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The two-sided infinite extension of the Mallows model for random permutations

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ABSTRACT

We introduce a probability distribution $\mathcal{Q}$ on the infinite group $\mathfrak{S}_{\mathbb{Z}}$ of permutations of the set of integers $\mathbb{Z}$. The distribution $\mathcal{Q}$ is a natural extension of the Mallows distribution on the finite symmetric group. A one-sided infinite counterpart of $\mathcal{Q}$, supported by the group of permutations of $\mathbb{N}$, was studied previously in our paper [A. Gnedin, G. Olshanski, q-Exchangeability via quasi-invariance, Ann. Probab. 38 (2010) 2103–2135, arXiv:0907.3275]. We analyze various features of $\mathcal{Q}$ such as its symmetries, the support, and the marginal distributions.

1. Introduction

Let $\mathfrak{S}_n$ be the group of permutations of $\{1,\dots,n\}$. A random permutation Σ_n with the probability distribution

$$\mathbb{P}(\Sigma_n=\sigma)=c_n^{-1}q^{\mathrm{inv}(\sigma)},\quad 0<q<1, \tag{1}$$

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is one of the Mallows models on $\mathfrak{S}_n$, see [5]. Here σ ranges over $\mathfrak{S}_n$,

$$\mathrm{inv}(\sigma) = \#\{(i, j)\colon 1 \leqslant i < j \leqslant n,\ \sigma(j) > \sigma(i)\}$$

is the number of inversions, and

$$c_n = [n!]_q = \prod_{i=1}^{n} \frac{1-q^i}{1-q}$$

is the normalizing constant (a q-analog of the factorial $n!$). Distribution (1) is qualitatively different from the uniform in that it favors the order: the probability is maximal at the identity permutation $\sigma(i) = i$, and falls off exponentially as the number of inversions increases.

The Mallows model was introduced in connection with ranking problems in statistics (see [18] and a more general definition in [3, p. 104, Example 3]) and recently it appeared in connection with random sorting algorithms [2,5], random trees [6], and q-exchangeability [11,12]. The distributions of displacements $\Sigma_n(i) - i$ are tight as $n \to \infty$ for each fixed i, which suggests understanding Σ_n as a random function with linear trend. To compare, the order of displacements in the uniform permutation is $O(n)$. See [19] and references therein for some large-n properties of the Mallows model when $q = q_n$ varies in such a way that $(1-q_n)n \to \text{const}$, so Σ_n approaches the uniform permutation in this regime.

In our previous paper [12] we observed that (1) is the unique distribution on $\mathfrak{S}_n$ with the property of q-exchangeability, which means that by swapping the values in any two adjacent positions i and $i+1$ the probability of permutation is multiplied by $q^{\mathrm{sgn}(\sigma(i+1)-\sigma(i))}$ where $\mathrm{sgn}(x) = \pm 1$ according to the sign of $x \neq 0$. In the course of generalizing this property to arbitrary infinite real-valued sequences we were lead to introducing a random permutation (bijection) $\Sigma_+ : \mathbb{Z}_+ \to \mathbb{Z}_+$ which extends (1) to the case "$n = \infty$", in the sense that, in suitable coordinates, the Mallows measures appear as consistent n-dimensional marginal distributions of Σ_+. One of the uses of Σ_+ is that many large-n features of Σ_n can be recognized as properties of Σ_+.

In this paper we introduce a further *two-sided infinite* extension of the Mallows model, which is a random q-exchangeable bijection $\Sigma : \mathbb{Z} \to \mathbb{Z}$. Permutations Σ_+ and Σ share many common features, among which are the invariance of the distribution under passing to the inverse permutation, and the related property of quasi-invariance under swapping positions of two adjacent values j and $j+1$. Both Σ_+ and Σ can be constructed from independent copies of the same geometric random variable, but Σ has more symmetries. A peculiar feature of the two-sided counterpart is that the process of displacements $(\Sigma(i) - i,\ i \in \mathbb{Z})$ is stationary. We shall describe the support of Σ and derive formulas for the joint distribution of the displacements in terms of some series of the q-hypergeometric type.

From a general perspective, (1) is a conditionally uniform distribution on $\mathfrak{S}_n$ obtained as deformation of the uniform distribution by exponential tilting of distribution of some statistic of permutations. Replacing $\mathrm{inv}(\sigma)$ in (1) by the number of cycles of σ will yield the familiar Ewens distribution [1], which is also an instance of the general Mallows model [4, Section 4.4]. See [10,9] for other choices of the statistic. Remarkably, the extended infinite counterparts of (1) live on the group of permutations, albeit these are no longer *finitary* permutations that move finitely many integers. To compare, the (one-sided or two-sided) infinite extension of the uniform and of Ewens' measures on $\mathfrak{S}_n$ are not supported by the space of permutations (see [15,16,20] for a realization of extended Ewens' measures in the space of virtual permutations). The extended Mallows and Ewens measures are quasi-invariant with respect to left and right shifts by finitary permutations, and both may be viewed as substitutes of the nonexisting finite Haar measure on the group of finitary permutations. The nice quasi-invariance properties of the measures make it possible to construct families of unitary representations of the group of finitary permutations, although it is still to be explored if the extended Mallows measures may play in the harmonic analysis the role similar to that of the Ewens measure on virtual permutations (see [15,16,20]).

2. Preliminaries on infinite permutations

This section is aimed to introduce various classes of permutations, and to distinguish the permutations considered as support of the to-be-constructed measures from the permutations considered as transformations acting on this support.

We use the standard notation $\mathbb{Z}$ for the set of integers. For $a \in \mathbb{Z}$ we set $\mathbb{Z}_{<a} := \{i \in \mathbb{Z}\colon\ i < a\}$. Likewise, we define the subsets $\mathbb{Z}_{\leqslant a}$, $\mathbb{Z}_{>a}$, and $\mathbb{Z}_{\geqslant a}$. A nonstandard convention of this paper is that

$$\mathbb{Z}_- := \mathbb{Z}_{\leqslant 0} = \{\ldots, -1, 0\}, \qquad \mathbb{Z}_+ := \mathbb{Z}_{\geqslant 1} = \{1, 2, \ldots\}.$$

Under *permutation of* $\mathbb{Z}$ we shall understand an arbitrary bijection $\sigma : \mathbb{Z} \to \mathbb{Z}$. Let $\mathfrak{S}$ denote the group of all permutations of $\mathbb{Z}$. We associate with $\sigma \in \mathfrak{S}$ an infinite 0–1 matrix $A = A(\sigma)$ of format $\mathbb{Z} \times \mathbb{Z}$ such that the (i, j)th entry of $A(\sigma)$ is $\mathbf{1}(\sigma(j) = i)$. Here and throughout $\mathbf{1}(\cdots)$ equals 1 if the condition $\cdots$ is true, and equals 0 otherwise. Observe that the group operation on permutations agrees with the matrix multiplication, that is $A(\sigma\tau) = A(\sigma)A(\tau)$. Write $A = A(\sigma)$ as a 2×2 block matrix

$$A = \begin{bmatrix} A_{--} & A_{-+} \\ A_{+-} & A_{++} \end{bmatrix}$$

according to the splitting $\mathbb{Z} = \mathbb{Z}_- \sqcup \mathbb{Z}_+$. For matrix B let $\mathrm{rk}(B)$ be the rank of B, which is equal to the number of 1's in B if B is a submatrix of $A(\sigma)$. We call σ *admissible* if both $\mathrm{rk}(A_{-+})$ and $\mathrm{rk}(A_{+-})$ are finite. The set of admissible permutations will be denoted $\mathfrak{S}^{\mathrm{adm}} \subset \mathfrak{S}$.

Remark 2.1. There is a similarity between our definition of admissibility and the concept of *restricted* matrix for infinite-dimensional classical matrix groups, as found e.g. in [21, Definition 6.2.1].

For $\sigma \in \mathfrak{S}^{\mathrm{adm}}$ we define the *balance* as

$$b(\sigma) := \mathrm{rk}(A_{-+}) - \mathrm{rk}(A_{+-}).$$

It is readily checked that the same value $b(\sigma)$ appears if instead of the splitting $\mathbb{Z} = \mathbb{Z}_- \sqcup \mathbb{Z}_+$ the difference of ranks is computed with respect to any other splitting of the form $\mathbb{Z} = \mathbb{Z}_{\leqslant a} \sqcup \mathbb{Z}_{>a}$ with $a \in \mathbb{Z}$.

Example 2.2. For a *shift permutation*

$$s^{(b)} : i \mapsto i - b, \quad b \in \mathbb{Z},$$

we have

$$\mathrm{rk}(A_{-+}) = \begin{cases} b, & b > 0, \\ 0, & b \leqslant 0 \end{cases} \quad \text{and} \quad \mathrm{rk}(A_{+-}) = \begin{cases} |b|, & b < 0, \\ 0, & b \geqslant 0. \end{cases}$$

It follows that $s^{(b)}$ is admissible and has balance b.

For $I \subset \mathbb{Z}$ let $\mathfrak{S}_I$ denote the set of permutations which satisfy $\sigma(j) = j$ for $j \in I$. The union

$$\mathfrak{S}^{\mathrm{fin}} := \bigcup_{\{I\colon \#I < \infty\}} \mathfrak{S}_I$$

is the group of *finitary* permutations σ which move only finitely many integers, i.e. fulfill $\sigma(j)=j$ for $|j|$ sufficiently large. The group $\mathfrak{S}^{\mathrm{fin}}$ is countable, and is generated by the elementary transpositions $\sigma_{i,i+1}$ that swap two adjacent integers $i, i+1 \in \mathbb{Z}$. Note that

$$b(\sigma'\sigma) = b(\sigma\sigma') = b(\sigma), \quad \sigma \in \mathfrak{S}^{\mathrm{adm}},\ \sigma' \in \mathfrak{S}^{\mathrm{fin}},$$

which is easy to check first for the elementary transpositions and then by induction for all $\sigma' \in \mathfrak{S}^{\mathrm{fin}}$. It follows that every finitary permutation is admissible and has balance 0.

Proposition 2.3.

(i) $\mathfrak{S}^{\mathrm{adm}}$ *is a subgroup in* $\mathfrak{S}$.
(ii) *The balance is an additive character on* $\mathfrak{S}^{\mathrm{adm}}$*, i.e.,*

$$b(\sigma\tau) = b(\sigma) + b(\tau), \quad \sigma, \tau \in \mathfrak{S}^{\mathrm{adm}}.$$

Proof. (i) This is obvious, for the matrix multiplication preserves finiteness of the ranks for the off-diagonal blocks.

