delta = 0$ *there exist the families* $\mathcal{H}_3$ *and* $\mathcal{H}_4$ *(we should substitute* $\delta = 0$ *in the corresponding formulae).*

If $\gamma = 0$, $\delta \neq 0$ *there exist two families of expansions*

$$\mathcal{H}_j^{(1)}: \ y = 1 + (-1)^j \sqrt{-2\delta} (\ln x + C) x + \sum_{p=2}^{\infty} \varphi_p x^p, \ j = 5, 6,$$

where C *is an arbitrary constant,* φ_p *are series in decreasing powers of* $\log x$ *with uniquely determined coefficients; and the family* $\mathcal{H}_4$ *(we should substitute* $\gamma = 0$ *in the corresponding formulae).*

Theorem 3.2. *In the neighbourhood of* $z = 0$ *there exist the following* 21 *families of asymptotic expansions corresponding to the edge* $G_3^{(1)}$ *and to the vertex* $G_3^{(0)} = (0, 3)$ *obtained from the corresponding families of expansions of solutions to the* *sixth Painlevé equation:* $\mathcal{B}_i$, $i = 1, 1', 2, 2', 3, 4, 5, 6, 8, 9, 10$, $\mathcal{B}_j^\tau$, $j = 0, 1, 2, 6, 7$, $\mathcal{A}_1$. *The family* $\mathcal{A}_2$ *is obtained by the symmetry* (1.3) *from the family* $\mathcal{A}_1$. *Families* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *are of type 1, but* $\mathcal{B}_{1'}$ *and* $\mathcal{B}_{2'}$ *are of type 2.*

4. Asymptotic expansions of solutions in the neighbourhood of the nonsingular point of the equation [5]

To explore the expansions near the nonsingular point $z = z_0$, $z_0 \neq 0$, $z_0 \neq \infty$ of the equation we perform the the transformation $z = t + z_0$ which permits us to apply to the transformed equation the algorithms of Power Geometry described above.

Theorem 4.1. *In the neighbourhood of the nonsingular point* $z = z_0$ *of the equation* (1.1) *there exist* 10 *families of asymptotic expansions of its solutions. The new one is*

$$\mathcal{O}_8: \ w = 1 - \frac{\gamma}{2z_0}(z - z_0)^2 + \sum_{s=4}^{\infty} c_s (z - z_0)^s,$$

where c_4 *is an arbitrary constant, other* c_s *are uniquely determined. The expansions exist when* $\gamma \neq 0$, $\delta = 0$. *One of* 10 *families is two-parameter, the rest families are one-parameter.*

Theorem 4.2. *The expansions of all* 10 *families converge in the deleted neighbourhood of* $z = z_0$.

Some of these results are not new – they can be found in [6,7].

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Gauge transformation of the sixth Painlevé equation

Yoshikatsu Sasaki

Abstract. This talk concerns the isomonodromic deformations of three linear ordinary differential equations, which governed the sixth Painlevé equation. We show that there exists a 2-dimensional system of LODEs s.t. (i) by eliminating an entry, the system reduces to one of the 3 LODEs; (ii) the IMD of the system is equivalent to the IMD of one of the 3 LODEs; (iii) the system is obtained from a linearization of the sixth Painlevé equation by use of a gauge transformation.

Mathematics Subject Classification (2000). Primary 34M55.

Keywords. the sixth Painlevé equation, gauge transformation.

1. 3 LODEs

Consider the following 3 LODEs:

$$
\begin{aligned}
L_{\mathrm{VI}}: \quad & \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + p_1 \frac{\mathrm{d}y}{\mathrm{d}x} + p_2 y = 0, \ \kappa = \rho(\rho + \kappa_\infty) \ (= \rho(\bar{\rho} + 1)), \\
& p_1 = \frac{1-\kappa_0}{x} + \frac{1-\kappa_1}{x-1} + \frac{1-\theta}{x-t} - \frac{1}{x-\lambda}, \\
& p_2 = \frac{\kappa}{x(x-1)} - \frac{t(t-1)H}{x(x-1)(x-t)} + \frac{\lambda(\lambda-1)\mu}{x(x-1)(x-\lambda)}.
\end{aligned}
$$

$$
\begin{aligned}
\bar{L}_{\mathrm{VI}}: \quad & \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + p_1 \frac{\mathrm{d}y}{\mathrm{d}x} + p_2 y = 0, \ \bar{\kappa} = \rho(\rho + \kappa_\infty - 1) \ (= \rho\bar{\rho}), \\
& p_1 = \frac{1-\kappa_0}{x} + \frac{1-\kappa_1}{x-1} + \frac{1-\theta}{x-t} - \frac{2}{x-\bar{\lambda}}, \\
& p_2 = \frac{\bar{\kappa}}{x(x-1)} - \frac{t(t-1)\bar{H}}{x(x-1)(x-t)} + \frac{\bar{\lambda}(\bar{\lambda}-1)\bar{\mu}}{x(x-1)(x-\bar{\lambda})}.
\end{aligned}
$$

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$$\tilde{L}_{\mathrm{VI}}: \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + p_1 \frac{\mathrm{d}y}{\mathrm{d}x} + p_2 y = 0,\ \bar{\kappa} = \rho(\rho + \kappa_\infty - 1)\ (= \rho\bar{\rho}),$$

$$p_1 = \frac{1-\kappa_0}{x} + \frac{1-\kappa_1}{x-1} + \frac{1-\theta}{x-t} - \sum_{k=1,2} \frac{1}{x-\lambda_k},$$

$$p_2 = \frac{\bar{\kappa}}{x(x-1)} - \frac{t(t-1)\tilde{H}}{x(x-1)(x-t)} + \sum_{k=1,2} \frac{\lambda_k(\lambda_k-1)\mu_k}{x(x-1)(x-\lambda_k)}.$$

We assume that

- the regular singularities of 3 equations lie in generic locations;
- the parameters $\kappa_0, \kappa_1, \theta, \kappa_\infty \in \mathbb{C}\backslash\mathbb{Z}$ (then ρ is defined by Fuchsian rel.);
- each equation has no logarithmic solutions (non-logarithmic condition).

Riemann scheme of each equation reads :

$$L_{\mathrm{VI}}: \begin{Bmatrix} x=0 & x=1 & x=t & x=\lambda & x=\infty \\ 0 & 0 & 0 & 0 & \rho \\ \kappa_0 & \kappa_1 & \theta & 2 & \rho+\kappa_\infty \end{Bmatrix},$$

$$\bar{L}_{\mathrm{VI}}: \begin{Bmatrix} x=0 & x=1 & x=t & x=\bar{\lambda} & x=\infty \\ 0 & 0 & 0 & 0 & \rho \\ \kappa_0 & \kappa_1 & \theta & 3 & \rho+\kappa_\infty-1 \end{Bmatrix},$$

$$\tilde{L}_{\mathrm{VI}}: \begin{Bmatrix} x=0 & x=1 & x=t & x=\lambda_1 & x=\lambda_2 & x=\infty \\ 0 & 0 & 0 & 0 & 0 & \rho \\ \kappa_0 & \kappa_1 & \theta & 2 & 2 & \rho+\kappa_\infty-1 \end{Bmatrix}.$$

2. Gauge transformation

IMD(=isomonodromic deformation) of the equations have been studied separately[3, 2, 5]. In this talk, we observe these 3 equations from a unific viewpoint.

Theorem 2.1. *(*[6]*) There exists a 2 dimensional system of LODEs:*

$$S_{\mathrm{VI}}: \quad \frac{\mathrm{d}\vec{y}}{\mathrm{d}x} = A(x)\vec{y},\ \vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},\ A(x) : \text{a } 2\times 2 \text{ matrix with function entries.}$$

s.t.
(i) by eliminating y_1 or y_2, S_{VI} reduces to L_{VI}, $\bar{L}_{\mathrm{VI}}$ or $\tilde{L}_{\mathrm{VI}}$;
(ii) IMD of S_{VI} is equivalent to IMD of L_{VI}, $\bar{L}_{\mathrm{VI}}$ or $\tilde{L}_{\mathrm{VI}}$;
(iii) S_{VI} is obtained by a Gauge transformation of the linearization of P_{VI} given in [1] *(also see* [4]*).*

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Antiquantization of quantum models as a tool for generating Painlevé equations

Sergey Yu. Slavyanov

The Painlevé equations can be derived and studied by different means. One of the most fruitful approaches is the use of linear equations as a basic tool. The conventional ones are 2×2 first order linear systems added by isomonodromie conditions [1] and second order equations with apparent singularity. Being quite straightforward from the analytical point of view these approaches lead, however, to very sophisticated practical computations. The author of the report recently proposed another approach which seems to be a tricky recipe. but leads to the result much more efficient [2, 3]. This approach was later termed as antiquantization procedure [5, 6]. It includes

Choice of quantum equation. It could be only equation belonging to Heun class.
Choice of the parameter which is termed as adiabatic and which plays the role of time.
Choice of "energy" or hamiltonian.
Normalization of the hamiltonian. This step is equivalent to introduction of isomonodromie condition.

