ISSN 0021-3640, JETP Letters, 2016, Vol. 103, No. 10, pp. 653–657. © Pleiades Publishing, Inc., 2016.

METHODS
OF THEORETICAL PHYSICS

Solitons in a Third-Order Nonlinear Schrödinger Equation with the Pseudo-Raman Scattering and Spatially Decreasing Second-Order Dispersion[1]

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Received March 24, 2016; in final form, April 22, 2016

Evolution of solitons is addressed in the framework of a third-order nonlinear Schrödinger equation (NLSE), including nonlinear dispersion, third-order dispersion and a pseudo-stimulated-Raman-scattering (pseudo-SRS) term, i.e., a spatial-domain counterpart of the SRS term, which is well known as a part of the temporal-domain NLSE in optics. In this context, it is induced by the underlying interaction of the high-frequency envelope wave with a damped low-frequency wave mode. In addition, spatial inhomogeneity of the second-order dispersion (SOD) is assumed. As a result, it is shown that the wavenumber downshift of solitons, caused by the pseudo-SRS, can be compensated with the upshift provided by decreasing SOD coefficients. Analytical results and numerical results are in a good agreement.

DOI: 10.1134/S0021364016100027

1. INTRODUCTION

The great interest to the dynamics of solitons is motivated by their ability to travel long distances keeping the shape and transferring the energy and information without losses. Soliton solutions are relevant to nonlinear models in various areas of physics which deal with the propagation of intensive wave fields in dispersive media: optical pulses and beams in fibers and spatial waveguides, electromagnetic waves in plasma, surface waves on deep water, etc. [1–7]. Recently, solitons have also drawn a great deal of interest in plasmonics [8–10].

Dynamics of long high-frequency (HF) wave packets is described by the second-order nonlinear dispersive wave theory. The fundamental equation of the theory is the nonlinear Schrödinger equation (NLSE) [11, 12], which includes the second-order dispersion (SOD) and cubic nonlinearity (self-phase modulation). Soliton solutions in this case arise as a result of the balance between the dispersive stretch and nonlinear compression of wave packets. In particular, invariant-shape solutions for damped solitons were found in the framework of the NLSE including linear losses of HF waves and spatially decreasing SOD [4, 13].

The dynamics of narrow HF wave packets is described by the third-order nonlinear dispersive wave theory [1], which takes into account the nonlinear dispersion (self-steeping) [14], stimulated Raman scattering (SRS) [15–17], and third-order dispersion (TOD). The basic equation of the theory is the extended (third-order) NLSE [17–21]. Soliton solutions in the framework of the extended NLSE with TOD and nonlinear dispersion were found in [22–29]. In [30, 31] stationary kink waves were found as solutions of the extended NLSE with SRS and nonlinear dispersion terms. This solution exists as the equilibrium between the nonlinear dispersion and SRS. For localized nonlinear wave packets (solitons), the SRS leads to the downshift of the soliton spectrum [15–17] and eventually to destabilization of solitons. The use of the balance between the SRS and the slope of the gain for the stabilization of solitons in long telecom links was proposed in [32]. The compensation of the SRS by emission of linear radiation fields from the soliton's core was considered in [33]. In addition, the compensation of the SRS in inhomogeneous media was considered in several situations: periodic SOD [34, 35], shifting zero-dispersion point of SOD [36], and dispersion-decreasing fibers [37].

Intensive short pulses of HF electromagnetic or Langmuir waves in plasmas, as well as HF surface waves on deep stratified water, suffer effective induced damping due to scattering on low-frequency (LF) waves, which, in turn, are subject to the action of viscosity. These LF modes are ion-sound waves in the plasma and internal waves in the stratified fluid. The first model for the damping induced by the interaction with the LF waves was proposed in [38, 39]. This model gives rise to an extended NLSE with the spatial-

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domain counterpart of the SRS term, that was call a *pseudo*-SRS one. The equation was derived from the system of the Zakharov's type equations [40] for the coupled Langmuir and ion-acoustic waves in plasmas. The pseudo-SRS leads to the self-wavenumber downshift, similar to what is well known in the temporal domain [1, 14–17] and, eventually, to destabilization of the solitons. The model equation elaborated in [38, 39] also included smooth spatial variation of the SOD, accounted for by a spatially decreasing SOD coefficient, which leads to an increase in the soliton's wavenumber, making it possible to compensate the effect of the pseudo-SRS on the soliton by the spatially inhomogeneous SOD. The equilibrium between the pseudo-SRS and decreasing SOD gives rise to stabilization of the soliton's wavenumber spectrum. To this time, the soliton dynamics was considered in the model neglecting the nonlinear dispersion and the TOD.

