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Advances in Mathematics 230 (2012) 1738–1779

ADVANCES IN Mathematics

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The boundary of the Gelfand–Tsetlin graph: A new approach

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Received 11 October 2011; accepted 8 April 2012
Available online 16 May 2012

Communicated by Andrei Zelevinsky

Abstract

The Gelfand–Tsetlin graph is an infinite graded graph that encodes branching of irreducible characters of the unitary groups. The boundary of the Gelfand–Tsetlin graph has at least three incarnations — as a discrete potential theory boundary, as the set of finite indecomposable characters of the infinite-dimensional unitary group, and as the set of doubly infinite totally positive sequences. An old deep result due to Albert Edrei and Dan Voiculescu provides an explicit description of the boundary; it can be realized as a region in an infinite-dimensional coordinate space.

The paper contains a novel approach to the Edrei–Voiculescu theorem. It is based on a new explicit formula for the number of semi-standard Young tableaux of a given skew shape (or of Gelfand–Tsetlin schemes of trapezoidal shape). The formula is obtained via the theory of symmetric functions, and new Schur-like symmetric functions play a key role in the derivation.

Keywords: Gelfand–Tsetlin graph; Gelfand–Tsetlin schemes; Unitary group characters; Totally positive sequences; Schur functions; Dual Schur functions

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doi:10.1016/j.aim.2012.04.005

Contents

1. Introduction

1.1. Finite characters of $S(\infty)$ and $U(\infty)$. A brief survey

The symmetric group $S(n)$ and the unitary group $U(N)$ are two model examples of finite and compact groups, respectively. Their irreducible characters are basic objects of representation theory that have numerous applications.

In two remarkable papers by Thoma [26] and Voiculescu [30] written independently and published twelve years apart from each other, the authors discovered that the theory of characters can be nontrivially generalized to groups $S(\infty)$ and $U(\infty)$ defined as inductive limits of the group chains

$$S(1) \subset S(2) \subset \cdots \quad \text{and} \quad U(1) \subset U(2) \subset \cdots.$$

The original idea of Thoma (for $S(\infty)$) and Voiculescu (for $U(\infty)$) consisted in replacing irreducible representations by factor representations (in the sense of von Neumann) with finite trace. Then characters are still ordinary functions on the group, and it turns out that for $S(\infty)$ and $U(\infty)$ they depend on countably many continuous parameters. This fact supports the intuitive feeling that these groups are “big”.

It was later discovered [27,28,6] that the classification of finite characters of $S(\infty)$ and $U(\infty)$ was obtained in a hidden form in earlier works at the beginning of the 1950s [1,2,9,10]. Those papers solved the problems of classification of totally positive sequences posed by Schoenberg at the end of 1940s [24].[1]

On the other hand, Vershik and Kerov [27–29] outlined a different approach to finite characters. Their approach was not based on total positivity and theory of functions of a complex variable, as Edrei's and Thoma's. Instead, it relied on the ideas of discrete potential theory and combinatorics of symmetric functions. In a broader context this approach was described in detail in [14,22], where the character problem was rephrased in the language of boundaries of two infinite graphs, the Young graph $\mathbb{Y}$ and the Gelfand–Tsetlin graph $\mathbb{GT}$. These are two model examples of the so-called branching graphs; they encode branching rules of the irreducible characters of symmetric and unitary groups, respectively.

Denote by χ_ν the irreducible character of $S(n)$ or $U(N)$. Here index ν is either a Young diagram with n boxes or a signature of length N (a highest weight for $U(N)$). In Vershik–Kerov's approach, one studies the limiting behavior of the normalized characters

$$\widetilde{\chi}_\nu := \frac{\chi_\nu}{\chi_\nu(e)}$$

when n or N becomes large, and the diagram/signature is n or N dependent. It turns out that possible limits of $\widetilde{\chi}_\nu$ are exactly the finite characters of $S(\infty)$ or $U(\infty)$, respectively.

1.2. A combinatorial formulation

In the language of branching graphs, the question of asymptotics of $\widetilde{\chi}_\nu$ can be reformulated in a purely combinatorial fashion. More exactly, one asks about the asymptotics of

$$\frac{\dim(\varkappa,\nu)}{\dim\nu} \quad \text{(for the symmetric group)} \tag{1.1}$$

$$\frac{\mathrm{Dim}_{K,N}(\varkappa,\nu)}{\mathrm{Dim}_N\,\nu} \quad \text{(for the unitary group),} \tag{1.2}$$

with the following notations.

- In the symmetric group case, $\varkappa$ is a Young diagram with $k < n$ boxes; $\dim\nu$ is the number of standard Young tableaux of shape ν; $\dim(\varkappa,\nu)$ is the number of the standard tableaux of skew shape $\nu/\varkappa$ if $\varkappa\subset\nu$, and 0 if $\varkappa\not\subset\nu$.
- In the unitary group case, $\varkappa$ is a signature of length $K < N$; $\mathrm{Dim}_N\,\nu$ is the number of triangular Gelfand–Tsetlin schemes with fixed top row $(\nu_1,\dots,\nu_N)$; $\mathrm{Dim}_{K,N}(\varkappa,\nu)$ is the number of truncated (trapezoidal) Gelfand–Tsetlin schemes with top row ν and bottom row $\varkappa$.

The "dimensions" dim and Dim count certain finite sets of monotone paths in $\mathbb{Y}$ and $\mathbb{GT}$. In both cases, the problem consists in classification of all possible ways for ν to approach infinity so that the "relative dimension" (1.1) or (1.2) has a finite limit for any fixed diagram/signature $\varkappa$. These possibilities are parameterized by the points of the branching graph's boundary.

Let us note that the denominator in (1.1) or (1.2) is given by a relatively simple formula, while computing the numerator is substantially harder. This basic difficulty results in nontriviality of the asymptotic analysis.

[1] Nowadays, largely due to the works of Lusztig and Fomin–Zelevinsky, total positivity is a popular subject. In the 1960s–70s the situation was different, and Thoma and Voiculescu were apparently unaware of the work of Schoenberg and his followers.

1.3. Motivations

In the present paper, we return to the problem of finding the boundary of $\mathbb{GT}$ and obtain a new proof of completeness of the list of characters of $U(\infty)$ given by Voiculescu in [30]. The reader would be fully justified to ask why we decided to reconsider an old theorem and produce its third proof, especially since our proof is not that simple. Here are our arguments.

(a) The boundary descriptions for $\mathbb{Y}$ and $\mathbb{GT}$ are strikingly similar. In terms of total positivity, the points of both boundaries correspond to infinite totally positive Toeplitz matrices, only in the first case the matrices are assumed to be triangular. We are confident that the parallelism between $\mathbb{Y}$ and $\mathbb{GT}$ is deeply rooted, and one should expect its appearance in other aspects as well.[2] However, if one compares the proofs given in [14] for $\mathbb{Y}$ and in [22] for $\mathbb{GT}$ then one would notice that they are substantially different.

More exactly, in the case of $\mathbb{Y}$ in [21, Theorem 8.1] the authors obtained a formula that expressed the relative dimension (1.1) through the shifted Schur functions. This formula is well adapted for the asymptotic analysis, and [14] was based on this formula (and on its generalization that includes the Jack parameter [20]). In the case of $\mathbb{GT}$ there was no analogous expression for (1.2). For that reason the authors of [22] had to follow a round-about path inspired by an idea from [28] of pursuing the asymptotics of the Taylor expansion of characters $\widetilde{\chi}_\nu$ at the unit element of the group.

In the present paper, in contrast to [22], we work directly with the relative dimension (1.2) and derive a formula for it that is suitable for asymptotic analysis. Hence, we achieve uniformity in the asymptotic approach to the boundaries of $\mathbb{Y}$ and $\mathbb{GT}$.

(b) We believe that our formula for the relative dimension (1.2) and its proof are of independent interest in algebraic combinatorics. The formula involves certain new symmetric functions of Schur type. In the proof we also use the so-called dual Schur functions that were thoroughly investigated in a recent paper by Molev [18].

(c) The description of $\mathbb{GT}$'s boundary is derived below from a new result that we call the Uniform Approximation Theorem. It is important to us as it allows to substantially strengthen our results in [4] on Markov dynamics on the boundary of $\mathbb{GT}$. This development will be described in a separate publication.

(d) To conclude, we believe that the classification of finite characters of $U(\infty)$ is a difficult and truly deep result, and already for that reason its third proof should not be dismissed as excessive.

1.4. Organization of the paper

Section 2 contains main definitions and auxiliary results. A part of those is contained in one form or another in [30]. At the end of the section, we give a description of $\mathbb{GT}$'s boundary (Theorem 2.15 and Corollary 2.16).

In Section 3, we state the Uniform Approximation Theorem (Theorem 3.1) and explain how it implies the results on the boundary of $\mathbb{GT}$.

Sections 4–8 contain the proof of the Uniform Approximation Theorem.

In Section 4, we give an auxiliary result on an identity of Cauchy type; here dual Schur functions come about (more general results in this direction can be found in [18]).

[2] New results in this direction are contained in our paper [5].

In Section 5, we prove a different identity of Cauchy type:

$$H^*(t_1;\nu)\cdots H^*(t_K;\nu)=\sum_{\varkappa\in\mathbb{GT}_K}\frac{\mathrm{Dim}_{K,N}(\varkappa,\nu)}{\mathrm{Dim}_N\nu}\,\mathfrak{S}_{\varkappa|N}(t_1,\dots,t_K). \tag{1.3}$$

Here $t_1,\dots,t_K$ are complex variables, ν is an arbitrary signature of length $N>K$,

$$H^*(t;\nu)=\prod_{i=1}^{N}\frac{t+i}{t+i-\nu_i},$$

the summation on the right-hand side of (1.3) is over signatures $\varkappa$ of length K, and $\mathfrak{S}_{\varkappa|N}(t_1,\dots,t_K)$ are certain new analogs of Schur functions in K variables. The coefficients in front of these functions are the relative dimensions (1.2) that we are interested in.

In Section 6, we show how (1.3) implies a Jacobi–Trudi type formula for the relative dimension. It expresses the relative dimension as a determinant of size $K\times K$ whose matrix elements are coefficients of the decomposition of $H^*(t;\nu)$ on certain rational functions.

Section 7 explains how to write those coefficients through residues of $H^*(t;\nu)$. As a result, we obtain an explicit formula for the relative dimension (Theorem 7.2). For comparison, we also give a different formula (Remark 3.2). In contrast to Theorem 7.2, its derivation is simple but the formula seems useless for our purposes.

In Section 8, using Theorem 7.2 we conclude the proof of the Uniform Approximation Theorem.

Together with the Uniform Approximation Theorem, the formula of Theorem 7.2 is one of our main results. It is plausible that this formula can be obtained in a simpler way, and we would be very interested in seeing how to do that. It often happens that combinatorial identities have different proofs which can be simpler than the original derivation. (For example, one could try to derive Theorem 7.2 from the formula of Remark 3.2 or from the binomial formula for the normalized characters $\widetilde{\chi}_\nu$ that [21] was based upon.)

Not only the Uniform Approximation Theorem provides a new derivation of $\mathbb{GT}$'s boundary, but it also immediately implies the main results of [28,22] on large N asymptotics of the normalized characters $\widetilde{\chi}_\nu$. In the Appendix, we demonstrate that conversely, the Uniform Approximation Theorem is not hard to prove using the results of [22] if one additionally employs the log-concavity of characters $\widetilde{\chi}_\nu$ discovered by Okounkov [19]. We emphasize however that this approach gives nothing for Theorem 7.2.

Let us finally mention a recent paper by Gorin [11] where the boundary of a "q-analog" of $\mathbb{GT}$ was described (the edges of the graph are supplied with certain formal q-dependent multiplicities). It would be interesting to extend the approach of the present paper to the q–$\mathbb{GT}$ case.

2. Preliminaries

2.1. The graph $\mathbb{GT}$

Following [31], for $N\ge 1$ define a *signature* of length N as an N-tuple of nonincreasing integers $\nu=(\nu_1\ge\cdots\ge\nu_N)$, and denote by $\mathbb{GT}_N$ the set of all such signatures.

Two signatures $\lambda\in\mathbb{GT}_{N-1}$ and $\nu\in\mathbb{GT}_N$ *interlace* if $\nu_{i+1}\le\lambda_i\le\nu_i$ for all meaningful values of indices; in this case we write $\lambda\prec\nu$.

Let $\mathbb{GT} = \bigsqcup_{N \geq 1} \mathbb{GT}_N$ be the set of signatures of arbitrary length, and equip $\mathbb{GT}$ with edges by joining λ and ν iff $\lambda \prec \nu$ or $\nu \prec \lambda$. This turns $\mathbb{GT}$ into a graph that is called the *Gelfand–Tsetlin graph*. We call $\mathbb{GT}_N \subset \mathbb{GT}$ the *level N subset* of the graph.

By a *path* between two vertices $\varkappa \in \mathbb{GT}_K$ and $\nu \in \mathbb{GT}_N$, $K < N$, we mean a sequence

$$\varkappa = \lambda^{(K)} \prec \lambda^{(K+1)} \prec \cdots \prec \lambda^{(N)} = \nu \in \mathbb{GT}_N.$$

Such a path can be viewed as an array of numbers

$$\{\lambda_i^{(j)}\}, \quad K \leq j \leq N,\ 1 \leq i \leq j,$$

satisfying the inequalities $\lambda_{i+1}^{(j+1)} \leq \lambda_i^{(j)} \leq \lambda_i^{(j+1)}$. It is called a *Gelfand–Tsetlin scheme*. If $K = 1$, the scheme has triangular form and if $K > 1$, it has trapezoidal form.

Let $\mathrm{Dim}_{K,N}(\varkappa, \nu)$ denote the number of paths between $\varkappa$ and ν, and let $\mathrm{Dim}_N \nu$ be the number of all paths starting at an arbitrary vertex of level 1 and ending at ν. Both these numbers are always finite; note that they count the lattice points in some bounded convex polyhedra. The number $\mathrm{Dim}_{K,N}(\varkappa, \nu)$ may be equal to 0, but $\mathrm{Dim}_N \nu$ is always strictly positive.

For $N \geq 2$ denote by Λ_{N-1}^N the matrix of format $\mathbb{GT}_N \times \mathbb{GT}_{N-1}$ with the entries

$$\Lambda_{N-1}^N(\nu, \lambda) = \begin{cases} \dfrac{\mathrm{Dim}_{N-1} \lambda}{\mathrm{Dim}_N \nu}, & \lambda \prec \nu, \\ 0, & \text{otherwise}. \end{cases}$$

By the very definition of the Dim function,

$$\mathrm{Dim}_N \nu = \sum_{\lambda: \lambda \prec \nu} \mathrm{Dim}_{N-1} \lambda.$$

It follows that Λ_{N-1}^N is a stochastic matrix:

$$\sum_{\lambda \in \mathbb{GT}_{N-1}} \Lambda_{N-1}^N(\nu, \lambda) = 1 \quad \forall \nu \in \mathbb{GT}_N.$$

More generally, for $N > K \geq 1$, the matrix product

$$\Lambda_K^N := \Lambda_{N-1}^N \Lambda_{N-2}^{N-1} \cdots \Lambda_K^{K+1} \tag{2.1}$$

is a stochastic matrix, too, and its entries are

$$\Lambda_K^N(\nu, \varkappa) = \frac{\mathrm{Dim}_K \varkappa \ \mathrm{Dim}_{K,N}(\varkappa, \nu)}{\mathrm{Dim}_N \nu}.$$

2.2. *The boundary of* $\mathbb{GT}$

We say that an infinite sequence $M_1, M_2, \ldots$ of probability distributions on the sets $\mathbb{GT}_1, \mathbb{GT}_2, \ldots$, respectively, forms a *coherent system* if the distributions are consistent with the transition matrices $\Lambda_1^2, \Lambda_2^3, \ldots$, meaning that

$$M_N \Lambda_{N-1}^N = M_{N-1} \quad \forall N \geq 2.$$

Here we interpret M_N as a row vector $\{M_N(\nu) : \nu \in \mathbb{GT}_N\}$, which makes it possible to define the multiplication on the left-hand side. In more detail, the relation means

$$\sum_{\nu \in \mathbb{GT}_N} M_N(\nu) \Lambda^N_{N-1}(\nu, \lambda) = M_{N-1}(\lambda) \quad \forall \lambda \in \mathbb{GT}_{N-1}.$$

Note that the set of all coherent systems is a convex set: if $\{M_N : N = 1, 2, \ldots\}$ and $\{M'_N : N = 1, 2, \ldots\}$ are two coherent systems, then for any $p \in [0, 1]$, the convex combination $\{pM_N + (1 - p)M'_N : N = 1, 2, \ldots\}$ is a coherent system, too. A coherent system is said to be *extreme* if it is an extreme point in this convex set.

Definition 2.1. The *boundary* $\partial(\mathbb{GT})$ *of the Gelfand–Tsetlin graph* $\mathbb{GT}$ is defined as the set of extreme coherent systems of distributions on $\mathbb{GT}$.