(ii) It is readily checked that for each $\sigma \in \mathfrak{S}^{\mathrm{adm}}$,

$$b\big(s^{(n)}\sigma\big) = b\big(\sigma s^{(n)}\big) = b(\sigma) + n, \quad n \in \mathbb{Z}. \tag{2}$$

Using this, one can show that σ can be written in the form

$$\sigma = s^{(k)}\sigma'\sigma'', \quad k = b(\sigma),$$

where $\sigma' \in \mathfrak{S}^{\mathrm{fin}}$ and $A(\sigma'')$ is a block-diagonal matrix. Likewise, τ is representable as the product

$$\tau = \tau''\tau's^{(l)}, \quad l = b(\tau),$$

where $\tau' \in \mathfrak{S}^{\mathrm{fin}}$ and $A(\tau'')$ is a block-diagonal matrix. It follows that the permutation

$$\sigma\tau = s^{(k)}\sigma'\sigma''\tau''\tau's^{(l)}$$

has the same balance as $s^{(k)}s^{(l)} = s^{(k+l)}$. Thus, $b(\sigma\tau) = k + l = b(\sigma) + b(\tau)$, which concludes the proof. □

We call a permutation *balanced* if it is admissible and has balance 0. We shall denote the set of balanced permutation $\mathfrak{S}^{\mathrm{bal}}$. By Proposition 2.3, $\mathfrak{S}^{\mathrm{bal}}$ is a normal subgroup in $\mathfrak{S}^{\mathrm{adm}}$, and the quotient group $\mathfrak{S}^{\mathrm{adm}}/\mathfrak{S}^{\mathrm{bal}}$ is isomorphic to $\mathbb{Z}$.

Every finitary permutation is balanced. An example of admissible permutation which is not finitary is the shift $s^{(n)}$ with $n \neq 0$. An example of a balanced permutation which is not finitary is the permutation which swaps each even integer $2i$ with $2i+1$.

We conclude the section with a brief discussion of topologies on infinite permutations. Let E denote the coordinate Hilbert space $\ell^2(\mathbb{Z})$ and let $U(E)$ be the group of unitary operators in E. This is a metrizable topological group with respect to the weak operator topology, which coincides (on unitary operators only!) with the strong operator topology. The assignment $\sigma \to A(\sigma)$ determines an embedding $\mathfrak{S} \to U(E)$, and we endow the group $\mathfrak{S}$ with the weak topology inherited from $U(E)$. This weak topology on $\mathfrak{S}$ is non-discrete and totally disconnected. (Note that taking the norm topology on $U(E)$ would produce on $\mathfrak{S}$ the discrete topology.)

Equivalently, the weak topology on $\mathfrak{S}$ can also be described as the topology inherited from the compact product space $\{0,1\}^{\mathbb{Z}\times\mathbb{Z}}$, where the embedding $\mathfrak{S}\to\{0,1\}^{\mathbb{Z}\times\mathbb{Z}}$ is determined by the entries of permutation matrices. Convergence $\sigma_n\to\sigma$ in the weak topology means that for every finite submatrix of $A(\sigma_n)$ the entries stabilize for sufficiently large n. An equivalent condition is that for any fixed $i\in\mathbb{Z}$ one has $\sigma_n(i)=\sigma(i)$ for all n large enough: this fact is a special instance of the coincidence of the weak and the strong operator topologies on $U(E)$.

Note that the group $\mathfrak{S}^{\mathrm{fin}}$ is dense in $\mathfrak{S}$.

The probability measures on $\mathfrak{S}$ or on its subgroups to be introduced in the sequel will be Borel with respect to the weak topology. Convergence of such measures will be understood in the weak sense. Specifically, with each measure μ and finite set $I\subset\mathbb{Z}$ we associate a discrete measure μ_I, which is the projection of μ on the finite space $\{0,1\}^{I\times I}\subset\{0,1\}^{\mathbb{Z}\times\mathbb{Z}}$ of 0–1 matrices over I. Convergence of a sequence measures $\mu^{(n)}$ to μ means convergence of the projections $\mu_I^{(n)}$ to μ_I, and the latter simply amounts to the natural convergence of measures on finite sets.

Finally, note that the subgroup $\mathfrak{S}^{\mathrm{adm}}\subset\mathfrak{S}$ can be endowed with a finer topology; it is defined by taking as a fundamental system of neighborhoods of the identity element the following system of subgroups in $\mathfrak{S}^{\mathrm{bal}}$ indexed by arbitrary finite integer intervals $[a,b]\subset\mathbb{Z}$, $a,b\in\mathbb{Z}$, $a\leqslant b$:

$$\mathfrak{S}^{[a,b]}:=\bigl\{\sigma\in\mathfrak{S}\colon\ \sigma(\mathbb{Z}_{<a})=\mathbb{Z}_{<a},\ \sigma(\mathbb{Z}_{>b})=\mathbb{Z}_{>b},\ \sigma(i)=i,\ i\in[a,b]\bigr\}.$$

For some reasons related to Remark 2.1 this finer topology on $\mathfrak{S}^{\mathrm{adm}}$ seems to be more natural than the weak topology, but for our purposes the weak topology is enough; we will exploit it in Section 7.

3. *q*-Exchangeability and the interlacing construction

The Mallows measures (1) and their extensions possess a fundamental quasi-invariance property. Recall that $\sigma_{i,i+1}$ denotes the elementary transposition swapping two adjacent indices $i,i+1\in\mathbb{Z}$.

Definition 3.1. Fix $q>0$ and let μ be a measure on the group $\mathfrak{S}$. Following [12] we say that μ is *right q-exchangeable* if for every $i\in\mathbb{Z}$, the pushforward $\mu_{i,i+1}$ of the measure μ under the transformation $\sigma\to\sigma\sigma_{i,i+1}$ is equivalent to μ and the value of the Radon–Nikodým derivative $d\mu_{i,i+1}/d\mu$ at an arbitrary point $\sigma\in\mathfrak{S}$ equals $q^{\operatorname{sgn}(\sigma(i+1)-\sigma(i))}$. Likewise, we call μ *left q-exchangeable* if the similar condition holds for transformations $\sigma\to\sigma_{i,i+1}\sigma$, with the Radon–Nikodým derivative $q^{\operatorname{sgn}(\sigma^{-1}(i+1)-\sigma^{-1}(i))}$.

Note that if μ is right q-exchangeable, then its pushforward μ' under the transformation $\sigma\to\sigma^{-1}$ is left q-exchangeable, and vice versa.

Definition 3.1 is obviously extended to the setting where $\mathfrak{S}$ is replaced by the group $\mathfrak{S}_I$ of permutations of a finite or semi-finite interval I of the ordered set $\mathbb{Z}$.

If I is finite, then $\mathfrak{S}_I$ is isomorphic to the symmetric group $\mathfrak{S}_n$ of degree $n=\#I$. Then it is readily seen that the notions of right and left versions of q-exchangeability coincide and mean that $\mu(\sigma)$ is proportional to $q^{\mathrm{inv}(\sigma)}$, as in (1). Therefore, if we additionally require that μ is a probability measure, such a measure is unique and coincides with (1), subject to the relabelling of the elements of I by increasing bijection with $\{1,\dots,n\}$.

In [12] we proved the following result:

Theorem 3.2. *Assume $0<q<1$. On the group $\mathfrak{S}_{\mathbb{Z}_+}$ the two notions of q-exchangeability coincide, and there exists a unique probability measure $\mathcal{Q}^+$, which is both right and left q-exchangeable.*

(Our notation for $\mathcal{Q}^+$ in [12] was $\mathcal{Q}$, but now we reserve $\mathcal{Q}$ for the two-sided extension.)

Obviously, the same result holds for permutations $\mathfrak{S}_I$ of any semi-infinite interval $I\subset\mathbb{Z}$ of the form $I=\mathbb{Z}_{\geqslant b}$ or $I=\mathbb{Z}_{\leqslant a}$. Following [12], we call $\mathcal{Q}^+$ the *Mallows measure on* $\mathfrak{S}_{\mathbb{Z}_+}$. The counterpart for $I=\mathbb{Z}_-$ will be called the *Mallows measure on* $\mathfrak{S}_{\mathbb{Z}_-}$ and denoted $\mathcal{Q}^-$.

Now we aim at proving an analog of Theorem 3.2 for permutations of the whole lattice $\mathbb{Z}$:

Theorem 3.3. *Assume $0 < q < 1$. On the group $\mathfrak{S}^{\mathrm{adm}} \subset \mathfrak{S}$ the two notions of q-exchangeability coincide. There exists a unique probability measure $\mathcal{Q}$ on $\mathfrak{S}^{\mathrm{bal}} \subset \mathfrak{S}$ which is both right and left q-exchangeable.*

We call $\mathcal{Q}$ the *Mallows measure on* $\mathfrak{S}^{\mathrm{bal}}$. Before proceeding to the proof we list some immediate corollaries and comments.

Corollary 3.4. *The measure $\mathcal{Q}$ is invariant under the inversion map $\sigma \to \sigma^{-1}$.*

Indeed, the inversion turns the left q-exchangeability into the right q-exchangeability and vice versa.

Corollary 3.5. *Let $\tau : \mathbb{Z} \to \mathbb{Z}$ stand for the reflection with respect to a given half-integer $n + \frac{1}{2}$. The measure $\mathcal{Q}$ is invariant under the conjugation $\sigma \to \tau\sigma\tau^{-1}$.*

Applying the shift transformations $\sigma \to s^{(b)}\sigma$ derives from $\mathcal{Q}$ a family $\{\mathcal{Q}^{(b)}\}_{b\in\mathbb{Z}}$ of q-exchangeable measures with pairwise disjoint supports, which leave on the larger subgroup $\mathfrak{S}^{\mathrm{adm}} \subset \mathfrak{S}$.

Corollary 3.6. *Each q-exchangeable probability measure on $\mathfrak{S}$ is a unique convex mixture of the measures $\mathcal{Q}^{(b)}$ over $b \in \mathbb{Z}$.*

Thus, on the group $\mathfrak{S}$ the uniqueness property is broken. The reason is that the symmetry group $\mathrm{Aut}(\mathbb{Z}, <)$ of the ordered set $(\mathbb{Z}, <)$ is the nontrivial group of shifts. However, the uniqueness is resurrected if we factorize the group $\mathfrak{S}$ modulo the subgroup $\mathrm{Aut}(\mathbb{Z}, <)$ (no matter, on the left or on the right). Finitary permutations still act on this quotient space from both sides, and there is again a unique q-exchangeable probability measure which is the push-forward of $\mathcal{Q}$.

The concept of q-exchangeability makes sense also for $q > 1$, but it is easily reduced to $\bar{q}$-exchangeability with $\bar{q} = 1/q \in (0, 1)$ by passing from σ to permutation $i \to -\sigma(i)$.

The plan of the proof of Theorem 3.3 is the following. We construct $\mathcal{Q}$ from the Mallows measures on $\mathbb{Z}_+$ and $\mathbb{Z}_-$, and a random interlacing pattern encoding how to fit the one-sided infinite extensions together. The group $\mathfrak{S}^{\mathrm{bal}}$ contains $\mathfrak{S}_{\mathbb{Z}_+} \times \mathfrak{S}_{\mathbb{Z}_-}$ as a subgroup, and there is a natural bijection between the quotient set $(\mathfrak{S}_{\mathbb{Z}_+} \times \mathfrak{S}_{\mathbb{Z}_-})\backslash\mathfrak{S}^{\mathrm{bal}}$ and the set $\mathbb{Y}$ of Young diagrams, which leads to a parametrization

$$\mathfrak{S}^{\mathrm{bal}} \ni \sigma \leftrightarrow \left(\sigma^+, \sigma^-, \lambda\right) \in \mathfrak{S}_{\mathbb{Z}_+} \times \mathfrak{S}_{\mathbb{Z}_-} \times \mathbb{Y}. \tag{3}$$

In these coordinates we define $\mathcal{Q}$ as a product measure

$$\mathcal{Q} := \mathcal{Q}^+ \otimes \mathcal{Q}^- \otimes \mathcal{P}, \tag{4}$$

where $\mathcal{P}$ is a probability measure on $\mathbb{Y}$ specified below in (8). A simple argument shows that $\mathcal{Q}$ is right q-exchangeable (Lemma 3.7). Next, we prove that $\mathcal{Q}$ is also left q-exchangeable, which is somewhat more complicated (Lemma 3.8). Note that these steps rely heavily on the q-exchangeability property of the measures $\mathcal{Q}^\pm$ established in [12]. Finally, we verify the uniqueness claim (Lemma 3.9 and Lemma 3.10), which is again an easy exercise.