In this work, the soliton dynamics is considered in the frame of third-order NLSE with a spatial stimulated Raman scattering, nonlinear dispersion, TOD and decreasing SOD. The equilibrium between the pseudo-SRS and decreasing SOD is considered. The equilibrium condition for pseudo-SRS, SOD gradient and soliton's amplitude is found. In the case of TOD, larger nonlinear dispersion multiplied by relative pseudo-SRS the equilibrium of the soliton state is stable, and it is unstable in inverse case.

2. BASIC EQUATION AND INTEGRAL RELATIONS

We consider the dynamics of the HF wave field $U(\xi,t)\exp(-i\omega t + ik\xi)$ in the frame of inhomogeneous third-order NLSE with pseudo-SRS, nonlinear dispersion, TOD, and inhomogeneous SOD:

$$2i\frac{\partial U}{\partial t} + \frac{\partial}{\partial \xi}\left[q(\xi)\frac{\partial U}{\partial \xi}\right] + 2U|U|^2 + 2i\beta\frac{\partial\left(U|U|^2\right)}{\partial \xi} + i\gamma\frac{\partial^3 U}{\partial \xi^3} + \mu U\frac{\partial\left(|U|^2\right)}{\partial \xi} = 0, \quad (1)$$

where $q(\xi)$ is the SOD, μ is the pseudo-SRS, β is the nonlinear dispersion, and γ is the TOD coefficients.

Equation (1) with zero boundary conditions on infinity, $U|_{\xi\to\pm\infty} \to 0$, has the integrals:

$$\frac{dN}{dt} \equiv \frac{d}{dt}\int_{-\infty}^{+\infty}|U|^2 d\xi = 0, \quad (2)$$

$$2\frac{d}{dt}\int_{-\infty}^{+\infty}K|U|^2 d\xi = -\mu\int_{-\infty}^{+\infty}\left[\frac{\partial\left(|U|^2\right)}{\partial\xi}\right]^2 d\xi - \int_{-\infty}^{+\infty}\frac{dq}{d\xi}\left|\frac{\partial U}{\partial \xi}\right|^2 d\xi, \quad (3)$$

$$N\frac{d\bar{\xi}}{dt} \equiv \frac{d}{dt}\int_{-\infty}^{+\infty}\xi|U|^2 d\xi = \int_{-\infty}^{+\infty}qK|U|^2 d\xi + \frac{3}{2}\beta\int_{-\infty}^{+\infty}|U|^4 d\xi - \frac{3}{2}\gamma\int_{-\infty}^{+\infty}\left|\frac{\partial U}{\partial \xi}\right|^2 d\xi, \quad (4)$$

where $U \equiv |U|\exp(i\varphi)$, $K \equiv \partial\varphi/\partial\xi$ is the wavenumber of wave packet, $\bar{\xi}(t)$ is coordinate of wave packet's center-mass.

3. ANALYTICAL RESULTS

For analytical solution of the system of Eqs. (2)–(4), we consider values of nonlinear dispersion, TOD, and wavenumber are small, β, γ, $K \sim \varepsilon \ll 1$. Neglecting values by order ε^2, imaginary part of (1) will be the following:

$$\frac{\partial|U|^2}{\partial t} + \frac{\partial}{\partial\xi}\left(qK|U|^2 + \frac{3}{2}\beta|U|^4\right) + \gamma|U|\frac{\partial^3(|U|)}{\partial\xi^3} = 0.$$

Assuming wave packets moving with keeping the shape, $\partial(|U|^2)/\partial t \approx -V\partial(|U|^2)/\partial\xi$, where V is packet's velocity, we have from last equation:

$$\frac{\partial}{\partial\xi}\left(-V|U|^2 + qK|U|^2 + \frac{3}{2}\beta|U|^4\right) + \gamma|U|\frac{\partial^3(|U|)}{\partial\xi^3} = 0.$$