This definition mimics the well-known definition of the *minimal part of the Martin entrance boundary* of a Markov chain (see, e.g. [13]). Indeed, consider the infinite chain

$$\mathbb{GT}_1 \dashleftarrow \mathbb{GT}_2 \dashleftarrow \mathbb{GT}_2 \dashleftarrow \cdots \tag{2.2}$$

where the dashed arrows symbolize the transition matrices Λ^N_{N-1}. One may regard (2.2) as a Markov chain with time parameter $N = 1, 2, \ldots$ ranging in the reverse direction, from infinity to 1, and with the state space varying with time. Although such a Markov chain looks a bit unusual, the conventional definition of the minimal entrance boundary can be adapted to our context, and this leads to the same space $\partial(\mathbb{GT})$. Note that the minimal entrance boundary may be a proper subset of the whole Martin entrance boundary, but for the concrete chain (2.2) these two boundaries coincide.

One more interpretation of the boundary $\partial(\mathbb{GT})$ is the following: it coincides with the *projective limit* of chain (2.2) in the category whose objects are measurable spaces and morphisms are defined as Markov transition kernels (stochastic matrices are just simplest instances of such kernels).

For more detail about the concept of entrance boundary employed in the present paper, see, e.g., [7,8,32].

2.3. *Representation-theoretic interpretation*

Let $U(N)$ denote the group of $N \times N$ unitary matrices or, equivalently, the group of unitary operators in the coordinate space $\mathbb{C}^N$. For every $N \geq 2$ we identify the group $U(N-1)$ with the subgroup of $U(N)$ that fixes the last basis vector. In this way we get an infinite chain of groups embedded into each other

$$U(1) \subset U(2) \subset U(3) \subset \cdots. \tag{2.3}$$

As is well known, signatures from $\mathbb{GT}_N$ parameterize irreducible characters of $U(N)$; given $\nu \in \mathbb{GT}_N$, let χ_ν denote the corresponding character. The *branching rule* for the irreducible characters of the unitary groups says that

$$\chi_\nu|_{U(N-1)} = \sum_{\lambda:\lambda \prec \nu} \chi_\lambda \quad \forall \nu \in \mathbb{GT}_N,\ N \geq 2, \tag{2.4}$$

where the vertical bar means the restriction map from $U(N)$ to $U(N-1)$. The graph $\mathbb{GT}$ just reflects the rule (2.4); for this reason one says that $\mathbb{GT}$ is the *branching graph* for the characters of the unitary groups.

It follows from (2.4) that $\mathrm{Dim}_N\,\nu$ equals $\chi_\nu(e)$, the value of χ_ν at the unit element of $U(N)$, which is the same as the dimension of the corresponding irreducible representation. This explains our notation.

Let $U(\infty)$ be the union of the groups (2.3). Although $U(\infty)$ is not a compact group, one can develop for it a rich theory of characters provided that the very notion of character is suitably revised:

Definition 2.2. By a *character* of $U(\infty)$ we mean a function $\chi : U(\infty) \to \mathbb{C}$ satisfying the following conditions:

- χ is continuous in the inductive limit topology on $U(\infty)$ (which simply means that the restriction of χ to every subgroup $U(N)$ is continuous);
- χ is a class function, that is, constant on conjugacy classes;
- χ is positive definite;
- $\chi(e) = 1$.

Next, observe that the set of all characters is a convex set and say that χ is an *extreme character* if it is an extreme point of this set.

The above definition makes sense for any topological group. In particular, the extreme characters of $U(N)$ are precisely the *normalized* irreducible characters

$$\widetilde{\chi}_\nu := \frac{\chi_\nu}{\chi_\nu(e)} = \frac{\chi_\nu}{\mathrm{Dim}_N\,\nu}, \quad \nu \in \mathbb{GT}_N,$$

and the set of all characters of $U(N)$ is an infinite-dimensional simplex; its vertices are the characters $\widetilde{\chi}_\nu$.

The extreme characters of $U(\infty)$ can be viewed as analogs of characters $\widetilde{\chi}_\nu$.

The representation-theoretic meaning of the extreme characters is that they correspond to *finite factor representations* of $U(\infty)$; see [30].

Proposition 2.3. *There is a natural bijective correspondence between the characters of the group $U(\infty)$ and the coherent systems on the graph $\mathbb{GT}$, which also induces a bijection between the extreme characters and the points of the boundary $\partial(\mathbb{GT})$.*

Proof. If χ is a character of $U(\infty)$, then for every $N = 1, 2, \dots$ the restriction χ to $U(N)$ is a convex combination of normalized characters $\widetilde{\chi}_\nu$. The corresponding coefficients, say $M_N(\nu)$, are nonnegative and sum to 1, so that they determine a probability distribution M_N on $\mathbb{GT}_N$. Further, the family $\{M_N : N = 1, 2, \dots\}$ is a coherent system. The correspondence $\chi \to \{M_N\}$ defined in this way is a bijection of the set of characters of $U(\infty)$ onto the set of coherent systems, which is also an isomorphism of convex sets. This entails a bijection between the extreme points of both sets, that is, the extreme characters and the points of $\partial(\mathbb{GT})$.

For more details, see [23] and especially Proposition 7.4 therein. $\square$

Informally, Proposition 2.3 says that the chain (2.2) is dual to the chain (2.3) and the boundary $\partial(\mathbb{GT})$ is a kind of dual object to $U(\infty)$.

2.4. The space Ω and the function $\Phi(u;\omega)$

Let $\mathbb{R}_+ \subset \mathbb{R}$ denote the set of nonnegative real numbers, $\mathbb{R}_+^\infty$ denote the product of countably many copies of $\mathbb{R}_+$, and set

$$\mathbb{R}_+^{4\infty+2} = \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+ \times \mathbb{R}_+.$$

Let $\Omega \subset \mathbb{R}_+^{4\infty+2}$ be the subset of sextuples

$$\omega = (\alpha^+, \beta^+; \alpha^-, \beta^-; \delta^+, \delta^-)$$

such that

$$\alpha^\pm = (\alpha_1^\pm \geq \alpha_2^\pm \geq \cdots \geq 0) \in \mathbb{R}_+^\infty, \qquad \beta^\pm = (\beta_1^\pm \geq \beta_2^\pm \geq \cdots \geq 0) \in \mathbb{R}_+^\infty,$$
$$\sum_{i=1}^\infty (\alpha_i^\pm + \beta_i^\pm) \leq \delta^\pm, \qquad \beta_1^+ + \beta_1^- \leq 1.$$

Equip $\mathbb{R}_+^{4\infty+2}$ with the product topology. An important fact is that, in the induced topology, Ω is a locally compact space. Moreover, it is metrizable and separable. Any subset in Ω of the form $\delta^+ + \delta^- \leq$ const is compact, which shows that a sequence of points in Ω goes to infinity if and only if the quantity $\delta^+ + \delta^-$ goes to infinity.

Set

$$\gamma^\pm = \delta^\pm - \sum_{i=1}^\infty (\alpha_i^\pm + \beta_i^\pm)$$

and note that γ^+, γ^- are nonnegative. For $u \in \mathbb{C}^*$ and $\omega \in \Omega$ set

$$\Phi(u; \omega) = e^{\gamma^+(u-1)+\gamma^-(u^{-1}-1)} \prod_{i=1}^\infty \frac{1 + \beta_i^+(u-1)}{1 - \alpha_i^+(u-1)} \frac{1 + \beta_i^-(u^{-1}-1)}{1 - \alpha_i^-(u^{-1}-1)}. \tag{2.5}$$

Here are some properties of $\Phi(u; \omega)$ as a function in variable u.

For any fixed ω, this is a meromorphic function in $u \in \mathbb{C}^*$ with poles on $(0, 1) \cup (1, +\infty)$. The poles do not accumulate to 1, so that the function is holomorphic in a neighborhood of the unit circle

$$\mathbb{T} := \{u \in \mathbb{C} : |u| = 1\}.$$

Obviously,

$$\Phi(1; \omega) = 1 \quad \forall \omega \in \Omega.$$

In particular, $\Phi(u; \omega)$ is well defined and continuous on $\mathbb{T}$.

Proposition 2.4. *One has*

$$|\Phi(u; \omega)| \leq 1 \quad \textit{for } u \in \mathbb{T}.$$

Proof. Indeed, the claim actually holds for every factor in (2.5):

$$|(1 - \alpha_i^\pm(u^{\pm 1} - 1))^{-1}| \leq 1, \qquad |1 + \beta_i^\pm(u^{\pm 1} - 1)| \leq 1,$$
$$|e^{\gamma^\pm(u^{\pm 1}-1)}| \leq 1. \quad \square \tag{2.6}$$

Proposition 2.5. *Different ω's correspond to different functions $\Phi(\,\cdot\,, \omega)$.*

Proof. See [22, Proof of Theorem 5.1, Step 3]. Here the condition $\beta_1^+ + \beta_1^- \leq 1$ plays the decisive role. $\square$

Proposition 2.6. *There exists a homeomorphism $S : \Omega \to \Omega$ such that*

$$\Phi(u; S\omega) = u\,\Phi(u; \omega).$$

Proof. Indeed, observe that

$$u(1 + \beta(u^{-1} - 1)) = 1 + (1 - \beta)(u - 1).$$

It follows that S has the following form: it deletes β_1^- from the list of the β^--coordinates of ω (so that β_2^- becomes coordinate number 1, β_3^- becomes coordinate number 2, etc.) and adds a new β^+ coordinate equal to $1 - \beta_1^-$. Note that this new coordinate is $\geq \beta_1^+$ (due to the condition $\beta_1^+ + \beta_1^- \leq 1$), so that it acquires number 1, β_1^+ becomes coordinate number 2, etc. All the remaining coordinates remain intact. □

2.5. The functions $\varphi_\nu(\omega)$ and the Markov kernels Λ_N^∞

Since $\Phi(\,\cdot\,; \omega)$ is regular in a neighborhood of $\mathbb{T}$, it can be expanded into a Laurent series:

$$\Phi(u; \omega) = \sum_{n=-\infty}^{\infty} \varphi_n(\omega) u^n,$$

where

$$\varphi_n(\omega) = \frac{1}{2\pi i} \oint_{\mathbb{T}} \Phi(u; \omega) \frac{du}{u^{n+1}}, \quad n \in \mathbb{Z}. \tag{2.7}$$

The Laurent coefficients $\varphi_n(\omega)$ play an important role in what follows.

More generally, we associate with every $\nu \in \mathbb{GT}_N$, $N = 1, 2, \dots$, the following function on Ω

$$\varphi_\nu(\omega) := \det[\varphi_{\nu_i - i + j}(\omega)]_{i,j=1}^N.$$

Recall that the *rational Schur function* in N variables is the Laurent polynomial indexed by a signature $\nu \in \mathbb{GT}_N$ and defined by

$$S_\nu(u_1, \dots, u_N) = \frac{\det\left[u_i^{\nu_j + N - j}\right]_{i,j=1}^N}{\prod\limits_{1 \leq i < j \leq N} (u_i - u_j)}.$$

These Laurent polynomials form a basis in the algebra $\mathbb{C}[u_1^{\pm 1}, \dots, u_N^{\pm 1}]^{\mathrm{sym}}$ of symmetric Laurent polynomials.

Proposition 2.7. *For $N = 1, 2, \dots$ the following expansion holds*

$$\Phi(u_1; \omega) \cdots \Phi(u_N; \omega) = \sum_{\nu \in \mathbb{GT}_N} \varphi_\nu(\omega) S_\nu(u_1, \dots, u_N), \tag{2.8}$$

where the series converges in a neighborhood of $\mathbb{T} \subset \mathbb{C}^*$ depending on $\omega \in \Omega$.

Proof. This is a very simple but fundamental fact. See, e.g., [30, Lemme 2]. □

Note that if (2.8) is interpreted as an identity of formal series, without addressing the question of convergence, then the result holds for an arbitrary two-sided infinite formal power series in u in place of $\Phi(u;\omega)$. Further, if the series is expanded on nonnegative powers of u only and the constant term is equal to 1, then the product on the left-hand side may be infinite. In that case the right-hand side becomes an expansion in Schur symmetric functions in infinitely many variables, indexed by arbitrary partitions. See, e.g., [16, pp. 99–100].

Proposition 2.8. *The functions $\varphi_\nu(\omega)$ are nonnegative.*

Proof. See [30, Proposition 2]. □

This fine property means that for any ω, all minors of the two-sided infinite Toeplitz matrix $[\varphi_{i-j}(\omega)]_{i,j\in\mathbb{Z}}$ extracted from several consecutive columns are nonnegative. But this actually implies that *all* minors are nonnegative (see [6, p. 218]). That is, the two-sided infinite sequence $\{\varphi_n(\omega)\}_{n\in\mathbb{Z}}$ is *totally positive*.

As is well known, the Laurent polynomials S_ν with $\nu\in\mathbb{GT}_N$ determine the irreducible characters of $U(N)$ in the sense that $\chi_\nu=S_\nu$ on the torus $\mathbb{T}^N=\mathbb{T}\times\cdots\times\mathbb{T}$ identified with the subgroup of diagonal matrices in $U(N)$. It follows that

$$S_\nu(\underbrace{1,\dots,1}_{N})=\operatorname{Dim}_N\nu \tag{2.9}$$

and, more generally,

$$S_\nu(u_1,\dots,u_K,\underbrace{1,\dots,1}_{N-K})=\sum_{\varkappa\in\mathbb{GT}_K}\operatorname{Dim}_{K,N}(\varkappa,\nu)S_\varkappa(u_1,\dots,u_K),\quad K<N. \tag{2.10}$$

Equalities (2.8)–(2.10) imply

Proposition 2.9. *Set*

$$\Lambda^\infty_N(\omega,\nu)=\operatorname{Dim}_N\nu\cdot\varphi_\nu(\omega), \tag{2.11}$$

where $N=1,2,\dots,\omega\in\Omega$, and $\nu\in\mathbb{GT}_N$.

(i) *Λ^∞_N is a Markov kernel, that is, $\Lambda^\infty_N(\omega,\nu)\ge 0$ for all ω and ν, and*

$$\sum_{\nu\in\mathbb{GT}_N}\Lambda^\infty_N(\omega,\nu)=1. \tag{2.12}$$

(ii) *For $N>K\ge 1$ there holds*

$$\Lambda^\infty_N\Lambda^N_K=\Lambda^\infty_K. \tag{2.13}$$

Or, in more detail,

$$\sum_{\nu\in\mathbb{GT}_N}\Lambda^\infty_N(\omega,\nu)\Lambda^N_K(\nu,\varkappa)=\Lambda^\infty_K(\omega,\varkappa),\quad\forall\omega\in\Omega,\quad\forall\varkappa\in\mathbb{GT}_K. \tag{2.14}$$

Proof. The property $\Lambda^\infty_N(\omega,\nu)\ge 0$ is ensured by Proposition 2.8.

Plug in $u_1=\cdots=u_N=1$ into (2.8) and use the fact that $\Phi(1;\omega)=1$. Then, because of (2.9), we get (2.12).

Likewise, plug in $u_{K+1} = \cdots = u_N = 1$ into (2.8) and apply (2.10). Comparing the result with the expansion

$$\Phi(u_1; \omega) \cdots \Phi(u_K; \omega) = \sum_{\varkappa \in \mathbb{GT}_K} \varphi_\varkappa(\omega) S_\varkappa(u_1, \dots, u_K)$$

we get (2.14). □

2.6. *The Feller property*

For a locally compact metrizable separable space X, denote by $C_0(X)$ the space of real-valued continuous functions vanishing at infinity. This is a separable Banach space with respect to the supremum norm. In particular, the definition makes sense for $X = \Omega$ and also for $\mathbb{GT}_N$, since this a countable discrete space. Let us interpret functions $f \in C_0(\mathbb{GT}_N)$ as column vectors.

Proposition 2.10. *The functions $\varphi_n(\omega), n \in \mathbb{Z}$, are continuous functions on Ω vanishing at infinity.*

An immediate consequence of this result is the following.

Corollary 2.11. *For every $N = 1, 2, \dots$ the Markov kernel Λ_N^∞ is a Feller kernel, meaning that the map $f \mapsto \Lambda_N^\infty f$ is a continuous (actually, contractive) linear operator $C_0(\mathbb{GT}_N) \to C_0(\Omega)$.*

Proof of Corollary 2.11. It follows from the proposition and the definition of the kernel that for $\nu \in \mathbb{GT}_N$ fixed, the function $\omega \mapsto \Lambda_N^\infty(\omega, \nu)$ is continuous and vanishes at infinity. This is equivalent to the Feller property. □

Proof of Proposition 2.10. The continuity is established in [23, Proof of Theorem 8.1, Step 1].

Now we must prove that for any fixed $n \in \mathbb{Z}$ and any sequence of points $\{\omega(k)\}$ in Ω converging to infinity one has $\lim_{k\to\infty} \varphi_n(\omega(k)) = 0$. It is enough to prove a weaker claim that the same limit relation holds for a subsequence in $\{\omega(k)\}$. Below we write $\alpha_i^\pm(k)$, $\beta_i^\pm(k)$, $\delta^\pm(k)$ for the coordinates of $\omega(k)$.