Proof of Theorem 3.3. We proceed with the construction of $\mathcal{Q}$. Given a permutation $\sigma \in \mathfrak{S}^{\mathrm{bal}}$, we represent it by the two-sided infinite permutation word

$$w = (\dots w_{-1}w_0w_1w_2\dots) = \big(\dots\sigma(-1)\sigma(0)\sigma(1)\sigma(2)\dots\big),$$

from which we derive a binary word

$$\varepsilon = (\dots\varepsilon_{-1}\varepsilon_0\varepsilon_1\varepsilon_2\dots) \in \{-1,+1\}^{\mathbb{Z}},$$

where

$$\varepsilon_i = \begin{cases} +1, & \sigma(i) \in \mathbb{Z}_+, \\ -1, & \sigma(i) \in \mathbb{Z}_-. \end{cases}$$

The binary word ε encodes the way in which positive and negative entries of w interlace. Obviously, ε is a full invariant of the coset $(\mathfrak{S}_{\mathbb{Z}_+} \times \mathfrak{S}_{\mathbb{Z}_-})\sigma$. The set E of interlacing patterns that stem from balanced permutations is characterized by the condition

$$\#\{i \in \mathbb{Z}_-\colon \varepsilon_i = +1\} = \#\{i \in \mathbb{Z}_+\colon \varepsilon_i = -1\} < \infty. \tag{5}$$

A well-known fact, often used in combinatorics, is that binary words $\varepsilon \in E$ can be conveniently encoded into Young diagrams $\lambda = (\lambda_1, \lambda_2, \dots)$ in the following way. Split $\mathbb{Z}$ into the disjoint union of the sets of positions occupied by -1's and $+1$'s, respectively

$$\mathbb{Z} = \sigma^{-1}(\mathbb{Z}_-) \sqcup \sigma^{-1}(\mathbb{Z}_+) = \{\dots < j_{-2} < j_{-1} < j_0\} \sqcup \{i_1 < i_2 < i_3 < \cdots\}. \tag{6}$$

Then λ is defined by

$$\lambda_1 = 1 - i_1, \quad \lambda_2 = 2 - i_2, \quad \lambda_3 = 3 - i_3, \quad \dots. \tag{7}$$

Here we identify Young diagrams and partitions, as in [17]. Note also that

$$\lambda'_1 = j_0, \quad \lambda'_2 = j_{-1} + 1, \quad \lambda'_3 = j_{-2} + 2, \quad \dots$$

where $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ is the transposed diagram; this is seen from [17, Chapter I, Eq. (1.7)].

We define $\mathcal{Q}$ by (4), taking for $\mathcal{P}$ the probability distribution on $\mathbb{Y}$ associated with Euler's partition generating function:

$$\mathcal{P}(\lambda) = \mathrm{const}^{-1} q^{|\lambda|}, \quad \lambda \in \mathbb{Y}, \ \mathrm{const} = \sum_{\lambda \in \mathbb{Y}} q^{|\lambda|} = \prod_{k=1}^{\infty} \left(\frac{1}{1-q^k}\right), \tag{8}$$

where $|\lambda| = \lambda_1 + \lambda_2 + \cdots$ is the number of boxes in the diagram λ. Note that $|\lambda| = |\lambda'|$ is equal to the number of inversions in the binary word ε, which is

$$\#\bigl\{(i,j)\colon i < j, \ \varepsilon_i = +1, \ \varepsilon_j = -1\bigr\}.$$

Together with permutations $\sigma^{\pm} \in \mathfrak{S}_{\mathbb{Z}_{\pm}}$ we will also consider the corresponding permutation words $w^{\pm}$. Here

$$w^+ := \bigl(\sigma^+(1)\sigma^+(2)\sigma^+(3)\dots\bigr) = (w_{i_1} w_{i_2} w_{i_3} \dots) \tag{9}$$

is a permutation of $\mathbb{Z}_+$ while

$$w^- := \big(\dots\sigma^-(-2)\sigma^-(-1)\sigma^-(0)\big) = (\dots w_{j_{-2}} w_{j_{-1}} w_{j_0}) \tag{10}$$

is a permutation of $\mathbb{Z}_-$.

To recover w from the triple (w^+, w^-, ε), one has to replace each $+1$ entry of the word ε left-to-right with the successive entries of w^+, and replace -1's right-to-left with the entries of w^-.

Lemma 3.7. *$\mathcal{Q}$ is right q-exchangeable.*

Proof. Fix $i \in \mathbb{Z}$ and examine the behavior of $\mathcal{Q}$ under the right shift $\sigma \to \sigma\sigma_{i,i+1}$. Encode σ (or rather the corresponding word w) by the triple (w^+, w^-, ε). Depending on the ε-component, there are three possible cases: $\varepsilon_i = \varepsilon_{i+1} = +1$, $\varepsilon_i = \varepsilon_{i+1} = -1$, and $\varepsilon_i \neq \varepsilon_{i+1}$.

In the first case, both w^- and ε remain intact, and the transformation $\sigma \to \sigma\sigma_{i,i+1}$ reduces to swapping two adjacent letters in w^+, whose positions depend only on ε. Then the desired transformation property of $\mathcal{Q}$ follows from the fact that measure $\mathcal{Q}^+$ is right q-exchangeable.

This kind of the argument also works in the second case, with appeal to the similar property of $\mathcal{Q}^-$.

Finally, in the third case, the transformation affects only the binary word ε and amounts to swapping of two distinct adjacent letters. This changes the total number of inversions in ε by ± 1. Then the desired transformation property follows from the very definition of distribution $\mathcal{P}$. $\square$

Lemma 3.8. *$\mathcal{Q}$ is left q-exchangeable.*

Proof. Let W denote the set of all words w corresponding to permutations $\sigma \in \mathfrak{S}^{\mathrm{bal}}$. As above, we represent W as the direct product $W^+ \times W^- \times E$, where $W^\pm$ is the set of words $w^\pm$ corresponding to permutations $\sigma^\pm \in \mathfrak{S}_{\mathbb{Z}_\pm}$ and E is the set of binary words satisfying (5).

A qualitative difference between the right shift $\sigma \to \sigma\sigma_{i,i+1}$ and the left shift $\sigma \to \sigma_{i,i+1}\sigma$ is that the former acts on the set of positions while the latter acts on the set of entries of word w.

Fix $i \in \mathbb{Z}$ and examine three possible cases: $(i, i+1) \subset \mathbb{Z}_+$, $(i, i+1) \subset \mathbb{Z}_-$, and $(i, i+1) = (0, 1)$.

In the first case, swapping letters i and $i+1$ in w reduces to swapping the same letters in w^+, the components w^- and ε remaining intact. Then we may use the fact that measure $\mathcal{Q}^+$ is left q-exchangeable.

The same argument is applicable in the second case.

In the third case, the transformation is more delicate, as it affects all three components w^+, w^-, and ε. To describe it in detail we need to introduce some notation.

Denote by $\varphi : W \to W$ the transformation in question (swapping $0 \leftrightarrow 1$). We write $W = W^{10} \sqcup W^{01}$, where the subset $W^{10} \subset W$ consists of the words in which 1 is on the left of 0, and its complement $W^{01} \subset W$ comprises the words in which 0 is on the left of 1. Next, we consider finer splittings

$$W^{10} = \bigsqcup_{p,k,r,l \geqslant 0} W^{10}(p, k; r, l), \qquad W^{01} = \bigsqcup_{p,k,r,l \geqslant 0} W^{01}(p, k; r, l),$$

according to the following constraint on the ε-component of w.

• Subset $W^{10}(p, k; r, l)$: on the left of the position of letter 1, the number of $+1$'s in ε equals p; between the positions of letters 1 and 0, the number of $+1$'s and -1's is k and l, respectively; on the right of 0, the number of -1's equals r.

• Subset $W^{01}(p, k; r, l)$: the same conditions, with letters 0 and 1 interchanged.

Obviously, φ maps $W^{10}(p, k; r, l)$ onto $W^{01}(p, k; r, l)$ and vice versa.

We need an extra notation: For $a \geqslant 0$, $W^+_a \subset W^+$ consists of the words w^+ in which letter 1 occupies position $a+1$ counting from the left. Likewise, for $b \geqslant 0$, $W^-_b \subset W^-$ comprises the words w^- in which letter 0 occupies position $b+1$ counting from the right. Then

$$W^{10}(p, k; r, l) = W^+_p \times W^-_r \times E^{10}(p, k; r, l)$$

where $E^{10}(p,k;r,l)$ is some subset in E. Likewise,

$$W^{01}(p,k;r,l) = W^+_{p+k} \times W^-_{r+l} \times E^{01}(p,k;r,l)$$

where $E^{01}(p,k;r,l)$ is some subset in E.

Now, the key fact is that, in this notation, the bijection $\varphi : W^{10}(p,k;r,l) \to W^{01}(p,k;r,l)$ factors through a triple of transformations

$$\varphi^+ : W^+_p \to W^+_{p+k}, \qquad \varphi^- : W^-_r \to W^-_{r+l}, \qquad \widetilde{\varphi} : E^{10}(p,k;r,l) \to E^{01}(p,k;r,l),$$

where φ^+ moves letter 1 in w^+ to k positions on the right, φ^- moves letter 0 in w^- to l positions on the left, and $\widetilde{\varphi}$ is a transformation that does not depend on the $w^\pm$ components.

By the virtue of q-exchangeability of measures $\mathcal{Q}^\pm$, φ^+ produces the factor q^k on every $w^+ \in W^+_p$, and φ^- produces factor q^l on every $w^- \in W^-_l$. On the other hand, $\widetilde{\varphi}$ diminishes the total number of inversions in every binary word $\varepsilon \in E^{10}(p,k;r,l)$ by $k+l+1$ and so produces the factor q^{-k-l-1}. Therefore, the resulting effect of swapping $10 \to 01$ is the constant factor q^{-1}, as it should be.

Likewise, the bijection $\varphi : W^{01}(p,k;r,l) \to W^{10}(p,k;r,l)$ produces the desired factor q. This concludes the proof of the lemma. □

To address the issue of uniqueness we shall change viewpoint and interpret permutation as order. This is almost tautological for finite permutations, but requires some care for permutations of infinite sets, because by far not every order corresponds to a permutation.

Formally, by an *order* on set X we understand a strict total order, which is a binary relation $x \prec y$ on X satisfying three conditions: $x \prec y$ implies $x \neq y$ (the order is strict), $x \prec y$ and $y \prec z$ implies $x \prec z$ (transitivity), if $x \neq y$ then either $x \prec y$ or $y \prec x$ (completeness).