Integrating the last equation for localize wave packets, $|U|_{\xi\to-\infty} \to 0$, and assuming the scale of the inhomogeneity of SOD is much larger than the inhomogeneity scale of the wave-packet envelope, $D \gg D_{|U|}$, gives rise to a relation for the wavenumber:

$$K = k(t) - \frac{3\beta|U|^2}{2q(\bar{\xi})} + \frac{\gamma}{2q(\bar{\xi})|U|^2}\left[\frac{\partial(|U|)}{\partial\xi}\right]^2 - \frac{\gamma}{q(\bar{\xi})|U|}\frac{\partial^2(|U|)}{\partial\xi^2},$$

where $k(t) = V/q(\bar{\xi})$. Solution of the system of Eqs. (2)–(4) will be found in adiabatic approximation, presenting solution in sech-like form with last wavenumber distribution:

$$U(\xi,t) = \frac{A(t)}{\cosh\left[\left(\xi-\bar{\xi}\right)/\Delta(t)\right]} \times \exp\left[i\int K(\xi,t)\,d\xi - \frac{i}{2}\int A^2(t)\,dt\right], \quad (5)$$

$$K(\xi,t) = k(t) - \frac{3}{2}\frac{\beta A^2(t)}{q\left(\bar{\xi}\right)\cosh^2\left[\left(\xi-\bar{\xi}\right)/\Delta(t)\right]} - \frac{3}{2}\frac{\gamma}{q\left(\bar{\xi}\right)\Delta^2(t)}\tanh^2\left[\frac{\xi-\bar{\xi}}{\Delta(t)}\right] + \frac{\gamma}{q\left(\bar{\xi}\right)\Delta^2(t)}, \quad (6)$$

where $\Delta(t) \equiv \sqrt{q\left(\bar{\xi}\right)}/A(t)$ and $A^2(t)\,\Delta(t) = \text{const}$. The solution specified by Eqs. (5) and (6) has two free parameters: an additional wavenumber $k(t)$ and a coordinate of center-mass $\bar{\xi}(t)$.

Substituting Eqs. (5) and (6) into Eqs. (2)–(4) and keeping terms by order ε, we obtain the system for k and $\bar{\xi}(t)$:

$$2\frac{dk}{dt} = -\frac{8q_0^2A_0^4\mu}{15q^3\left(\bar{\xi}\right)} - \frac{q_0A_0^2q'\left(\bar{\xi}\right)}{3q^2\left(\bar{\xi}\right)} + \frac{2q_0\gamma A_0^2q'\left(\bar{\xi}\right)k}{q^3\left(\bar{\xi}\right)} - \frac{2\beta A_0^2q'\left(\bar{\xi}\right)k}{q^2\left(\bar{\xi}\right)} - q'\left(\bar{\xi}\right)k^2, \quad (7)$$
$$\frac{d\bar{\xi}}{dt} = q\left(\bar{\xi}\right)k,$$

where $q_0 = q(0)$, $A_0 = A(0)$, $q'(\bar{\xi}) = (dq/d\xi)_{\bar{\xi}}$. System (7) gives rise to an obvious equilibrium state (alias fixed point, FP): $8q_0A_0^2\mu = -5q'\left(\bar{\xi}_*\right)q\left(\bar{\xi}_*\right)$, $k_* = 0$. In particular, for

$$\mu = \mu_* \equiv 5q'(0)/\left(8A_0^2\right), \quad (8)$$

the FP corresponds to initial soliton parameters: $\bar{\xi} = 0$, $k = 0$. For $\mu \neq \mu_*$ soliton's parameters are time varying. To analyze the evolution around the FP, we assume linearly decreasing SOD, $q' = \text{const} < 0$, and rescale the variables by defining $\tau \equiv tq'A_0/\sqrt{3q_0}$, $y \equiv k\sqrt{3q_0}/A_0$, and $n \equiv q\left(\bar{\xi}\right)/q_0$. Then, system (7) is reduced to

$$2\frac{dy}{d\tau} = -\frac{\lambda}{n^3} + \frac{1}{n^2} + y^2 - \sigma\frac{y}{n^3} + \delta\frac{y}{n^2}, \quad (9)$$
$$\frac{dn}{dt} = -ny,$$