Step 1. We may assume that $\sup_{k\geq 1} \alpha_1^\pm(k) < \infty$. Indeed, if there is a subsequence $\{k_m\}_{m\geq 1}$ such that $\alpha_1^\pm(k_m) \to \infty$, then along this subsequence $(1 - \alpha_1^\pm(u^{\pm 1} - 1))^{-1}$ tends to zero uniformly on any compact subset of $\mathbb{T} \setminus \{u = 1\}$, which implies that the right-hand side of (2.7) tends to zero.

Let us fix $A > 0$ such that $\sup_k \alpha_1^\pm(k) \leq A$.

Step 2. Assume ω ranges over the subset of elements of Ω with $\alpha_1^\pm \leq A$ and $\beta_1^\pm \leq \frac{1}{2}$. Then for any $\epsilon > 0$,

$$\lim_{\delta^+ + \delta^- \to \infty} \Phi(u; \omega) = 0 \quad \text{uniformly on } \{u \in \mathbb{T},\ \Re u \leq 1 - \epsilon\}.$$

Indeed, assume $0 \leq \beta \leq \frac{1}{2}$ and $0 \leq \alpha \leq A$. For u on the unit circle with $\Re u \leq 1 - \epsilon$ we have elementary estimates

$$\begin{aligned} |1 + \beta(u-1)|^2 &= (1-\beta)^2 + \beta^2 + 2\beta(1-\beta)\Re u \\ &= 1 - 2\beta(1-\beta)(1 - \Re u) \leq 1 - 2\beta(1-\beta)\epsilon \leq 1 - \beta\epsilon \leq e^{-\beta\epsilon}, \\ |1 - \alpha(u-1)|^{-2} &= (1 + 2\alpha(1+\alpha)(1 - \Re u))^{-1} \\ &\leq (1 + 2\alpha(1+\alpha)\epsilon)^{-1} \leq (1 + 2\alpha\epsilon)^{-1} \leq e^{-\mathrm{const}\,\alpha\epsilon} \end{aligned}$$

with a suitable constant const > 0 that depends only on A,

$$|e^{\gamma^+(u-1)+\gamma^-(u^{-1}-1)}|^2 = e^{-2(\gamma^++\gamma^-)(1-\Re u)} \le e^{-2(\gamma^++\gamma^-)\epsilon}.$$

Thus, if

$$\delta^+ + \delta^- = \gamma^+ + \gamma^- + \sum_{i=1}^{\infty}(\alpha_i^+ + \beta_i^+ + \alpha_i^- + \beta_i^-) \to \infty$$

then at least one of the right-hand sides in these estimates yields an infinitesimally small contribution, and consequently $\Phi(u;\omega)$ must be small.

Thus, under the above assumptions on ω, we see that $\omega \to \infty$ implies $\varphi_n(\omega) \to 0$ *uniformly* on $n \in \mathbb{Z}$.

Step 3. Now we get rid of the restriction $\beta_1^\pm \le \frac{1}{2}$. Set

$$B^\pm(k) = \#\left\{i \ge 1 \mid \beta_i^\pm(k) > \frac{1}{2}\right\}.$$

Since for any $k \ge 1$ we have $\beta_1^+(k) + \beta_1^-(k) \le 1$, at least one of the numbers $B^\pm(k)$ is equal to 0. For inapplicability of the Step 2 argument, for any subsequence $\{\omega_{k_m}\}$ of our sequence $\{\omega(k)\}$, we must have $B^+(k_m)+B^-(k_m) \to \infty$. Hence, possibly passing to a subsequence and switching $+$ and $-$, we may assume that $B^+(k) \to \infty$ as $k \to \infty$.

Set

$$\tilde\omega(k) := S^{-B^+(k)}\omega(k),$$

where S is the homeomorphism from Proposition 2.6. In words, $\tilde\omega(k)$ is obtained from $\omega(k)$ as follows: each β^+-coordinate of $\omega(k)$ that is $>1/2$ is transformed into a β^- coordinate of $\tilde\omega(k)$ equal to 1 minus the original β^+-coordinate; all other coordinates are the same (equivalently, the function $\Phi(u;\omega)$is multiplied by $u^{-B^+(k)}$). Let $(\tilde\alpha^\pm(k), \tilde\beta^\pm(k), \tilde\gamma^\pm(k), \tilde\delta^\pm(k))$ be the coordinates of $\tilde\omega(k)$.

Step 4. Since no β-coordinates of $\tilde\omega(k)$ are greater than $1/2$, the result of Step 2 implies that if $\sup(\tilde\delta^+(k) + \tilde\delta^-(k)) = \infty$ then $\varphi_n(\omega(k)) = \varphi_{n-B^+(k)}(\tilde\omega(k)) \to 0$ as $k \to \infty$ along an appropriate subsequence (because the conclusion of that step holds uniformly on $n \in \mathbb{Z}$). Hence, it remains to examine the case when $\tilde\delta^+(k) + \tilde\delta^-(k)$ is bounded.

Let us deform the integration contour in (2.7) to $|u| = R$ with $A/(1+A) < R < 1$. Using the estimates (for $|u| = R, 0 \le \alpha \le A, 0 \le \beta \le \frac{1}{2}$)

$$\begin{aligned}
&|1+\beta(u^{\pm1}-1)| \le 1+\beta|u^{\pm1}-1| \le e^{\mathrm{const}_1\,\beta},\\
&|1-\alpha(u^{\pm1}-1)|^{-1} \le |1-\alpha(R^{\pm1}-1)|^{-1} \le e^{\mathrm{const}_2\,\alpha},\\
&|e^{\gamma(u^{\pm1}-1)}| \le e^{\mathrm{const}_3\,\gamma}
\end{aligned}$$

with suitable $\mathrm{const}_j > 0$, $j = 1, 2, 3$, we see that

$$|\Phi(u;\tilde\omega(k))| \le e^{\mathrm{const}_4(\tilde\delta^+(k)+\tilde\delta^-(k))}$$

for a $\mathrm{const}_4 > 0$, which remains bounded.

On the other hand, as $k \to \infty$, the factor $u^{-n-1+B^+(k)}$ in the integral representation (2.7) for $\varphi_{n-B^+(k)}(\tilde\omega(k))$ tends to 0 uniformly in u, because $B^+(k) \to +\infty$ and $|u| = R < 1$. Hence, $\varphi_n(\omega(k)) = \varphi_{n-B^+(k)}(\tilde\omega(k)) \to 0$ as $k \to \infty$, and the proof of the proposition is complete. $\square$

The following proposition is an analog of Corollary 2.11 for the stochastic matrices Λ^N_K. It is much easier to prove.

Proposition 2.12. *Let $K < N$. If $\varkappa \in \mathbb{GT}_K$ is fixed and ν goes to infinity in the countable discrete space $\mathbb{GT}_N$, then $\Lambda^N_K(\nu, \varkappa) \to 0$. Equivalently, the map $f \mapsto \Lambda^N_K f$ is a continuous (actually contractive) operator $C_0(\mathbb{GT}_K) \to C_0(\mathbb{GT}_N)$, so that Λ^N_K is Feller.*

Proof. Because of (2.1) it suffices to prove the assertion of the proposition in the particular case when $K = N-1$. The classic Weyl's dimension formula says that

$$\operatorname{Dim}_N \nu = \prod_{1\le i<j\le N} \frac{\nu_i - \nu_j + j - i}{j-i}. \tag{2.15}$$

Therefore, for $\varkappa \prec \nu$

$$\Lambda^N_{N-1}(\nu,\varkappa) = \frac{(N-1)! \prod\limits_{1\le i<j\le N-1} (\varkappa_i - \varkappa_j + j - i)}{\prod\limits_{1\le i<j\le N} (\nu_i - \nu_j + j - i)}, \tag{2.16}$$

otherwise $\Lambda^N_{N-1}(\nu,\varkappa) = 0$.

Fix $\varkappa$ and assume ν is such that $\varkappa \prec \nu$. Then $\nu \to \infty$ is equivalent to either $\nu_1 \to +\infty$, or $\nu_N \to -\infty$, or both; all other coordinates of ν must remain bounded because of the interlacing condition $\varkappa \prec \nu$. But then it is immediate that at least one of the factors in the denominator of (2.16) tends to infinity. Thus, the ratio goes to 0 as needed. $\square$

2.7. *Totality of* $\{\varphi_\nu\}$

Given $\nu \in \mathbb{GT}_N$, write the expansion of $S_\nu(u_1,\dots,u_N)$ in monomials,

$$S_\nu(u_1,\dots,u_N) = \sum c(\nu; n_1,\dots,n_N) u_1^{n_1},\dots,u_N^{n_N},$$

where the sum is over N-tuples of integers $(n_1,\dots,n_N)$ with

$$n_1 + \dots + n_N = \nu_1 + \dots + \nu_N.$$

Obviously, the sum is actually finite. Further, the coefficients are nonnegative integers: they are nothing else than the weight multiplicities of the irreducible representation of $U(N)$ indexed by ν. In a purely combinatorial way, this can be also deduced from the branching rule for the characters: it follows that $c(\nu; n_1,\dots,n_N)$ equals the number of triangular Gelfand–Tsetlin schemes $\{\lambda^{(j)}_i : 1 \le i \le j \le N\}$ with the top row $\lambda^{(N)} = \nu$ and such that

$$\left(\lambda^{(j)}_1 + \dots + \lambda^{(j)}_j\right) - \left(\lambda^{(j-1)}_1 + \dots + \lambda^{(j-1)}_{j-1}\right) = n_j, \quad j = 2,\dots,N;\ \lambda^{(1)}_1 = n_1.$$

By virtue of Proposition 2.10, the functions $\varphi_n(\omega)$ lie in $C_0(\Omega)$. Therefore, the same holds for the functions $\varphi_\nu(\omega)$.

The results of the next proposition and its corollary are similar to [30, Lemme 3], and the main idea of the proof is the same.

Proposition 2.13. *For any $N = 1, 2, \dots$ and any N-tuple $(n_1,\dots,n_N) \in \mathbb{Z}^N$,*

$$\varphi_{n_1}(\omega)\cdots\varphi_{n_N}(\omega) = \sum_\nu c(\nu; n_1,\dots,n_N)\varphi_\nu(\omega), \tag{2.17}$$

where the series on the right-hand side converges in the norm topology of the Banach space $C_0(\Omega)$.

Proof. First, let us show that (2.17) holds pointwise. Indeed, this follows from the comparison of the following two expansions:

$$\begin{aligned}\Phi(u_1;\omega)\cdots\Phi(u_n;\omega) &= \sum_{(n_1,\dots,n_N)\in\mathbb{Z}^N} \varphi_{n_1}(\omega)\cdots\varphi_{n_N}(\omega)u_1^{n_1}\cdots u_N^{n_N}\\ &= \sum_{\nu\in\mathbb{GT}_N}\varphi_\nu(\omega)S_\nu(u_1,\dots,u_N).\end{aligned}$$

Next, as all the functions in (2.17) are continuous and the summands on the right-hand side are nonnegative, the series converges uniformly on compact subsets of Ω.

Finally, as all the functions vanish at infinity, monotone convergence also implies convergence in norm. □

Corollary 2.14. *The family* $\{\varphi_\nu : \nu\in\mathbb{GT}\}$ *is total in the Banach space* $C_0(\Omega)$*, that is, the linear span of these functions is dense.*

Proof. Let $\Omega\cup\{\infty\}$ denote the one-point compactification of Ω. It suffices to show that the family $\{\varphi_\nu : \nu\in\mathbb{GT}\}$ together with the constant 1 is total in the real Banach algebra $C(\Omega\cup\{\infty\})$. By Proposition 2.13, the linear span of the family contains the subalgebra generated by 1 and the functions $\varphi_n(\omega)$. By virtue of Proposition 2.5, this subalgebra separates points of Ω. Next, for any fixed $\omega\in\Omega$, the function $u\to\Phi(u;\omega)$ cannot be identically equal to 0, which implies that all the functions φ_n cannot vanish at ω simultaneously. On the other hand, recall that they vanish at ∞. This means that our subalgebra separates points of $\Omega\cup\{\infty\}$, too. Therefore, we may apply the Stone–Weierstrass theorem. □

2.8. *Description of the boundary*

Theorem 2.15. *For an arbitrary coherent system* $\{M_K : K=1,2,\dots\}$ *of distributions on the graph* $\mathbb{GT}$ *there exists a probability Borel measure* M *on* Ω *such that*

$$M_K = M\Lambda_K^\infty,\quad K=1,2,\dots,\tag{2.18}$$

that is,

$$M_K(\varkappa)=\int_\Omega M(d\omega)\Lambda_K^\infty(\omega,\varkappa),\quad \varkappa\in\mathbb{GT}_K,\ K=1,2,\dots,$$

where $\Lambda_K^\infty:\Omega\dashrightarrow\mathbb{GT}_K$ *is the Markov kernel defined in Section* 2.5.

Such a measure is unique, and any probability Borel measure M *on* Ω *gives rise in this way to a coherent system.*

In Section 3, we reduce Theorem 2.15 to Theorem 3.1 whose proof in turn is given in the subsequent sections.

Let us say that M is the *boundary measure* of a given coherent system $\{M_K\}$.

By virtue of the theorem, the boundary measures of the extreme coherent systems are exactly the delta measures on Ω. Therefore, the theorem implies the following corollary.

Corollary 2.16. *There exists a bijection $\partial(\mathbb{GT}) \leftrightarrow \Omega$, under which the extreme coherent system $\{M_K^{(\omega)} : K = 1, 2, \dots\}$ corresponding to a point $\omega \in \Omega$ is given by the formula*

$$M_K^{(\omega)}(\varkappa) = \Lambda_K^{\infty}(\omega, \varkappa), \quad \varkappa \in \mathbb{GT}_K,\ K = 1, 2, \dots.$$

Conversely, the theorem can be derived from the result of the corollary: the necessary arguments can be found in [30, Théorème 2] and [23, Theorems 9.1 and 9.2].

3. The Uniform Approximation Theorem

Recall the definition of the *modified Frobenius coordinates* of a Young diagram λ (see [27]): first, introduce the conventional Frobenius coordinates of λ:

$$p_i = \lambda_i - i, \qquad q_i = (\lambda')_i - i, \quad i = 1, \dots, d(\nu),$$

where λ' stands for the transposed diagram and $d(\lambda)$ denotes the number of diagonal boxes of a Young diagram λ. The modified Frobenius coordinates differ from the conventional ones by the addition of one-halves:

$$a_i = p_i + \frac{1}{2}, \qquad b_i = q_i + \frac{1}{2}.$$

Next, it is convenient to set

$$a_i = b_i = 0, \quad i > d(\lambda),$$

which makes it possible to assume that index i ranges over $\{1, 2, \dots\}$. Note that $\sum_{i=1}^{\infty}(a_i + b_i) = |\lambda|$, where $|\lambda|$ denotes the total number of boxes in λ.

Using the modified Frobenius coordinates we define for every $N = 1, 2, \dots$ an embedding $\mathbb{GT}_N \hookrightarrow \Omega$ in the following way. Let $\nu \in \mathbb{GT}_N$ be given. We represent ν as a pair (ν^+, ν^-) of partitions or, equivalently, Young diagrams: ν^+ consists of positive ν_i's, ν^- consists of minus negative ν_i's, and zeros can go in either of the two:

$$\nu = (\nu_1^+, \nu_2^+, \dots, -\nu_2^-, -\nu_1^-).$$

Write $a_i^{\pm}, b_i^{\pm}$ for the modified Frobenius coordinates of $\nu^{\pm}$. Then we assign to ν the point $\omega(\nu) \in \Omega$ with coordinates

$$\alpha_i^{\pm} = \frac{a_i^{\pm}}{N}, \qquad \beta_i^{\pm} = \frac{b_i^{\pm}}{N} \quad (i = 1, 2, \dots), \qquad \delta^{\pm} = \frac{|\nu^{\pm}|}{N}.$$

Clearly, the correspondence $\mathbb{GT}_N \ni \nu \mapsto \omega(\nu)$ is indeed an embedding. The image of $\mathbb{GT}_N$ under this embedding is a locally finite set in Ω: its intersection with any relatively compact subset is finite.

Note also that for points $\omega = \omega(\nu)$, $\delta^{\pm}$ exactly equals $\sum(\alpha_i^{\pm} + \beta_i^{\pm})$.

Theorem 3.1 (*Uniform Approximation Theorem*)**.** *For any fixed $K = 1, 2, \dots$ and $\varkappa \in \mathbb{GT}_K$*

$$\lim_{N\to\infty} \sup_{\nu\in\mathbb{GT}_N} \left| \Lambda_K^N(\nu, \varkappa) - \Lambda^{\infty}(\omega(\nu), \varkappa) \right| = 0. \tag{3.1}$$

Derivation of Theorem 2.15 from Theorem 3.1. We will verify the assertions of Theorem 2.15 in the reverse order.

