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On Nonoscillating Solutions of Emden–Fowler-Type Equations

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Abstract—The asymptotic properties of nonoscillating solutions of Emden–Fowler-type equations of arbitrary order are considered. The paper contains the results of the study of the asymptotic properties of solutions with integer-valued asymptotics as well as of solutions arising from the rapid decrease of the coefficient of the equation. To analyze the asymptotic behavior of solutions of the equations, methods of power geometry are used.

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1. INTRODUCTION

Consider the equation

$$y^{(n)} = \frac{d^n y}{dx^n} = p(x)|y|^\sigma \operatorname{sgn} y, \qquad n \ge 2, \quad \sigma > 1,$$
$$y = y(x), \qquad p(x) \in C^0, \quad x, y \in \mathbb{R}^1. \tag{1}$$

For $n = 2$ and $p(x) = \pm x^\beta$, $x > 0$, $\beta = \mathrm{const}$, this is the well-known Emden–Fowler equation encountered in the study of many physical processes. Equation (1) was studied in numerous papers (see, in particular, [1]–[9]).

The present paper aims at a systematic description, as $x \to \pm\infty$, of nonoscillating (i.e., sign-preserving) solutions of Eq. (1) depending on the growth parameters of the function $p(x)$. We shall also consider solutions tending to infinity as $x \to a \ne \pm\infty$.

In this paper, we study nonoscillating solutions of Eq. (1), provided that the following condition holds:

$$|p(x)| \le c|x|^{-n-\delta}, \qquad c, \delta = \mathrm{const} > 0, \quad |x| \ge x_0 > 0. \tag{2}$$

For comparison, note that a complete description of nonoscillating solutions of Eq. (1) is given in [1], provided that the following conditions hold:

$$|p(x)| \ge c|x|^{-n}, \qquad c = \mathrm{const} > 0, \quad |x| \ge x_0 > 0. \tag{3}$$

Recall the following definition.

Definition 1. The solution $y(x)$ of Eq. (1) is said to be *continuable to the right* (*to the left*) if it is defined in a neighborhood of $+\infty$ ($-\infty$).

Solutions that are not continuable in a particular direction are said to be noncontinuable in this direction.

In this paper, we consider the sign-preserving, continuable, and noncontinuable to the right (to the left), solutions of Eq. (1) and present the asymptotic estimates of such solutions as $x \to \pm\infty$.

The following theorem (see [1], [3]) establishes the existence of noncontinuable solutions of Eq. (1) for $p(x) > 0$.

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Theorem 1. *If $p(x) > 0$, then, for any number a_1, there exists a noncontinuable (to the right) solution $y(x)$ of Eq. (1) satisfying the condition*

$$\lim_{x\to a_1-0} |y^{(i)}(x)| = +\infty, \qquad 0 \le i \le n-1. \tag{4}$$

Theorem 2. *If, for an integer l, $0 \le l \le n-1$, the function $p(x)$ satisfies the condition*

$$|p(x)| \le cx^{-n-l(\sigma-1)-\delta}, \qquad c, \delta = \text{const} > 0, \quad x \ge x_0 > 0, \tag{5}$$

then, for all numbers $c_m \ne 0$ and $\sigma \ge 1$, Eq. (1) has an continuable (to the right) solution $y_m(x)$, $m \in \{0, 1, \dots, l\}$ such that, as $x \to +\infty$,

$$\begin{aligned} &y_m(x) = x^m(c_m + o(x^{-\widetilde{\delta}})), \\ &y_m^{(i)}(x) = m(m-1)\cdots(m-i+1)x^{m-i}(c_m + o(x^{-\widetilde{\delta}})), && 0 \le i \le m, \\ &y_m^{(i)}(x) = o(x^{-i+m-\widetilde{\delta}}), && m+1 \le i \le n-1, \\ &\widetilde{\delta} = \text{const} > 0. \end{aligned} \tag{6}$$

If condition (5) holds, then any nontrivial solution of Eq. (1) satisfying the constraint

$$|y(x)| \le Dx^l, \qquad D = \text{const} > 0, \tag{7}$$

for $x \ge x_0$ is one of the solutions of the form (6) where m is an integer, $m \in \{0, 1, \dots, l\}$.

In the linear case ($\sigma = 1$), the existence of solutions of the form (6) was established in [4], the nonlinear case for $n = 2$ was studied in [2], and the general nonlinear case was studied in [5]. In proving Theorem 2, methods of power geometry turn out to be very effective (see, for example, [9], [10]). In the present paper, we present such a proof.

Theorem 2 readily implies the following statement.

Theorem 3. *If*

$$0 > p(x) \ge -c_1x^{-(n-1)\sigma-1-\delta}, \qquad c_1, \delta = \text{const} > 0, \quad x \ge x_0 > 0, \tag{8}$$

then Eq. (1) possesses an continuable (to the right) solution $y(x)$ of the form (6), in which $m \in \{0, 1, \dots, n-1\}$.

Equation (1) has no other sign-preserving (for $x \ge x_0$) solutions under condition (8).

The following theorem is based on results from [3]–[7].

Theorem 4. *If*

$$0 < p(x) \le c_1x^{-(n-1)\sigma-1-\delta}, \qquad c_1, \delta = \text{const} > 0, \quad x \ge x_0 > 0, \tag{9}$$

then

1) *Equation (1) has a noncontinuable (to the right) solution $y(x)$ satisfying condition (4);*

2) *Equation (1) has an continuable (to the right) solution of the form (6), in which $m \in \{0, 1, \dots, n-1\}$;*

3) *Equation (1) has an continuable (to the right) solution satisfying the condition*

$$|y(x)| \ge c_2x^{n-1+\delta/(\sigma-1)}, \qquad c_2 = \text{const} > 0 \tag{10}$$

for large x.