where $\lambda \equiv -8\mu A_0^2/(5q') \equiv \mu/\mu_*$, $\sigma \equiv 2\sqrt{3}\gamma A_0/\sqrt{q_0^3}$, $\delta \equiv 2\sqrt{3}\beta A_0/\sqrt{q_0^3}$. The FP of the (9) in rescale variables is $y_* = 0$, $n_* = \lambda$. For $\mathrm{I} \equiv \sigma - \lambda\delta > 0$ the FP is the stable focus, $\mathrm{I} = 0$: center, $\mathrm{I} < 0$: unstable focus. For $\mu = \mu_* \equiv 5q'/\left(8A_0^2\right)$, corresponding to $\lambda = 1$, the FP coincide with initial soliton's parameter $n_0 \equiv 1$, $y_0 = 0$. In this case, soliton's parameters are constant at time.

4. NUMERICAL RESULTS

We now aim to solve the initial-value problem for the dynamics of the wave packet, $U(\xi, t = 0) = \exp[i\phi(\xi)]/\cosh\xi$, with spatial phase distribution $\phi(\xi) = -(3/2)\beta\tanh\xi - (3/2)\gamma(\xi - \tanh\xi) + \gamma\xi$ (corresponding to wavenumber $K(\xi) = d\phi/d\xi \equiv -3\beta/(2\cosh^2\xi) - (3/2)\gamma\tanh^2\xi + \gamma)$, in the framework of (1), for $q(\xi) = 1 - \xi/10$ and different values of μ, β, and γ numerically. The analytically predicted equilibrium value of strength of the pseudo-SRS term from (9) for the initial pulse is $\mu_* = 1/16$. In direct simulations the initial pulse for $\mu = \mu_*$ and $\mathrm{I} = 2\sqrt{3}[\gamma - \beta] \equiv 2\sqrt{3}\left[\gamma - \left(\mu/\mu_*\right)\beta\right] \geq 0$ is transformed into a stationary localized distribution. For $\mathrm{I} < 0$ the initial pulse is unstable at time.

Variation of the parameter μ leads to the soliton's parameters variation (wavenumber and amplitude). For example, for $\mu = 5/64 \equiv (5/4)\mu_*$ and $\mathrm{I} = 2\sqrt{3}[\gamma - (5/4)\beta] \equiv 2\sqrt{3}\left[\gamma - \left(\mu/\mu_*\right)\beta\right] > 0$ the soliton's parameters make several fluctuations and tend to constant values (asymptotically stable soliton). For $\mu = 5/64$ and $\mathrm{I} = 0$ the soliton's parameters vary periodically with constant period and scope (dynamically stable soliton). For $\mu = 5/64$ and $\mathrm{I} < 0$ the soliton's parameters change with increasing scale (unstable soliton).

In Fig. 1, numerical simulation of the value of coordinate point of the maximum modulus of the wave-packet's shape ξ_m ($\max\left|U(\xi,t)\right| = \left|U(\xi_m,t)\right|$) as a time function, are compared with the analytical counterparts of the mass-center wave-packet envelope $\bar{\xi} \equiv q_0(n-1)/q'$ obtained from (7) for $\mu = 5/64 \equiv (5/4)\mu_*$ and different values of $\mathrm{I} = 2\sqrt{3}[\gamma - (5/4)\beta] \equiv 2\sqrt{3}\left[\gamma - \left(\mu/\mu_*\right)\beta\right]$.

In Fig. 2, numerical results produced, as functions of time, by the simulations for the value $k_{\text{num}}(t) = K(\xi_m,t) + (3/2)\beta|U(\xi_m,t)|^2/q(\xi_m)$ at the point of maximum modulus wave-packet's shape ξ_m, are compared with the analytical counterparts of additional wavenumber $k(t)$ (see relation (6)) obtained from (7) for $\mu = 5/64 \equiv (5/4)\mu_*$ and different values of $\mathrm{I} = 2\sqrt{3}[\gamma - (5/4)\beta] \equiv 2\sqrt{3}\left[\gamma - \left(\mu/\mu_*\right)\beta\right]$. Close