The fact that any probability Borel measure M on Ω serves as the boundary measure of a coherent system $\{M_K : K = 1, 2, \dots\}$ is obvious from (2.13).

Next, if a coherent system $\{M_K\}$ has a boundary measure on Ω, then its uniqueness directly follows from Corollary 2.14. Indeed, here we use the fact that the space of finite signed measures on Ω is dual to the Banach space $C_0(\Omega)$.

Now let us deduce from Theorem 3.1 the existence of the boundary measure for every coherent system $\{M_K\}$.

Write the compatibility relation for our coherent system,

$$M_N \Lambda^N_K = M_K, \quad N > K,$$

in the form

$$\langle M_N, \Lambda^N_K(\,\cdot\,, \varkappa)\rangle = M_K(\varkappa), \quad N > K,\ \varkappa \in \mathbb{GT}_K,$$

where $\Lambda^N_K(\,\cdot\,, \varkappa)$ is viewed as the function $\nu \mapsto \Lambda^N_K(\nu, \varkappa)$ on $\mathbb{GT}_N$ and the angle brackets denote the canonical pairing between measures and functions.

Denote by $\widetilde{M}_N$ the pushforward of M_N under the embedding $\mathbb{GT}_N \hookrightarrow \Omega$ defined by $\nu \mapsto \omega(\nu)$; this is a probability measure on Ω supported by the subset

$$\widetilde{\mathbb{GT}_N} := \{\omega(\nu) : \nu \in \mathbb{GT}_N\} \subset \Omega.$$

Next, regard $\Lambda^N_K(\nu, \varkappa)$ as a function of variable ω ranging over $\widetilde{\mathbb{GT}_N}$ and denote this function by $\widetilde{\Lambda}^N_K(\,\cdot\,, \varkappa)$. Then we may write the above compatibility relation as

$$\langle \widetilde{M}_N, \widetilde{\Lambda}^N_K(\,\cdot\,, \varkappa)\rangle = M_K(\varkappa), \quad N > K.\ \varkappa \in \mathbb{GT}_K.$$

By virtue of Theorem 3.1, for any $\omega \in \widetilde{\mathbb{GT}_N}$

$$\widetilde{\Lambda}^N_K(\omega, \varkappa) = \Lambda^\infty_K(\omega, \varkappa) + o(1), \quad N \gg K,$$

where the remainder term $o(1)$ depends on $\varkappa$ but is uniform on $\omega \in \widetilde{\mathbb{GT}_N}$. Since $\widetilde{M}_N$ is a probability measure, we get

$$\langle \widetilde{M}_N, \Lambda^\infty_K(\,\cdot\,, \varkappa)\rangle = M_K(\varkappa) + o(1), \quad N \gg K. \tag{3.2}$$

The space Ω is not only locally compact but also metrizable and separable. Therefore, any sequence of probability measures on Ω always has a nonvoid set of partial limits in the vague topology (which is nothing else than the weak-* topology of the Banach dual to $C_0(\Omega)$). Note that, in general, it may happen that such limits are sub-probability measures (the total mass is strictly less than 1).

So, let M be any partial vague limit of the sequence $\{\widetilde{M}_N\}$. Passing to the limit in (3.2) along an appropriate subsequence of indices N we get

$$\langle M, \Lambda^\infty_K(\,\cdot\,, \varkappa)\rangle = M_K(\varkappa), \quad K = 1, 2, \dots,\ \varkappa \in \mathbb{GT}_K, \tag{3.3}$$

which is equivalent to the desired relation (2.18).

Finally, once relation (2.18) is established, M must be a probability measure because otherwise the total mass of M_K would be strictly less than 1, which is impossible. □

The rest of the section is a comment to Theorem 3.1, and the proof of the theorem is given next in Sections 4–8.

Recall that both $\Lambda^N_K(\nu,\varkappa)$ and $\Lambda^\infty_K(\omega,\varkappa)$ involve one and the same common factor $\operatorname{Dim}_K \varkappa$:

$$\Lambda^N_K(\nu,\varkappa) = \operatorname{Dim}_K \varkappa \cdot \frac{\operatorname{Dim}_{K,N}(\varkappa,\nu)}{\operatorname{Dim}_N \nu}, \qquad \Lambda^\infty_K(\omega,\varkappa) = \operatorname{Dim}_K \varkappa \cdot \varphi_\nu(\omega).$$

As ν varies together with N, this factor remains intact. For this reason, in what follows, we ignore it and study the relative dimension

$$\frac{\operatorname{Dim}_{K,N}(\varkappa,\nu)}{\operatorname{Dim}_N \nu}. \tag{3.4}$$

Incidentally, we get explicit formulas for this quantity (see Theorem 7.2 and its modification, Proposition 7.3).

Remark 3.2. Recall that for the denominator in (3.4) there is a simple expression, (2.15). Let us also write down an expression for the numerator. Assume that $\varkappa$ and ν are as in Theorem 3.1, and assume additionally that $\varkappa_K \ge \nu_N$ (otherwise $\operatorname{Dim}_{K,N}(\varkappa,\nu) = 0$). Define partitions $\bar\nu$ and $\bar\varkappa$ as follows:

$$\bar\nu = (\nu_1 - \nu_N, \dots, \nu_{N-1} - \nu_N, 0, 0, \dots)$$
$$\bar\varkappa = (\varkappa_1 - \nu_N, \dots, \varkappa_K - \nu_N, 0, 0, \dots).$$

We will also assume that $\bar\nu_i \ge \bar\varkappa_i$ for all $i = 1, 2, \dots$ (otherwise $\operatorname{Dim}_{K,N}(\varkappa,\nu) = 0$). This enables us to define the skew Schur function $S_{\bar\nu/\bar\varkappa}$. Then one has

$$\operatorname{Dim}_{K,N}(\varkappa,\nu) = S_{\bar\nu/\bar\varkappa}(\underbrace{1,\dots,1}_{N-K}) = \det\left[h_{\bar\nu_i - \bar\varkappa_j - i + j}(\underbrace{1,\dots,1}_{N-K})\right] \tag{3.5}$$

where the order of the determinant is any number greater than or equal to $\ell(\bar\nu)$ (the number of nonzero coordinates in $\bar\nu$) and

$$h_m(\underbrace{1,\dots,1}_{N-K}) = \begin{cases} \dfrac{(m+N-K-1)!}{m!(N-K-1)!}, & m \ge 0 \\ 0, & m < 0. \end{cases} \tag{3.6}$$

(The proof of the first equality in (3.5) is an easy exercise, and the remaining equalities are standard facts.)

Combining (2.15), (3.5) and (3.6) we get a closed expression for the relative dimension (3.4). However, it is unclear how one could use it for the problem of asymptotic analysis that we need. The formulas of Section 7, on the contrary, are difficult to prove, but they have the advantage to be well adapted to asymptotic analysis. Another advantage is that they involve determinants of order K, while the order of determinant in (3.5) is generically $N-1$. Because of this, for $N \gg K$ and generic ν the formulas of Section 7 seem to be more efficient than (3.5) from the purely computational viewpoint, too.

4. A Cauchy-type identity

The classical Cauchy identity for the Schur symmetric functions is

$$\prod_{i,j}\frac{1}{1-x_iy_j}=\sum_{\mu}S_\mu(x_1,x_2,\dots)S_\mu(y_1,y_2,\dots),$$

see e.g. [17, Section I.4]. Here summation is over all partitions μ and $S_\mu(x_1,x_2,\dots)$ denotes the Schur function indexed by μ. For finitely many indeterminates the identity takes the form

$$\prod_{i=1}^{N}\prod_{j=1}^{K}\frac{1}{1-x_iy_j}=\sum_{\ell(\mu)\le\min(N,K)}S_\mu(x_1,\dots,x_N)S_\mu(y_1,\dots,y_K). \tag{4.1}$$

Here the Schur functions turn into the Schur polynomials and $\ell(\mu)$ denotes the length of partition μ, i.e. the number of its nonzero parts.

The purpose of this section is to derive an analog of identity (4.1) where the Schur polynomials in x's are replaced by the shifted Schur polynomials [21], and the Schur polynomials in y's are replaced by other Schur-type functions, the dual symmetric Schur functions [18]. Let us give their definition.

The *shifted Schur polynomial* with N variables and index μ is given by the formula

$$S^*_\mu(x_1,\dots,x_N)=\frac{\det[(x_i+N-i)^{\downarrow\mu_j+N-j}]}{\prod\limits_{i<j}(x_i-x_j-i+j)}.$$

Here indices i and j range over $\{1,\dots,N\}$, and $x^{\downarrow m}$ is our notation for the mth falling factorial power of variable x,

$$x^{\downarrow m}=\frac{\Gamma(x+1)}{\Gamma(x+1-m)}=x(x-1)\cdots(x-m+1). \tag{4.2}$$

The polynomial $S^*_\mu(x_1,\dots,x_N)$ is symmetric in shifted variables $x'_i:=x_i-i$, and one has

$$S^*_\mu(x_1,\dots,x_N)=S_\mu(x'_1,\dots,x'_N)+\text{lower degree terms}.$$

This implies that, as functions in shifted variables $x'_1,\dots,x'_N$, the polynomials S^*_μ form a basis in the ring $\mathbb{C}[x'_1,\dots,x'_N]^{\mathrm{sym}}$ of N-variate symmetric polynomials. For more details, see [21].

By the *dual Schur symmetric function* in K variables with index μ we mean the following function

$$\sigma_\mu(t_1,\dots,t_K)=(-1)^{K(K-1)/2}\,\frac{\det\left[\frac{\Gamma(t_i+j-\mu_j)}{\Gamma(t_i+1)}\right]}{\prod\limits_{i<j}(t_i-t_j)}, \tag{4.3}$$

where i and j range over $\{1,\dots,K\}$ and the matrix in the numerator is of order K. The (i,j) entry of this matrix is a rational function in variable t_i, so that σ_μ is a rational function in $t_1,\dots,t_K$. Clearly, it is symmetric.

Let $\mathbb{C}(t_1,\dots,t_K)^{\mathrm{sym}}\subset\mathbb{C}(t_1,\dots,t_K)$ denote the subfield of symmetric rational functions and $\mathbb{C}(t_1,\dots,t_K)^{\mathrm{sym}}_{\mathrm{reg}}\subset\mathbb{C}(t_1,\dots,t_K)^{\mathrm{sym}}$ be the subspace of functions regular about the point $(t_1,\dots,t_K)=(\infty,\dots,\infty)$. We will also regard the space $\mathbb{C}(t_1,\dots,t_K)^{\mathrm{sym}}_{\mathrm{reg}}$ as a subspace in

$\mathbb{C}[[t_1^{-1},\dots,t_K^{-1}]]^{\mathrm{sym}}$, the ring of symmetric formal power series in variables $t_1^{-1},\dots,t_K^{-1}$. There is a canonical topology in this ring: the I-adic topology determined by the ideal I of the series without the constant term. The Schur polynomials in $t_1^{-1},\dots,t_K^{-1}$ form a topological basis in $\mathbb{C}[[t_1^{-1},\dots,t_K^{-1}]]^{\mathrm{sym}}$, meaning that every element of the ring is uniquely represented as an infinite series in these polynomials.

We claim that functions σ_μ belong to $\mathbb{C}(t_1,\dots,t_K)^{\mathrm{sym}}_{\mathrm{reg}}$ and form another topological basis in the ring $\mathbb{C}[[t_1^{-1},\dots,t_K^{-1}]]^{\mathrm{sym}}$. Indeed, σ_μ is evidently symmetric. To analyze its behavior about $(\infty,\dots,\infty)$, set $y_i:=t_i^{-1}$ and observe that

$$\frac{(-1)^{K(K-1)/2}}{\prod\limits_{i<j}(t_i-t_j)}=\frac{(y_1\cdots y_K)^{K-1}}{\prod\limits_{i<j}(y_i-y_j)}$$

and

$$y_i^{K-1}\frac{\Gamma(t_i+j-\mu_j)}{\Gamma(t_i+1)}=y_i^{\mu_j+K-j}+\text{higher degree terms in } y_i.$$

It follows that

$$\sigma_\mu(t_1,\dots,t_K)=S_\mu(y_1,\dots,y_K)+\text{higher degree terms in } y_1,\dots,y_K,$$

which entails our claim.

Note that functions σ_μ are a special case of more general *multi-parameter* dual Schur functions defined in [18].

In the definitions above we tacitly assumed that $\ell(\mu)$ does not exceed the number of variables; otherwise the corresponding function is set to be equal to zero. Under this convention the following *stability property* holds:

$$\begin{aligned}&S^*_\mu(x_1,\dots,x_N)|_{x_N=0}=S^*_\mu(x_1,\dots,x_{N-1}),\\&\sigma_\mu(t_1,\dots,t_K)|_{t_K=\infty}=\sigma_\mu(t_1,\dots,t_{K-1}).\end{aligned}$$

Both relations are verified in the same way as the stability property for the ordinary Schur polynomials. The detailed argument for the first relation can be found in [21, Proposition 1.3].

Proposition 4.1 (*Cauchy-Type Identity, Cf.* (4.1))**.** *One has*

$$\prod_{i=1}^{N}\prod_{j=1}^{K}\frac{t_j+i}{t_j+i-x_i}=\sum_{\ell(\mu)\le\min(N,K)}S^*_\mu(x_1,\dots,x_N)\sigma_\mu(t_1,\dots,t_K). \tag{4.4}$$

Here the infinite series on the right-hand side is the expansion with respect to the topological basis $\{\sigma_\mu\}$ of $(\mathbb{C}[x_1,\dots,x_N]^{\mathrm{sym}})[[t_1^{-1},\dots,t_K^{-1}]]^{\mathrm{sym}}$, the topological ring of symmetric formal power series in variables $t_1^{-1},\dots,t_K^{-1}$ with coefficient ring $\mathbb{C}[x_1,\dots,x_N]^{\mathrm{sym}}$. A more general form of the identity can be found in [18].

Proof. It suffices to prove (4.4) for $N=K$. Indeed, the general case is immediately reduced to this one by making use of the stability property by adding a few extra variables x_i or t_j and then specializing them to 0 or ∞, respectively. Thus, in what follows we will assume $N=K$.

In the simplest case $N = K = 1$, (4.4) takes the form

$$\frac{t+1}{t+1-x} = \sum_{m=0}^{\infty} \frac{x^{\downarrow m}}{t^{\downarrow m}}, \tag{4.5}$$

which is just formula (12.3) in [21].

Using (4.5), we will reduce the case $N = K > 1$ of (4.4) to Cauchy's determinant formula. Indeed, set $x_i' = x_i + K - i$, $m_i = \mu_i + K - i$, and denote by symbol $V(\cdot)$ the Vandermonde in K variables. Multiplying the right-hand side of (4.4) by $V(x_1', \dots, x_K')V(t_1, \dots, t_K)$ we transform it to

$$(-1)^{K(K-1)/2} \prod_{j=1}^{K} \frac{\Gamma(t_j+K)}{\Gamma(t_j+1)} \sum_{m_1 > \cdots > m_K \geq 0} \det\left[x_i'^{\downarrow m_j}\right] \det\left[\frac{1}{(t_i+K-1)^{\downarrow m_j}}\right], \tag{4.6}$$

where both determinants are of order K.

A well-known trick allows one to write the sum on the right-hand side as a single determinant:

$$\sum_{m_1 > \cdots > m_K \geq 0} \det\left[x_i'^{\downarrow m_j}\right] \det\left[\frac{1}{(t_i+K-1)^{\downarrow m_j}}\right] = \det[A(i,j)]$$

with

$$A(i,j) = \sum_{m=0}^{\infty} \frac{x_i'^{\downarrow m}}{(t_j+K-1)^{\downarrow m}} = \frac{t_j+K}{t_j+K-x_i'},$$

where the last equality follows from (4.5).

By Cauchy's determinant formula,

$$\det[A(i,j)] = (-1)^{K(K-1)/2} \prod_{j=1}^{K} (t_j+K) \cdot \frac{V(x_1', \dots, x_K')V(t_1, \dots, t_K)}{\prod_{i,j}(t_j+K-x_i')}. \tag{4.7}$$

Observe that $t_j + K - x_i' = t_j + i - x_i$. Taking this into account and plugging in (4.7) instead of the sum in (4.6) we see that the plus–minus sign disappears and the resulting expression for (4.6) coincides with the left-hand side of (4.4) (for $N = K$) multiplied by the same product of two Vandermonde determinants. This concludes the proof. $\square$

5. A generating function for the relative dimension

Throughout this section, we assume that $N \geq K$ are two natural numbers, $\varkappa$ ranges over $\mathbb{GT}_K$ and ν ranges over $\mathbb{GT}_N$.