This is the content of a quantum model. Examples are discussed in the publication [7]. Further the quantum variables in the hamiltonian coordinate and momentum are substituted for classical coordinate and momentum. The classical Euler-Lagrange equations appear to be one of the Painlevé's equation.

In conventional antiquantization procedures the role of order of quantum operators is important. In our procedure is not. Normalization of the wave function is also not crucial.

The general Heun equation in essence can be studied as following [4]

$$\sigma(z)y''(z)+\sum_{j=1}^{3}(1-b_j)\sigma_j(z)y'(z)+\left[\sum_{j=1}^{3}\frac{a_j\sigma_j(z)}{(z-z_j)}+\delta(z-z_3)-\right.$$
$$\left.\left(\frac{\lambda\sigma_3(z_3)}{(z_2-z_1)}+\frac{1}{2}\sum_{j=1}^{2}(1-b_3)(1-b_j)\frac{\sigma_3(z_3)}{z_3-z_j}\right)\right]y(z)=0. \qquad (1)$$

Here

$$b_j=(\rho_{1j}+\rho_{2j}),\quad a_j=\rho_{1j}\rho_{2j},\ j=1,2,3,\quad a_\infty=\kappa_1\kappa_2,$$
$$\sigma(z)=\prod_{j=1}^{3}(z-z_j),\quad \sigma_j(z)=\frac{\sigma(z)}{z-z_j},$$
$$\delta=a_\infty-\sum_{j=1}^{3}a_j,$$

where ρ_{mj} are characteristic exponents at finite singular points and κ_1,κ_2 are characteristic exponents at infinity. It can be shown that the quantity λ stays invariant under linear transform of independent variable and s-homotopic transformation of the function y. The other invariants are squares of differences between characteristic exponents

$$\Delta_j=(\rho_{1j}-\rho_{2j})^2,\ j=1,2,3\quad \Delta_\infty=(\kappa_1-\kappa_2)^2.$$

Applying the antiquantization procedure we arrive to the following equation [4]

$$\frac{2\sigma_3(t)}{\sqrt{\sigma(q)}}\frac{d}{dt}\frac{\dot q\sigma_3(t)}{\sqrt{\sigma(q)}}+\frac{\dot q\sigma_3^2(t)}{\sigma(q)(q-t)}+$$
$$\left[-\Delta_\infty+\sum_{j=1}^{2}\frac{(\Delta_j+1-2b_j)\sigma_j(z_j)}{(q-z_j)^2}+\frac{(\Delta_3-1)\sigma_3(t)}{(q-t)^2}\right]=0. \qquad (2)$$

This is a general form of the Painlevé equation P^6 generated by general form of Heun equation. Two important features of equation (2) should be emphasized.

1. The role of the singular point $z_3=t$ is specific in (2) compared to the other points z_1,z_2.
2. The only influence of generalization due to s-homotopic transformation is a slight dependence on b_j $j=1,2$ in (2).

The general approach to Hamiltonian structure for Painlevé equations is as following. Each equation belonging to the Heun class in its canonical form may be presented in a form

$$\frac{1}{f(t)}\left[P_0(z,t)D^2+P_1(z,t)D+P_2(z,t)\right]y(z)=\lambda y(z) \qquad (3)$$

In eq. (3) $P_0(z,t)$, $P_1(z,t)$, $P_2(z,t)$ are polynomials in z of order not higher than third, t is the scaling parameter, λ is the accessory parameter or "energy" in formulation above. The factor $f(t)$ is due to normalization of energy. If quantum observables $\hat{q}$, $\hat{p}$, ($\hat{q}$ the coordinate and $\hat{p}$ the momentum) are associated with z and D in eq. (3) it becomes the Hamiltonian structure and can be rewritten as

$$H(\hat{q},\hat{p},t)y = \lambda y \tag{4}$$

In eq. (4) the function H is the Hamiltonian adiabatically depending on the parameter t which can be considered as time, λ is energy. The corresponding Hamiltonian in classical machanics is quadratic in the classical momentum p.

$$H(q,p,t) = \frac{1}{f(t)}\left[P_0(q,t)p^2 + P_1(q,t)p + P_2(q,t)\right] \tag{5}$$

The Legendre transform can be applied to this Hamiltonian turning from momentum p to velocity q_t. The following Euler-Lagrange equation of motion relates to this Lagrangian

$$\begin{aligned} q_{tt} = {} & \frac{1}{2}\frac{\partial}{\partial q}(\ln P_0(q,t))q_t^2 - \left(\frac{\partial}{\partial t}(\ln f(t)) - \frac{\partial}{\partial t}(\ln P_0(q,t))\right) q_t \\ & + \frac{P_0(q,t)}{f^2(t)}\left(\frac{\partial}{\partial q}\frac{P_1^2(q,t)}{2P_0(q,t)} + f(t)\frac{\partial}{\partial t}\frac{P_1(q,t)}{P_0(q,t)} - 2\frac{\partial P_2(q,t)}{\partial q}\right) \end{aligned} \tag{6}$$

The following theorem holds.

Theorem. *Each type of equations belonging to Heun class is generic for one of the equations of Painlevéé class in the sense that a Schrödinger type equation corresponds via antiquantization recipe to an equation of motion in classical dynamics. The inverse statement holds also. Each type of Painlevé equations may be generated by the corresponding Heun equation.*

Specially chosen solutions of Painleve equations constitute the class of special functions related to nonlinear mathematical physics – the so-called Painlevé transcendents.

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Integral transformation of Heun's equation and some applications

Kouichi Takemura

Abstract. Kazakov and Slavyanov established that Heun's equation admits Euler's integral transformation which changes the parameters. In the talk we investigate the transformation in detail. Existence of polynomial-type solutions corresponds to apparency (non-branching) of a singularity by the transformation, and we obtain integral representations of solutions of Heun's equation which has an apparent singularity.

Mathematics Subject Classification (2000). Primary 34M35; Secondary 33E10.

1. Heun's equation and Integral transformation

Heun's equation is a standard form of a second-order Fuchsian equation with four singularities, which is given by

$$\frac{d^2y}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-t}\right)\frac{dy}{dz}+\frac{\alpha\beta z-q}{z(z-1)(z-t)}y=0, \tag{1.1}$$

with the condition $\gamma+\delta+\epsilon=\alpha+\beta+1$ [2, 3]. The parameter q is independent from the local exponents and is called an accessory parameter. It is known that Heun's equation appears as a degenerate case of the linear differential equations which produce Painlevé VI equation by monodromy preserving deformation.

Kazakov and Slavyanov established that Heun's equation admits Euler's integral transformation which changes the parameters;

Proposition 1.1 ([1]). *Set*

$$\begin{aligned}&(\eta-\alpha)(\eta-\beta)=0,\ \gamma'=\gamma-\eta+1,\ \delta'=\delta-\eta+1,\ \epsilon'=\epsilon-\eta+1, \\ &\{\alpha',\beta'\}=\{2-\eta,\alpha+\beta-2\eta+1\},\ q'=q+(1-\eta)(\epsilon+\delta t+(\gamma-\eta)(t+1)).\end{aligned} \tag{1.2}$$

Let $v(w)$ be a solution of

$$\frac{d^2v}{dw^2}+\left(\frac{\gamma'}{w}+\frac{\delta'}{w-1}+\frac{\epsilon'}{w-t}\right)\frac{dv}{dw}+\frac{\alpha'\beta'w-q'}{w(w-1)(w-t)}v=0. \tag{1.3}$$

Then the function

$$y(z) = \int_{[\gamma_z,\gamma_p]} v(w)(z-w)^{-\eta} dw \tag{1.4}$$

is a solution of Eq. (1.1) *for* $p \in \{0,1,t,\infty\}$*, where* $[\gamma_z,\gamma_p] = \gamma_z\gamma_p\gamma_z^{-1}\gamma_p^{-1}$ *is the Pochhammer contour which turns around the points* $w = z$ *and* $w = p$*.*

Note that Proposition 1.1 was rediscovered by considering middle convolutions of 2×2 Fuchsian differential equations with four singularities (see [6]).