Equation (1) *has no other sign-preserving* (*for* $x \geq x_0$) *solutions under condition* (9).

If the function $p(x)$ *satisfies the additional condition*

$$p(x) \geq c_3 x^{-(n-1)\sigma-1-\delta_1}, \qquad c_3, \delta_1 = \text{const} > 0, \quad x \geq x_0, \tag{11}$$

then, for any continuable (*to the right*) *sign-preserving solution of Eq.* (1) *the following estimate holds*:

$$|y(x)| \leq c_4 x^{n-1+\delta_1/(\sigma-1)}, \qquad c_4 = \text{const} > 0. \tag{12}$$

Theorem 5. *If, for some* k, $0 \leq k \leq n-1$, (*k can be noninteger*), *the function* $p(x)$ *satisfies the condition*

$$|p(x)| \geq c x^{-n-k(\sigma-1)}, \qquad c = \text{const} > 0, \quad x \geq x_0 > 0, \tag{13}$$

then any sign-preserving (*for* $x \geq x_0$) *solution* $y(x)$ *of Eq.* (1) *satisfies either condition* (4) *or the condition*

$$|y(x)| \leq D x^k, \qquad D = \text{const} > 0. \tag{14}$$

If the number k *is an integer, then, instead of* (14), *the following stronger estimate holds*:

$$\lim_{x\to+\infty} \frac{y(x)}{x^k} = 0. \tag{15}$$

The following result related to the case $0 < \sigma < 1$ was established in [6]. In what follows, we shall show that the proof of Theorem 2 based on methods of power geometry automatically justifies the theorem.

Theorem 6. *If* $0 < \sigma < 1$ *in Eq.* (1)*and the function* $p(x)$ *satisfies condition* (5) *for an integer* l, $0 \leq l \leq n-1$, *then, for all numbers* $c_m \neq 0$, *Eq.* (1) *has continuable* (*to the right*) *solutions* $y_m(x)$, $m \in \{l, \dots, n-1\}$, *such that, as* $x \to +\infty$, *conditions* (6) *hold.*

Remark. Solutions of the form (6) of the linear equation (1) (where $\sigma = 1$) were first discovered in [4]. For the nonlinear case, the existence of such solutions (see Theorem 2) was established in [2], [5]. The use of methods of power geometry [9], [10] makes the discovery and analysis of such and other solutions possessing an exact power asymptotics a standard procedure.

The result given in Theorem 5, is new.

Note that the results given above make it also possible to describe solutions of Eq. (1) in a neighborhood of the left boundary of their domains, which is accomplished by the replacement $x = x_0 - u$. Here equations of even order preserve their form, while, in equations of odd order, the function $p(x)$ changes its sign.

2. EXAMPLES

Examples illustrating Theorem 1 were given in [1].

Example 1 (Theorem 2). Consider the equation

$$y^{(3)} = x^{-6} y^3, \qquad x \geq 1, \tag{16}$$

subject to condition (5) in which $l = \delta = 1$. Using methods of power geometry (see [9, pp. 261–263], we shall calculate the terms of the asymptotic representation of solutions having the form (6). Here and elsewhere, we use terminology from [9].

The Newton polyhedron of this equation is the closed interval

$$[Q_1, Q_2], \qquad \text{where} \quad Q_1 = (-3, 1), \quad Q_2 = (-6, 3).$$

The vertex Q_1 corresponds to the truncated equation $y^{(3)} = 0$ whose solutions are the functions

$$y_m = c_m x^m, \qquad m = 0, 1, 2, \quad c_m = \text{const} \neq 0.$$

The vector orders of these solutions $P_m = (1, m)$ belong to the normal cone of the vertex Q_1 if $(P_m, R) < 0$, $R = Q_2 - Q_1 = (-3, 2)$, which is only possible for $m = 0$ or $m = 1$.

First, let $m = 0$. Then, in (16), let us replace $y = c_0 + u$. For $u = u(x)$, we obtain the equation

$$u^{(3)} = x^{-6}(c_0 + u)^3.$$

The support of the Newton polyhedron of this equation consists of five points. The polyhedron itself is the triangle with vertices $Q_3 = (-6, 0)$, $Q_4 = (-6, 3)$, $Q_5 = (-3, 1)$. We are interested in the truncated equation corresponding to the edge $[Q_3, Q_5]$,

$$u^{(3)} = c_0^3 x^{-6},$$

which has the solution $u = -c_0^3/60x^3$. We have obtained the second term of the expansion of the solution under consideration as $x \to +\infty$. This process can be continued, but here we restrict ourselves to the result obtained:

$$y_0(x) = c_0 - \frac{c_0^3}{60x^3} + \cdots, \qquad c_0 = \text{const} \neq 0. \tag{17}$$

Now consider the case $m = 1$. Replacing $y = c_1 x + \nu$ in (16), we obtain the equation

$$\nu^{(3)} = x^{-6}(c_1 x + \nu)^3.$$

The support of the Newton polyhedron of this equation consists of five points and the polyhedron itself is the triangle with vertices $Q_6 = (-3, 0)$, $Q_7 = (-6, 3)$, $Q_8 = (-3, 1)$. Consider the truncated equation corresponding to edge $[Q_6, Q_8]$,

$$\nu^{(3)} = c_1^3 x^{-3},$$

whose solution is $\nu = 0.5c_1^3 \ln x$. Accordingly, Eq. (16) has a solution of the following form:

$$y_1(x) = c_1 x + 0.5c_1^3 \ln x + \cdots, \qquad c_1 = \text{const} \neq 0. \tag{18}$$

A detailed analysis of solutions of Eq. (1) using methods of power geometry will be given in one of the following papers.