Let X be a finite or countable set, and let $\mathrm{Ord}(X)$ be the set of all orders on X. The group $\mathfrak{S}_X$ of all permutations of X acts on $X \times X$, hence acts on $\mathrm{Ord}(X)$, provided we identify the order with its graph. The identification also enables us to topologize $\mathrm{Ord}(X)$ by viewing it as a subset of $\{0,1\}^{X\times X}$. The group $\mathfrak{S}_X$ acts on $\mathrm{Ord}(X)$ by homeomorphisms.

Orders can be restricted from larger sets to smaller, hence for $Y \subset X$ there is a natural projection map $\mathrm{Ord}(X) \to \mathrm{Ord}(Y)$. For X countable the space $\mathrm{Ord}(X)$ can be identified with the projective limit $\varprojlim \mathrm{Ord}(Y)$, where Y ranges over finite subsets of X.

Let I be an interval in $\mathbb{Z}$, possibly coinciding with $\mathbb{Z}$ itself. Every permutation $\sigma \in \mathfrak{S}_I$ determines an order $\prec$ on I by setting

$$i \prec j \quad \text{iff} \quad \sigma^{-1}(i) < \sigma^{-1}(j). \tag{11}$$

Equivalently, writing σ as permutation word with positions labelled by I, $i \prec j$ means that letter i occurs in w before letter j. In this way, we get a natural map $\mathfrak{S}_I \to \mathrm{Ord}(I)$ that intertwines the left action of $\mathfrak{S}_I$ on itself with its canonical action on $\mathrm{Ord}(I)$. For $I \neq \mathbb{Z}$ this map is an embedding while for $I = \mathbb{Z}$ it is not. The reason is that the order set $(\mathbb{Z}, <)$ has a nontrivial group of symmetries formed by shifts, which implies that two permutations $\sigma, \tau \in \mathfrak{S}_\mathbb{Z} = \mathfrak{S}$ induce the same order if and only if $\sigma = s^{(b)}\tau$ with some $b \in \mathbb{Z}$. In the case $I \neq \mathbb{Z}$ this effect disappears, for then the ordered set $(I, <)$ has only trivial automorphisms.

If I is finite, the map $\mathfrak{S}_I \to \mathrm{Ord}(I)$ is a bijection. But for $I = \mathbb{Z}$ or a semi-infinite interval $I = Z_{\geqslant a}, \mathbb{Z}_{\leqslant a}$ the orders coming from permutations constitute a very small subset in $\mathrm{Ord}(I)$.

For arbitrary interval $I \subseteq \mathbb{Z}$ and any $q > 0$, the notion of a q-exchangeable measure on $\mathrm{Ord}(I)$ is introduced in complete analogy with Definition 3.1, only now we do not need to distinguish between right and left actions of transpositions, since the action is unique.

Lemma 3.9. *For every interval $I \subseteq \mathbb{Z}$ and $q > 0$, there exists a unique q-exchangeable probability measure $\mu_{I,q}$ on $\mathrm{Ord}(I)$.*

Proof. For finite I this is a property of the Mallows measure (1), since $(I,<)$ is isomorphic to $(\{1,\dots,n\},<)$, for $n=\#I$.

Assume now $I=\mathbb{Z}$. The space $\mathrm{Ord}(\mathbb{Z})$ coincides with the projective limit space $\varprojlim \mathrm{Ord}(I)$, where I ranges over the set of finite intervals. For two finite intervals $I\subset J$, the restriction map $\mathrm{Ord}(J)\to\mathrm{Ord}(I)$ is consistent with q-exchangeability and so maps $\mu_{J,q}$ to $\mu_{I,q}$. Appealing to Kolmogorov's extension theorem shows that $\mu_{\mathbb{Z},q}$ exists and is unique.

For semi-infinite interval $I\subset\mathbb{Z}$ the argument is exactly the same. □

Lemma 3.10. *There exists at most one left q-exchangeable probability measure on $\mathfrak{S}^{\mathrm{bal}}$.*

Proof. Restricting the map $\mathfrak{S}\to\mathrm{Ord}(\mathbb{Z})$ to $\mathfrak{S}^{\mathrm{bal}}$ we get an embedding $\mathfrak{S}^{\mathrm{bal}}\to\mathrm{Ord}(\mathbb{Z})$. The latter map is continuous and hence Borel, and the image of $\mathfrak{S}^{\mathrm{bal}}$ is a Borel subset in $\mathrm{Ord}(\mathbb{Z})$ (the latter claim follows, e.g., from the fact that both $\mathfrak{S}^{\mathrm{bal}}$ and $\mathrm{Ord}(\mathbb{Z})$ are standard Borel spaces). Therefore, two distinct left q-exchangeable probability measures on $\mathfrak{S}^{\mathrm{bal}}$ would give rise to two distinct q-exchangeable probability measures on $\mathrm{Ord}(\mathbb{Z})$, which is impossible by the virtue of Lemma 3.9. □

The proof of Theorem 3.3 is thus completed. □

Remark 3.11. We have seen that the Mallows measure $\mathcal{Q}$ on $\mathfrak{S}^{\mathrm{bal}}$ is obtainable from the q-exchangeable measure $\mu_{\mathbb{Z},q}$ on the space $\mathrm{Ord}(\mathbb{Z})$. However, the latter measure exists for every $q>1$ while $\mathcal{Q}$ is defined only for $0<q<1$. The obvious explanation is that for $q\geqslant 1$ the measure $\mu_{\mathbb{Z},q}$ is no longer supported by $\mathfrak{S}^{\mathrm{bal}}\subset\mathrm{Ord}(\mathbb{Z})$.

For $q>1$, $\mu_{\mathbb{Z},q}$ is still supported by permutations. Namely, by non-balanced permutations of the type $\tau\sigma$, where σ ranges over $\mathfrak{S}^{\mathrm{bal}}$ and $\tau\in\mathfrak{S}$ is the reflection map $\tau(i)=-i$.

For $q=1$, the measure $\mu_{\mathbb{Z},q}=\mu_{\mathbb{Z},1}$ is the only exchangeable probability measure on $\mathrm{Ord}(\mathbb{Z})$. In this case, the order structure on $\mathbb{Z}$ plays no role, we simply regard $\mathbb{Z}$ as a countable set. Following a familiar recipe [8,13], the random exchangeable order can be defined by declaring $i\prec j$ iff $\xi_i<\xi_j$, where ξ_i's are independent, uniform $[0,1]$ random variables. It follows that the type of the exchangeable order is almost surely $(\mathbb{Q},<)$. Remarkably, for $q\neq 1$ the situation is different, in that the q-exchangeable order is of the type $(\mathbb{Z},<)$.

4. A construction from independent geometric variables

Fix $q\in(0,1)$ and let Σ denote the random q-exchangeable balanced permutation of $\mathbb{Z}$, i.e. the random element of $\mathfrak{S}^{\mathrm{bal}}$ distributed according to Mallows' measure $\mathcal{Q}$ with parameter q. Likewise, let Σ_+ and Σ_n denote the random q-exchangeable permutations of $\mathbb{Z}_+$ and $\{1,\dots,n\}$, respectively. There is a simple algorithm to construct Σ_n from independent truncated geometric variables. In [12], we described a "one-sided infinite" extension of this algorithm, called *q-shuffle*, to generate Σ_+. As an application of the q-shuffle, we obtained an isomorphism of the measure space $(\mathfrak{S}_{\mathbb{Z}_+},\mathcal{Q}^+)$ with the space $\{0,1,2,\dots\}^{\mathbb{Z}_+}$ equipped with a product of geometric distributions. The aim of this section is to derive a "two-sided infinite" analog of these results: we shall introduce an algorithm of generating Σ from infinitely many copies of a geometric random variable. In comparison with the interlacing construction, one advantage of the new approach is that it makes transparent some stationarity property of $\mathcal{Q}$.

For $\sigma\in\mathfrak{S}$, a pair of positions $(i,j)\in\mathbb{Z}\times\mathbb{Z}$ is an *inversion* in σ if $i<j$ and $\sigma(i)>\sigma(j)$. If (i,j) is an inversion, we say that it is a *left inversion* for j, and a *right inversion* for i. Introduce the counts of left and right inversions,

$$\ell_i:=\#\{j:\ j<i,\ \sigma(j)>\sigma(i)\},\qquad r_i:=\#\{j:\ j>i,\ \sigma(j)<\sigma(i)\},\quad i\in\mathbb{Z},$$

respectively (of course, for the general $\sigma\in\mathfrak{S}$ these quantities may be infinite). The following easy proposition relates these notions to the admissibility condition.

Proposition 4.1. *If σ is admissible, then $\ell_i, r_i < \infty$ for all $i \in \mathbb{Z}$. Conversely, if this condition holds for some index $i \in \mathbb{Z}$, then it holds for all i and σ is admissible.*

Proof. Consider the permutation matrix $A = A(\sigma)$. We assume that its rows are enumerated from top to bottom, and the columns from left to right. Then ℓ_i equals the number of 1's in the lower left corner of A formed by positions (a, b) satisfying inequalities $a > \sigma(i)$, $b < i$. Likewise, r_i is the number of 1's in the upper right corner determined by the opposite inequalities $a < \sigma(i)$, $b > i$. For any two lower left corners, the corresponding numbers of 1's always differ by a finite quantity, and the same holds for upper right corners. This makes the claim of the proposition evident. □

Obviously,

$$\operatorname{inv}(\sigma) = \sum_{i\in\mathbb{Z}} \ell_i = \sum_{i\in\mathbb{Z}} r_i \tag{12}$$

is the total number of inversions in σ; for balanced permutations this quantity is finite if and only if $\sigma \in \mathfrak{S}^{\mathrm{fin}}$.

The left and right inversion counts are defined similarly for finite permutations. A well-known fact is that a permutation $\sigma \in \mathfrak{S}_n$ can be uniquely encoded in the sequence of its right inversion counts

$$(r_1, \dots, r_n) \in \{0, \dots, n-1\} \times \dots \times \{0, 1\} \times \{0\}.$$

For instance, the permutation word $(3, 1, 2, 4)$ (i.e. permutation in the one-row notation) corresponds to the sequence $(2, 0, 0, 0)$. Similarly, $\sigma \in \mathfrak{S}_n$ can be uniquely encoded in the sequence of its left inversion counts

$$(\ell_1, \dots, \ell_n) \in \{0\} \times \{0, 1\} \times \dots \times \{0, \dots, n-1\}.$$

The correspondence inverse to the latter amounts to the following well-known algorithm:

Elimination algorithm.

(1) Given a sequence $(r_1, \dots, r_n)$, construct a permutation word $w = w_1 \dots w_n$ recursively, as follows. At step 1 set $w_1 = r_1 + 1$, and eliminate integer $r_1 + 1$ from the list $1, \dots, n$. At step 2 let w_2 be equal to the $(r_2 + 1)$th smallest entry of the reduced list, and eliminate the entry from the list, etc. For instance, from $(2, 0, 0, 0)$ we derive $w_1 = 2 + 1 = 3$, and remove 3 from the initial list $1, 2, 3, 4$ to obtain $1, 2, 4$. At step 2 we find $w_2 = 1$ (which is the $(0 + 1)$st smallest element $1, 2, 4$) and further reduce the list to $2, 4$, etc.

(2) Likewise, one can construct w from $(\ell_1, \dots, \ell_n)$. In this case, we first determine w_n from ℓ_n, then find w_{n-1}, etc.