2. Apparent singularity and Integral representation of solutions

Let us consider local solutions of Heun's equation. The exponents of Eq.(1.1) about $z = t$ are 0 and $1-\epsilon$. If $\epsilon \notin \mathbb{Z}$, then we have a basis of local solutions as follows;

$$f(z) = \sum_{j=0}^{\infty} c_j (z-t)^j, \quad g(z) = (z-t)^{1-\epsilon} \sum_{j=0}^{\infty} \tilde{c}_j (z-t)^j, \ (c_0 \neq 0 \neq \tilde{c}_0). \tag{2.1}$$

We now define an apparent singurality in the case $\epsilon \in \mathbb{Z}$. For simplicity, we only consider the case $\epsilon \in \mathbb{Z}_{\leq 0}$. If $\epsilon \in \mathbb{Z}_{\leq 0}$, then we have a basis of local solutions as

$$f(z) = \sum_{j=0}^{\infty} c_j (z-t)^j + A\, g(z) \log(z-t), \ g(z) = (z-t)^{1-\epsilon} \sum_{j=0}^{\infty} \tilde{c}_j (z-a)^j. \tag{2.2}$$

If the logarithmic term in Eq.(2.2) disappear, i.e. $A = 0$, then the singularity $z = t$ is called apparent. Note that the apparency of a regular singularity is equivalent to that the monodromy about $z = t$ is trivial.

Set $n = 1 - \epsilon$. Then the condition that the singularity $z = t$ is apparent is written as $P^{\mathsf{app}}(q) = 0$, where $P^{\mathsf{app}}(q)$ is a polynomial of the variable q of order n.

Example. If $\epsilon = -2$ $(n = 3)$, then the condition that the regular singularity $z = t$ is apparent is written as

$$\begin{aligned} 0 = P^{\mathsf{app}}(q) = {} & q^3 + \{(-3\alpha\beta - 3\alpha - 3\beta - 1)t + (3\gamma - 4)\}q^2 \\ & + \{(3\alpha^2\beta^2 + 6\alpha\beta(\alpha+\beta) + 10\alpha\beta + 2(\alpha^2+\beta^2) + 2\alpha + 2\beta)t^2 \\ & + ((-6\alpha\beta - 4\alpha - 4\beta)\gamma + 4\alpha\beta + 4\alpha + 4\beta)t + 2(\gamma-1)(\gamma-2)\}q \\ & - \alpha\beta t\{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)t^2 - \gamma(3\alpha\beta + 4\alpha + 4\beta + 4)t + 2\gamma(\gamma-1)\}. \end{aligned} \tag{2.3}$$

By the integral transformation, polynomial-type solutions of Heun's equation corresponds to solutions which has an apparent singularity ([7]). Moreover we have the following proposition;

Theorem 2.1. *[*[7]*] If* $\epsilon \in \mathbb{Z}_{\leq 0}$*,* $\alpha, \beta, \beta-\gamma, \beta-\delta \notin \mathbb{Z}$ *and the singularity* $z = t$ *of Eq.* (1.1) *is apparent, then there exists a non-zero solution of Eq.*(1.3) *which can* *be written as* $w^{\beta-\gamma}(w-1)^{\beta-\delta}h(w)$ *where* $h(w)$ *is a polynomial of degree* $-\epsilon$ *and the functions*

$$\int_{[\gamma_z,\gamma_p]} w^{\beta-\gamma}(w-1)^{\beta-\delta}h(w)(z-w)^{-\beta}dw, \tag{2.4}$$

$(p=0,1)$ *are non-zero solutions of Eq.* (1.1).

Example. Set $\epsilon=-2$. The condition that the singularity $z=t$ of Eq. (1.1) is apparent is written as Eq.(2.3). Then there exists a non-zero solution of Eq. (1.3) written as $w^{1-\gamma'}(w-1)^{1-\delta'}h(w)=w^{\beta-\gamma}(w-1)^{\beta-\delta}h(w)$ where

$$\begin{aligned} h(w) &= 2\alpha(\alpha+1)w^2+2(\alpha+1)\{q-\alpha(\beta+2)t\}w \\ &+ q^2-\{2\alpha\beta+3\alpha+\beta+1)t-\gamma+2\}q+\alpha t\{t(\alpha+1)(\beta+1)(\beta+2)-\beta\gamma\}, \end{aligned} \tag{2.5}$$

and the functions in Eq.(2.4) $(p=0,1)$ are non-zero solutions of Eq.(1.1).

We have similar results for linear equations which produce Painlevé VI ([7]).

3. Elliptical representation of Heun's equation and Integral transformation

It is known that Heun's equation has an elliptical representation. Let $\wp(x)$ be the Weierstrass doubly periodic function with periods $(2\omega_1, 2\omega_3)$, $\omega_0(=0)$, ω_1, $\omega_2(=-\omega_1-\omega_3)$, ω_3 be the half-periods and $e_i=\wp(\omega_i)$ $(i=1,2,3)$. Then Heun's equation (Eq. (1.1)) is transformed to

$$H^{(l_0,l_1,l_2,l_3)}f(x)=Ef(X), \quad H^{(l_0,l_1,l_2,l_3)}=-\frac{d^2}{dx^2}+\sum_{i=0}^{3} l_i(l_i+1)\wp(x+\omega_i), \tag{3.1}$$

by setting $z=(\wp(x)-e_1)/(e_2-e_1)$, $t=(e_3-e_1)/(e_2-e_1)$. The eigenvalue E corresponds to the accessory parameter q.

We rewrite the integral transformation of Heun's equation (i.e. Proposition 1.1) in elliptical representation form. It is remarkable that the eigenvalue E is unchanged by the integral transformation.

Proposition 3.1 ([7, 8]). *Let* $\sigma(x)$, $\sigma_i(x)$ $(i=1,2,3)$ *be the Weierstrass sigma and co-sigma functions and* I *be a suitable cycle. Let* α'_i *be a number such that* $\alpha'_i=-l'_i$ *or* $\alpha'_i=l'_i+1$ *for each* $i\in\{0,1,2,3\}$. *Set* $d=-\sum_{i=0}^{3}\alpha'_i/2$ *and* $\eta=d+2$. *If* $\tilde{f}(x)$ *satisfies*

$$H^{(l'_0,l'_1,l'_2,l'_3)}\tilde{f}(x)=E\tilde{f}(x), \tag{3.2}$$

then the function

$$\begin{aligned} f(x) &= \sigma(x)^{\alpha'_0+d+1}\sigma_1(x)^{\alpha'_1+d+1}\sigma_2(x)^{\alpha'_2+d+1}\sigma_3(x)^{\alpha'_3+d+1}\cdot \\ &\int_I \tilde{f}(\xi)\sigma(\xi)^{1-\alpha'_0}\sigma_1(\xi)^{1-\alpha'_1}\sigma_2(\xi)^{1-\alpha'_2}\sigma_3(\xi)^{1-\alpha'_3}(\sigma(x+\xi)\sigma(x-\xi))^{-\eta}d\xi \end{aligned} \tag{3.3}$$

satisfies

$$H^{(\alpha'_0+d,\alpha'_1+d,\alpha'_2+d,\alpha'_3+d)}f(x) = Ef(x). \tag{3.4}$$

Theorem 3.2 ([7, 8])**.** *Let $k \in \{1,3\}$ and $M^{(l_0,l_1,l_2,l_3)}_{2\omega_k}(E)$ be the monodromy matrix by the shift of the period $x \to x + 2\omega_k$ with respect to a certain basis of solutions to $H^{(l_0,l_1,l_2,l_3)}f(x) = Ef(x)$. Then*

$$\text{trace}\, M^{(l'_0,l'_1,l'_2,l'_3)}_{2\omega_k}(E) = \text{trace}\, M^{(\alpha'_0+d,\alpha'_1+d,\alpha'_2+d,\alpha'_3+d)}_{2\omega_k}(E). \tag{3.5}$$

In other word, periodicity is preserved by the integral transformation.

We apply the integral transformation for the case where Heun's equation has the finite-gap property, i.e. the case where $l_0, l_1, l_2, l_3 \in \mathbb{Z}$ ($\gamma, \delta, \epsilon, \alpha - \beta \in \mathbb{Z} + 1/2$). For the case $l_0, l_1, l_2, l_3 \in \mathbb{Z}$ we can calculate the monodomy in principle for all E by means of hyperelliptic integrals [4] and by the Hermite-Krichever Ansatz [5]. By applying monodromy invariance, we can calculate the monodromy of Heun's equation for the case $l_0, l_1, l_2, l_3 \in \mathbb{Z} + 1/2$ and $l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z} + 1$ ($\gamma, \delta, \epsilon, \alpha + 1/2, \beta + 1/2 \in \mathbb{Z}$), which have not been studied so far.