In the next example, we show that, in the cases described in Theorems 2 and 5 Eq. (1) can have solutions of the form different from (6). A general estimate for such solutions is given in (14) and (15).

Example 2 (Theorems 2 and 5)**.** Consider the equation

$$y^{(3)} = \frac{y^2 \operatorname{sgn} y}{x^3\sqrt{x}}, \qquad x \geq 1, \tag{19}$$

that satisfies condition (5) for $l = 0$, $\delta = 0.5$ and condition (13) for $k = 0.5$. We study positive solutions of this equation which have the form (6). The Newton polyhedron is the closed interval $[Q_1, Q_2]$, where $Q_1 = (-3, 1)$, $Q_2 = (-3.5, 2)$. The vertex Q_1 corresponds to the truncated equation $y^{(3)} = 0$ having a unique family of positive solutions $y_0 = c_0 = \text{const} > 0$ whose vector order belongs to the normal cone of the vertex Q_1. Replacing $y = c_0 + u$, we obtain the equation

$$u^{(3)} = \frac{(c_0 + u)^2}{x^3\sqrt{x}}.$$

Arguing just as in Example 1, we extract the truncated equation $u^{(3)} = c_0^2/(x^3\sqrt{x})$ whose solution is the function $u = -8c_0^2/(15\sqrt{x})$, which is the second term of the expansion for the solutions under consideration. Thus, we find that the equation in question has a solution of the form (6), namely,

$$y_0(x) = c_0 - \frac{8c_0^2}{15\sqrt{x}} + \cdots, \qquad c_0 = \text{const} > 0.$$

Obviously, Eq. (19) also has the solution $y(x) = 3\sqrt{x}/8$, which does not satisfy condition (6) but satisfies estimate (14) for $k = 0.5$.

Example 3 (Theorem 4). The equation

$$y^{(3)} = p(x)y^2 \operatorname{sgn} y, \qquad p(x) = \begin{cases} 1, & x \le 1, \\ x^{-6}, & x > 1, \end{cases} \tag{20}$$

satisfies conditions (9) and (11) for $\delta = \delta_1 = 1$. It has the solution $y(x) = -60/(x-1)^3$, $x < 1$, which satisfies (4) for $a_1 = 1$ as well as the solution $y(x) = 6x^3$, $x \ge 1$, for which (10) and (12) hold. In addition, note that this equation has the solutions

$$\begin{gathered} y_0(x) = c_0 - \frac{c_0^2}{60x^3} + \cdots, \\ y_1(x) = c_1 x - \frac{c_1^2}{6x} + \cdots, \\ y_2(x) = c_2 x^2 - c_2^2(x \ln x - x) + \cdots, \\ c_m = \text{const} > 0, \qquad 0 \le m \le 2, \quad x \ge 1, \end{gathered}$$

which satisfy (6). These expansions of the solutions were obtained by using the methods of power geometry presented in the previous examples.

3. PROOFS OF THE THEOREMS

The proofs of many statements closely related to the theorems presented above and given in earlier papers contain, in our view, too many technical details and references to other papers, which makes it difficult to fully grasp the mathematical meaning and may lead to misunderstandings. For such cases, we present proofs allowing us to show more fully the relationship between the solutions of Eq. (1) and the characteristics of the function $p(x)$.

In a number of cases, we shall use the "universal" constant $D > 0$ for which we assume that $\mu D = D$, $D^\mu = D$, $\mu = \text{const} > 0$.

We have already noted above that Theorem 1 was proved in [1].

Proof of Theorem 2. Note that the fact that Eq. (1) has solutions of the form (6) is indicated by the analysis of the Newton polyhedron of this equation (see, for example, [9, p. 262]).

Let condition (5) hold for an integer l, $0 \le l \le n-1$. First, consider the case $\sigma \ge 1$ related to Theorem 2. Let us prove that Eq. (1) has solutions of the form (6) for all integer m, $m \in \{0, 1, \dots, l\}$, the c_m are arbitrary nonzero constants, and $\widetilde{\delta}$ is the number satisfying the condition $0 < \widetilde{\delta} < \delta$. It suffices to consider only positive solutions (otherwise, we replace $y = -\widetilde{y}$), i.e., below we shall assume that $c_m > 0$.

Let us fix the number m, $m \in \{0, 1, \dots, l\}$, and, in (1), replace $y = x^m(c_m + z)$. Equation (1) takes the form

$$\sum_{j=0}^{m} a_j x^{m-j} z^{(n-j)} = (c_m + z)^\sigma x^{m\sigma} p(x). \tag{21}$$

In view of [9, p. 259], we perform the logarithmic transformation $t = \ln x$, as a result of which we obtain the equation

$$\begin{gathered} z_t^{(n)} + b_1 z_t^{(n-1)} + \cdots + b_{n-1} z_t' = (c_m + z)^\sigma p_m(t), \\ z_t^{(i)} = \frac{d^i z}{dt^i}, \quad z = z(e^t), \quad b_i = \text{const}, \qquad 1 \le i \le n, \\ |p_m(t)| \le De^{-((l-m)(\sigma-1)+\delta)t} \le De^{-\delta t}, \qquad t \ge 0. \end{gathered}$$

Denoting $z_t^{(i)} = u_{i+1}$, $0 \le i \le n-1$ and $u = (u_1, \dots, u_n)$, we obtain the following system of equations for the function $u(t)$ in a small neighborhood of zero:

$$\begin{gathered} \dot{u} = Au + F(u)f(t) + g(t), \quad \|f(t)\| + \|g(t)\| \le D_1 e^{-\delta t}, \qquad t \ge 0, \\ F(u) \in C^\infty, \quad F(0) = 0, \quad D_1 = \text{const} > 0. \end{gathered} \tag{22}$$