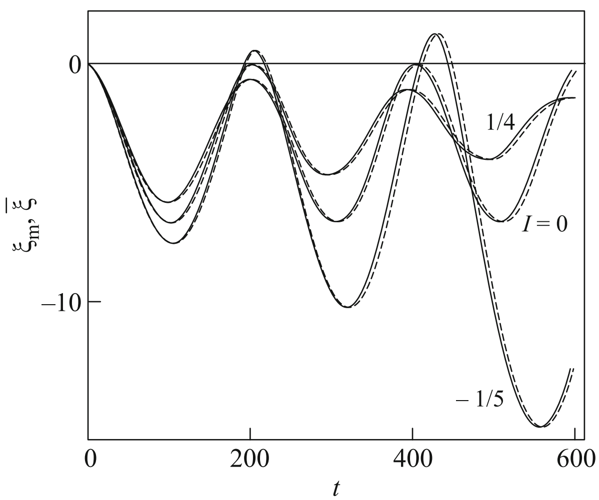

Fig. 1. (Solid curves) Numerical results for ξ_m and (dotted curves) analytical results for $\bar{\xi}$ versus time for $\mu = 5/64 \equiv (5/4)\mu_*$, and different values of I.

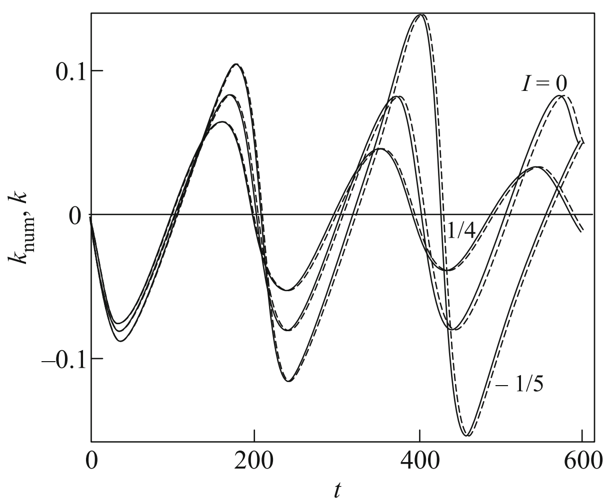

Fig. 2. (Solid curves) Numerical results for the k_{num} and (dotted curves) analytical results for the value k versus time for $\mu = 5/64 \equiv (5/4)\mu_*$, and different values of I.

agreement between the analytical and numerical results is demonstrated in the figure. A similar results agreement at other parameter's values is observed too.

5. CONCLUSIONS

In this work, the soliton's dynamics is studied in the framework of third-order NLSE with pseudo-SRS term (induced by the interaction of the HF waves with damped LF modes), nonlinear dispersion, TOD and linearly decreasing SOD. The solitons exist due to the balance between the self-wavenumber downshift, caused by the pseudo-SRS term, and the upshift induced by the inhomogeneous SOD. The equilibrium value of the pseudo-SRS, depended on soliton amplitude and SOD gradient, is found. Exactly in the equilibrium state the soliton's parameters is constant in time. This equilibrium state can be stable or unstable. It is found: for TOD larger nonlinear dispersion multiplied by relative pseudo-SRS, initial wave packet tends to stable soliton form (soliton's asymptotical stability); for TOD equal to nonlinear dispersion multiplied by relative pseudo-SRS, soliton parameters are periodically vary (soliton's dynamical stability); for TOD smaller nonlinear dispersion multiplied by relative pseudo-SRS, initial wave packet is destroyed (soliton's instability). The results were obtained by means of numerical and analytical methods. Results of the numerical experiments prove: the analytical results are correct.

This work was supported by the Russian Foundation for Basic Research, project no. 15-02-01919 a.