For $1 \leqslant i \leqslant n$ let $L_{n,i}$ and $R_{n,i}$ be the random counts of left and right inversions in the Mallows permutation Σ_n. It is immediate from the above bijections and (1) that:

(i) $R_{n,i}$ are independent for $i = 1, \dots, n$, and have the truncated geometric distributions

$$\mathbb{P}(R_{n,i} = k) = \frac{q^k}{[n-i]_q}, \quad k = 0, \dots, n - i.$$

(ii) $L_{n,i}$ are independent for $i = 1, \dots, n$, and have the truncated geometric distributions

$$\mathbb{P}(L_{n,i} = k) = \frac{q^k}{[i]_q}, \quad k = 0, \dots, i-1,$$

where $[m]_q := \sum_{k=0}^{m-1} q^k$ denotes the q-number.

Sending $n \to \infty$ in (i) leads to the q-shuffle algorithm [12] mentioned above:

The *q*-shuffle algorithm. Let $R_1, R_2, \dots$ be independent, identically distributed random variables with the geometric distribution on $\{0, 1, \dots\}$ with parameter q,

$$\mathbb{P}(R_i = k) = (1-q)q^k, \quad k = 0, 1, 2, \dots, \ i = 1, 2, \dots.$$

Set $w_1 = R_1 + 1$. Inductively, choose for w_i the $(R_i + 1)$th smallest element of $\mathbb{Z}_+ \setminus \{w_1, \dots, w_{i-1}\}$. Eventually every element of $\mathbb{Z}_+$ will be chosen at some step, because $R_i = 0$ for infinitely many i almost surely. Therefore the procedure yields almost surely a word $w = w_1 w_2 \dots$ corresponding to a permutation $\sigma \in \mathfrak{S}_{\mathbb{Z}_+}$. Note that its right inversion counts are just $R_1, R_2, \dots$.

Remark 4.2. One can prove that the left inversion counts of Σ_+ are independent random variables with truncated geometric distributions, as in (ii). However, generation of Σ_+ through the left inversion counts is a more difficult task, for a direct extension of the second version of the elimination algorithm no longer works: the right-most letter does not exist. A similar difficulty arises with two-sided infinite words, since for them there is no qualitative difference between right and left inversion counts. To resolve this difficulty we will exploit below a more sophisticated procedure.

Consider the product space $X := \{0, 1, 2, \dots\}^{\mathbb{Z}}$ and equip it with the product measure $\mathcal{G} := \bigotimes_{i \in \mathbb{Z}} G$, where G stands for the geometric distribution on $\{0, 1, 2, \dots\}$ with a fixed parameter $q \in (0, 1)$, that is,

$$G(n) = (1-q)q^n, \quad n = 0, 1, 2, \dots. \tag{13}$$

Let $\psi : \mathfrak{S}^{\mathrm{bal}} \to X$ be the map defined by the right inversion counts: $\psi(\sigma) = (r_i\colon\ i \in \mathbb{Z})$.

Theorem 4.3. *The map ψ provides an isomorphism of the measure space $(\mathfrak{S}^{\mathrm{bal}}, \mathcal{Q})$ onto the product measure space $(X, \mathcal{G})$.*

Comment. Before proceeding to the proof, note that under the correspondence $\psi : \sigma \to (r_i)$, the alternative "either $w_i < w_{i+1}$ or $w_i > w_{i+1}$" for the word $w \leftrightarrow \sigma$ translates as "either $r_i \leqslant r_{i+1}$ or $r_i > r_{i+1}$", and the transformation $T_{i,i+1} : \sigma \to \sigma\sigma_{i,i+1}$ turns into the transformation $T'_{i,i+1} : X \to X$ that does not affect coordinates r_j with $j \neq i, i+1$ and reduces to

$$(r_i, r_{i+1}) \to \begin{cases} (r_{i+1} + 1, r_i), & r_i \leqslant r_{i+1}, \\ (r_{i+1}, r_i - 1), & r_i > r_{i+1}. \end{cases}$$

Thus, $\mathcal{G}$ is transformed under the action of $T'_{i,i+1}$ in the same way as $\mathcal{Q}$ is transformed under $T_{i,i+1}$, i.e. the Radon–Nikodým derivative equals $q^{\pm 1}$, depending on whether $r_i \leqslant r_{i+1}$ or $r_i > r_{i+1}$. This observation agrees with the claim of the theorem but is not yet enough for the proof.

Proof of Theorem 4.3. Regard (r_i) as a two-sided infinite sequence of random variables defined on the probability space $(\mathfrak{S}^{\mathrm{bal}}, \mathcal{Q})$.

Lemma 4.4. *The random sequence (r_i) is stationary with respect to shifts $i \to i+n$.*

Proof. Use for a moment a more detailed notation $r_i(\sigma)$. We have

$$r_{i+n}(\sigma) = r_i\big(s^{(n)}\sigma s^{(-n)}\big), \quad n \in \mathbb{Z}.$$

On the other hand, the map $\sigma \to s^{(n)}\sigma s^{(-n)}$ leaves invariant the subgroup $\mathfrak{S}^{\mathrm{bal}} \subset \mathfrak{S}^{\mathrm{adm}}$ and preserves the q-exchangeability property. Due to uniqueness of measure $\mathcal{Q}$, it is invariant under this map, whence the assertion of the lemma. $\square$

Lemma 4.5. *For every finite sequence of integers $i_1 < \cdots < i_k$, the joint law of $(r_{i_1}, \ldots, r_{i_k})$ is the product measure $G \otimes \cdots \otimes G$.*

Proof. By virtue of Lemma 4.4, it suffices to prove that, as $n \to +\infty$, the limit law for $(r_{i_1+n}, \ldots, r_{i_k+n})$ exists and coincides with $G \otimes \cdots \otimes G$.

Use the encoding $\sigma \leftrightarrow w \leftrightarrow (w^+, w^-, \varepsilon)$. Given ε, the conditional distribution of $(r_{i_1+n}, \ldots, r_{i_k+n})$ stabilizes as $n \to +\infty$ and coincides with $G \otimes \cdots \otimes G$. Indeed, this immediately follows from the definition of the encoding and the q-shuffle algorithm generating w^+. This implies the desired claim. $\square$

By virtue of Lemma 4.5, it suffices to prove that the map ψ is injective. We do this in the next two lemmas: Lemma 4.6 says how to reconstruct a balanced permutation from the two sequences (r_i) and (ℓ_i), and Lemma 4.7 describes an algorithm which expresses (ℓ_i) through (r_i).

Lemma 4.6. *For permutation $\sigma \in \mathfrak{S}^{\mathrm{bal}}$ we have*

$$\sigma(i) = i + r_i - \ell_i, \quad i \in \mathbb{Z}. \tag{14}$$

Proof. It is convenient to prove a slightly more general claim: for each $\sigma \in \mathfrak{S}^{\mathrm{adm}}$

$$r_i - \ell_i + i - \sigma(i) = b(\sigma), \quad i \in \mathbb{Z} \tag{15}$$

(recall that $b(\sigma)$ denotes the balance of σ). Since $\mathrm{bal}(\sigma) = 0$ for $\sigma \in \mathfrak{S}^{\mathrm{bal}}$, (15) will imply (14). Indeed, by (2), for each $n \in \mathbb{Z}$ we have

$$b\big(s^{(n)}\sigma\big) = b(\sigma) + n, \quad \sigma \in \mathfrak{S}^{\mathrm{adm}}.$$

On the other hand, if we replace σ by $s^{(n)}\sigma$, then all counts r_i, ℓ_i will not change, while $i - \sigma(i)$ will transform to $i - \sigma(i) + n$. Thus, (15) is consistent with the replacement $\sigma \to s^{(n)}\sigma$. Now fix an arbitrary i, take $n = \sigma(i) - i$, and replace σ by $\sigma' := s^{(n)}\sigma$. Then $\sigma'(i) = i$, which in turn implies that $b(\sigma') = r_i - \ell_i$, by the very definition of balance and the left and right inversion counts. Consequently, (15) holds true for σ'. $\square$

Fix $\sigma \in \mathfrak{S}^{\mathrm{bal}}$ and let, as usual, w stand for the corresponding two-sided infinite permutation word. Observe that $r_j + 1$ is the rank of w_j in the left-truncated word $w_j w_{j+1} \ldots$, with respect to the canonical order on $\mathbb{Z}$. (That is, the rank equals k if w_j is the kth minimal element among w_j, w_{j+1}, $\ldots$.) More generally, for $i \leqslant j$, define $r_j^{(i)}$ to be one smaller the rank of w_j among $w_i \ldots w_j \ldots$. Evidently, $r_j^{(j)} = r_j$.

Lemma 4.7.

(i) *Fix $j \in \mathbb{Z}$. For $i \leqslant j$, the quantity $r_j^{(i)}$ can be determined from the finite sequence $r_j, r_{j-1}, \dots, r_i$ by means of the recursion*

$$r_j^{(j)} = r_j, \qquad r_j^{(i-1)} = r_j^{(i)} + \mathbf{1}\bigl(r_j^{(i)} \geqslant r_{i-1}\bigr). \tag{16}$$

(ii) *The left inversion counts can be determined from the quantities $(r_j^{(i)}\colon i \leqslant j)$ by*

$$\ell_j = \sum_{i\colon i<j} \mathbf{1}\bigl(r_j^{(i)} < r_i\bigr) = \sum_{i\colon i<j} \mathbf{1}\bigl(r_j^{(i)} = r_j^{(i+1)}\bigr). \tag{17}$$

Proof. (i) We have

$$r_j^{(i-1)} = r_j^{(i)} + \mathbf{1}(w_{i-1} < w_j).$$

But $w_{i-1} < w_j$ is equivalent to $r_{i-1}^{(i-1)} \leqslant r_j^{(i)}$, and $r_{i-1}^{(i-1)} = r_{i-1}$.

(ii) We have

$$\ell_j = \sum_{i\colon i<j} \mathbf{1}(w_i > w_j),$$

but $w_i > w_j$ is equivalent to $r_i^{(i)} > r_j^{(i)}$ (i.e., $r_i > r_j^{(i)}$) and also to $r_j^{(i)} = r_j^{(i+1)}$. □

This concludes the proof of Theorem 4.3. □

Corollary 4.8. *The Mallows measures on $\mathfrak{S}^{\mathrm{bal}}$ corresponding to any two distinct values of parameter q are disjoint* (*mutually singular*).

Proof. Immediate by virtue of classical Kakutani's theorem [14] about product measures. □

The latter can also be seen from the law of large numbers: $\ell_1 + \dots + \ell_n$ under $\mathcal{Q}$ is asymptotic to $nq/(1+q)$.

5. The distribution of displacements

Let as above $\mathcal{Q}$ be the Mallows measure on $\mathfrak{S}^{\mathrm{bal}}$ with parameter $q \in (0,1)$ and let Σ be the random permutation of $\mathbb{Z}$ with law $\mathcal{Q}$. Consider the two-sided infinite random sequence of *displacements*

$$D_i := \Sigma(i) - i, \quad i \in \mathbb{Z}. \tag{18}$$

The sequence (D_i) is stationary in "time" $i \in \mathbb{Z}$, because a shift of parameter i amounts to a measure preserving transformation of the basic probability space $(\mathfrak{S}^{\mathrm{bal}}, \mathcal{Q})$ – conjugation of a balanced permutation by a shift $s^{(n)}$, cf. Lemma 4.4.