For a sufficiently small $0 < \widetilde{\delta} < \delta$, this system of equations has the solution

$$u = u(t), \qquad \|u(t)\| = o(e^{-\widetilde{\delta} t}) \quad \text{as } t \to +\infty$$

(see, for example, [8]). This yields

$$y^{(i)}(x) = m(m-1)\cdots(m-i+1)x^{m-i}\biggl(c_m + \sum_{j=1}^{i+1} d_{ij}u_j(\ln x)\biggr), \qquad 0 \le i \le m,$$

$$y^{(i)}(x) = x^{m-i}\sum_{j=2}^{i+1} d_{ij}u_j(\ln x), \qquad m+1 \le i \le n-1, \quad d_{ij} = \text{const}\,.$$

Therefore, in view of the equality

$$\frac{d^q u_1(\ln x)}{dx^q} = \frac{\sum_{j=1}^{q+1} a_{qj}u_j(\ln x)}{x^q}, \qquad a_{qj} = \text{const}, \quad 0 \le q \le n-1,$$

we see that $y = x^m(c_m + u_1(\ln x))$ is a solution of Eq. (1) with the required properties (6).

Now, let the solution $y(x)$ satisfy condition (7). In [5], it was noted that the function $y = y(x)$ satisfies the linear equation

$$y^{(n)} = p_1(x)y, \qquad p_1(x) = p(x)|y(x)|^{\sigma-1},$$

and it follows from (5) and (7) that

$$|p_1(x)| \le D_2 x^{-n-\delta}, \qquad D_2 = \text{const} > 0.$$

It follows from the previous part of the proof of our theorem that the given linear equation has solutions $y_m(x)$ of the form (6) for all integer m, $m \in \{0, 1, \dots, n-1\}$, the c_m are arbitrary nonzero constants, and $\widetilde{\delta} > 0$. But, in that case, the functions $y_m(x)$, $m \in \{0, 1, \dots, n-1\}$, constitute the fundamental system of solutions of the given linear equation. Therefore, the function $y(x)$ is their linear combination:

$$y(x) = \sum_{j=0}^{n-1} a_j y_j(x), \qquad a_j = \text{const}.$$

Since, by assumption, $|y(x)| \le Dx^l$, $D = \text{const} > 0$, it follows that

$$a_j = 0, \quad j > l, \qquad y(x) = \sum_{j=0}^{l} a_j y_j(x),$$

whence we see that $y(x)$ is a solution of the form (6), where $m \in \{0, 1, \dots, l\}$. The assertion of Theorem 2 is proved. □

Let us now turn to the proof of Theorem 6 and consider the case $0 < \sigma < 1$.

Proof of Theorem 6. We must prove that Eq. (1) has solutions of the form (6) for all integer m, $m \in \{l, \dots, n-1\}$, where the c_m are arbitrary nonzero constants and $\widetilde{\delta}$ is a number satisfying the condition $0 < \widetilde{\delta} < \delta$.

Just as above, we fix the number m, $m \in \{l, \dots, n-1\}$, and, in (1), replace $y = x^m(c_m + z)$, $t = \ln x$. Equation (1) takes the form of system (22) and, since $m \ge l$ and $0 < \sigma < 1$, it follows that the estimate for the function $p_m(t)$ remains valid. As a result, all the foregoing can be reduced to the fact that system (22) has solutions with the required properties. The proof of Theorem 6 is complete. □

Proof of Theorem 3. Condition (8) implies (5), where $l = n-1$, and, by Theorem 2, Eq. (1) has solutions of the form (6), where $m \in \{0, 1, \dots, n-1\}$. Let $y(x)$ be a sign-preserving solution of Eq. (1). Without loss of generality, we assume that $y(x) > 0$. It follows from (8) that $y^{(n)}(x) < 0$, i.e., the function $y^{(n-1)}(x)$ decreases and, therefore, is bounded above by a constant. Then

$$y(x) \le Dx^{n-1}, \quad x \ge x_0, \qquad D = \text{const} > 0,$$

and conditions (5) and (7) hold for $l = n-1$. From Theorem 2, we see that the solution $y(x)$ is of the form (6), where m is an integer, $m \in \{0, 1, \dots, n-1\}$. Theorem 3 is proved. □

Proof of Theorem 4. The existence of noncontinuable solutions of the form (4) was proved for Theorem 1. The existence of continuable (to the right) solutions $y = y_m(x)$, $m \in \{0, 1, \dots, n-1\}$ satisfying condition (6) follows from Theorem 2 for $l = n-1$. Let us now pass to the proof of the fact that, under condition (9), there exist solutions of the form (10) and also verify that all possible nonoscillating solutions of Eq. (1), are stated in the theorem.