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Asymptotics at the gradient catastrophe points and Painlevé 1 transcendents: semiclassical focusing NLS and orthogonal polynomials cases

Alexander Tovbis and Marco Bertola

We study the asymptotics at the points of gradient catasrophe for: A) semiclassical limit of the focusing Nonlinear Schrödinger equation (NLS); B) large N asymptotics of monic orthogonal polynomials $\pi_n(z)$ with the quartic weight $e^{-N(z^2/2+tz^4/4)}$. Our technique is based on the nonlinear steepest descent method for matrix Riemann-Hilbert problems.

1. Semiclassical limit of the focusing NLS

We consider the integrable ($x \in \mathbb{R}$) focusing NLS

$$i\varepsilon\partial_t q + \varepsilon^2 \partial_x^2 q + 2|q|^2 q = 0, \tag{1-1}$$

with rapidly decaying analytic initial data $q(x,0,\varepsilon) = A(x)e^{i\Phi(x)/\varepsilon}$ in the semiclassical (zero dispersion) limit $\varepsilon \to 0$. Generically, in different regions of the spacetime, there are different asymptotic (as $\varepsilon \to 0$) regimes separated by **breaking curves** (or **nonlinear caustics**), see Fig. 1. Separating the amplitude and the phase

$$q(x,t,\varepsilon) = b(x,t,\varepsilon)\mathrm{e}^{i\Phi(x,t,\varepsilon)/\varepsilon}, \quad U := b^2, \quad V = \Phi_x, \tag{1-2}$$

the NLS equation can be recast as

$$U_t + (UV)_x = 0\ , \qquad V_t + VV_x - U_x + \frac{\varepsilon^2}{2}\left(\frac{1}{2}\frac{U_x^2}{U^2} - \frac{U_{xx}}{U}\right)_x = 0 \tag{1-3}$$

Neglecting the ε^2-term yields an elliptic system, which develops singularities in the derivatives at some finite (x_0, t_0). This point is called a gradient catastrophe point. On Fig. 1, (x_0, t_0) corresponds to the tip of the breaking curve (right in front of the very first spike).

A solution $q(x,t,\varepsilon)$ of the NLS (1-1) is defined by its initial data $q(x,0,\varepsilon)$. At the same time, since (1-1) is an integrable equation, its solutions can be defined through the scattering data: the reflection coefficient $r(z,\varepsilon)$, $z \in \mathbb{R}$, and the points of the discrete spectrum (solitons) together with their norming constants. The

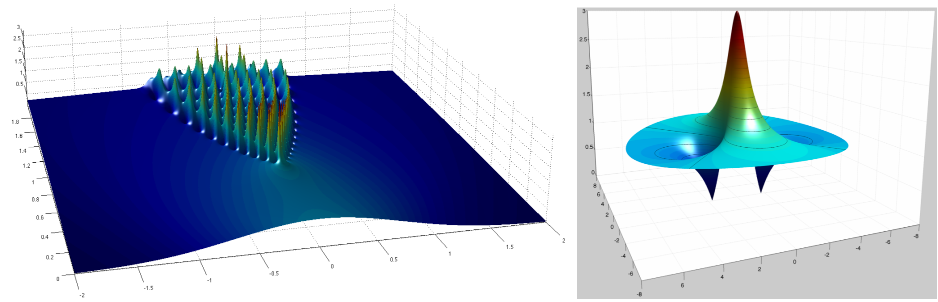

FIGURE 1. Left: the case of $A(x) = e^{-x^2}$, $\Phi'(x) = \tanh x$ and $\varepsilon = 0.03$; Right:the shape of a spike is the scaled Peregrine's breather.

initial and the scattering data of a solution to the NLS are related through direct and inverse scattering transformations for the corresponding Zakharov-Shabat system of linear equations. For example, the initial data with $A(x) = -\text{sech}\ x$ and $\Phi_x = -\mu \tanh x$, $\mu > 0$, corresponds to ([5])

$$r(z,\varepsilon) = -i\varepsilon 2^{-\frac{i\mu}{\varepsilon}} \frac{\Gamma(1-w+w_+ + w_-)\Gamma(w-w_+)\Gamma(w-w_-)}{\Gamma(w_+)\Gamma(w_-)\Gamma(w-w_+-w_-)}\ , \tag{1-4}$$

where

$$w_+ = -\frac{i}{\varepsilon}\left(T + \frac{2}{\mu}\right), w_- = \frac{i}{\varepsilon}\left(T - \frac{2}{\mu}\right), w = -z\frac{i}{\varepsilon} - \mu\frac{i}{2\varepsilon} + \frac{1}{2} \text{ and } T = \sqrt{\frac{\mu^2}{4} - 1}. \tag{1-5}$$

For the sake of simplicity, we consider $\mu \geq 2$, when there are no solitons. *The semi-classical limit of the scattering transformations is the subject of ongoing research.* Using the Stirling asymptotics in the above example with $\mu = 2$, we can calculate the leading order term r_0 of r as

$$r_0(z,\varepsilon) = e^{-2if_0(z)/\varepsilon} \text{ where } f_0(z) = (1-z)\left[i\frac{\pi}{2} + \ln(1-z)\right] + z\ln z + \ln 2 + \frac{\pi}{2}\varepsilon, \tag{1-6}$$

when $\Im z \geq 0$. Casting the inverse scattering problem for the NLS-evolution of r_0 as the Riemann-Hilbert problem (RHP) and using the nonlinear steepest descent method, we recover the corresponding solution q of (1-1) with $O(\varepsilon)$ accuracy. In particular, we recover $|q(x,0,\varepsilon)| = \text{sech}\ x$ and $\Phi(x,0,\varepsilon) = -2\tanh x$. In general, solutions to (1-1), corresponding to a wide class of reflection coefficients of the form $r_0(z,\varepsilon) = e^{-2if_0(z)/\varepsilon}$, were described in [6]. It was assumed there that $f_0(z)$ are analytic in the upper halfplane and $\Im f_0(z) < 0$ everywhere on $\mathbb{R}$ except one finite interval, where $\Im f_0(z) > 0$. It was also shown that, subject to additional technical assumptions, the corresponding solutions q will evolve as modulated plane waves (1-2) until undergoing the gradient catastrophe at some finite points x_0, t_0.

The results of Subsection 1.1 below were proven (see [1]) for solutions q defined through their reflection coefficients $r_0(z,\varepsilon) = e^{-2if_0(z)/\varepsilon}$, where the requirements to $f_0(z)$ were described in [6]. There are little doubts that these results are valid for a significantly wider class of solutions q, including solutions with solitons. However, in order to describe (and prove) the essence of the gradient catastrophe phenomenon for the focusing NLS , we consider the most technically convenient and yet a wide enough class of solutions q, as described above.

1.1. The results

Let us introduce scaling variables $x = x_0 + \varepsilon^{4/5}X$, $t = t_0 + \varepsilon^{4/5}T$ in an $O(\varepsilon^{4/5})$-size neighborhood D of the point of gradient catastrophe x_0, t_0. We have established the following *universal behaviors* of the NLS solutions near the point of *generic* gradient catastrophe (x_0, t_0):

1. there exists a mapping $v = v(X,T)$ of D of into a finite region V, $0 \in V$, of the complex v-plane of the form
$$v = -i\sqrt{\frac{2ib_0}{C}}\,(X + 2(2a_0 + ib_0)T)\,(1 + \mathcal{O}(\varepsilon^{2/5})), \tag{1-7}$$
such that the center of each spike is given by $(X,T) = v^{-1}(v_p)$, where $v_p \in V$ is a pole of the *tritronquée* solution $y(v)$ of the Painlevé I (P1) $y'' = 6y^2 - v$; here $b_0 = |q(x_0,t_0,\varepsilon)|$ and $a_0 = \Phi_x(x_0,t_0)$, see Fig. 1.1;
2. the maximum amplitude of each spike in D is $3|q(x_0,t_0,\varepsilon)| + O(\varepsilon^{1/5})$;
3. the shape of each spike is universally the one of the *rational* (Peregrine) *breather* solution (aka rogue wave, see Figure 1) to the NLS , scaled to the size $O(\varepsilon)$;
4. if $(x,t) \in D$ but lies $O(\varepsilon)$ away from the spikes, the asymptotics of the NLS solution is
$$q(x,t,\varepsilon) = \left(b_0 - 2\varepsilon^{2/5}\Im\left(\tfrac{y(v)}{C}\right) + \mathcal{O}(\varepsilon^{3/5})\right) \times$$
$$\exp\frac{2i}{\varepsilon}\left[\tfrac{1}{2}\Phi(x_0,t_0) - \left(a_0(x-x_0) - (2a_0^2 - b_0^2)(t-t_0)\right) + \varepsilon^{\frac{6}{5}}\Re\left(\sqrt{\tfrac{2i}{Cb_0}}H_I(v)\right)\right] \tag{1-8}$$
where $H_I = (y'(v))^2/2 + vy(v) - 2y^3(v)$ and C is a nonzero constant that can be calculated explicitly.