REFERENCES

1. E. Infeld and G. Rowlands, *Nonlinear Waves, Solitons, and Chaos* (Cambridge Univ. Press, Cambridge, 2000).
2. G. P. Agrawal, *Nonlinear Fiber Optic* (Academic, San Diego, 2001).
3. J. Yang, *Solitons in Field Theory and Nonlinear Analysis* (Springer, New York, 2001).
4. Y. S. Kivshar and G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals* (Academic, San Diego, 2003).
5. L. A. Dickey, *Soliton Equation and Hamiltonian Systems* (World Scientific, New York, 2005).
6. B. A. Malomed, *Soliton Management in Periodic Systems* (Springer, New York, 2006).
7. T. Dauxois and M. Peyrard, *Physics of Solitons* (Cambridge Univ. Press, Cambridge, 2006).
8. M. Sich, D. N. Krizhanovskii, M. S. Skolnick, A. V. Gorbach, R. Hartley, D. V. Skryabin, E. A. Cerda-Mendez, K. Biermann, R. Hey, and P. V. Santos, Nature Photon. **6**, 50 (2012).
9. M. Kauranen and A. V. Zayats, Nature Photon. **6**, 737 (2012).
10. E. A. Cerda-Mendez, D. Sarkar, D. N. Krizhanovskii, S. S. Gavrilov, K. Biermann, M. S. Skolnick, and P. V. Santos, Phys. Rev. Lett. **111**, 146401 (2013).
11. V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP **34**, 62 (1972).
12. A. Hasegawa and F. Tappert, Appl. Phys. Lett. **23**, 142 (1973).
13. K. Tajima, Opt. Lett. **12**, 54 (1987).
14. J. R. Oliviera and M. A. Moura, Phys. Rev. E **57**, 4751 (1998).
15. F. M. Mitschke and L. F. Mollenauer, Opt. Lett. **11**, 659 (1986).
16. J. P. Gordon, Opt. Lett. **11**, 662 (1986).
17. Y. Kodama, J. Stat. Phys. **39**, 597 (1985).
18. Y. Kodama and A. Hasegawa, IEEE J. Quant. Electron. **23**, 510 (1987).

19. C. E. Zaspel, Phys. Rev. Lett. **82**, 723 (1999).
20. B. Hong and D. Lu, Int. J. Nonlin. Sci. **7**, 360 (2009).
21. V. I. Karpman, Eur. Phys. J. B **39**, 341 (2004).
22. E. M. Gromov and V. I. Talanov, J. Exp. Theor. Phys. **83**, 73 (1996).
23. E. M. Gromov and V. I. Talanov, Chaos **10**, 551 (2000).
24. E. M. Gromov, L. V. Piskunova, and V. V. Tyutin, Phys. Lett. A **256**, 153 (1999).
25. M. A. Obregon and Yu. A. Stepanyants, Phys. Lett. A **249**, 315 (1998).
26. M. Scalora, M. Syrchin, N. Akozbek, E. Y. Poliakov, G. D. Aguanno, N. Mattiucci, M. J. Bloemer, and A. M. Zheltikov, Phys. Rev. Lett. **95**, 013902 (2005).
27. S. C. Wen, Y. Wang, W. Su, Y. Xiang, X. Fu, and D. Fan, Phys. Rev. E **73**, 036617 (2006).
28. M. Marklund, P. K. Shukla, and L. Stenflo, Phys. Rev. E **73**, 037601 (2006).
29. N. L. Tsitsas, R. I. Kourakis, P. G. Kevrekidis, and D. J. Frantzeskakis, Phys. Rev. E **79**, 037601 (2009).
30. Y. S. Kivshar, Phys. Rev. A **42**, 1757 (1990).
31. Y. S. Kivshar and B. A. Malomed, Opt. Lett. **18**, 485 (1993).
32. B. A. Malomed and R. S. Tasgal, J. Opt. Soc. Am. B **15**, 162 (1998).
33. F. Biancalama, D. V. Skrybin, and A. V. Yulin, Phys. Rev. E **70**, 011615 (2004).
34. R.-J. Essiambre and G. P. Agrawal, J. Opt. Soc. Am. B **14**, 314 (1997).
35. R.-J. Essiambre and G. P. Agrawal, J. Opt. Soc. Am. B **14**, 323 (1997).
36. A. Andrianov, S. Muraviev, A. Kim, and A. Sysoliatin, Laser Phys. **17**, 1296 (2007).
37. S. Chernikov, E. Dianov, D. Richardson, and D. Payne, Opt. Lett. **18**, 476 (1993).
38. E. M. Gromov and B. A. Malomed, J. Plasma Phys. **79**, 1057 (2013).
39. E. M. Gromov and B. A. Malomed, Opt. Commun. **320**, 88 (2014).
40. V. E. Zakharov, Sov. Phys. JETP **35**, 908 (1972).