As usual, we restrict ourselves to the study of positive (for $x \ge x_0$) solutions of Eq. (1). Let $y(x)$ be such a solution, and let $\widetilde{x}$ be the right boundary of its domain. Assume that $\widetilde{x} < +\infty$. Since $y^{(n)}(x) > 0$, $x \ge x_0$, it follows that $y^{(n-1)}(x)$ increases for $x \ge x_0$. If

$$\lim_{x \to \widetilde{x}-0} y^{(n-1)}(x) = D < +\infty,$$

then $|y(x)| \le D < +\infty$, and hence $|y^{(n)}(x)| \le D < +\infty$; therefore, for $x_0 \le x < \widetilde{x}$, we have

$$|y^{(i)}(x)| \le D < +\infty, \qquad 0 \le i \le n-1.$$

But this contradicts the fact that the solution is noncontinuable. Therefore,

$$\lim_{x \to \widetilde{x}-0} y^{(n-1)}(x) = +\infty.$$

Then, near the point $\widetilde{x}$, the function $y^{(n-2)}(x)$ will increase and, similarly,

$$\lim_{x \to \widetilde{x}-0} y^{(n-2)}(x) = +\infty.$$

Arguing as above, we conclude that

$$\lim_{x \to \widetilde{x}-0} y^{(i)}(x) = +\infty, \qquad 0 \le i \le n-1,$$

i.e., the solution $y(x)$ satisfies condition (4). Thus, for $\widetilde{x} < +\infty$, the solution is noncontinuable to the right and has the form (4).

Now let $y(x)$ be defined for $x_0 \le x < +\infty$. Then if

$$\lim_{x \to +\infty} y^{(n-1)}(x) = D < +\infty,$$

then $y(x) \le Dx^{n-1}$, $D = \text{const} > 0$, $x \ge x_0$, and assumption (7) of Theorem 2 holds for $l = n-1$. Therefore, the solution $y(x)$ is a solution of the form (6), where m is an integer, $m \in \{0, 1, \dots, n-1\}$.

It remains to consider the case in which

$$\lim_{x \to +\infty} y^{(n-1)}(x) = +\infty.$$

Let us show that such solutions really exist. This fact was proved in [6, pp. 274–276], but we restrict ourselves to the exposition of the main points of the proof.

Consider any solution $y(x)$ of our equation satisfying condition (6) where $m = n-1$ and $c_m > 0$. All the derivatives of this solution satisfy the inequalities $y^{(i)}(x) > 0$, $0 \le i \le n-1$, for $x \ge \widetilde{x}_0$, where $\widetilde{x}_0 \ge x_0$ is some number. Let us fix the numbers $b_i = y^{(i)}(\widetilde{x}_0)$, $0 \le i \le n-1$, and consider the solutions $y_A(x)$ of our equation satisfying the following initial conditions:

$$y_A^{(i)}(\widetilde{x}_0) = b_i, \qquad 1 \le i \le n-1, \quad y_A(\widetilde{x}_0) = A \ge b_0. \tag{23}$$

Here we consider numbers $A \geq b_0$ for which $y_A^{(n-1)}(x) \leq D < +\infty$. The set of such numbers A is bounded above, because, for a sufficiently large A, the solution $y_A(x)$ will be noncontinuable to the right. This follows from the proof of Theorem 1 from [1]).

Let B be the supremum for this set. Consider the solution $y_B(x)$. This solution cannot satisfy condition (4), because, by the remark to the proof of Theorem 1 from [1], if the solution $y_B(x)$ satisfies (4), then any solution with sufficiently close initial conditions will also satisfy (4), possibly, with another value of a_1. But, in that case, the number B cannot be the supremum for this set. Thus, the solution $y_B(x)$ does not satisfy condition (4) and, therefore, it is continuable to the right.

It follows from Theorem 2 that if $y_B^{(n-1)}(x) \leq D < +\infty$, then $y_B(x)$ is a solution of the form (6). We can show that, in this case, solutions with close initial conditions will also satisfy conditions of the form (6). However, this contradicts the definition of the number B. This yields

$$\lim_{x \to +\infty} y_B^{(n-1)}(x) = +\infty.$$

Let us now show that, under condition (9), the solutions $y(x)$ for which

$$\lim_{x \to +\infty} y^{(n-1)}(x) = +\infty,$$

satisfy estimate (10).

Note that all the functions $y^{(i)}(x)$, $0 \leq i \leq n-1$, are increasing and positive for large values of the argument. For simplicity, we can assume that this is so for $x \geq x_0 > 0$. Then, for $x \geq x_0$,

$$y^{(i)}(x) = y^{(i)}(x_0) + \int_{x_0}^{x} y^{(i+1)}(t)\,dt \leq y^{(i)}(x_0) + (x - x_0)y^{(i+1)}(x) \leq Dxy^{(i+1)}(x),$$
$$0 \leq i \leq n-2, \qquad D = D(x_0) = \text{const} > 0.$$

This implies the estimate

$$y(x) \leq D_1 x^{n-1} y^{(n-1)}(x), \qquad x \geq x_0, \quad D_1 = D_1(x_0) = \text{const} > 0. \tag{24}$$

Substituting (24) into Eq. (1) and denoting $z = y^{(n-1)}(x)$, we find that, under condition (9), the function $z(x)$ satisfies the inequality $z' \leq Dz^{\sigma}x^{-1-\delta}$, $D = \text{const} > 0$. Integrating this inequality on the interval $[x, +\infty)$, $x \geq x_0$, we obtain

$$z(x) = y^{(n-1)}(x) \geq Dx^{\delta/(\sigma-1)}, \qquad x \geq x_0, \quad D = \text{const} > 0.$$

Integrating this inequality $n-1$ times, we obtain the required estimate (10).

Let us now pass to the proof of estimate (12), provided that inequality (11) holds. As usual, without loss of generality, we consider the positive solution $y(x)$ of Eq. (1). If the limit as $x \to +\infty$ of the increasing function $y^{(n-1)}(x)$ is finite, then (11) is obvious. Consider the case in which this limit is $+\infty$. For large values of x, all the functions $y^{(i)}(x)$, $0 \leq i \leq n-1$, will be positive and increasing as well as

$$y^{(i)}(x) \leq xy^{(i+1)}(x), \qquad 0 \leq i \leq n-2;$$

this implies, in particular, that

$$y(x) \leq x^i y^{(i)}(x), \qquad 0 \leq i \leq n-1. \tag{25}$$

The construction on which the proof of estimate (12) is based is due to Izobov [7].