This asymptotics is uniform on V away fromthe poles of $y(v)$.

Eq. (1-8) confirms the conjecture of Dubrovin, Grava and Klein [3] about the form of the leading order correction to the semiclassical asymptotics in the non-oscillatory region around (x_0,t_0), whereas the first 3 items go well beyond the prediction of this conjecture, establishing simple universal behavior of NLS-solutions near the point of gradient catastrophe. We further conjecture that the P1 hierarchy occurs at higher degenerate catastrophe points and that the amplitudes of the spikes are odd multiples of the amplitude at the corresponding catastrophe point.

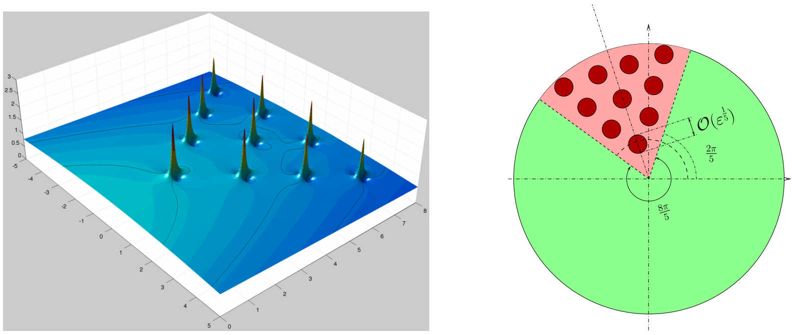

FIGURE 2. Spikes correspond to the poles of the *tritronquée* solution $y(v)$, shown as red (dark) circles on the right.

2. Double scaling limit of orthogonal polynomials with quartic weight

Asymptotic analysis of the recurrence coefficients $\alpha_n(t,N), \beta_n(N,t)$ for orthogonal polynomials with varying quartic weight $e^{-N(\frac{1}{2}z^2+\frac{1}{4}tz^4)}$, $t \in \mathbb{R}$ and $n = N \to +\infty$ was considered in [2], [4]. The asymptotics in the case $t > 0$ is well known

$$\alpha_n(t) = \frac{\sqrt{1+12t}-1}{6t} + O(N^{-1}), \;\; \beta_n(t) \to 0, \tag{2-1}$$

It can be continued into some negative interval $(t_0, 0)$ provided the integrals are taken along the rays with the slope ± 1 in $\mathbb{C}$ instead of $\mathbb{R}$. We put different constant weight factors ν_j, $j = 1, 2, 3, 4$, for integral along each of the above four rays. The corresponding monic orthogonal polynomials π_n satisfy the 3-term reccurence relation $\pi_{n+1} = (z - \beta_n)\pi_n(z) - \alpha_n \pi_{n-1}(z)$, where α_n, β_n are the recurrence coefficients.

Similarly to the point of gradient catastrophe (x_0, t_0) of the NLS, this asymptotics cannot (in general) be continued beyond the critical $t = t_0 = -1/12$. Moreover, in the double scaling limit $N^{4/5}(t - t_0) = \frac{v}{2^{9/5}3^{6/5}}$, where $v \in \mathbb{C}$ is a constant, the correction to the leading order constant term (2-1) is given in terms of a P1 solution $y_\nu(v)$ as long as v is away from the poles of $y_\nu(v)$ ([DK], [FIK]). The solution $y_\nu(v)$ is defined through the weights ν_j. This sounds similar to the [3] conjecture for the NLS.

2.1. Results for orthogonal polynomials, generic case

1. there exists a mapping $v = v(t) = cN^{\frac{4}{5}}(t - t_0)(1 + O(N^{-\frac{2}{5}}))$ of a $O(N^{-\frac{4}{5}})$ size neighborhood $D \subset \mathbb{C}$ of the critical point $t = t_0$ into an $O(1)$ size open set $V \subset \mathbb{C}$, $0 \in V$ (with $c \in \mathbb{R}$), such that the center if each spike is given by $t_p = v^{-1}(v_p)$, where $v_p \in V$ is a pole of the solution $y_\nu(v)$ of the Painlevé I (P1);

2. if the weights are even, the hight of each spike at its center t_p is 9 times the hight of the background, (with accuracy $O(N^{-\frac{1}{5}})$);
3. we calculated the (complex) *universal shape* of each spike;
4. for some special choices of the weights ν_j there is an additional point of gradient catastrophe at $t_1 = \frac{1}{15}$, which is *closer* to the origin than t_0. We calculated the shape of the spikes near t_1, as well as near t_0 in nonsymmetric cases.

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Analytical Properties of Solutions to Some Nonlinear Differential Equations and Their Systems Associated with Models of Random-matrix Type

Vladimir Tsegel'nik

Abstract. Certain analytical properties of solutions to some differential equations and their systems associated with models of random-matrix type are investigated.

Mathematics Subject Classification (2000). Primary 34M55; Secondary 35Q51.

Keywords. Painlevé test, Painlevé property, model of random-matrix type.

1. Introduction

The last three decades witnessed an increasing interest in studying certain classes of continuous and discrete probability models by the name "models of random-matrix type". The origins of such models can be very different [1].

One of the most important characteristics of these models is the emptiness formation probability, i.e., the probability that particles are absent from a given interval or union of intervals.

The emptiness formation probabilities, as a rule, can be represented in the form of the Fredholm determinant $det\,(\mathbf{1}-K)_J$, where K is an integral operator and J is the set inside which no particles must be present. The kernel of the operator K usually has the form

$$K(x,y)=\frac{A(x)B(y)-B(x)A(y)}{x-y}\sqrt{\psi(x)\psi(y)} \tag{1}$$

with appropriate functions ψ, A, and B. The only currently known way to calculate such Fredholm determinants is to present them as solutions of an ordinary differential equation or a system of partial differential equations.

2. Model of Random-matrix Type with Airy Kernal

This paper deals with the Hamilton system

$$q' = p - qu + \alpha s, \qquad v' = -pq - \alpha sq, \tag{2}$$

$$p' = sq - 2qv + pu + \alpha su, u' = -q^2, \tag{3}$$

where s is an independent variable, α is a parameter. The hamiltonian of system (2) has the form

$$H = \frac{p^2}{2} - \frac{sq^2}{2} + q^2 v - pqu + \alpha sp - \alpha squ.$$

In case of $\alpha = 0$ system (2), (3) corresponds to a model of random-matrix type with the Airy kernel [2]. We have carried out Painlevé-analysis of solutions to system (2), (3) and have proved

Theorem 2.1. *System (2), (3) is a Painlevé type system. Its solutions are expressed in terms of solutions to the second Painlevè equation*

$$q'' = 2q^3 + (s + C)q + \alpha, \tag{P_2}$$

where C is an arbitrary constant of integration.

The correctness of this theorem follows from the existence of the first integral $u^2 - 2v - q^2 = C$ of system (2), (3) and the next formulas $u(s) = -\int q^2(s)ds, p = q' + qu - 2s, 2v = u^2 - q^2 - C$.

Example. As equation (P_2) in case of $\alpha = -1$ has a solution $q = \frac{1}{s+C}$, system (2), (3) has a solution $u = \frac{1}{s+C} + C_1, v = \frac{C_1}{s+C} + \frac{C_1^2 - C}{2}, p = \frac{C_1}{s+C} + s$, where C_1 is an arbitrary constant.

3. System of Differential Equations Associated with Dyson Process

We have carried out Painlevé-analysis of solutions to the system of differential equations

$$\begin{aligned} q'' &= \left(s^2 - 2n - 1\right) q + 2q^2 p, \\ p'' &= \left(s^2 - 2n + 1\right) p + 2p^2 q, \end{aligned} \tag{4}$$

associated with so called Dyson process [3].

Theorem 3.1. *System (4) satisfies the Painlevé test.*

4. Solutions of Travelling Wave Form of a Partial Differential Equation

We have studied a class of selfsimilar solutions to the partial differential equations

$$\left(A_1^3 - 4\left(A_3 - \frac{1}{2}\right)\right) f + 6\left(A_1 f\right)^2 = 0, \tag{5}$$

where $A_n = \sum\limits_{i=1}^{2r} x_i^{\frac{n-1}{2}} \cdot \frac{\partial}{\partial x_i}$, $n = 1, 3$; f is an unknown function of independent variables x_1, $x_2, \ldots x_{2r}$.

Equation (5) is also associated with a model [4] of random-matrix type.