Below we use a nonstandard notation for some particular q-Pochhammer symbols:

$$\langle n\rangle_q := (q;q)_n = \prod_{k=1}^{n}\bigl(1-q^k\bigr), \qquad \langle\infty\rangle_q := (q;q)_\infty = \prod_{k=1}^{\infty}\bigl(1-q^k\bigr).$$

In the present section, we will compute the one-dimensional marginal distribution of (D_i). In the next section, we will describe the finite-dimensional distributions.

Theorem 5.1. *For any fixed $j \in \mathbb{Z}$, the distribution of displacement $D := D_j$ is given by*

$$\mathbb{P}(D = d) = (1-q)\langle\infty\rangle_q \sum_{\{r,\ell \geqslant 0:\ r-\ell=d\}} \frac{q^{r\ell+r+\ell}}{\langle r\rangle_q \langle \ell\rangle_q}, \quad d \in \mathbb{Z}. \tag{19}$$

This distribution is symmetric about 0. It can be expressed through the basic geometric series ${}_0\phi_1$ (see [7]),

$$\mathbb{P}(D = d) = \frac{(1-q)(q;q)_\infty q^d}{(q;q)_d}\, {}_0\phi_1\big(-; q^{d+1}; q, q^{d+3}\big), \quad d = 0, 1, 2, \ldots,$$

or through a q-Bessel function, see [7, Exercise 1.24].

We will give two proofs of the theorem.

First proof. Let (R_i) and (L_i) be the sequences of right and left inversion counts for Σ. By Lemma 4.6, $D_j = R_j - L_j$. To compute the distribution of $R_j - L_j$ we apply Lemma 4.7. Set

$$x_0 = r_j^{(j)}, \quad x_1 = r_j^{(j-1)}, \quad x_2 = r_j^{(j-2)}, \quad \ldots,$$

where we use the notation of Lemma 4.7. Since $R_j, R_{j-1}, R_{j-2}, \ldots$ are independent random variables with geometric distribution G, claim (i) of Lemma 4.7 implies that, given $R_j = r$, the sequence $x_0, x_1, \ldots$ forms a Markov chain on $\{r, r+1, r+2, \ldots\}$ with initial state $x_0 = r$, 0–1 increments and transition probabilities

$$\mathbb{P}(k \to k) = q^{k+1}, \qquad \mathbb{P}(k \to k+1) = 1 - q^{k+1}, \quad k = r, r+1, r+2, \ldots. \tag{20}$$

Next, by claim (ii) of the lemma, L_j equals the total number of 0-increments (this number is finite almost surely by Lemma 7.4). It follows that the (conditional) probability generating function for L_j has the form

$$\sum_{\ell=0}^{\infty} \mathbb{P}(L_j = \ell \mid R_j = r) x^l = \prod_{k=r}^{\infty} \frac{1-q^{k+1}}{1-xq^{k+1}}. \tag{21}$$

The coefficient by x^ℓ is extracted from (21) using the Eulerian identity (see e.g. [7, (1.3.15)])

$$\prod_{m=0}^{\infty} \big(1 - yq^m\big)^{-1} = \sum_{n=0}^{\infty} \frac{y^n}{\langle n\rangle_q},$$

where we substitute $y = xq^{r+1}$. This gives

$$\mathbb{P}(L_j = \ell \mid R_j = r) = q^{\ell(r+1)} \frac{\langle\infty\rangle_q}{\langle r\rangle_q \langle \ell\rangle_q}. \tag{22}$$

Since the distribution of R_j is geometric we find the joint distribution of (R_j, L_j):

$$\mathbb{P}(R_j = r, L_j = \ell) = (1-q)q^{r\ell+r+\ell}\frac{\langle\infty\rangle_q}{\langle r\rangle_q\langle\ell\rangle_q}, \quad r, \ell \in \mathbb{Z}. \tag{23}$$

Finally, the distribution of $D = D_j = R_j - L_j$ follows by summation. $\square$

Second proof. By stationarity, it suffices to find the distribution of $\Sigma(1) - 1$. Next, it will be convenient for us to replace Σ with the inverse permutation Σ^{-1}, which has the same law, see Corollary 3.4. Thus, we will deal with $\Sigma^{-1}(1) - 1$.

Let $w = (w_i)_{i\in\mathbb{Z}}$ be the random word corresponding to Σ. We encode w by the triple (w^+, w^-, λ), see the definitions in (6), (7), (9), and (10) above. Denote by $B + 1 = 1, 2, \dots$ the position of 1 in w^+. Then, in the notation of (6), the position of 1 in w is i_{B+1}, which is the same as $B + 1 - \lambda_{B+1}$, as is seen from (7). On the other hand, this position is just $\Sigma^{-1}(1)$. Therefore, we may write

$$\Sigma^{-1}(1) = B + 1 - C, \quad C := \lambda_{B+1}.$$

To find the distribution of

$$\Sigma^{-1}(1) - 1 = B - C$$

we will compute the probabilities

$$\mathbb{P}(B = b, C = c), \quad b, c = 0, 1, 2, \dots.$$

Recall that w^+ and λ are independent. Therefore,

$$\mathbb{P}(B = b, C = c) = \mathbb{P}\big(1 \text{ occupies in } w^+ \text{ position } b+1\big)\cdot\mathbb{P}(\lambda_{b+1} = c).$$

The first factor in the right-hand side is determined from the q-shuffle algorithm generating w^+. This gives

$$\mathbb{P}\big(1 \text{ occupies in } w^+ \text{ position } b+1\big) = q^b(1-q).$$

In the second factor $\mathbb{P}(\lambda_{b+1} = c)$, the probability is understood according to the distribution (8). Therefore,

$$\mathbb{P}(\lambda_{b+1} = c) = \langle\infty\rangle_q \sum_{\lambda} q^{|\lambda|}$$

summed over diagrams λ of shape specified in Fig. 1. The sum in question is computed using the generating function for Young diagrams in a strip:

$$\sum_{\nu=(\nu_1\geqslant\cdots\geqslant\nu_a\geqslant 0)} q^{|\nu|} = \frac{1}{\langle a\rangle_q};$$

we apply this formula for $\nu = \lambda^+$ and $\nu = (\lambda^-)'$ (the transposed diagram). This gives

$$\mathbb{P}(\lambda_{b+1} = c) = \langle\infty\rangle_q \frac{q^{(b+1)c}}{\langle b\rangle_q\langle c\rangle_q}.$$

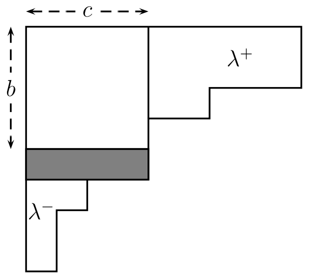

Fig. 1. One-row decomposition of the diagram.

Thus,

$$\mathbb{P}(B=b, C=c) = (1-q)\langle\infty\rangle_q \frac{q^{bc+b+c}}{\langle b\rangle_q \langle c\rangle_q}$$

and finally

$$\mathbb{P}\big(\Sigma^{-1}(1)-1=d\big) = \mathbb{P}(B-C=d) = (1-q)\langle\infty\rangle_q \sum_{b,c\geqslant 0:\ b-c=d} \frac{q^{bc+b+c}}{\langle b\rangle_q \langle c\rangle_q},$$

which is the same as (19). □

Remark 5.2. The distribution (19) has exponentially decaying tails

$$\mathbb{P}\big(|D|>m\big) \asymp q^m, \quad m\to\infty.$$

Indeed, the lower bound follows from (19) while the upper bound is easy from

$$\mathbb{P}\big(|D_j|>m\big) \leqslant \max\big(\mathbb{P}(R_j>m), \mathbb{P}(L_j>m)\big) \leqslant 2\mathbb{P}(R_j>m) = 2q^m.$$

This allows us to estimate the size of fluctuation of Σ about the identity permutation. Using

$$\mathbb{P}\big(|D_n| > (1+\epsilon)\log_{1/q}|n|\big) < 2|n|^{-1-\epsilon}$$

and applying the Borel–Cantelli lemma we obtain

$$\limsup_{|n|\to\infty} \frac{|D_n|}{\log_{1/q}|n|} \leqslant 1 \quad \text{a.s.} \tag{24}$$

In the first version of the paper, we conjectured that bound (24) is sharp. The conjecture has been settled by an anonymous referee.

6. Finite-dimensional distributions

We proceed with deriving a multivariate distribution for the displacements (18).

Theorem 6.1. *For $k = 1, 2, \dots$ and integers $d_1 \leqslant \dots \leqslant d_k$*

$$\begin{aligned}&\mathbb{P}(D_1 = d_1, \dots, D_k = d_k)\\ &\quad = (1-q)^k q^{-k(k+1)/2} \langle\infty\rangle_q \prod_{m=2}^{k} \langle d_m - d_{m-1}\rangle_q \sum \frac{q^{\sum_{1\leqslant i\leqslant j\leqslant k}(b_i+1)(a_j+1)}}{\langle b_1\rangle_q \dots \langle b_k\rangle_q \langle a_1\rangle_q \dots \langle a_k\rangle_q},\end{aligned}$$

where the summation is over all nonnegative integers $a_1, b_1, \dots, a_k, b_k$ which satisfy the constraints

$$(b_1 + \dots + b_m) - (a_m + \dots + a_k) = d_m, \quad m = 1, \dots, k. \tag{25}$$

Comments.

(i) In the case $k = 1$ the product over m is empty and the result agrees with Theorem 5.1.

(ii) The constraints $d_1 \leqslant \dots \leqslant d_k$ are not substantial and only introduced to simplify the formula. These inequalities are equivalent to $\Sigma(1) < \dots < \Sigma(k)$, so that the general case can be handled by introducing the additional factor

$$q^{\mathrm{inv}(d_1+1,\dots,d_k+k)} = q^{\mathrm{inv}(\Sigma(1),\dots,\Sigma(k))}$$

implied by the q-exchangeability; here "inv" stands for the number of inversions.

(iii) By stationarity, the distribution does not change if we simultaneously shift all indices $1, \dots, k$ by a constant. However, we did not manage to write a reasonable formula for $(D_{i_1}, \dots, D_{i_k})$ with arbitrary indices $i_1 < \dots < i_k$.

(iv) Excluding variables $b_1, \dots, b_k$ from relations (25), the resulting inequalities on the remaining k variables $a_1, \dots, a_k$ take the form

$$\begin{gathered} 0 \leqslant a_1 \leqslant d_2 - d_1,\\ 0 \leqslant a_2 \leqslant d_3 - d_2,\\ \vdots\\ 0 \leqslant a_{k-1} \leqslant d_k - d_1,\\ a_k \geqslant \max(0, -a_1 - \dots - a_{k-1} - d_1), \end{gathered}$$

which shows that the summation runs over a domain with only one of free variables assuming infinitely many values.