Consider the function

$$h(x) = \prod_{i=0}^{n-1} y^{(i)}(x).$$

Note that

$$h'(x) = h(x)\biggl(\frac{y'}{y} + \frac{y''}{y'} + \cdots + \frac{y^{(n)}}{y^{(n-1)}}\biggr). \tag{26}$$

In what follows, we shall use the well-known inequality

$$\sum_{i=1}^{n} \beta_i z_i \ge \prod_{i=1}^{n} z_i^{\beta_i}, \qquad z_i \ge 0, \quad \beta_i > 0, \quad \sum_{i=1}^{n} \beta_i = 1. \tag{27}$$

We shall need the numbers β and $\mu > 0$ satisfying the conditions

$$\beta_n = \beta, \quad \beta_i = \beta_{i+1} + \mu, \qquad 1 \le i \le n-1, \qquad \sum_{i=1}^{n} \beta_i = 1.$$

To this end, we set $\mu = 2(1 - n\beta)/(n(n-1))$, and choose the number $\beta > 0$ we choose so that $1 - n\beta > 0$. In view of (11) and (27), using (26), we obtain

$$\begin{aligned} h' &\ge h\biggl(\beta_1 \frac{y'}{y} + \beta_2 \frac{y''}{y'} + \dots + \beta_n \frac{y^{(n)}}{y^{(n-1)}}\biggr) \ge h\biggl(\frac{y'}{y}\biggr)^{\beta_1} \biggl(\frac{y''}{y'}\biggr)^{\beta_2} \cdot \dots \cdot \biggl(\frac{y^{(n)}}{y^{(n-1)}}\biggr)^{\beta_n} = h^{1+\mu} \frac{(y^{(n)})^{\beta}}{y^{\beta+n\mu}} \\ &\ge D h^{1+\mu} y^{\beta(\sigma-1)-n\mu} x^{-\beta((n-1)\sigma+1+\delta_1)}, \qquad D = \text{const} > 0. \end{aligned} \tag{28}$$

Now let us refine the number β so that

$$\beta(\sigma - 1) - n\mu > 0 \qquad \text{and} \qquad \beta((n-1)\sigma + 1 + \delta_1) > 1. \tag{29}$$

It is readily seen that this does not contradict the inequality $1 - n\beta > 0$, because $\sigma > 1$. Let us now divide both sides of inequality (28) by $h^{1+\mu}$ and integrate it on the interval $[x, +\infty)$, obtaining

$$\begin{aligned} (h(x))^{-\mu} &\ge D \int_x^{+\infty} t^{-\beta((n-1)\sigma+1+\delta_1)} (y(t))^{\beta(\sigma-1)-n\mu}\, dt \\ &\ge D (y(x))^{\beta(\sigma-1)-n\mu} \int_x^{+\infty} t^{-\beta((n-1)\sigma+1+\delta_1)}\, dt \\ &\ge D (y(x))^{\beta(\sigma-1)-n\mu} x^{-\beta((n-1)\sigma+1+\delta_1)+1}, \qquad D = \text{const} > 0. \end{aligned} \tag{30}$$

It follows from (29) and the equality $\lim_{x\to+\infty} h(x) = +\infty$, that all the integrals converge.

Using (25), we obtain the inequality $(h(x))^{-\mu} \le x^{1-n\beta} y^{-n\mu}$. Substituting it into (30), we obtain the required estimate (12), namely,

$$y(x) \le D x^{n-1+\delta_1/(\sigma-1)}, \qquad D = \text{const} > 0.$$

Theorem 4 is proved. □

Proof of Theorem 5. Consider the sign-preserving (for $x \ge x_0$) solution $y(x)$ of Eq. (1). Without loss of generality, we can assume that $y(x) > 0$ (otherwise, replacing $y = -z$). Denote by $\widetilde{x}$ the right boundary of the domain of the solution. The following standard argument shows that if $\widetilde{x} < +\infty$, then $y(x)$ will satisfy condition (4). Indeed, in the case under consideration, the function $p(x)$ is positive, because, otherwise, for $y(x) > 0$ we have $y^{(n)}(x) < 0$, and $y^{(n-1)}(x)$ is a decreasing function; this implies that, for $x_0 \le x < \widetilde{x}$, we have $|y^{(i)}(x)| \le D$, $0 \le i \le n-1$, $D = \text{const} > 0$. However, this contradicts the assumption the solution is noncontinuable to the right. Thus, $p(x) > 0$ and $y^{(n)}(x) > 0$, i.e., $y^{(n-1)}(x)$ is an increasing function. In that case, if

$$\lim_{x\to\widetilde{x}-0} y^{(n-1)}(x) = D_1 < +\infty,$$

then $|y(x)| \le D_2 < +\infty$, and hence $|y^{(n)}(x)| \le D_3 < +\infty$; therefore, for $x_0 \le x < \widetilde{x}$, we have

$$|y^{(i)}(x)| \le D_4 < +\infty, \qquad 0 \le i \le n-1.$$

But, in that case, the point $\widetilde{x}$ cannot be the boundary of the domain of the solution. Therefore,

$$\lim_{x\to\widetilde{x}-0} y^{(n-1)}(x) = +\infty.$$

Then, near the point $\widetilde{x}$, the function $y^{(n-2)}(x)$ will increase and, similarly,

$$\lim_{x\to\widetilde{x}-0} y^{(n-2)}(x) = +\infty.$$

Arguing as above, we conclude that

$$\lim_{x\to\widetilde{x}-0} y^{(i)}(x) = +\infty, \qquad 0 \le i \le n-1,$$

i.e., the solution $y(x)$ satisfies condition (4).