The set $E = \bigcup\limits_{i=1}^{r} [x_{2i-1}, x_{2i}] \subset R$ is associated with equation (5). The transformation $w_1 = f(\tau), \tau = x_1 + x_2 + \ldots + x_{2r}$ reduces equation (5) to the equation

$$rw_1''' - 2rw_1' + w_1 + 6rw_1'^2 = 0, w_1' = \frac{dw_1}{d\tau}, w_1''' = \frac{d^3 w_1}{d\tau^3}, \tag{6}$$

that has the first integral

$$w_1''^2 + 4w_1'^3 - \frac{2}{r}\tau w_1'^2 + \frac{2}{r} w_1 w_1' = K_1, \tag{7}$$

where K_1 ia an arbitrary constant. The substitution $w_1 = \lambda_1 y, \tau = \mu_1 s, \lambda_1 \mu_1 = 1, \mu_1^3 = r$ transforms [5] equation (7) into the equation

$$y''^2 + 4y'^3 - 2sy'^2 + 2yy' - \left(\alpha - \frac{\varepsilon}{2}\right)^2 = 0, \tag{8}$$

where α is an arbitrary parameter; $\varepsilon^2 = 1$.

In case of $C = 0$ there is a correspondence between solutions of equation (8) and solutions of equation (P_2) given by the next formulas

$$y = w'^2 - \left(w^2 + \frac{s}{2}\right)^2 - 2\left(\alpha - \frac{\varepsilon}{2}\right) w, \tag{8}$$

$$w = \left(y'' + \frac{1}{2} - \alpha\varepsilon\right) (-2\varepsilon y')^{-1}. \tag{10}$$

Theorem 4.1. *Let $y = y(s)$ be a solution of equation (8) with fixed parameter values α and ε. Then the function $f(\tau) = \lambda_1 y\left(\frac{\tau}{\mu_1}\right), \tau = x_1 + x_2 + \ldots + x_{2r}, \lambda_1 \mu_1 = 1, \mu_1^3 = 1$ is a solution of equation (1).*

In case $E = (x, +\infty)$ the function $f(x)$ satisfies the following equation

$$f''' - 4xf'' + 2f + 6f'^2 = 0,$$

that has the first integral

$$f''^2 + 4f'^3 - 4xf'^2 + 4ff' = K_2, \tag{11}$$

where K_2 is an arbitrary constant.

A scale transformation of unknown function f and independent variable x reduces equation (11) to the form (8).

Equation (11) for K_2 is obtained in [2].

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Local expansions for solutions of the Schlesinger equation

Ilya Vyugin

Abstract. A local behavior of solutions of the Schlesinger equation is studied. We obtain expansions for this solutions, which converge in some neighborhood of a singular point. As a corollary the similar result for the sixth Painlevé equation was obtained. In our analysis, we use the isomonodromic approach to solve this problem.

Mathematics Subject Classification (2000). Primary 34M35; Secondary 34M35.

Keywords. Schlesinger equation, isomonodromic deformation, sixth Painlevé equation, Fuchsian system.

1. Introduction

We study a local behavior of solutions of the Schlesinger equation. We present solutions of this equation in the form of power series or logarithmic-power series. This series are converge in some neighborhood of a singular point. As a corollary we obtain a similar result for description of the behavior of solutions of the sixth Painlevé equation in some sectorial neighborhood. We use the isomonodromic approach to solve this problem.

Let us consider the following system of analytical partial differential equations

$$dB_i = -\sum_{j=1, j\neq i}^{n} \frac{[B_i, B_j]}{a_i - a_j} d(a_i - a_j), \qquad i = 1, \dots, n, \tag{1.1}$$

where B_i $(i = 1, \dots, n)$ — are analytical $p \times p$-matrix functions of the variable $a = (a_1, \dots, a_n)$, $[B_i, B_j]$ denotes the commutator of matrices B_i and B_j. The matrix-functions $B_i(a)$ are defined and meromorphic (see B. Malgrange [1], R.

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Gontsov and I. Vyugin [4]) on the space

$$\{a|a = (a_1, \dots, a_n) \in \overline{\mathbb{C}}^n \setminus \bigcup_{i,j} A_{ij}\}, \qquad A_{ij} = \{a \,|\, a_i = a_j\}.$$

This system is called *Schlesinger equation* (read more in A.A. Bolibruch [7]). Divisor of the Schlesinger equation is the following set $\Omega = \bigcup_{i,j} A_{ij}$. We are going to describe a local form of solutions of Schlesinger equation (1.1) in a neighborhood of the point $a^0 = (a_1^0, \dots, a_n^0)$, which belongs to the following singular set

$$a^0 \in \Omega' = \Omega \setminus \left(\bigcup_{i,j,k} A_{ijk} \right), \qquad A_{ijk} = \{a \,|\, a_i = a_j = a_k\}.$$

We obtain the local expansions of the solutions of the system (1.1) in the form of power and logarithmic-power series of $(a_s - a_r)$ (if $a^0 \in A_{sr}$), which converges in some neighborhood of the point $z = a^0$ (the first version of these results see [6]). These series have terms of complex degrees.

Theorem 1.1. *Any solution of two dimensional Schlesinger equation (*1.1*) can be represented in the neighborhood of a point $a^0 = (a_1^0, \dots, a_n^0) \in \Omega'$, where $a_r^0 = a_s^0$, $r \neq s$, in one of two following forms:*

- $b_{kl}^i(a) = F_1(a) + (a_s - a_r)^{\varphi} F_2(a) + (a_s - a_r)^{-\varphi} F_3(a)$, $\varphi \in \mathbb{C}$ *in the general case;*
- $b_{kl}^i(a) = F_1(a) + F_2(a) \ln(a_s - a_r) + F_3(a) \ln^2(a_s - a_r)$ *in the degenerate case,*

where $F_1^{kli}(a), F_2^{kli}(a), F_3^{kli}(a)$ are meromorphic (holomorphic in the generic case) functions, $i = 1, \dots, n$, and $k, l \in \{1, 2\}$.

The notions of "general case" and "non-general case" are explained below. Notice that the measure of the systems of non-general case is equal to zero.

Now consider the case $n = 4$, $p = 2$, which is equivalent to case of the sixth Painlevé equation (1.2). Without loss of generality, let us fix three variable $a_1 = 0$, $a_2 = 1$, $a_3 = \infty$ and denote a_4 by t. We obtain the system of ordinary differential equations with variable t and unknown matrix-functions

$$B_i(t) = \begin{pmatrix} b_{11}^i(t) & b_{12}^i(t) \\ b_{21}^i(t) & b_{22}^i(t) \end{pmatrix}, \qquad i = 0, t, 1, \infty.$$

With restrictions above the following corollary holds.

Corollary 1.2. *Any solution of the Schlesinger equation under the above constraints can be represented in the neighborhood of $t = 0$ in one of two forms:*

- $b_{kl}^i(t) = F_1^{kli}(t) + t^{\varphi} F_2^{kli}(t) + t^{-\varphi} F_3^{kli}(t)$, $\varphi \in \mathbb{C}$ *in the general case;*
- $b_{kl}^i(t) = F_1^{kli}(t) + F_2^{kli}(t) \ln t + F_3^{kli}(t) \ln^2 t$ *in the degenerate case,*

where $F_1^{kli}(t), F_2^{kli}(t), F_3^{kli}(t)$ are meromorphic in $t = 0$ functions, $i = 0, t, 1, \infty$, and $k, l \in \{1, 2\}$.

Note that the well-known sixth Painlevé equation

$$\frac{d^2w}{dt^2} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-t}\right)\left(\frac{dw}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{w-t}\right)\frac{dw}{dt} + \\ + \frac{w(w-1)(w-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{w^2} + \gamma\frac{t-1}{(w-1)^2} + \delta\frac{t(t-1)}{(w-t)^2}\right), \tag{1.2}$$

α, β, γ, $\delta \in \mathbb{C}$ is equivalent to the system (1.1), where

$$w(t) = \frac{tb_{12}^0}{(t+1)b_{12}^0 + tb_{12}^1 + b_{12}^t}. \tag{1.3}$$

Corollary 1.2 and (1.3) give the power expansions for solutions of the sixth Painlevé equation. A different asymptotics for sixth Painlevé equation was obtained in D. Guzzetti [2], A. Bruno and I. Goryuchkina [5], M. Mazzocco [3] and others.

For the sixth Painlevé equation, we have an analogue of Corollary 1.2.