Set

$$\mathfrak{S}_{\varkappa|N}(t_1, \dots, t_K) = (-1)^{K(K-1)/2} \prod_{i=1}^{K} \frac{(N-K)!}{(N-K+i-1)!} \times \frac{\det\left[\frac{\Gamma(t_i+1+N)\Gamma(t_i+j-\varkappa_j)}{\Gamma(t_i+1)\Gamma(t_i+j-\varkappa_j+N-K+1)}\right]}{V(t_1, \dots, t_K)}, \tag{5.1}$$

where the determinant is of order K and $V(t_1, \dots, t_K) = \prod_{i<j}(t_i - t_j)$, as above. The (i,j) entry of the matrix in the numerator is a rational function in t_i, which entails that $\mathfrak{S}_{\varkappa|N}(t_1, \dots, t_K)$ is

an element of $\mathbb{C}(t_1,\dots,t_K)^{\mathrm{sym}}$. Moreover, it is contained in $\mathbb{C}(t_1,\dots,t_K)^{\mathrm{sym}}_{\mathrm{reg}}$; this is readily verified by passing to variables $y_i = t_i^{-1}$, as we already did in the case of σ_μ; see Section 4.

Next, in accordance with [21, (12.3)], we set

$$H^*(t;\nu) = \prod_{j=1}^{N} \frac{t+j}{t+j-\nu_j}$$

and more generally

$$H^*(t_1,\dots,t_K;\nu) = H^*(t_1;\nu)\cdots H^*(t_K;\nu).$$

For ν fixed, $H^*(t_1,\dots,t_K;\nu)$ is obviously an element of $\mathbb{C}(t_1,\dots,t_K)^{\mathrm{sym}}_{\mathrm{reg}}$, too.

Finally, recall the notation $\mathrm{Dim}_{K,N}(\varkappa,\nu)$ and $\mathrm{Dim}_N\,\nu$ introduced in Section 2.1. We agree that $\mathrm{Dim}_{K,K}(\varkappa,\nu)$ is the Kronecker delta $\delta_{\varkappa\nu}$.

The purpose of this section is to prove the following claim.

Proposition 5.1. *Let $N \ge K$. For any fixed $\nu \in \mathbb{GT}_N$, the function $H^*(t_1,\dots,t_K;\nu)$ can be uniquely expanded into a finite linear combination of the functions $\mathfrak{S}_{\varkappa|N}(t_1,\dots,t_K)$, and this expansion takes the form*

$$H^*(t_1,\dots,t_K;\nu) = \sum_{\varkappa\in\mathbb{GT}_K} \frac{\mathrm{Dim}_{K,N}(\varkappa,\nu)}{\mathrm{Dim}_N\,\nu}\,\mathfrak{S}_{\varkappa|N}(t_1,\dots,t_K). \tag{5.2}$$

We regard this as a generating function for the quantities $\mathrm{Dim}(\varkappa,\nu)/\mathrm{Dim}\ \nu$. In the case $K=1$, $\varkappa$ is simply an integer k, and the above expansion turns into

$$\begin{aligned} H^*(t;\nu) &= \sum_{k\in\mathbb{Z}} \frac{\mathrm{Dim}_{1,N}(k,\nu)}{\mathrm{Dim}_N\,\nu}\,\frac{(t+1)\cdots(t+N)}{(t+1-k)\cdots(t+N-k)} \\ &= \sum_{k\in\mathbb{Z}} \frac{\mathrm{Dim}_{1,N}(k,\nu)}{\mathrm{Dim}_N\,\nu}\,H^*(t;(k^N)), \end{aligned}$$

where $(k^N) = (k,\dots,k) \in \mathbb{GT}_N$.

Proof. The proof is rather long and will be divided into a few steps. In what follows, μ always stands for an arbitrary partition with $\ell(\mu) \le K$.

Step 1. Set

$$(N)_\mu = \prod_{i=1}^{\ell(\mu)} (N-i+1)_{\mu_i} = \prod_{i=1}^{\ell(\mu)} (N-i+1)\cdots(N-i+\mu_i) \tag{5.3}$$

and note that $(N)_\mu \ne 0$ because $N \ge K \ge \ell(\mu)$.

Let

$$D_{N,K} : \mathbb{C}[\![t_1^{-1},\dots,t_K^{-1}]\!]^{\mathrm{sym}} \to \mathbb{C}[\![t_1^{-1},\dots,t_K^{-1}]\!]^{\mathrm{sym}}$$

denote the linear operator defined on the topological basis $\{\sigma_\mu\}$ by

$$D_{N,K} : \sigma_\mu \to \frac{(N)_\mu}{(K)_\mu}\,\sigma_\mu. \tag{5.4}$$

We claim that

$$H^*(t_1,\dots,t_K;\nu)=\sum_{\varkappa\in\mathbb{GT}_K}\Lambda_K^N(\nu,\varkappa)D_{N,K}H^*(t_1,\dots,t_K;\varkappa). \tag{5.5}$$

This is interpreted as an equality in $\mathbb{C}[[t_1^{-1},\dots,t_K^{-1}]]^{\mathrm{sym}}$. Note that the sum is finite because for ν fixed, the quantity $\Lambda_K^N(\nu,\varkappa)$ does not vanish only for finitely many $\varkappa$'s.

Indeed, by virtue of (4.4) we have

$$H^*(t_1,\dots,t_k;\nu)=\sum_\mu S_\mu^*(\nu_1,\dots,\nu_N)\sigma_\mu(t_1,\dots,t_K)$$

and likewise

$$H^*(t_1,\dots,t_k;\varkappa)=\sum_\mu S_\mu^*(\varkappa_1,\dots,\varkappa_K)\sigma_\mu(t_1,\dots,t_K).$$

Therefore, (5.5) is equivalent to

$$\frac{S_\mu^*(\nu_1,\dots,\nu_N)}{(N)_\mu}=\sum_\varkappa\Lambda_K^N(\nu,\varkappa)\frac{S_\varkappa^*(\varkappa_1,\dots,\varkappa_K)}{(K)_\mu}. \tag{5.6}$$

But (5.6) follows from the *coherence relation* for the shifted Schur polynomials, which says that

$$\frac{S_\mu^*(\nu_1,\dots,\nu_N)}{(N)_\mu}=\sum_{\lambda:\lambda\prec\nu}\frac{\mathrm{Dim}_{N-1}\lambda}{\mathrm{Dim}_N\nu}\,\frac{S_\mu^*(\lambda_1,\dots,\lambda_{N-1})}{(N-1)_\mu}. \tag{5.7}$$

See [21, (10.30)], which coincides with (5.7) within an obvious change of notation. To deduce (5.6) from (5.7) we use induction on N. For the initial value $N=K$, (5.6) is trivial (with the understanding that Λ_K^K is the identity matrix), and the induction step is implemented by (5.7), because $\Lambda_K^N(\nu,\varkappa)$ satisfies the same recursion

$$\Lambda_K^N(\nu,\varkappa)=\sum_{\lambda:\lambda\prec\nu}\frac{\mathrm{Dim}_{N-1}\lambda}{\mathrm{Dim}_N\nu}\Lambda_K^{N-1}(\lambda,\varkappa),\qquad N>K.$$

This completes the proof of (5.5).

Step 2. Our next goal is to prove the equality

$$\frac{\mathfrak{S}_{\varkappa|N}(t_1,\dots,t_K)}{\mathrm{Dim}_K\varkappa}=D_{N,K}H^*(t_1,\dots,t_K;\varkappa). \tag{5.8}$$

Then (5.2) will immediately follow from (5.5). Note that (5.8) does not involve ν.

On this step we will check that (5.8) holds for $N=K$, that is

$$\frac{\mathfrak{S}_{\varkappa|K}(t_1,\dots,t_K)}{\mathrm{Dim}_K\varkappa}=H^*(t_1,\dots,t_K;\varkappa). \tag{5.9}$$

By virtue of (5.1), the left-hand side of (5.9) equals

$$\frac{(-1)^{K(K-1)/2}}{\prod_{i=1}^K(i-1)!\cdot\mathrm{Dim}_K\varkappa\cdot V(t_1,\dots,t_k)}\det\left[\frac{\Gamma(t_i+1+K)\Gamma(t_i+j-\varkappa_j)}{\Gamma(t_i+1)\Gamma(t_i+j-\varkappa_j+1)}\right].$$

Setting $k_j = \varkappa_j - j,\ j = 1, \dots, K$, this expression can be easily transformed to

$$\frac{(-1)^{K(K-1)/2} \prod_{i,j=1}^{K} (t_i + j)}{V(k_1, \dots, k_K) V(t_1, \dots, t_k)} \det\left[\frac{1}{t_i - k_j}\right].$$

Since

$$\det\left[\frac{1}{t_i - k_j}\right] = \frac{(-1)^{K(K-1)/2} V(k_1, \dots, k_K) V(t_1, \dots, t_k)}{\prod_{i,j}(t_i - k_j)},$$

the final result is

$$\prod_{i,j=1}^{K} \frac{t_i + j}{t_i + j - \varkappa_j} = H^*(t_1, \dots, t_K; \varkappa),$$

as desired.

Step 3. By virtue of Step 2, to prove (5.8) it suffices to show that

$$\frac{\mathfrak{S}_{\varkappa|N}}{\mathrm{Dim}_K \varkappa} = D_{N,K}\left(\frac{\mathfrak{S}_{\varkappa|K}}{\mathrm{Dim}_K \varkappa}\right),$$

or, equivalently,

$$\mathfrak{S}_{\varkappa|N} = D_{N,K} \mathfrak{S}_{\varkappa|K}. \tag{5.10}$$

A possible approach would consist in computing explicitly the expansion

$$\mathfrak{S}_{\varkappa|N}(t_1, \dots, t_K) = \sum_{\mu} C(\mu; N) \sigma_\mu(t_1, \dots, t_K)$$

from which one could see that the coefficients satisfy the relation

$$C(\mu; N) = \frac{(N)_\mu}{(K)_\mu} C(\mu; K).$$

However, we did not work out this approach. Instead of it we adopt the following strategy. From the definition of $D_{N,K}$, see (5.4), it is clear that it suffices to prove that

$$\mathfrak{S}_{\varkappa|N} = D_{N,N-1} \mathfrak{S}_{\varkappa|N-1}, \quad \forall N > K. \tag{5.11}$$

To do this we will show that $D_{N,N-1}$ can be implemented by a certain difference operator in variables $(t_1, \dots, t_K)$. Then this will allow us to easily verify (5.11).

On this step we find the difference operator in question:

$$D_{N,N-1} = \frac{1}{(N-1)^{\downarrow K}} \frac{1}{V} \circ \prod_{i=1}^{K} (t_i + N - (t_i + 1)\tau_i) \circ V, \tag{5.12}$$

where V is the operator of multiplication by $V(t_1, \dots, t_K)$, and τ_i is the shift operator

$$(\tau f)(t) := f(t+1)$$

acting on variable t_i.

To verify that (5.12) agrees with the initial definition of $D_{N,N-1}$, see (5.4), we have to prove that the difference operator on the right-hand side of (5.12) acts on σ_μ as multiplication by $(N)_\mu/(N-1)_\mu$.

By the very definition of $(N)_\mu$, see (5.3),

$$\frac{(N)_\mu}{(N-1)_\mu}=\prod_{j=1}^{K}\frac{N-j+\mu_j}{N-j}=\frac{1}{(N-1)^{\downarrow K}}\prod_{j=1}^{N}(N-j+\mu_j).$$

Taking into account the same factor $1/(N-1)^{\downarrow K}$ in front of (5.12) and the definition of σ_μ given in (4.3), we see that the desired claim reduces to the following one: the action of the difference operator

$$\prod_{i=1}^{K}(t_i+N-(t_i+1)\tau_i)$$

on the function

$$\det\left[\frac{\Gamma(t_i+j-\mu_j)}{\Gamma(t_i+1)}\right]$$

amounts to multiplication by $\prod_{j=1}^{N}(N-j+\mu_j)$. This in turn reduces to the following claim, which is easily verified:

$$(t+N-(t+1)\tau)\,\frac{\Gamma(t-m)}{\Gamma(t+1)}=(N+m)\,\frac{\Gamma(t-m)}{\Gamma(t+1)},\qquad \forall m\in\mathbb{Z}.$$

This completes the proof of (5.12).

Step 4. Here we will establish (5.11) with the difference operator defined by (5.12). By the definition of $\mathfrak{S}_{\mu|N}$, see (5.1), we have to prove that operator

$$\prod_{i=1}^{K}(t_i+N-(t_i+1)\tau_i)$$

sends function

$$\det\left[\frac{\Gamma(t_i+N)\Gamma(t_i+j-\varkappa_j)}{\Gamma(t_i+1)\Gamma(t_i+j-\varkappa_j+N-K)}\right]$$

to

$$(N-K)^K\cdot\det\left[\frac{\Gamma(t_i+1+N)\Gamma(t_i+j-\varkappa_j)}{\Gamma(t_i+1)\Gamma(t_i+j-\varkappa_j+N-K+1)}\right].$$

This reduces to the following claim, which is easily verified: for any $k\in\mathbb{Z}$,

$$\begin{aligned}(t+N-(t+1)\tau)&\frac{\Gamma(t+N)\Gamma(t-k)}{\Gamma(t+1)\Gamma(t-k+N-K)}\\ &=(N-K)\frac{\Gamma(t+1+N)\Gamma(t-k)}{\Gamma(t+1)\Gamma(t-k+N-K+1)}.\end{aligned}$$

Thus we have completed the proof of expansion (5.2).

Step 5. It remains to prove the uniqueness claim of the proposition. That is, to prove that the functions $\mathfrak{S}_{\varkappa|N}(t_1,\dots,t_K)$ with N fixed and parameter $\varkappa$ ranging over $\mathbb{GT}_K$ are linearly independent. It suffices to do this for the minimal value $N=K$, because of relation (5.10) and the fact that operator $D_{N,K}$ is invertible. Next, by virtue of (5.9), this is equivalent to the claim that the functions $H^*(t_1,\dots,t_K;\varkappa)$ are linearly independent.

Recall that

$$H^*(t_1, \dots, t_K; \varkappa) = H^*(t_1; \varkappa) \cdots H^*(t_K; \varkappa),$$

where

$$H^*(t; \varkappa) = \prod_{j=1}^{K} \frac{t+j}{t+j-\varkappa_j}.$$

The numerators of the fractions do not depend on $\varkappa$ and so may be ignored. Set $k_j = \varkappa_j - j$ and observe that $k_1 > \cdots > k_K$. Thus, we are led to the claim that the family of the functions

$$f_{k_1,\dots,k_K}(t_1, \dots, t_K) := \prod_{i=1}^{K} \prod_{j=1}^{K} \frac{1}{t_i - k_j}$$

depending on an arbitrary K-tuple $k_1 > \cdots > k_K$ of integers is linearly independent. But this is obvious, because for a given K-tuple of parameters, the corresponding function $f_{k_1,\dots,k_K}(t_1, \dots, t_K)$ can be characterized as the only function of the family that has a nonzero multidimensional residue at $t_1 = k_1, \dots, t_K = k_K$. □

The next proposition is used in informal Remark 5.3 and then in the proof of Proposition 8.1.