Proof of Theorem 6.1. We generalize the second proof of Theorem 5.1. Let us compute the probability

$$\mathbb{P}\big(\Sigma^{-1}(1) - 1 = d_1, \dots, \Sigma^{-1}(k) - k = d_k\big). \tag{26}$$

Consider again the encoding (w^+, w^-, λ) of the random word w associated with Σ. Let $x_1, \dots, x_k$ be the positions of letters $1, \dots, k$ in w^+. Then the positions of the same letters in w are

$$\Sigma^{-1}(1) = x_1 - y_1, \quad \dots, \quad \Sigma^{-1}(k) = x_k - y_k,$$

where we set

$$y_1 = \lambda_{x_1}, \quad \ldots, \quad y_k = \lambda_{x_k}.$$

The assumption $d_1 \leqslant \cdots \leqslant d_k$ means $1 \leqslant x_1 < \cdots < x_k$, which entails

$$y_1 \geqslant \cdots \geqslant y_k \geqslant 0.$$

Now pass to new variables $b_1, \ldots, b_k, a_1, \ldots, a_k$ by setting

$$\begin{aligned} x_1 &= b_1 + 1, & y_1 &= a_1 + \cdots + a_k, \\ x_2 &= b_1 + b_2 + 2, & y_2 &= a_2 + \cdots + a_k, \\ &\vdots & &\vdots \\ x_k &= b_1 + \cdots + b_k + k, & y_k &= a_k. \end{aligned} \tag{27}$$

Then the above inequalities imposed on $x_1, \ldots, x_k, y_1, \ldots, y_k$ just mean that the new variables are nonnegative, and conditions

$$\Sigma^{-1}(m) - i = d_m, \quad m = 1, \ldots, k$$

take the form (25).

Introduce the set of Young diagrams

$$\Lambda(x_1, \ldots, x_k; y_1, \ldots, y_k) = \{\lambda\colon \lambda_{x_1} = y_1, \ldots, \lambda_{x_k} = y_k\}. \tag{28}$$

The probability (26) can be written in the form

$$\sum_{b_1, \ldots, b_k, a_1, \ldots, a_k} P_1(x_1, \ldots, x_k) P_2(x_1, \ldots, x_k; y_1, \ldots, y_k), \tag{29}$$

with summation over nonnegative integers $b_1, \ldots, b_k, a_1, \ldots, a_k$ subject to constraints (25), where $P_1(x_1, \ldots, x_k)$ is the probability that letters $1, \ldots, k$ occupy positions $x_1, \ldots, x_k$ in w^+, and

$$P_2(x_1, \ldots, x_k; y_1, \ldots, y_k) = \langle\infty\rangle_q \sum_{\lambda \in \Lambda(x_1, \ldots, x_k; y_1, \ldots, y_k)} q^{|\lambda|}$$

is the probability that the random diagram λ with law (8) falls into the subset (28). This probability is computed in the next lemma.

Lemma 6.2. *For $k = 1, 2, \ldots$ and integers*

$$1 \leqslant x_1 < x_2 < \cdots < x_k, \qquad y_1 \geqslant y_2 \geqslant \cdots \geqslant y_k \geqslant 0$$

written in form (27)*, one has*

$$\begin{aligned} &P_2(x_1, \ldots, x_k; y_1, \ldots, y_k) \\ &\quad = \langle\infty\rangle_q \frac{\langle b_2 + a_1\rangle_q \ldots \langle b_k + a_{k-1}\rangle_q}{\langle b_1\rangle_q \ldots \langle b_k\rangle_q \langle a_1\rangle_q \ldots \langle a_k\rangle_q} q^{\sum_{\{(i,j)\colon 1 \leqslant i \leqslant j \leqslant k\}} (b_i+1)(a_j+1)}. \end{aligned} \tag{30}$$

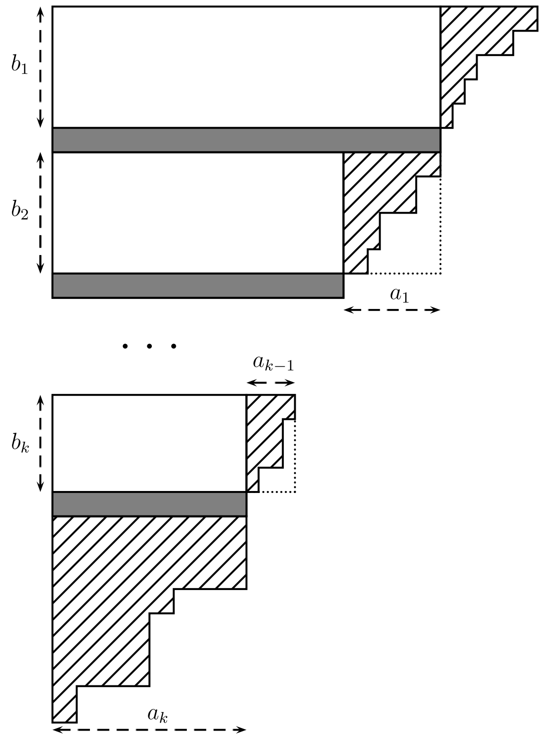

Fig. 2. Multirow decomposition of a diagram.

Proof. Fig. 2 shows that a diagram $\lambda \in \Lambda(x_1, \dots, x_k; y_1, \dots, y_k)$ is comprised of

(i) k rectangles of size $b_m \times (a_m + \cdots + a_k)$, $m = 1, \dots, k$,
(ii) k rows $a_m + \cdots + a_k$, $m = 1, \dots, k$,
(iii) $k - 1$ diagrams enclosed in rectangles $b_2 \times a_1, b_3 \times a_2, \dots, b_k \times a_{k-1}$,
(iv) two edge diagrams, one with at most b_1 rows, another with at most a_k columns.

The contribution (i) of the rectangles is the factor $q^{\sum_{\{(i,j):\ 1 \leqslant i \leqslant j \leqslant k\}} b_i a_j}$. The contribution (ii) of distinguished rows is $q^{\sum_{m=1}^{k} m a_m}$. It is known (see Proposition 1.3.19 in [22]) that the generating function of diagrams enclosed in rectangle $b \times a$ is the q-binomial coefficient

$$\frac{\langle b + a \rangle_q}{\langle b \rangle_q \langle a \rangle_q}.$$

From this, the contribution of (iii) is

$$\frac{\langle b_2 + a_1 \rangle_q}{\langle b_2 \rangle_q \langle a_1 \rangle_q} \cdots \frac{\langle b_k + a_{k-1} \rangle_q}{\langle b_k \rangle_q \langle a_{k-1} \rangle_q}.$$

Finally, the contribution of edge diagrams is

$$\frac{1}{\langle b_1 \rangle_q \langle a_k \rangle_q}.$$

Multiplying the factors out and recalling the normalizing factor $\langle \infty \rangle_q$ yields the result. $\square$

Now we can quickly complete the proof. The probability $P_1(x_1,\dots,x_k)$ is easily found from the q-shuffle algorithm:

$$P_1(x_1,\dots,x_k) = (1-q)^k q^{\sum_{m=1}^k (k-m+1)b_m}. \tag{31}$$

Here we substantially used the feature that the letters $1,\dots,k$ are the first successive letters in the alphabet $\mathbb{Z}_+$. We could not do the same with arbitrary indices $i_1 < \dots < i_k$, that is, to get a closed formula for the probability that given generic letters $i_1,\dots,i_k$ occupy given generic positions $x_1,\dots,x_k$ in the random word w^+.

Finally, substitute (30) and (31) into (29), and note that

$$a_m + b_{m-1} = d_m - d_{m-1}, \quad m = 2,\dots,k.$$

Because of these relations, the factor

$$\prod_{m=2}^k \langle a_m + b_{m-1}\rangle_q = \prod_{m=2}^k \langle d_m - d_{m-1}\rangle_q,$$

is a constant and so can be taken out of the sum. This gives the desired formula. □

In the first proof of Theorem 5.1 we used formula (22) for the conditional distribution of the inversion count L_j given R_j. Similarly, the right-hand side of (30) is the conditional probability $\mathbb{P}(L_1 = \ell_1,\dots,L_k = \ell_k \mid R_1 = r_1,\dots,R_k = r_k)$ with $\ell_m = y_m, r_m = x_m - m$ and x_m, y_m as in (27), for $1 \leqslant m \leqslant k$. Multiplying the conditional probability by the right-hand side of (31) yields the joint distribution of $(L_m, R_m;\ 1 \leqslant m \leqslant k)$, and the result of Theorem 6.1 can be recast in terms of this distribution.

7. Complements

7.1. A characterization of the inversion counts

In the course of proving Theorem 4.3, we established in Lemmas 4.6 and 4.7 a correspondence between balanced permutations and sequences $(r_i,\ i \in \mathbb{Z})$ of their right inversion counts. By far not every nonnegative integer sequence $(r_i,\ i \in \mathbb{Z})$ can occur in this way, thus it is of some interest to describe possible sequences in some detail.

Theorem 7.1. *A nonnegative integer sequence $(r_i,\ i \in \mathbb{Z})$ occurs as a sequence of right inversion counts for some permutation $\sigma \in \mathfrak{S}^{\mathrm{bal}}$ if and only if the following two conditions hold*:

(i) *the values $(\ell_i,\ i \in \mathbb{Z})$ determined from* (17) *are finite,*
(ii) *$r_i = 0$ for infinitely many $i \in \mathbb{Z}_+$.*

Under these conditions such σ is unique, and it has left inversion counts $(\ell_i,\ i \in \mathbb{Z})$.

Proof. Condition (i) is necessary by the definition of admissible permutation. We prove the remaining assertions in three steps.

Step 1. The starting point is the following extension of the elimination algorithm.

The one-sided infinite elimination algorithm. Given a nonnegative integer sequence $(r_1, r_2, \dots)$, construct a word $w = w_1 w_2 \dots$ by setting first $w_1 = r_1 + 1$, and for $k > 1$ defining w_k recursively as the $(r_k + 1)$st minimal element of the reduced list $\mathbb{Z}_+ \setminus \{w_1,\dots,w_{k-1}\}$.

In general, the output w need not be a permutation, and we need a condition to guarantee that.

Lemma 7.2. *A nonnegative integer sequence $(r_i,\ i \in \mathbb{Z}_+)$ is a sequence of the right inversion counts for some permutation $\sigma \in \mathfrak{S}_{\mathbb{Z}_+}$ if and only if $r_i = 0$ for infinitely many $i \in \mathbb{Z}_+$. In this case σ is the output of the infinite elimination algorithm applied to $(r_i,\ i \in \mathbb{Z}_+)$.*

Proof. Suppose $(r_i,\ i \in \mathbb{Z}_+)$ are the right inversion counts of $\sigma \in \mathfrak{S}_{\mathbb{Z}_+}$. For each $i \in \mathbb{Z}_+$ there exist $r_i + 1$ positions $j \geqslant i$ with $\sigma(j) \leqslant \sigma(i)$. Since $1 \leqslant r_i + 1 < \infty$, there exists position $j^* \geqslant i$ with $\sigma(j^*) = \min(\sigma(j)\colon\ j \geqslant i)$, but then obviously $r_{j^*} = 0$. Since i was arbitrary, we conclude by induction that $r_j = 0$ for infinitely many j.

Conversely, suppose a sequence (r_i) satisfies $r_{i_k} = 0$ for $i_1 < i_2 < \cdots$. At step i_k the algorithm selects $w_{i_k} = \min(\mathbb{Z}_+ \setminus \{w_1, \ldots, w_{i_{k-1}}\})$, hence the generic $k \in \mathbb{Z}_+$ is eliminated from the original list $12\ldots$ in at most i_k steps. Thus eventually every positive integer is chosen, and the output w is a permutation word encoding some permutation $\sigma \in \mathfrak{S}_{\mathbb{Z}_+}$.