Corollary 1.3. *Any solution $w(t)$ of sixth Painlevé equation (1.2) in the intersection of the given sector for t sufficiently close to singular point $t = 0, 1, \infty$ can be represented as a converged power series or as a converged logarithmic-power series:*

- *if G_1G_∞ is digonalizable, then $w(t) = S(t, t^\lambda, t^{-\lambda})$, where $\lambda = \lambda(\alpha, \beta, \gamma, \delta, t_0, w(t_0), w'(t_0))$ can be found approximately;*
- *if G_1G_∞ is a Jordan block, then $w(t) = S(t, \ln t)$.*

2. Schlesinger equation and isomonodromic deformations

In this section we give a description of the Schlesinger equation (1.1) as an isomonodromy condition for a family of Fuchsian systems. Let us consider a Fuchsian system

$$\frac{dy}{dz} = \left(\sum_{i=1}^n \frac{B_i^0}{z - a_i^0}\right)y, \qquad B_i^0 \in \mathrm{Mat}_{p\times p}(\mathbb{C}), \quad y(z) \in \mathbb{C}^p. \tag{2.1}$$

The family of such systems

$$\frac{dy}{dz} = \left(\sum_{i=1}^n \frac{B_i(a)}{z - a_i}\right)y \tag{2.2}$$

is called *isomonodromic* if the following conditions hold:

- $B_i(a)$ are continuous matrix-functions of $a = (a_1, \dots, a_n)$;
- The Fuchsian system (2.2) with any fixed a has fixed monodromy representation $\chi : \pi_1(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}, z_0) \longrightarrow GL(p, \mathbb{C})$.

Schlesinger isomonodromic family is a family defined by the equation (1.1). An isomonodromic fundamental matrix $Y(z, a)$ of the Schlesinger isomonodromic family (2.2) satisfies the following condition

$$Y(\infty, a) \equiv Y(\infty, a_0).$$

The initial data of such family are the coefficients $B_i(a^0) = B_i^0$, $i = 1, \dots, n$ of system (2.1). It is known that the solutions of Schlesinger equation are meromorphic functions on the space $a \in \overline{\mathbb{C}}^n \setminus \Omega$.

Let us consider the Painlevé VI case ($n = 4, p = 2$, $a_1 = 0$, $a_2 = 1$, $a_3 = \infty$, $a_4 = t$). Usually the following family

$$\frac{dy}{dz} = \left(\frac{B_0(t)}{z} + \frac{B_t(t)}{z-t} + \frac{B_1(t)}{z-1}\right) y \tag{2.3}$$

is considered, where

$$\mathrm{tr} B_0 = \mathrm{tr} B_t = \mathrm{tr} B_1 = \mathrm{tr} B_\infty = 0, \qquad B_\infty = -(B_0 + B_t + B_1),$$

and the matrices B_0, B_t, B_1, $B_\infty = \mathrm{diag}(\delta, -\delta)$ are diagonalizable.

The formula (1.3) gives a solution of sixth Painlevé equation (1.2) with the following constants

$$\alpha = \frac{(2\lambda_\infty - 1)^2}{2}, \quad \beta = -2\lambda_0^2, \quad \gamma = 2\lambda_1^2, \quad \delta = \frac{1}{2} - 2\lambda_t^2,$$

where λ_0, λ_t,λ_1, λ_∞ are eigenvalues of matrices B_0, B_t, B_1, B_∞.

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Reductions on the lattice and Painlevé equations

Pavlos Xenitidis

Abstract. The symmetry analysis of the discrete integrable systems of the Adler-Bobenko-Suris classification is reviewed and symmetry reductions of the latter are discussed. In particular, reductions of the discrete potential KdV equation to discrete Painlevé equations are presented, and continuous symmetric reductions leading to integrable systems of partial differential equations are considered. As a byproduct, an interpretation of solutions of continuous Painlevé equations as particular solutions of the discrete potential KdV and a discrete Painlevé II equations is demonstrated.

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Keywords. Integrable equations, symmetries, reductions, Painlevé equations.

1. Introduction

Adler, Bobenko and Suris (ABS) classified recently [2] all the integrable scalar difference equations which are defined on an elementary quadrilateral of the lattice and possess the following two properties : *i*) they are affine linear and *ii*) they are multidimensionally consistent[1]. Their classification led to the following list of seven equations which includes some new cases (H2, $H3_{\delta=1}$, $Q1_{\delta=1}$, Q2, $Q3_{\delta=1}$), as well as some well known equations (H1, $H3_{\delta=0}$, $Q1_{\delta=0}$, $Q3_{\delta=0}$, Q4), see e.g. [1, 3, 6].

H1 $\quad (u-z)\,(x-y)\,-\,\alpha\,+\,\beta\,=\,0$

H2 $\quad (u-z)(x-y)+(\beta-\alpha)(u+x+y+z)-\alpha^2+\beta^2=0$

H3 $\quad \alpha(ux+yz)-\beta(uy+xz)+\delta(\alpha^2-\beta^2)=0$

[1] This means that these equations can be extended to a multidimensional lattice in a self consistent way. This property serves as an integrability criterion which played central role in the ABS classification.

$$
\begin{array}{ll}
\text{Q1} & \alpha(u-y)(x-z)-\beta(u-x)(y-z)+\delta^2\alpha\beta(\alpha-\beta)=0\\
\text{Q2} & \alpha(u-y)(x-z)-\beta(u-x)(y-z)+\\
 & \alpha\beta(\alpha-\beta)(u+x+y+z)-\alpha\beta(\alpha-\beta)(\alpha^2-\alpha\beta+\beta^2)=0\\
\text{Q3} & (\beta^2-\alpha^2)(uz+xy)+\beta(\alpha^2-1)(ux+yz)\\
 & -\alpha(\beta^2-1)(uy+xz)-\delta^2(\alpha^2-\beta^2)(\alpha^2-1)(\beta^2-1)/(4\alpha\beta)=0\\
\text{Q4} & a_0uxyz+a_1(uxy+xyz+yzu+zux)+a_2(uz+xy)\\
 & +\bar{a}_2(ux+yz)+\tilde{a}_2(uy+xz)+a_3(u+x+y+z)+a_4=0
\end{array}
$$

In the above list we have used the following shorthand notation for the values of the function $u:\mathbb{Z}^2\to\mathbb{R}$

$$
u:=u_{n,m},\quad x:=u_{n+1,m},\quad y:=u_{n,m+1},\quad z:=u_{n+1,m+1}\,,
$$

and α, β denote the *lattice parameters* assigned to the n and the m direction of the lattice, respectively. In Adler's equation Q4 the parameters are functions of the lattice parameters given in terms of the Weierstrass $\wp$ function [1].

The study of equations H1, H3_0 and Q1_0 led to the derivation of special type of solutions, namely similarity solutions [5], which have been shown to be related to discrete Painlevé equations [4, 7]. Moreover, a new type of connection among integrable discrete and continuous equations was demonstrated in [4] where a system of partial differential equations was constructed in relation with the discrete potential KdV equation, i.e. equation H1.

In this talk we will formulate this analysis in terms of symmetries and symmetry reductions, and consequently we will derive these results in a systematic and simple manner which can be applied to any equation of the ABS classification. In particular, the symmetries of the above equations have been studied [10] and it has been demonstrated how they can be used effectively in the construction of group invariant solutions [10] and continuously symmetric solutions [11]. The former are related to solutions of discrete Painlevé equations while the latter to solutions of the continuous Painlevé equations. Finally, relations among the solutions of particular discrete and continuous Painlevé equations follow from these considerations.

2. Symmetries of the ABS equations

All of the ABS equations

$$
Q(u_{n,m},u_{n+1,m},u_{n,m+1},u_{n+1,m+1},\alpha,\beta)\;=\;0
$$

admit a pair of symmetries generated by the following vector fields

$$
\begin{aligned}
V_1 &= \left(\frac{f(u_{n,m},u_{n+1,m},\alpha)}{u_{n+1,m}-u_{n-1,m}}-\frac{1}{2}\partial_{u_{n+1,m}}f(u_{n,m},u_{n+1,m},\alpha)\right)\partial_{u_{n,m}}\\
G_1 &= n\,V_1\,+\,\xi(\alpha)\,\partial_\alpha\,,
\end{aligned}
$$

where the symmetric and biquadratic polynomial f is determined completely by the polynomial Q and its derivatives. Moreover, the invariance of the equation under the interchange of the lattice directions, i.e. the mutual interchange $(n, m, u_{n+i,m+j}, \alpha, \beta) \longrightarrow (m, n, u_{n+j,m+i}, \beta, \alpha)$, implies the existence of two similar symmetries V_2 and G_2 in the m-direction [10].

Symmetries generated by V_i are referred to as generalized symmetries, and they will be used in the derivation of similarity solutions. The other two symmetries G_i are referred to as extended generalized symmetries, and the reduction of the ABS equations under the action of both of these symmetries leads to systems of partial differential equations, also known as generating PDEs.