Proposition 5.2. *We have*

$$H^*(t; \nu) = \Phi(u; \omega(\nu)), \tag{5.13}$$

provided that variables t and u are related by the mutually inverse linear-fractional transformations

$$t = -\frac{1}{2} + \frac{N}{u-1}, \quad u = 1 + \frac{N}{t+\frac{1}{2}}. \tag{5.14}$$

Proof. Recall that $\omega(\nu)$ is defined in terms of the modified Frobenius coordinates $\{a_i^\pm, b_i^\pm : 1 \le i \le d^\pm\}$ of the Young diagrams $\nu^\pm$; see the beginning of Section 3. Set

$$\widetilde{\nu}_i = \nu_i + \frac{N+1}{2} - i.$$

That is,

$$(\widetilde{\nu}_i, \dots, \widetilde{\nu}_N) = (\nu_1, \dots, \nu_N) + \left(\frac{N-1}{2}, \frac{N-1}{2} - 1, \dots, -\frac{N-1}{2} + 1, -\frac{N-1}{2}\right).$$

The next identity follows from [3, Proposition 4.1] (cf. [12, Proposition 1.2]):

$$\prod_{i=1}^{N} \frac{s - \frac{N+1}{2} + i}{s - \widetilde{\nu}_i} = \prod_{i=1}^{d^+} \frac{s - \frac{N}{2} + b_i^+}{s - \frac{N}{2} - a_i^+} \cdot \prod_{i=1}^{d^-} \frac{s + \frac{N}{2} - b_i^-}{s + \frac{N}{2} + a_i^-}. \tag{5.15}$$

Plug in $s = t + \frac{N+1}{2}$ into (5.15), then the left-hand side equals $H^*(t; \nu)$. Let us transform the right-hand side. Variables s and u are related to each other via

$$s = \frac{N}{2} \cdot \frac{u+1}{u-1}, \quad u = \frac{s+\frac{N}{2}}{s-\frac{N}{2}}.$$

Recall also that the coordinates of $\omega(\nu)$ are given by

$$\alpha_i^\pm = \frac{a_i^\pm}{N}, \qquad \beta_i^\pm = \frac{b_i^\pm}{N}, \qquad \delta^\pm = \frac{|\nu^\pm|}{N} = \sum(\alpha_i^\pm + \beta_i^\pm).$$

From this it is easy to check that the right-hand side of (5.15) equals

$$\prod_{i=1}^{d^+} \frac{1+\beta_i^+(u-1)}{1-\alpha_i^+(u-1)} \cdot \prod_{i=1}^{d^-} \frac{1+\beta_i^-(u^{-1}-1)}{1-\alpha_i^-(u^{-1}-1)} = \Phi(u;\omega(\nu)),$$

as desired. □

Remark 5.3. Let variables $t_1, \dots, t_K$ be related to variables $u_1, \dots, u_K$ via (5.14). Assume that variables u_i are fixed and N goes to infinity, so that variables t_i grow linearly in N. Then it is easy to check that in this limit regime

$$\mathfrak{S}_{\varkappa|N}(t_1, \dots, t_K) \to S_\varkappa(u_1, \dots, u_K).$$

Taking into account (5.13) we see that expansion (5.2) mimics expansion (2.8). (We recall that the latter expansion has the form

$$\begin{aligned}\Phi(u_1;\omega)\cdots\Phi(u_K;\omega) &= \sum_{\varkappa\in\mathbb{GT}_K} \varphi_\varkappa(\omega) S_\varkappa(u_1,\dots,u_K)\\ &= \sum_{\varkappa\in\mathbb{GT}_K} \det[\varphi_{\varkappa_i-i+j}(\omega)]_{i,j=1}^K S_\varkappa(u_1,\dots,u_K).)\end{aligned}$$

This observation makes it plausible that if $N \to \infty$ and $\nu \in \mathbb{GT}_N$ varies together with N in such a way that $\omega(\nu)$ converges to a point $\omega \in \Omega$, then the relative dimension $\mathrm{Dim}_{K,N}(\varkappa,\nu)/\mathrm{Dim}_N\,\nu$ tends to $\varphi_\varkappa(\omega)$. However, the rigorous proof of this assertion (and of the stronger one stated in the Uniform Approximation Theorem) requires substantial efforts. The first step made in the next section is to obtain a determinantal formula for the relative dimension mimicking the determinantal formula

$$\varphi_\varkappa(\omega) = \det[\varphi_{\varkappa_i-i+j}(\omega)]_{i,j=1}^K.$$

6. A Jacobi–Trudi-type formula

The classical Jacobi–Trudi formula expresses the Schur function S_μ as a determinant composed of the complete symmetric functions h_m, which are special cases of the Schur functions:

$$S_\mu = \det[h_{\mu_i-i+j}].$$

This formula can be obtained in various ways (see e.g. [17, Chapter I, (3.4)], [25, Section 7.16]). In particular, it can be easily derived from the Cauchy identity (4.1): to do this one multiplies both sides of identity (4.1) by $V_K(y_1,\dots,y_K)$ and then $S_\mu(x_1,\dots,x_N)$ is computed as the coefficient of the monomial $y_1^{\mu_1+K-1} y_2^{\mu_2+K-2}\cdots y_K^{\mu_K}$ (cf. the second proof of Theorem 7.16.1 in [25]). The same idea, albeit in a somewhat disguised form, is applied in the proof of Proposition 6.2.

Observe that the structure of formula (5.1) for the functions $\mathfrak{S}_{\varkappa|N}$ is similar to that for the Schur polynomials. This suggests the idea that identity (5.2) may be viewed as a kind of Cauchy identity, so that one may expect a Jacobi–Trudi formula for the quantities

$\mathrm{Dim}_{K,N}(\varkappa,\nu)/\mathrm{Dim}_N\,\nu$. The purpose of the present section is to derive such a formula. But first we have to introduce necessary notation.

For a finite interval $\mathbb{L}$ of the lattice $\mathbb{Z}$, let $V_{\mathbb{L}}$ denote the space of rational functions in variable $t\in\mathbb{C}\cup\{\infty\}$, regular everywhere including $t=\infty$, except possible simple poles at some points in $\mathbb{Z}\setminus\mathbb{L}$. Thus, $V_{\mathbb{L}}$ is spanned by 1 and the fractions $(t-m)^{-1}$, where m ranges over $\mathbb{Z}\setminus\mathbb{L}$.

Lemma 6.1. *The functions*

$$f_{\mathbb{L},k}(t)=\frac{\prod\limits_{x\in\mathbb{L}}(t-x)}{\prod\limits_{x\in\mathbb{L}}(t-x-k)},\quad k\in\mathbb{Z},\tag{6.1}$$

form one more basis in $V_{\mathbb{L}}$.

Proof. Obviously, $f_{\mathbb{L},k}$ is in $V_{\mathbb{L}}$ for every $k\in\mathbb{Z}$. In particular, $f_{\mathbb{L},0}$ is the constant function 1. On the other hand, given $k=1,2,\dots$, any function in $V_{\mathbb{L}}$ with the only possible poles on the right of $\mathbb{L}$, at distance at most k from the right endpoint of $\mathbb{L}$, can be expressed through $f_{\mathbb{L},0},\dots,f_{\mathbb{L},k}$, as is easily verified by induction on k. Moreover, such an expression is unique. Likewise, the same holds for functions with poles located on the left of $\mathbb{L}$. $\square$

By the lemma, any function $f\in V_{\mathbb{L}}$ is uniquely written as a finite linear combination

$$f=\sum_{k\in\mathbb{Z}}c_k f_{\mathbb{L},k}.$$

For the coefficients c_k we will use the notation

$$c_k=(f:f_{\mathbb{L},k}).$$

Set

$$\mathbb{L}(N)=\{-N,\dots,-1\}.$$

From the very definition of the function $H^*(t;\nu)$ one sees that it lies in $V_{\mathbb{L}(N)}$ for every $\nu\in\mathbb{GT}_N$. Consequently, the coefficients $(H^*(\,\cdot\,;\nu):f_{\mathbb{L}(N),k})$ are well defined. We also need more general coefficients $(H^*(\,\cdot\,;\nu):f_{\mathbb{L},k})$, where $\mathbb{L}$ is a subinterval in $\mathbb{L}(N)$. They are well defined, too, because $V_{\mathbb{L}}\supseteq V_{\mathbb{L}(N)}$.

The coefficients $(H^*(\,\cdot\,;\nu):f_{\mathbb{L}(N),k})$ will play the role of the h_k-functions in variables $\nu=(\nu_1,\dots,\nu_N)$, while more general coefficients $(H^*(\,\cdot\,;\nu):f_{\mathbb{L},k})$ should be interpreted as some modification of those "h_k-functions". It is worth noting that the conventional complete homogeneous symmetric functions are indexed by nonnegative integers, while in our situation the index ranges over the set $\mathbb{Z}$ of all integers.

The purpose of the present section is to prove the following proposition.

Proposition 6.2 (*Jacobi–Trudi-Type Formula*)**.** *Let $N\ge K\ge 1$, $\nu\in\mathbb{GT}_N$, and $\varkappa\in\mathbb{GT}_K$. For $j=1,\dots,K$, set*

$$\mathbb{L}(N,j)=\{-N+K-j,\dots,-j\}.$$

One has

$$\frac{\mathrm{Dim}_{K,N}(\varkappa,\nu)}{\mathrm{Dim}_N\,\nu}=\det\left[\left(H^*(\,\cdot\,;\nu):f_{\mathbb{L}(N,j),\ \varkappa_i-i+j}\right)\right]_{i,j=1}^{K}.\tag{6.2}$$

Note that the interval $\mathbb{L}(N, j) \subset \mathbb{Z}$ comprises $N - K + 1$ points and is entirely contained in $\mathbb{L}(N)$. As j ranges from 1 to K, this interval moves inside $\mathbb{L}(N)$ from the rightmost possible position to the leftmost one.

In the simplest case $K = 1$, (6.2) agrees with (5.2). Indeed, then the signature $\varkappa$ is reduced to a single integer $k \in \mathbb{Z}$, and formula (6.2) turns into the following one

$$\frac{\operatorname{Dim}_{1,N}(k, \nu)}{\operatorname{Dim}_N \nu} = \left(H^*(\,\cdot\,; \nu) : f_{\mathbb{L}(N),k}\right). \tag{6.3}$$

On the other hand, $f_{\mathbb{L}(N),k}$ coincides with $\mathfrak{S}_{k|N}$, so that (6.3) is a special case of (5.2) corresponding to the univariate case $K = 1$:

$$H^*(t; \nu) = \sum_{k\in\mathbb{Z}} \frac{\operatorname{Dim}_{1,N}(k, \nu)}{\operatorname{Dim}_N \nu} \mathfrak{S}_{k|N}(t).$$

A naive Jacobi–Trudi-type generalization of (6.3) to the case $K > 1$ would consist in taking the determinant

$$\det\left[\left(H^*(\,\cdot\,; \nu) : f_{\mathbb{L}(N),\,\varkappa_i - i + j}\right)\right].$$

But this does not work, and it turns out that we have to appropriately modify the univariate coefficients by shrinking $\mathbb{L}(N)$ to a subinterval which varies together with the column number j.

Note that a similar effect arises in the Jacobi–Trudi-type formula for the shifted Schur functions or other variations of the Schur functions; see [17, Chapter I, Section 3, Example 21], [21, Section 13]. Namely, in the Jacobi–Trudi determinant, the h-functions need to be appropriately modified according to the column number.

Proof of Proposition 6.2. *Step* 1. Parameter ν being fixed, we will omit it from the notation below. In particular, we abbreviate $H^*(t) = H^*(t; \nu)$. Assume that we dispose of an expansion into a finite sum, of the form

$$H^*(t_1) \cdots H^*(t_K) = \sum_{\varkappa} C(\varkappa) \mathfrak{S}_{\varkappa|N}(t_1, \dots, t_K), \tag{6.4}$$

with some coefficients $C(\varkappa)$. Then, due to the uniqueness claim of Proposition 5.1, the coefficients $C(\varkappa; \nu)$ must be the same as the quantities $\operatorname{Dim}_{K,N}(\varkappa, \nu)/\operatorname{Dim}_N \nu$.

The functions $\mathfrak{S}_{\varkappa|N}(t_1, \dots, t_K)$ can be written in the form

$$\mathfrak{S}_{\varkappa|N}(t_1, \dots, t_K) = \operatorname{const}(N, K)\, \frac{\det\left[g_{k_j}(t_i)\right]_{i,j=1}^K}{V(t_1, \dots, t_K)},$$

where

$$k_1 = \varkappa_1 - 1, \dots, k_K = \varkappa_K - K,$$
$$g_k(t) = \frac{\Gamma(t + 1 + N)\Gamma(t - k)}{\Gamma(t + 1)\Gamma(t - k + N - K + 1)}, \quad k \in \mathbb{Z},$$
$$\operatorname{const}(N, K) = (-1)^{K(K-1)/2} \prod_{i=1}^K \frac{(N - K)!}{(N - K + i - 1)!}.$$

Assume that we have found some rational functions $\varphi_1(t), \dots, \varphi_K(t)$ with the following two properties.

- First, for every $a = 1, \dots, K$ there exists a finite expansion

$$H^*(t)\varphi_a(t) = \sum_{k\in\mathbb{Z}} C_k^a\, g_k(t) \tag{6.5}$$

with some coefficients C_k^a.
- Second,

$$\det\left[\varphi_a(t_i)\right]_{a,i=1}^K = \frac{V(t_1,\dots,t_K)}{\mathrm{const}(N,K)}. \tag{6.6}$$

We claim that then (6.4) holds with coefficients

$$C(\varkappa) := C(k_1,\dots,k_K) := \det\left[C_{k_b}^a\right]_{a,b=1}^K. \tag{6.7}$$

Indeed, first of all, note that these coefficients vanish for all but finitely many $\varkappa$'s (because of finiteness of expansion (6.5)), so that the future expansion (6.4) will be finite. Next, applying (6.5) and (6.6), we have

$$\begin{aligned}\sum_{k_1>\dots>k_K} C(k_1,\dots,k_K)\det\left[g_{k_j}(t_i)\right]_{i,j=1}^K &= \sum_{k_1>\dots>k_K}\det\left[C_{k_b}^a\right]_{a,b=1}^K\det\left[g_{k_j}(t_i)\right]_{i,j=1}^K\\ &= \det\left[\sum_{k\in\mathbb{Z}} C_k^a g_k(t_i)\right]_{i,a=1}^K\\ &= \det\left[H^*(t_i)\varphi_a(t_i)\right]_{i,a=1}^K\\ &= H^*(t_1)\cdots H^*(t_K)\det\left[\varphi_a(t_i)\right]_{a,i=1}^K\\ &= H^*(t_1)\cdots H^*(t_K)\,\frac{V(t_1,\dots,t_K)}{\mathrm{const}(N,K)},\end{aligned}$$

which is equivalent to the desired equality

$$H^*(t_1)\cdots H^*(t_K) = \sum_{\varkappa}\det\left[C_{k_b}^a\right]_{a,b=1}^K \mathfrak{S}_{\varkappa|N}(t_1,\dots,t_K).$$

Step 2. Now we exhibit the functions $\varphi_a(t)$:

$$\varphi_a(t) = g_{-a}(t) = \frac{\Gamma(t+a)\Gamma(t+N+1)}{\Gamma(t+1)\Gamma(t+N-K+a+1)}, \qquad a = 1,\dots,K. \tag{6.8}$$

Let us examine what (6.5) means. Dividing both sides of (6.5) by $\varphi_a(t)$ we get

$$H^*(t) = \sum_{k\in\mathbb{Z}} C_k^a\,\frac{g_k(t)}{\varphi_a(t)}.$$

But

$$\frac{g_k(t)}{\varphi_a(t)} = \frac{\Gamma(t+N-K+a+1)\Gamma(t-k)}{\Gamma(t+a)\Gamma(t-k+N-K+1)} = \frac{(t+a)(t+a+1)\cdots(t+a+N-K)}{(t-k)(t-k+1)\cdots(t-k+N-K)}.$$

In the notation of (6.1), this fraction is nothing else than $f_{\mathbb{L},k+a}$, where $\mathbb{L}$ denotes the interval $\{-N+K-a,\dots,-a\}$ in $\mathbb{Z}$. It follows that the desired expansion (6.5) does exist and (restoring the detailed notation $H^*(t;\nu)$) the corresponding coefficients are

$$C_k^a = \left(H^*(\,\cdot\,;\nu) : f_{\{-N+K-a,\dots,-a\},k+a}\right).$$

Then the prescription (6.7) gives us

$$C(\varkappa) = \det\left[C^a_{k_b}\right]_{a,b=1}^K = \det\left[C^a_{\varkappa_b - b}\right]_{a,b=1}^K$$
$$= \det\left[\left(H^*(\,\cdot\,;\nu) : f_{\{-N+K-a,\dots,-a\},\varkappa_b-b+a}\right)\right]_{a,b=1}^K .$$

This is exactly (6.2), within the renaming of indices $(b,a)\to(i,j)$.

Step 3. It remains to check that the functions (6.8) satisfy (6.6). That is, renaming a by j,

$$\det\left[\frac{\Gamma(t_i+j)\Gamma(t_i+N+1)}{\Gamma(t_i+1)\Gamma(t_i+N-K+j+1)}\right]_{i,j=1}^K$$
$$= V(t_1,\dots,t_K)\,(-1)^{K(K-1)/2}\prod_{i=1}^K \frac{(N-K+i-1)!}{(N-K)!}$$

or

$$\det\left[(t_i+1)\cdots(t_i+j-1)(t_i+N-K+j+1)\cdots(t_i+N)\right]_{i,j=1}^K$$
$$= V(t_1,\dots,t_K)\,(-1)^{K(K-1)/2}\prod_{i=1}^K \frac{(N-K+i-1)!}{(N-K)!}. \tag{6.9}$$

This identity is a particular case of Lemma 3 in Krattenthaler's paper [15]. For the reader's convenience we reproduce the statement of this lemma in the original notation.

Let $X_1,\dots,X_n$, $A_2,\dots,A_n$, and $B_2,\dots,B_n$ be indeterminates. Then

$$\det\Big[(X_i+A_n)(X_i+A_{n-1})\cdots(X_i+A_{j+1})(X_i+B_j)$$
$$\times\ (X_i+B_{j-1})\cdots(X_i+B_2)\Big]_{i,j=1}^n = \prod_{1\le i<j\le n}(X_i-X_j)\prod_{2\le i\le j\le n}(B_i-A_j). \tag{6.10}$$

Setting $n=K$, $X_i=t_i$, $A_j=N-K+j$ and

$$(B_2,\dots,B_n)=(1,\dots,K-1)$$

one sees that the determinant in (6.10) turns into that in (6.9). Next,

$$\prod_{1\le i<j\le n}(X_i-X_j)=V(t_1,\dots,t_K)$$

and

$$\prod_{2\le i\le j\le n}(B_i-A_j) = \prod_{1\le i<j\le K}(i-(N-K+j))$$
$$= (-1)^{K(K-1)/2}\prod_{1\le i<j\le K}(N-K+j-i)$$
$$= (-1)^{K(K-1)/2}\prod_{j=1}^K\frac{(N-K+j-1)!}{(N-K)!},$$

which agrees with (6.9).