Finally, suppose the output is a permutation. The elimination algorithm implies that exactly r_i integers smaller w_i remain in the list at step i, and these are eventually chosen at later steps. Hence there are r_i right inversions (i, j), $j > i$ for every $i \in \mathbb{Z}_+$. $\square$

Step 2. We turn next to the connection between permutations and strict orders. Note that the definition of inversion counts for permutations is applicable to orders as well. For instance, given a strict order $\lhd$, the corresponding left inversion count ℓ_i of a number i is the cardinality of the set $\{j\colon\ j < i,\ i \lhd j\}$. For $\sigma \in \mathfrak{S}_{\mathbb{Z}_+}$ we define a strict order $\lhd$ on $\mathbb{Z}_+$ by setting $i \lhd j$ iff $\sigma(i) < \sigma(j)$[2]. The elimination algorithm has an obvious modification which derives a word $(w_j,\ j \in \mathbb{Z}_{\geqslant i})$ from nonnegative integer input sequence $(r_j,\ j \in \mathbb{Z}_{\geqslant i})$. If $(r_i,\ i \in \mathbb{Z}_+)$ satisfies condition (ii) of Theorem 7.1 then for every $i \in \mathbb{Z}$ the elimination algorithm transforms $(r_j,\ j \in \mathbb{Z}_{\geqslant i})$ in permutation word $(w_j,\ j \in \mathbb{Z}_{\geqslant i})$, which in turn corresponds to some order $\lhd$ on $\mathbb{Z}_{\geqslant i}$. For two positions $i < j$, the relation according to the order $\lhd$ only depends on $r_i, \ldots, r_j$, thus there is a unique order $\lhd$ on $\mathbb{Z}$ compatible with all $(r_i,\ i \in \mathbb{Z})$. It follows that condition (ii) holds if and only if $(r_i,\ i \in \mathbb{Z})$ are the right inversion counts of some order $\lhd$.

Step 3. Assuming order $\lhd$ with finite right inversion counts $(r_i,\ i \in \mathbb{Z})$, the left inversion counts of $\lhd$ are determined recursively from (17). If all $\ell_i < \infty$, then the order $\lhd$ is *admissible*, meaning that the order has finite left and right inversion counts.

Lemma 7.3. *Let $(\ell_j), (r_i)$ be the inversion counts of admissible order on $\mathbb{Z}$. The formula $\sigma(i) = i - \ell_i + r_i$ establishes a bijection between the set of admissible orders on $\mathbb{Z}$ and $\mathfrak{S}^{\mathrm{bal}}$.*

Proof. Consider admissible order $\lhd$ with the inversion counts $(\ell_i), (r_i)$. The relation $i \lhd j$ entails that the interval $\{k \in \mathbb{Z}\colon\ i \lhd k \lhd j\}$ is finite, for otherwise at least one of the inversion counts ℓ_i, r_i, ℓ_j r_j were infinite. For similar reason $(\mathbb{Z}, \lhd)$ has neither minimal nor maximal element. It follows that the ordered set $(\mathbb{Z}, \lhd)$ is isomorphic to $(\mathbb{Z}, <)$, but then there exists a bijection $\sigma : \mathbb{Z} \to \mathbb{Z}$ such that $i \lhd j$ iff $\sigma(i) < \sigma(j)$. By Proposition 4.1, such permutation σ is admissible. The only freedom in the choice of σ stems from the observation that $s^{(n)}\sigma$ yields the same order, where $s^{(n)}$ is an arbitrary shift permutation. Since the shift affects the balance we may choose σ balanced, and then it is defined uniquely. Finally, for balanced σ the relation $\sigma(i) = i - \ell_i + r_i$ is guaranteed by Lemma 4.6. $\square$

We have verified that conditions (i) and (ii) of Theorem 7.1 are also sufficient. This completes the proof of the theorem. $\square$

We did not succeed to replace condition (i) by a substantially simpler one. For $\sigma \in \mathfrak{S}^{\mathrm{bal}}$ there must be also infinitely many $i < 0$ with $r_i = 0$, however this together with (ii) still does not suffice. For instance, the permutation defined by

[2] Note that here we step away from the convention (11) of Section 3, where the order was defined through σ^{-1}.

$$\sigma(-2k) = -k, \qquad \sigma(-2k-1) = 2k+1, \qquad \sigma(k) = 2k \quad \text{for } k \in \mathbb{Z}_+,$$

has r_i finite for all i, $r_i = 0$ for $i \geqslant 0$ and even $i < 0$, but $\ell_0 = \infty$.

The following result follows from Theorem 4.3. We feel nevertheless that a direct derivation adds insight. As before, G stands for the geometric distribution (13) with parameter $q \in (0, 1)$.

Lemma 7.4. *The random collection $(r_i\colon i \in \mathbb{Z})$ obtained by independent sampling from G satisfies both conditions of Theorem* 7.1 *almost surely.*

Proof. Condition (ii) is obvious, since G gives positive mass to 0.

To check (i), we fix $j \in \mathbb{Z}$ and prove that for every $r \in \{0, 1, 2, \ldots\}$, conditionally given $r_j = r$, the quantity ℓ_j determined from (17) is finite almost surely. Indeed, the sequence

$$(x_0, x_1, x_2, \ldots) := \left(r_j^{(j)}, r_j^{(j-1)}, r_j^{(j-2)}, \ldots\right)$$

defined by recursion (16) is a nondecreasing Markov chain on $\{r, r+1, \ldots\}$ with 0–1 increments, the initial state $x_0 = r$, and the one-step transition probabilities (20). As we have already pointed out, (17) entails that ℓ_j is the total number of the 0-increments of this Markov chain. To show that $\ell_j < \infty$ almost surely it suffices to prove that the expectation of ℓ_j is finite. The expectation is

$$\mathbb{E}\ell_j = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \mathbb{P}(x_m = r+k) q^{r+k+1} = q^{r+1} \sum_{k=0}^{\infty} q^k \sum_{m=0}^{\infty} \mathbb{P}(x_m = r+k),$$

so it is enough to check that the last sum over $m \geqslant 0$ is bounded by a constant independent of k. Note that $\mathbb{P}(x_m = r+k)$ vanishes unless $m \geqslant k$, so that we may start summation from $m = k$. Then the event $x_m = r+k$ means that among the first m moves of the chain there are exactly $m-k$ trivial transitions. The probability of this event does not exceed q^{m-k}, because the probability of any trivial transition is at most q. Therefore,

$$\sum_{m=k}^{\infty} \mathbb{P}(x_m = r+k) \leqslant \sum_{m=k}^{\infty} \mathbb{P}(x_m = r+k) q^{m-k} = (1-q)^{-1},$$

which concludes the proof. $\square$

7.2. Approximation by finite permutations

We proceed with the convention of Section 7.1, which implies that a bijection between $\sigma \in \mathfrak{S}^{\mathrm{bal}}$ and admissible orders on $\mathbb{Z}$ is established via $i \lhd j \Leftrightarrow \sigma(i) < \sigma(j)$. As above, for an interval $I \subset \mathbb{Z}$, we denote by $\mathfrak{S}_I \subset \mathfrak{S}$ the subgroup of permutations that do not move integers outside I. Below we will deal with finite intervals only; then $\mathfrak{S}_I$ is naturally isomorphic to the finite symmetric group of degree $\#I$. For $\sigma \in \mathfrak{S}^{\mathrm{bal}}$ we have a unique increasing bijection between $\{\sigma(i),\ i \in I\}$ and I. Replacing each $\sigma(i)$ with its counterpart by this bijection, and otherwise setting $\sigma_I(i) = i$ for $i \notin I$ maps $\sigma \in \mathfrak{S}^{\mathrm{bal}}$ to some $\sigma_I \in \mathfrak{S}_I$, a *truncation* of σ. In other words, the orders induced by σ and σ_I coincide on I. For instance, for $I = \{1, 2, 3\}$, the truncation of any permutation with pattern $\ldots \mid 4\ 3\ 2\ 1\ 6 \ldots$ gives $\ldots 0 \mid 3\ 2\ 1\ 4 \ldots$ (here the vertical bar is used to separate positions 0 and 1).

With reference to the discussion at the end of Section 2, we endow $\mathfrak{S}^{\mathrm{bal}}$ with the topology inherited from $\mathfrak{S}$. Recall that in this topology, the convergence $\sigma_n \to \sigma$ means $\sigma_n(i) = \sigma(i)$ for each i and all n larger than some $n(i)$. In what follows I_n ($n \in \mathbb{N}$) stands for arbitrary increasing sequence of finite intervals in $\mathbb{Z}$ whose union is the whole $\mathbb{Z}$. One obvious choice is $I_n = \{-n, -n+1, \ldots, n-1, n\}$.

Lemma 7.5. *For every $\sigma \in \mathfrak{S}^{\mathrm{bal}}$ we have $\sigma_{I_n} \to \sigma$.*

Proof. Let (ℓ_i) and (r_i) be the inversion counts of σ, and let $(\ell_i^{(n)})$ and $(r_i^{(n)})$ be the similar quantities for σ_{I_n}. For every fixed $i \in \mathbb{Z}$, the sequences $(\ell_i^{(n)})$ and $(r_i^{(n)})$ are nondecreasing. Indeed, the larger set I_n the larger the set of inversions in σ_{I_n} associated with i. Next, observe that these sequences stabilize for large n to values ℓ_i and r_i, respectively. Since $\sigma(i) = i - \ell_i + r_i$ and $\sigma_{I_n}(i) = i - \ell_i^{(n)} + r_i^{(n)}$, this just means that $\sigma_{I_n} \to \sigma$. $\square$

Immediately from the lemma and considerations in Section 3 we derive:

Proposition 7.6. *Let Σ be the random Mallows permutation of $\mathbb{Z}$, and let Σ_{I_n} be the truncation of Σ corresponding to I_n. Then $\Sigma_{I_n} \to \Sigma$ with probability one.*

Note that the law of Σ_{I_n} is essentially the Mallows distribution on the finite symmetric group of degree $\#I_n$.

Proposition 7.6 enables us to give alternative derivations to Corollary 3.4 and to the most technical part of Theorem 3.3 (Lemma 3.8) from the analogous properties of the Mallows distributions (1) on finite symmetric groups. To that end, one just needs to observe that the inversion mapping $\sigma \to \sigma^{-1}$ is a homeomorphism of $\mathfrak{S}^{\text{bal}}$ endowed with the weak topology. Now, $\Sigma_{I_n} \to \Sigma$ taken together with the equality in the law $\Sigma_{I_n} \stackrel{d}{=} (\Sigma_{I_n})^{-1}$ for finite permutations imply $\Sigma \stackrel{d}{=} \Sigma^{-1}$.

We emphasize that it is nowhere stated, nor is true, that $(\sigma_I)^{-1} = (\sigma^{-1})_I$. Examples are easily designed to show that the truncation and the inversion operation do not commute.

Finally, we note that the Mallows measures on $\mathfrak{S}_I$ with finite I, and the system of the joint distributions of $(\Sigma(i),\ i \in I)$ (computed in Section 6) are two competing families of "finite-dimensional distributions" for Σ considered as a random function. The suitability of one or another system in concrete situation depends on the nature of the statistic of permutation under study.

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