Now consider the positive solution defined for all $x \ge x_0$. First, note that, for large values of x, all the derivatives $y^{(i)}(x)$, $0 \le i \le n-1$, are monotone functions. Without loss of generality, we can assume that this is so for $x \ge x_0$. Denote

$$\lim_{x\to+\infty} y^{(i)}(x) = D_i, \qquad 0 \le D_i \le +\infty.$$

Let us show that $D_{n-1} = 0$. Indeed, if $0 < D_{n-1} \le +\infty$, then, for $p(x) > 0$, this contradicts Lemma 1 from [1]. But if $p(x) < 0$, then, noting that, for large x, $y(x) \ge Dx^{n-1}$, $D = \text{const} > 0$, we obtain

$$y^{(n)}(x) \le -Dx^{(n-k-1)(\sigma-1)-1}.$$

Hence, after integration, we have $\lim_{x\to+\infty} y^{(n-1)}(x) = -\infty$, which cannot be true. In addition, note that if all $D_i = 0$, $0 \le i \le n-1$, then the assertion of the theorem obvious holds.

It remains to consider the case in which there exists a number $0 \le m \le n-2$ such that

$$\lim_{x\to+\infty} y^{(i)}(x) = 0, \qquad m < i \le n-1, \tag{31}$$

and

$$\lim_{x\to+\infty} y^{(m)}(x) = D_m, \qquad 0 < D_m \le +\infty.$$

Assuming for simplicity that $y^{(i)}(x) > 0$, $0 \le i \le m$, $x \ge x_0$ (if necessary, making a shift along the axis x), we then have

$$y(x) \ge Dx^m, \quad x \ge x_0, \qquad D = \text{const} > 0. \tag{32}$$

Substituting (32) into Eq. (1), integrating it $n-(m+1)$ times on the interval $[x, +\infty)$, and using (13) and (31), we obtain

$$|y^{(m+1)}(x)| \ge Dx^{(m-k)(\sigma-1)-1}, \quad x \ge x_0, \qquad D = \text{const} > 0. \tag{33}$$

If $y^{(m+1)}(x) < 0$ and $m \ge k$, then the last inequality implies that

$$\lim_{x\to+\infty} y^{(m)}(x) = -\infty,$$

which cannot be true. If $y^{(m+1)}(x) < 0$ and $m < k$, then the assertion of the theorem is obviously fulfilled.

Let $y^{(m+1)}(x) > 0$.

First, consider the case in which $m > k$, where k is the number from formula (13).

Integrating (33) $m+1$ times on the interval $[x_0, x]$, we obtain the inequality

$$y(x) \ge Dx^{\beta+m}, \quad x \ge x_0, \qquad \beta = (m-k)(\sigma-1), \quad D = \text{const} > 0.$$

Repeating a similar argument j times, we obtain the estimate

$$y(x) \ge Dx^{\beta_j+m}, \quad x \ge x_0, \qquad \beta_j = \beta\sigma^{j-1}, \quad D = \text{const} > 0.$$

But, in that case, for a sufficiently large j, we have $|y^{(n)}(x)| \ge x^h$, $h > 0$, $x \ge x_0 > 0$, which contradicts the equality $\lim_{x\to+\infty} y^{(n-1)}(x) = 0$. Thus, the case $m > k$ is impossible.

Now consider the case in which $m = k$. Integrating inequality (33) on the interval $[x_0, x]$, we obtain

$$y^{(k)}(x) \geq D \ln \frac{x}{x_0}, \quad x \geq x_0 \qquad D = \text{const} > 0. \tag{34}$$

Here and elsewhere, we assume without loss of generality that

$$y^{(i)}(x) > 0, \qquad 0 \leq i \leq m, \qquad x \geq x_0.$$

In what follows, we shall need the following inequality (it was used in [1] in the proof of Lemma 1):

$$\begin{gathered} \int_a^x t^p \ln^q \frac{t}{a}\, dt \geq \frac{1}{1+p+q^2}\, x^{p+1} \ln^q \frac{x}{\widetilde{a}}, \\ x \geq \widetilde{a} = a\sqrt[q]{e}, \quad a \geq 1, \quad p \geq -1, \quad q \geq 1. \end{gathered} \tag{35}$$

The validity of this estimate follows from the fact that, at the point $\widetilde{a}$, the difference of the functions on the left- and right-hand sides of the inequality is positive, while the difference of their derivatives for $x \geq a\sqrt[q]{e}\ \ a, q \geq 1$ is nonnegative.

Let us integrate inequality (34) k times, taking (35) into account. As a result, we obtain the estimate

$$y(x) \geq Dx^k \ln \frac{x}{x_1}, \quad x \geq x_1 = x_0 e^k, \qquad D = \text{const} > 0.$$

Recall that $D > 0$ is the so-called "universal" constant.

Let us argue by induction.