This completes the proof of the proposition. □

7. Expansion in rational fractions

In this section, we derive an expression for the coefficients $(H^*(\,\cdot\,;\nu) : f_{\mathbb{L},k})$ making formula (6.2) available for practical use.

Fix a finite interval $\mathbb{L} = \{a, a+1, \dots, b-1, b\} \subset \mathbb{Z}$. As explained in the beginning of Section 6, the space $V_{\mathbb{L}}$ has a basis consisting of the rational fractions

$$f_{\mathbb{L},k}(t) = \frac{(t-b)(t-b+1)\cdots(t-a)}{(t-b-k)(t-b-k)\cdots(t-a-k)}, \quad k \in \mathbb{Z}.$$

For a rational function $G(t)$ from $V_{\mathbb{L}}$, we write its expansion in the basis $\{f_{\mathbb{L},k}\}_{k\in\mathbb{Z}}$ as

$$G(t) = \sum_{k\in\mathbb{Z}} (G : f_{\mathbb{L},k}) f_{\mathbb{L},k}(t)$$

and denote by $\operatorname{Res}_{t=x} G(t)$ the residue of $G(t)$ at a point $x \in \mathbb{Z}$.

Proposition 7.1. *Assume $n := b-a+1 \geq 2$. In the above notation*

$$(G : f_{\mathbb{L},k}) = \begin{cases} (n-1)\displaystyle\sum_{m\geq k} \frac{(m-k+1)_{n-2}}{(m)_n} \operatorname{Res}_{t=b+m} G(t), \quad k \geq 1, \\ -(n-1)\displaystyle\sum_{m\geq |k|} \frac{(m-|k|+1)_{n-2}}{(m)_n} \operatorname{Res}_{t=a-m} G(t), \quad k \leq -1, \\ G(\infty) + \displaystyle\sum_{m\geq 1} \frac{1}{m+n-1} \left(-\operatorname{Res}_{t=b+m} G(t) + \operatorname{Res}_{t=a-m} G(t)\right), \\ \quad k = 0. \end{cases}$$

Proof. It is easy to write the expansion of $G(t)$ in another basis of $V_{\mathbb{L}}$, formed by 1 and the fractions $(t-x)^{-1}$, where x ranges over $\mathbb{Z} \setminus \mathbb{L}$:

$$G(t) = G(\infty) + \sum_{m\geq 1} \left(\frac{\operatorname{Res}_{t=b+m} G(t)}{t-(b+m)} + \frac{\operatorname{Res}_{t=a-m} G(t)}{t-(a-m)} \right). \tag{7.1}$$

Thus, to find the coefficients $(G : f_{\mathbb{L},k})$ it suffices to compute the expansion of the elements of the second basis on the fractions $f_{\mathbb{L},k}$.

Obviously,

$$1 = f_{\mathbb{L},0}. \tag{7.2}$$

Thus, the problem is to expand the functions $(t-(b+m))^{-1}$ and $(t-(a-m))^{-1}$ with $m = 1, 2, \dots$. We are going to prove that

$$\frac{1}{t-(b+m)} = -\frac{1}{m+n-1} f_{\mathbb{L},0} + \frac{n-1}{(m)_n} \sum_{k=1}^{m} (m-k+1)_{n-2} f_{\mathbb{L},k} \tag{7.3}$$

$$\frac{1}{t-(a-m)} = \frac{1}{m+n-1} f_{\mathbb{L},0} - \frac{n-1}{(m)_n} \sum_{k=1}^{m} (m-k+1)_{n-2} f_{\mathbb{L},-k}. \tag{7.4}$$

The claim of the proposition immediately follows from (7.1)–(7.4).

Observe that (7.4) is reduced to (7.3) by making use of reflection $t \to -t$. Indeed, under this reflection the basis formed by 1 and $\{f_{\mathbb{L},k}\}$ is transformed into the similar basis with $\mathbb{L}$

replaced with $-\mathbb{L}$ (that is, parameters a and b are replaced by $-b$ and $-a$, respectively), while the fractions from the second basis are transformed into the similar fractions but multiplied by -1. This explains the change of sign on the right-hand side of (7.4) as compared to (7.3).

Thus, it suffices to prove identity (7.3). Since it is invariant under the simultaneous shift of t, a, and b by an integer, we may assume, with no loss of generality, that $a = 1$, $b = n$. Then the identity takes the form

$$\frac{1}{t-n-m} = -\frac{1}{m+n-1} + \frac{n-1}{(m)_n}\sum_{k=1}^{m}(m-k+1)_{n-2} \times \frac{(t-1)\cdots(t-n)}{(t-1-k)\cdots(t-n-k)}. \tag{7.5}$$

The left-hand side vanishes at $t = \infty$. Let us check that the same holds for the right-hand side. Indeed, this amounts to the identity

$$\frac{n-1}{(m)_n}\sum_{k=1}^{m}(m-k+1)_{n-2} = \frac{1}{m+n-1}, \quad n \geq 2.$$

Renaming $n-1$ by n, the identity can be rewritten as

$$n\sum_{k=1}^{m}(m-k+1)\cdots(m-k+n-1) = m\cdots(m+n-1), \quad n \geq 1,$$

and then it is easily proved by induction on m.

Next, the only singularity of the left-hand side of (7.5) is the simple pole at $t = n+m$ with residue 1. Let us check that the right-hand side has the same singularity at this point. Indeed, the only contribution comes from the mth summand, which has a simple pole at $t = n+m$ with residue

$$\frac{(n-1)!}{(m)_n}\left.\frac{(t-1)\cdots(t-n)}{(t-1-m)\cdots(t-n-m+1)}\right|_{t=n+m} = 1,$$

as desired.

It remains to check that the right-hand side of (7.5) is regular at points $t = n+1, \dots, n+m-1$. All possible poles are simple, so that it suffices to check that the residue at every such point vanishes. In the corresponding identity, we may formally extend summation to $k = 1, \dots, m+n-2$, because the extra terms actually vanish. This happens due to the factor $(m-k+1)_{n-2}$.

Thus, compute the residue at a given point $s \in \{n+1, \dots, n+m-1\}$. The terms that contribute to the residue are those with $k = s-n, s-n+1, \dots, s-1$ (a total of n summands). Setting $j = k-(s-n)$, the sum of the residues has the form

$$\frac{n-1}{(m)_n}\sum_{j=0}^{n-1}(m-j-s+n+1)_{n-2}\,\frac{(-1)^j}{j!(n-1-j)!}\,(s-1)\cdots(s-n).$$

The fact that this expression vanishes follows from a more general claim: For any polynomial P of degree $\leq n-2$,

$$\sum_{j=0}^{n-1}P(j)\,\frac{(-1)^j}{j!(n-1-j)!} = 0.$$

Finally, to prove the last identity, apply the differential operator $(x\frac{d}{dx})^\ell$ to $(1-x)^{n-1}$ and then set $x=1$. For $\ell = 0, 1, \dots, n-2$ this gives

$$\sum_{j=0}^{n-1} j^\ell \, \frac{(-1)^j}{j!(n-1-j)!} = 0. \quad \square$$

Propositions 6.2 and 7.1 together give the following explicit formula.

Theorem 7.2. *Let $\varkappa \in \mathbb{GT}_K$ and $\nu \in \mathbb{GT}_N$, where $N > K \geq 1$, and recall the notation*

$$H^*(t;\nu) = \frac{(t+1)\cdots(t+N)}{(t+1-\nu_1)\cdots(t+N-\nu_N)}.$$

One has

$$\frac{\mathrm{Dim}_{K,N}(\varkappa,\nu)}{\mathrm{Dim}_N \nu} = \det\left[A_N(i,j)\right]_{i,j=1}^K, \tag{7.6}$$

where the entries of the $K \times K$ matrix $A_N = [A_N(i,j)]$ are defined according to the following rule, which depends on the column number $j = 1, \dots, K$ and the integer

$$k := k(i,j) = \varkappa_i - i + j.$$

- *If $k \geq 1$, then*

$$A_N(i,j) = (N-K)\sum_{m\geq k} \frac{(m-k+1)_{N-K-1}}{(m)_{N-K+1}} \operatorname{Res}_{t=-j+m} H^*(t;\nu). \tag{7.7}$$

- *If $k \leq -1$, then*

$$A_N(i,j) = -(N-K)\sum_{m\geq |k(i,j)|} \frac{(m-|k|+1)_{N-K-1}}{(m)_{N-K+1}} \operatorname{Res}_{t=-N+K-j-m} H^*(t;\nu). \tag{7.8}$$

- *If $k = 0$, then*

$$\begin{aligned} A_N(i,j) = 1 &- \sum_{m\geq 1} \frac{1}{m+N-K} \operatorname{Res}_{t=-j+m} H^*(t;\nu) \\ &+ \sum_{m\geq 1} \frac{1}{m+N-K} \operatorname{Res}_{t=-N+K-j-m} H^*(t;\nu). \end{aligned} \tag{7.9}$$

Proof. By virtue of Proposition 6.2, formula (7.6) holds with the $K \times K$ matrix $A_N = [A_N(i,j)]$ defined by

$$A_N(i,j) = \left(H^*(\,\cdot\,;\nu) : f_{\mathbb{L}(N,j),\,\varkappa_i - i + j}\right), \quad i,j = 1,\dots,K,$$

where

$$\mathbb{L}(N,j) := \{-N+K-j, \dots, -j\} \subseteq \mathbb{L}(N) := \{-N,\dots,-1\} \subset \mathbb{Z}.$$

To compute the entry $A_N(i,j)$ we apply Proposition 7.1, where we substitute $G(t) = H^*(t;\nu)$ and

$$\mathbb{L} = \mathbb{L}(N,j), \qquad a = -N+K-j, \qquad b = -j, \qquad n = N-K+1.$$

This leads to the desired formulas. $\square$

Proposition 7.3. *Assume that N is large enough, where the necessary lower bound depends on $\varkappa$. Then the formulas of Theorem 7.2 can be rewritten in the following equivalent form.*

- *If $k \ge 1$, then*

$$A_N(i,j) = (N-K)\sum_{\ell=0}^{\infty} \frac{(\ell+j-k+1)_{k-1}}{(\ell+j-k+N-K)_{k+1}} \operatorname{Res}_{t=\ell} H^*(t;\nu). \tag{7.10}$$

- *If $k \le -1$, then*

$$A_N(i,j) = -(N-K)\sum_{\ell=-\infty}^{-N-1} \frac{(\ell+j+N-K+1)_{|k|-1}}{(\ell+j)_{|k|+1}} \operatorname{Res}_{t=\ell} H^*(t;\nu). \tag{7.11}$$

- *If $k = 0$, then*

$$\begin{aligned} A_N(i,j) = 1 &- \sum_{\ell=0}^{\infty} \frac{1}{\ell+j+N-K} \operatorname{Res}_{t=\ell} H^*(t;\nu) \\ &- \sum_{\ell=-\infty}^{-N-1} \frac{1}{-\ell-j} \operatorname{Res}_{t=\ell} H^*(t;\nu). \end{aligned} \tag{7.12}$$

Proof. Examine formula (7.7). Its transformation to (7.10) involves three steps.

Step 1. The key observation is that the summation in (7.7) can be formally extended by starting it from $m = 1$. The reason is that the extra terms with $1 \le m < k$ actually vanish. Indeed, the vanishing comes from the product

$$(m-k+1)_{N-K-1} = (m-k+1)\cdots(m-k+N-K-1).$$

Since $1 \le m < k$, the first factor of the product is ≤ 0 while the last factor is positive (here the assumption that N is large enough is essential!). Therefore, one of the factors is 0.

Step 2. A simple transformation shows that

$$\frac{(m-k+1)_{N-K-1}}{(m)_{N-K+1}} = \frac{\Gamma(m-k+N-K)\Gamma(m)}{\Gamma(m-k+1)\Gamma(m+N-K+1)} = \frac{(m-k+1)_{k-1}}{(m-k+N-K)_{k+1}}.$$

Step 3. Observe that the possible poles of $H^*(t;\nu)$ are located in

$$\mathbb{Z} \setminus \mathbb{L} = \{\dots, -N-3, -N-2, -N-1\} \cup \{0, 1, 2, \dots\}.$$

All possible poles at points $t = -j + m$, where $m = 1, 2, \dots,$ are entirely contained in $\{0, 1, 2, \dots\}$. Therefore, we may assume that m ranges over $\{j, j+1, j+2, \dots\}$. Setting $m = j+\ell$ we finally arrive at (7.10).

To transform (7.8) to (7.11) we apply the similar argument.

To transform the sums in (7.9) we need to apply only the last step of the above argument. □

8. Contour integral representation

We keep the notation of the preceding section: the number $K = 1, 2, \dots$ and the signature $\varkappa \in \mathbb{GT}_K$ are fixed, and we are dealing with the $K \times K$ matrix $[A_N(i,j)]$ that depends on $\varkappa \in \mathbb{GT}_K$ and $\nu \in \mathbb{GT}_N$, and is defined by the formulas of Proposition 7.3. We denote by $\mathbb{T}$ the unit circle $|u| = 1$ in $\mathbb{C}$ oriented counterclockwise.

Proposition 8.1. *Every entry $A_N(i,j)$ can be written in the form*

$$A_N(i,j)=\frac{1}{2\pi i}\oint_{\mathbb{T}}\Phi(u;\omega(\nu))R^{(j)}_{\varkappa_i-i+j}(u;N)\frac{du}{u},\tag{8.1}$$

where, for any $k\in\mathbb{Z}$, $j=1,\dots,K$, and natural $N>K$, the function $u\to R^{(j)}_k(u;N)$ is continuous on $\mathbb{T}$ and such that

$$\lim_{N\to\infty}R^{(j)}_k(u;N)=\frac{1}{u^k}\tag{8.2}$$

uniformly on $u\in\mathbb{T}$.

The explicit expression for $R^{(j)}_k(u;N)$ is the following.

- *If $k\ge 1$, then*

$$R^{(j)}_k(u;N)=\frac{N-K}{N}\,\frac{u\prod_{m=1}^{k-1}\left(1+\frac{\left(j-k-\frac12+m\right)(u-1)}{N}\right)}{\prod_{m=1}^{k+1}\left(u+\frac{\left(j-k-K-\frac32+m\right)(u-1)}{N}\right)}.\tag{8.3}$$

- *If $k\le -1$, then*

$$R^{(j)}_k(u;N)=\frac{N-K}{N}\,\frac{u\prod_{m=1}^{|k|-1}\left(u+\frac{\left(j-K-\frac12+m\right)(u-1)}{N}\right)}{\prod_{m=1}^{|k|+1}\left(1+\frac{\left(j-\frac32+m\right)(u-1)}{N}\right)}.\tag{8.4}$$

- *If $k=0$, then*

$$R^{(j)}_0(u;N)=\frac{N-K}{N}\,\frac{u}{\left(u+\frac{\left(j-K-\frac12\right)(u-1)}{N}\right)\left(1+\frac{\left(j-\frac12\right)(u-1)}{N}\right)}.\tag{8.5}$$

Proof. Recall (see Proposition 5.2) that

$$H^*(t;\nu)=\Phi(u;\omega(\nu)),$$

where t and u are related by the mutually inverse linear-fractional transformations

$$t=-\frac12+\frac{N}{u-1},\qquad u=1+\frac{N}{t+\frac12}.$$

The transformation $t\to u$ maps the right half-plane $\Re t>-\frac{N+1}{2}$ onto the exterior of the unit circle $|u|=1$, and the left half-plane $\Re t<-\frac{N+1}{2}$ is mapped onto the interior of the circle. The vertical line $\Re t=-\frac{N+1}{2}$ just passes through the midpoint of the interval $[-N,-1]$, which is free of the poles of $H^*(t;\nu)$. Note also that

$$dt=-\frac{N}{(u-1)^2}du=-\frac{Nu}{(u-1)^2}\,\frac{du}{u}.\tag{8.6}$$

Consider separately the three cases depending on whether $k \geq 1$, $k \leq -1$ or $k = 0$.

Case $k \geq 1$. We can write (7.10) as the contour integral

$$A(i, j) = \frac{1}{2\pi i} \oint_C \rho(t) H^*(t; \nu) dt,$$

where C is a simple contour in the half-plane $\Re t > -\frac{N+1}{2}$, oriented in the positive direction and encircling all the poles of $H^*(t; \nu)$ located in this half-plane and

$$\rho(t) = (N - K) \frac{(t + j - k + 1)_{k-1}}{(t + j - k + N - K)_{k+1}}.$$

Passing to variable u we get, after a simple transformation,

$$\rho(t) = \widetilde{\rho}(u) := \frac{(N - K)(u - 1)^2}{N^2} \frac{\prod_{m=1}^{k-1} \left(1 + \frac{\left(j-k-\frac{1}{2}+m\right)(u-1)}{N}\right)}{\prod_{m=1}^{k+1} \left(u + \frac{\left(j-k-K-\frac{3}{2}+m\right)(u-1)}{N}\right)}.$$

Without loss of generality we can assume that contour C also encircles the special point $t = -\frac{1}{2}$ corresponding to $u = \infty$. This means that its image in the u-plane goes around the unit circle $|u| = 1$ in the negative direction. Thus, we can deform it, in the u-plane, to the unit circle. The change of orientation to the positive one produces the minus sign, which cancels the minus sign in formula (8.6) for the transformation of the differential. Note that the deformation of the contour is justified, because $\widetilde{\rho}(u)$ has no singularity in the exterior of the unit circle (this is best seen from the expression for $\rho(t)$, which obviously has no singularity in the half-plane $\Re t > -\frac{N+1}{2}$). As for the factor $(u - 1)^2$ in the denominator of (8.6), it is canceled by the same factor in the numerator of $\widetilde{\rho}(u)$. Finally we get the desired integral representation (8.1) with $R_k^{(j)}(u; N)$ given by (8.3).