3. Reduction on the lattice and discrete Painlevé equations

The solutions of an equation which remain invariant under the action of a symmetry of the equation are called group invariant solutions [8]. These particular solutions of the equation satisfy, not only the equation, but certain *constraints* following from their invariance.

In the case of the ABS equations which admit a generalized symmetry of the form $X = P\partial_{u_{n,m}} := c_1V_1 + c_2V_2$, $c_i \in \mathbb{R}$, such group invariant solutions can be derived systematically. More precisely, they are solutions of the equation which in addition satisfy the constraint $P = 0$. From the symmetry analysis of the ABS equations [10], it is known that the numerator and the denominator of the rational function P depends linearly on $u_{n\pm1,m}$ and $u_{n,m\pm1}$, which implies that the equation $P = 0$ can be solved uniquely with respect to any of these values of u. Using this observation, the system constituted from the original equation and the above constraint leads, in general, to a fourth order dimensional map, the derivation of which is straightforward. If the equation admits point symmetries which commute with X, then the order of the map can be reduced further. In the case of H1 and a specific form of X, one finds that these group invariant solutions are determined by solutions of the asymmetric, alternate discrete Painlevé II equation [10].

4. Continuous symmetry reductions

In the same fashion, the solutions of the ABS equation, which remain invariant under the action of the extended generalized symmetries G_1 and G_2, are the ones which additionally satisfy the system of differential-difference equations

$$\xi(\alpha)\partial_\alpha u_{n,m} = nR(u_{n,m}, u_{n+1,m}, u_{n-1,m}, \alpha),$$

$$\xi(\beta)\partial_\beta u_{n,m} = mR(u_{n,m}, u_{n,m+1}, u_{n,m-1}, \beta)\,.$$

From the compatible system of the discrete equation and the above two differential-difference equations one can systematically derive a system of partial differential equations involving only $u_{n,m}$, $u_{n+1,m}$, and $u_{n,m+1}$ [11].

The continuous system corresponding to equation H1 generalizes the Ernst equation [9] and some of its similarity solutions are determined by solutions of Painlevé V and VI equations. From the derivation of this system follows that solutions of the continuous Painlevé V equation can be regarded as solutions of the asymmetric, alternate discrete Painlevé II equation [11].

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Preliminary Conference Program

June 17

Arrival day

11:00 – 17:00 **Registration in Euler Institute**

June 18

9:00 – 10:30 **Registration, coffee**
10:30 – 11:00 **OPENING THE CONFERENCE**

Morning session

11:00 – 12:00 **Sergey Yu. Slavyanov** Antiquantization of quantum models as a tool for generating Painlevé equations

12:00 – 12:30 **Coffee break**

12:30 – 13:30 **Alexander D. Bruno** Plane Power Geometry for single ODE and Painlevé equations

13:30 – 15:00 **Lunch**

Afternoon session

15:00 – 15:30 **Irina V. Goryuchkina** On convergence of a formal solution to an ODE
15:30 – 16:00 **Marco Bertola** Fredholm determinants and noncommutative Painlevé II
16:00 – 16:30 **Alexander D. Bruno** Space Power Geometry for an ODE and Painlevé equations
16:30 – 17:00 **Coffee break**
17:00 – 17:30 **Alexander Ya. Kazakov and Sergey Yu. Slavyanov** Integral Euler symmetries for confluent Heun equation and symmetries of Painlevé equation P^5
17:30 – 18:00 **Kouichi Takemura** Integral transformation of Heun's equation and some applications

18:00 **WELCOME PARTY**

June 19

Morning session

10:00 – 11:00 **Irina V. Goryuchkina** Asymptotic expansions and forms of solutions to the sixth Painlevé equation

11:00 – 11:30 **Coffee break**

11:30 – 12:30 **Davide Guzzetti** Solving PVI by Isomonodromy Deformations

12:30 – 13:00 **Ilya Vyugin** Local expansions for solutions of the Schlesinger equation

13:00 – 13:30 **Yurii V. Brezhnev** The sixth Painlevé transcendent as a generator of uniformizable orbifolds

13:30 – 15:00 **Lunch**

Afternoon session

15:00 – 15:30 **Alexander B. Batkhin and Natalia V. Batkhina** Exact simple solutions to PVI

15:30 – 16:00 **Dmitrii P. Novikov** A monodromy problem and some functions connected with Painlevé VI

16:00 – 16:30 **Yoshikatsu Sasaki** Gauge transformation of the sixth Painlevé equation

16:30 – 17:00 **Coffee break**

17:00 – 17:30 **Anastasya V. Parusnikova** Asymptotic expansions of solutions to the fifth Painlevé equation

17:30 – 18:00 **Alexander Ya. Kazakov and Sergey Yu. Slavyanov** Integral Euler symmetries for confluent Heun equation and symmetries of Painlevé equation P^5

June 20
PETRODVORETS EXCURSION
BANQUETTE (with dances)

June 21

Morning session

10:00 – 11:00 **Mikhail V. Babich** On rational canonical parametrization of isomonodromic deformation equations phase space

11:00 – 11:30 **Coffee break**

11:30 – 12:30 **Philip Boalch** Simply–laced isomonodromy systems

12:30 – 13:00 **Yulia P. Bibilo and Renat R. Gontsov** On the Malgrange isomonodromic deformations of non-resonant meromorphic connections

13:00 – 13:30 **Dmitry V. Artamonov** The Schlesinger system and isomonodromic deformations of bundles with connections on Riemann surfaces

13:30 – 15:00 **Lunch**

Afternoon session

15:00 – 15:30 **Vladimir Tsegel'nik** Analytical Properties of Solutions to Some Nonlinear Differential Equations and Their Systems Associated with Models of Random-matrix Type

15:30 – 16:00 **Dmitry Korotkin and Peter Zograf** From the tau function of Painlevé VI equation to the geometry of moduli spaces

16:00 – 16:30 **Vladimir G. Lysov** Asymptotics of Angelesco polynomials and double scaling limit at pushing point

16:30 – 17:00 **Coffee break**

17:00 – 17:30 **Vladimir P. Leksin** Isomonodromic deformations and Jordan-Pochhammer systems

17:30 – 18:00 **Valentina A. Goloubeva** On Fuchsian reduction of differential equations

June 22

Morning session

10:00 – 10:30 **Viktor Novokshenov** Tronquée solutions of the Painlevé II equation

10:30 – 11:00 **Vladimir Matveev** Quasi-rational solutions to the focusing NLS equation and multiple rogue-waves generation

11:00 – 11:30 **Coffee break**

11:30 – 12:00 **Kohei Iwaki** Parametric Stokes phenomenon for the second Painlev**é** equation

12:00 – 12:30 **A. Kessi and Y. Adjabi** Third order differential equation with Painlevé property

12:30 – 13:00 **Pantelis A. Damianou** Lotka–Volterra equations in three and four dimensions satisfying the Kowalevski-Painlevé property

13:00 – 13:30 **Rustem N. Garifullin**
Phase shift for some special solution Korteweg–de Vries equation

13:30 – 15:00 **Lunch**

Afternoon session

15:00 – 15:30 **I. P. Martynov, V. A. Pronko and T. K. Andreeva** Third Order Equation with an Irrational Right-Hand Side with the Painlevé Property

15:30 – 16:00 **Yousuke Ohyama** Particular solutions of q-Painlevé equations and q-hypergeometric equations

16: 00 – 16:30 **Pavlos Xenitidis** Reductions on the lattice and Painlevé equations

16:30 – 17:00 **Coffee break**

17:00 – 17:30 **Alexander Tovbis and Marco Bertola** Asymptotics at the gradient catastrophe points and Painlevé 1 transcendents: semiclassical focusing NLS and orthogonal polynomials cases

18:00 **BOAT TOUR**

June 23

Morning session

10:00 – 11:00 **Irina Astashova** Asymptotic Classification of Solutions to 3rd and 4th Order Emden–Fowler Type Differential Equations

11:00 – 11:30 **Coffee break**

11:30 – 12:00 **Nataliya Dilna and Michal Fečkan** About Parametric Weakly Nonlinear ODE with Time-reversal Symmetries

12:00 – 12:30 **Svetlana Ezhak** On Dependence of the First Eigenvalue of the Sturm – Liouville Problem with Dirichlet Boundary Conditions on Parameter of Integral Condition

12:30 – 13:00 **Vera V. Kartak** Solution of the equivalence problem for the second order ODE's with the degenerate Cartan's invariants

13:00 – 13:30 **Elena Karulina** On some estimates of the minimal eigenvalue for the Sturm — Liouville problem with third-type boundary conditions and integral condition

14:00 **CLOSING THE CONFERENCE**

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