Suppose that the number $r \geq 1$ is an integer and $x_r \geq x_0$, $c_r > 0$, $\omega_r \geq 1$ are given numbers. Suppose that, for $x \geq x_r$, the following inequality holds:

$$y(x) \geq \frac{1}{c_r}\, x^k \ln^{\omega_r} \frac{x}{x_r}, \qquad x \geq x_r. \tag{36}$$

Substituting this inequality into Eq. (1) and taking (13) into account, we can write

$$|y^{(n)}(x)| \geq \frac{c}{c_r^\sigma x^{n-k}} \ln^{\sigma\omega_r} \frac{x}{x_r}, \qquad x \geq x_r.$$

Integrating this inequality $n-k$ times on the interval $[x, +\infty)$ and using (31), we obtain the estimate

$$y^{(k)}(x) \geq \frac{c}{(n-k-1)!\, c_r^\sigma} \ln^{\sigma\omega_r+1} \frac{x}{x_r} \geq \frac{c}{n!\, c_r^\sigma} \ln^{\sigma\omega_r+1} \frac{x}{x_r}, \qquad x \geq x_r.$$

Let us now integrate the last inequality k times and use (35). As a result, we obtain the following estimate:

$$\begin{gathered} y(x) \geq \frac{1}{\widetilde{c}_r}\, x^k \ln^{\sigma\omega_r+1} \frac{x}{x_{r+1}}, \qquad x \geq x_{r+1}, \\ \widetilde{c}_r = c^{-1} n!\, c_r^\sigma (1 + k + (\sigma\omega_r + 1)^2)^k \leq A\sigma_1^r c_r^\sigma, \\ \sigma_1 = \sigma^{2n}, \quad A = A(\sigma, n, c) = \text{const} > 0, \quad x_{r+1} = x_r e^{k/(\sigma\omega_r+1)}. \end{gathered}$$

Setting

$$\omega_{r+1} = \sigma\omega_r + 1, \quad c_{r+1} = A\sigma_1^r c_r^\sigma, \qquad \sigma_1 = \sigma^{2n}, \quad A = A(\sigma, n, c),$$

we can easily see that

$$y(x) \geq \frac{1}{c_{r+1}}\, x^k \ln^{\omega_{r+1}} \frac{x}{x_{r+1}}, \qquad x \geq x_{r+1}.$$

This implies

$$\begin{gathered} y(x) \geq \frac{1}{B^{\sigma^r}} \ln^{\beta_r} \frac{x}{\widetilde{x}_0}, \qquad x \geq \widetilde{x}_0, \\ B = B(n, \sigma, c_1, c) = \text{const} > 0, \end{gathered}$$

$$\beta_r = \frac{\sigma^r - 1}{\sigma - 1}, \qquad \widetilde{x}_0 = \mu x_0, \quad \mu = \sum_{i=1}^{\infty} \frac{\sigma - 1}{\sigma^i - 1}.$$

However, the resulting inequality leads to a contradiction for a sufficiently large fixed x (for example, for $\ln\ln x/\widetilde{x}_0 > 2(\sigma-1)\ln B$) as $r \to \infty$. Thus, the equality $m = k$ contradicts the assumptions of the theorem.

It remains to consider the case $m < k$.

Let us integrate Eq. (1) on the interval $[x, +\infty)$. Taking into account (13) and (31) as well as the fact that the function $y(x)$ is increasing, we obtain the inequality

$$y^{(n-1)}(x) \le -D\frac{y^\sigma}{x^{n+k(\sigma-1)-1}}, \qquad D = \text{const} > 0.$$

Again integrating the resulting inequality on the given interval, we can write

$$y^{(n-2)}(x) \ge D\frac{y^\sigma}{x^{n+k(\sigma-1)-2}}, \qquad D = \text{const} > 0.$$

Continuing this process, we finally obtain the inequality

$$y^{(m+1)}(x) \ge D\frac{y^\sigma}{x^{k(\sigma-1)+m+1}}, \qquad D = \text{const} > 0. \tag{37}$$

Note that all the functions $y^{(i)}(x)$, $0 \le i \le m$, are increasing and positive for large values of the argument. For simplicity, we can assume that this is so for $x \ge x_0 > 0$. Then, for $x \ge x_0$, we obtain the following analog of inequalities (25):

$$y(x) \le x^i y^{(i)}(x), \qquad 0 \le i \le m. \tag{38}$$

Further, let us use the construction due to Izobov (see [7]) which was described in the proof of Theorem 4. Consider the function $h(x) = \prod_{i=0}^{m} y^{(i)}(x)$. Just as in the proof of Theorem 4, we use the numbers β and $\mu > 0$ satisfying the conditions

$$\beta_{m+1} = \beta, \qquad \beta_i = \beta_{i+1} + \mu, \quad 1 \le i \le m, \qquad \sum_{i=1}^{m+1} \beta_i = 1.$$

To this end, we set $\mu = 2(1-(m+1)\beta)/(m(m+1))$, and the number $\beta > 0$ is chosen so as to have $1-(m+1)\beta > 0$. As a result, in view of (13) and (37), we obtain the following inequality similar to (28):

$$h' \ge Dh^{1+\mu} y^{\beta(\sigma-1)-\mu(m+1)} x^{-\beta(m+1+k(\sigma-1))}, \qquad D = \text{const} > 0. \tag{39}$$

Let us define the number β so that

$$\beta(\sigma-1) - \mu(m+1) > 0 \qquad \text{and} \qquad \beta(m+1+k(\sigma-1)) > 1.$$

It is readily seen that this does not contradict the inequality $1-(m+1)\beta > 0$. Let us now divide both sides of inequality (39) by $h^{1+\mu}$ and integrate it on the interval $[x, +\infty)$, obtaining

$$h^{-\mu} \ge Dy^{\beta(\sigma-1)-\mu(m+1)} x^{-\beta(m+1+k(\sigma-1))+1}, \qquad D = \text{const} > 0. \tag{40}$$

Using (38), we find the inequality

$$(h(x))^{-\mu} \le x^{1-(m+1)\beta} y^{-(m+1)\mu}.$$

Substituting it into (40), we obtain the required estimate (14).

It remains to prove (15) for an integer k. Since m is an integer and $m < k$, it follows that $m+1 \le k$. But, obviously, relation (31) implies (15).

Theorem 5 is proved. □

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