Case $k \leq -1$. This case is analyzed in the same way.

Case $k = 0$. The same argument as above allows one to write the expression in (7.12) as

$$1 - \frac{1}{2\pi i} \oint_{C_+} \frac{\Phi(u; \omega(\nu)) du}{(u - 1)(u + \varepsilon_1(u - 1))} + \frac{1}{2\pi i} \oint_{C_-} \frac{\Phi(u; \omega(\nu)) du}{(u - 1)(1 + \varepsilon_2(u - 1))} \tag{8.7}$$

where

$$\varepsilon_1 = \frac{j - K - \frac{1}{2}}{N}, \qquad \varepsilon_2 = \frac{j - \frac{1}{2}}{N},$$

and C_+ and C_- are two circles close to the unit circle $|u| = 1$, both oriented in the positive direction, such that C_+ is outside the unit circle while C_- is inside it. Since $\Phi(u; \omega(\nu))$ takes value 1 at $u = 1$, we have

$$1 - \frac{1}{2\pi i} \oint_{C_+} \frac{\Phi(u; \omega(\nu)) du}{(u - 1)(1 + \varepsilon_2(u - 1))} + \frac{1}{2\pi i} \oint_{C_-} \frac{\Phi(u; \omega(\nu)) du}{(u - 1)(1 + \varepsilon_2(u - 1))} = 0.$$

Subtracting this from (8.7) we get the contour integral with the integrand equal to $\Phi(u; \omega(\nu)) du$ multiplied by

$$\frac{1}{(u - 1)(1 + \varepsilon_2(u - 1))} - \frac{1}{(u - 1)(u + \varepsilon_1(u - 1))} = \frac{1 + \varepsilon_1 - \varepsilon_2}{(u + \varepsilon_1(u - 1))(1 + \varepsilon_2(u - 1))}.$$

This leads to (8.5).

The asymptotics (8.2) is obvious from the explicit expressions (8.3)–(8.5). □

Let $\mathbb{T}^K = \mathbb{T} \times \cdots \times \mathbb{T}$ be the K-fold product of unit circles. Theorem 7.2, Propositions 7.3 and 8.1 together imply the following result.

Theorem 8.2. *Given $K = 1, 2, \dots$ and $\varkappa \in \mathbb{GT}_K$, one can exhibit a sequence $\{R_\varkappa(u_1, \dots, u_K; N) : N > K\}$ of continuous functions on the torus $\mathbb{T}^K$ such that:*

(i) *For all N large enough and every $\nu \in \mathbb{GT}_N$*

$$\frac{\operatorname{Dim}_{K,N}(\varkappa, \nu)}{\operatorname{Dim}_N \nu} = \frac{1}{(2\pi i)^K} \oint_{\mathbb{T}} \cdots \oint_{\mathbb{T}} \Phi(u_1; \omega(\nu)) \cdots \Phi(u_K; \omega(\nu)) \times R_\varkappa(u_1, \dots, u_K; N) \frac{du_1}{u_1} \cdots \frac{du_K}{u_K}, \tag{8.8}$$

where each copy of $\mathbb{T}$ is oriented counterclockwise.

(ii) *As N goes to infinity,*

$$R_\varkappa(u_1, \dots, u_K; N) \to \det\left[u_j^{-(\varkappa_i - i + j)}\right]_{i,j=1}^K$$

uniformly on $(u_1, \dots, u_K) \in \mathbb{T}^K$.

Proof. (i) Indeed, set

$$R_\varkappa(u_1, \dots, u_K; N) = \det\left[R^{(j)}_{\varkappa_i - i + j}(u_j; N)\right]_{i,j=1}^K, \tag{8.9}$$

where the functions $R^{(j)}_{\varkappa_i - i + j}(u, N)$ are defined in Proposition 8.1. Recall that Theorem 7.2 expresses the relative dimension $\operatorname{Dim}_{K,N}(\varkappa, \nu)/\operatorname{Dim}_N \nu$ as the determinant of a matrix $A_N = [A_N(i, j)]$; Proposition 7.3 provides a more convenient expression for the matrix entries that works for large N; finally, Proposition 8.1 says that this expression can be written as a contour integral involving the functions $R^{(j)}_{\varkappa_i - i + j}(u, N)$. Now we plug in the determinant (8.9) into the K-fold contour integral (8.8) and expand the determinant on columns. Applying (8.1) we get $\det[A_N(i, j)]$, as desired.

(ii) This follows directly from (8.9) and (8.2). □

Remark 8.3. The graph $\mathbb{GT}$ possesses the *reflection symmetry* $\nu \mapsto \widehat{\nu}$, where, given a signature $\nu \in \mathbb{GT}_N$, $N = 1, 2, \dots$, we set

$$\widehat{\nu} = (\widehat{\nu}_1, \dots, \widehat{\nu}_N) := (-\nu_N, \dots, -\nu_1).$$

The corresponding symmetry $\omega \mapsto \widehat{\omega}$ of Ω amounts to switching the plus- and minus-coordinates:

$$\alpha_i^+ \leftrightarrow \alpha_i^-, \qquad \beta_i^+ \leftrightarrow \beta_i^-, \qquad \delta^+ \leftrightarrow \delta^-.$$

Note also that $\widehat{\omega(\nu)} = \omega(\widehat{\nu})$ and

$$\Phi(u; \omega) = \Phi(u^{-1}; \widehat{\omega}).$$

Evidently, the reflection symmetry preserves the relative dimension:

$$\frac{\operatorname{Dim}_{K,N}(\varkappa, \nu)}{\operatorname{Dim}_N \nu} = \frac{\operatorname{Dim}_{K,N}(\widehat{\varkappa}, \widehat{\nu})}{\operatorname{Dim}_N \widehat{\nu}}.$$

Therefore, the expression given in (8.8) must satisfy this identity. This is indeed true and can be readily verified using the relation

$$R_k^{(j)}(u;N) = R_{-k}^{(K+1-j)}(u^{-1};N),$$

which follows directly from (8.3)–(8.5).

The Uniform Approximation Theorem (Theorem 3.1) is a direct consequence of Theorem 8.2:

Proof of the Uniform Approximation Theorem. As was already pointed out in the end of Section 3, both quantities $\Lambda_K^N(\nu,\varkappa)$ and $\Lambda_K^\infty(\omega,\varkappa)$ entering (3.1) involve one and the same constant factor $\operatorname{Dim}_K \varkappa$. Therefore, (3.1) is equivalent to

$$\lim_{N\to\infty} \sup_{\nu\in\mathbb{GT}_N} \left| \frac{\operatorname{Dim}_{K,N}(\varkappa,\nu)}{\operatorname{Dim}_N \nu} - \varphi_\varkappa(\omega(\nu)) \right| = 0. \tag{8.10}$$

To estimate the deviation

$$\frac{\operatorname{Dim}_{K,N}(\varkappa,\nu)}{\operatorname{Dim}_N \nu} - \varphi_\varkappa(\omega(\nu)) \tag{8.11}$$

we observe that both quantities in (8.11) can be written as K-fold contour integrals of the same type. Indeed, for the relative dimension we apply (8.8). Next, by the very definition $\varphi_\varkappa(\omega) = \det[\varphi_{\varkappa_i - i + j}(\omega)]$ and

$$\varphi_k(\omega) = \frac{1}{2\pi i} \oint_{\mathbb{T}} \Phi(u;\omega) \frac{1}{u^k} \frac{du}{u},$$

so that $\varphi_\varkappa(\omega(\nu))$ admits a similar integral representation, only $R(u_1,\dots,u_K;N)$ has to be replaced by

$$\det\left[u_j^{-(\varkappa_i - i + j)}\right]_{i,j=1}^K.$$

It follows that for any $\nu \in \mathbb{GT}_N$ the modulus of (8.11) is bounded from above by the following integral over the torus $\mathbb{T}^K$ taken with respect to the normalized Lebesgue measure $m(du)$, where we abbreviate $u = (u_1,\dots,u_K)$:

$$\int_{\mathbb{T}^K} |\Phi(u_1;\omega(\nu)) \cdots \Phi(u_K;\omega(\nu))| \\ \times \left| R_\varkappa(u_1,\dots,u_K;N) - \det\left[u_j^{-(\varkappa_i - i + j)}\right]_{i,j=1}^K \right| m(du).$$

By Proposition 2.4,

$$|\Phi(u_1;\omega(\nu)) \cdots \Phi(u_K;\omega(\nu))| \le 1.$$

Therefore, the above integral does not exceed

$$\int_{\mathbb{T}^K} \left| R_\varkappa(u_1,\dots,u_K;N) - \det\left[u_j^{-(\varkappa_i - i + j)}\right]_{i,j=1}^K \right| m(du)$$

and the desired uniform bound follows from the second assertion of Theorem 8.2. $\square$

Acknowledgments

A.B. was partially supported by NSF-grant DMS-1056390. G.O. was partially supported by a grant from Simons Foundation (Simons—IUM Fellowship), the RFBR-CNRS grant 10-01-93114, and the project SFB 701 of Bielefeld University.

Appendix

Let $\{\nu(N) \in \mathbb{GT}_N : N = 1, 2, \dots\}$ be a sequence of signatures of growing length. We say that it is *regular* if for any fixed $K = 1, 2, \dots$ the sequence of probability measures $\Lambda^N_K(\nu(N), \,\cdot\,)$ weakly converges to a probability measure on $\mathbb{GT}_K$. (This means that for every $\varkappa \in \mathbb{GT}_K$ there exists a limit $\lim_{N\to\infty} \Lambda^N(\nu(N), \varkappa)$ and the sum over $\varkappa \in \mathbb{GT}_K$ of the limit values equals 1.) This definition is equivalent to regularity of the sequence of normalized characters $\widetilde{\chi}_{\nu(N)}$ as defined in [22].

A particular case of the results of [22] is the following theorem.

Theorem A.1. *A sequence $\{\nu(N) \in \mathbb{GT}_N : N = 1, 2, \dots\}$ is regular if and only if the corresponding sequence $\{\omega(\nu(N))\}$ of points in Ω converges to a point $\omega \in \Omega$.*

Moreover, if $\{\nu(N) \in \mathbb{GT}_N : N = 1, 2, \dots\}$ is regular, then the limit measure $\lim_{N\to\infty} \Lambda^N_K(\nu(N), \,\cdot\,)$ coincides with $\Lambda^\infty_K(\omega, \,\cdot\,)$, where $\omega = \lim_{N\to\infty} \omega(\nu(N))$.

The aim of this section is to discuss the interrelations between this assertion and the Uniform Approximation Theorem (Theorem 3.1). Recall that this theorem says that for any fixed $\varkappa \in \mathbb{GT}_K$

$$\lim_{N\to\infty} \sup_{\nu\in\mathbb{GT}_N} \left| \Lambda^N_K(\nu, \varkappa) - \Lambda^\infty(\omega(\nu), \varkappa) \right| = 0. \tag{A.1}$$

Derivation of Theorem A.1 from Theorem 3.1. Combining (A.1) with continuity of $\Lambda^\infty_K(\omega, \varkappa)$ in the first argument we see that if the sequence $\omega(\nu(N))$ converges to a point $\omega \in \Omega$, then for any fixed K, the measure $\Lambda^N_K(\nu(N), \,\cdot\,)$ weakly converges to the probability measure $\Lambda^\infty_K(\omega, \,\cdot\,)$, so that $\{\nu(N)\}$ is regular.

Conversely, assume that $\{\nu(N)\}$ is regular and prove that $\{\omega(\nu(N))\}$ has a limit $\omega \in \Omega$. Since Ω is locally compact, it suffices to prove that $\{\omega(\nu(N))\}$ cannot have two distinct limit points in Ω and cannot contain a subsequence converging to infinity.

The existence of distinct limit points is excluded by virtue of the argument above and the fact that different points of Ω generate different measures on $\mathbb{GT}_1$, which in turn follows from Proposition 2.5.

The escape to infinity for a subsequence is also impossible, as is seen from (A.1) and the fact that $\Lambda^\infty_K(\omega, \varkappa) \to 0$ as ω goes to infinity.

This completes the proof. □

Derivation of Theorem 3.1 from Theorem A.1 and results from [19]. It suffices to prove the following assertion. If $N(1) < N(2) < \cdots$ and $\nu(1) \in \mathbb{GT}_{N(1)}$, $\nu(2) \in \mathbb{GT}_{N(2)}, \dots$ are such that for any fixed K and $\varkappa \in \mathbb{GT}_K$ there exists a limit

$$\lim_{n\to\infty} \left(\Lambda^{N(n)}_K(\nu(n), \varkappa) - \Lambda^\infty(\omega(\nu(n)), \varkappa) \right) = c_\varkappa,$$

then $c_\varkappa = 0$ for all $\varkappa$.

Passing to a subsequence we are led to the following two cases: either the sequence $\{\omega(\nu(N))\}$ converges to a point $\omega \in \Omega$ or this sequence goes to infinity.

In the first case, the desired assertion follows from Theorem A.1. Indeed, it says that $\Lambda_K^{N(n)}(\nu(n), \varkappa) \to \Lambda_K^{\infty}(\omega, \varkappa)$. On the other hand, $\Lambda_K^{\infty}(\omega(\nu(n)), \varkappa) \to \Lambda_K^{\infty}(\omega, \varkappa)$ by continuity of $\Lambda_K^{\infty}(\omega, \varkappa)$.

In the second case, we know that $\Lambda_K^{\infty}(\omega(\nu(n)), \varkappa) \to 0$ for any $\varkappa$ (see Corollary 2.11). Therefore, we have to prove that for any K, the measures $M_K^{(n)} := \Lambda_K^{N(n)}(\nu(n), \cdot)$ weakly converge to 0.

Without loss of generality we may assume that for every K the sequence $\{M_K^{(n)}\}$ weakly converges to a measure $M_K^{(\infty)}$. It follows (here we also use the Feller property of the stochastic matrices Λ_K^{K+1}, see Proposition 2.12) that the limit measures are compatible with these matrices:

$$M_{K+1}^{(\infty)} \Lambda_K^{K+1} = M_K^{(\infty)}, \quad K = 1, 2, 3, \ldots.$$

Therefore, the total mass of $M_K^{(\infty)}$ does not depend on K. If this mass equals 1, that is, the limit measures are probability measures, then Theorem A.1 implies that the sequence $\omega(\nu(N(n)))$converges in Ω, which is impossible. If the total mass is equal to 0, the limit measures are zero measures and we are done. Thus, it remains to prove that the total mass of M_K^{∞} cannot be equal to a number strictly contained between 0 and 1. It suffices to prove this assertion for $K = 1$, and then it is the subject of the proposition below, which relies on results of [19]. □

Proposition A.2. *Let $M^{(1)}, M^{(2)}, \ldots$ be a sequence of probability measures on $\mathbb{Z}$ such that every $M^{(n)}$ has the form $\Lambda_1^N(\nu, \cdot)$, where $N \geq 2$ and $\nu \in \mathbb{GT}_N$ depend on n. Then $\{M^{(n)}\}$ cannot weakly converge to a nonzero measure of total mass strictly less than* 1.

In other words, such a sequence of probability measures cannot escape to infinity *partially*.

Proof. A measure M on $\mathbb{Z}$ is said to be *log-concave* if for any two integers k, l of the same parity

$$M(k)M(l) \leq \left(M\left(\frac{1}{2}(k+l)\right)\right)^2.$$

Each measure of the form $M = \Lambda_1^N(\nu, \cdot)$ is log-concave: this nontrivial fact is a particular case of the results of [19].

Furthermore, such a measure has no *internal zeros*, that is, its support is a whole interval in $\mathbb{Z}$. Indeed, it is not hard to check that the support of $\Lambda_1^N(\nu, \cdot)$ is the interval $\{\nu_N, \ldots, \nu_1\} \subset \mathbb{Z}$.

Thus, our probability measures $M^{(n)}$ are log-concave and have no internal zeros. Assume that they weakly converge to a *nonzero* measure $M^{(\infty)}$. Then we may apply the argument of [19, p. 276]. It provides a uniform on n bound on the tails of measures $M^{(n)}$, which shows that for any $r = 1, 2, \ldots,$ the rth moment of $M^{(n)}$ converges to the rth moment of $M^{(\infty)}$. The convergence of the second moments already suffices (via Chebyshev's inequality) to conclude that $M^{(\infty)}$ is a probability measure. □

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