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Exotic Chekanov tori in toric symplectic varieties

Nikolay A. Tyurin

Bogolyubov Lab. of Theoretical Physics, JINR (Dubna), Joint Institute for Nuclear research, Joliot - Curie, 6, Moscow region, Dubna 141980, Russia and Bogomolov Lab. of Algebraic Geometry, HSE (Moscow), National Research University Higher School of Economics, 20 Myasnitskaya Ulitsa, Moscow 101000, Russia

Abstract. We present a generalization of toric structures on compact symplectic manifolds called pseudotoric structure. In the present talk we show that every toric manifold admits pseudotoric structures and then we show that the construction of exotic Chekanov tori can be peformed in terms of pseudotoric structures.

Let (X, ω) be a symplectic manifold of real dimension $2n$ so it can be understood as the phase space of a classical mechanical system. Lagrangian geometry of X is focused on the questions about lagrangian submanifolds of X namely: which homology classes from $H_n(X, \mathbb{Z})$ can be realized by smooth lagrangian submanifolds; what are the topological types of these lagrangian submanifolds; classification up to lagrangian deformations of lagrangian submanifolds of the same topological type and homology class; classification up to Hamiltonian isotopy of lagrangian submanifolds of the same deformation type; unification of all lagrangian submanifolds in an appropriate category.

Recall that $S \subset X$ is lagrangian if the restriction $\omega|_S$ vanishes identically and real dimension of S is maximal, equal to n. Thus at least the homology class of S must be perpendicular to the cohomology class $[\omega]$. Two lagrangian submanifolds $S_0, S_1 \subset X$ are of the same deformation type if there is a family of lagrangina submanifolds $S_t, t \in [0,1]$ which ends at S_0 and S_1. Hamiltonian isotopy of lagrangian submanifold $S_0 \subset X$ is given by a time dependent Hamiltonian function $H(x,t) : X \times \mathbb{R} \to \mathbb{R}$ which generates the flow ϕ^t_H, and $S_t = \phi^t_H(S_0)$ is the corresponding isotopy.

Toy example: dim $= 2$. Let Σ be a Riemann surface equipped with a symplectic form. Then since every loop is lagrangian (dimensional reason): every primitive homology class from $H_1(\Sigma, \mathbb{Z})$ is realizable by a smooth lagrangian submanifold; every smooth lagrangian submanifold is isomorphic to S^1; two loops from the same homology class are deformation equivalent; two loops are Hamiltonian isotopic if the symplectic area of the oriented film bounded by the loops is zero; the Fukaya category for a curve of any genus exists.

Thus for this case the problem is completely solved!

But making one new step we face already highly nontrivial situation. Consider "the simplest and basic" 4- dimensional compact symplectic manifold — the projective plane $\mathbb{CP}^2$. Fot the projective plane we have that: since the cohomology group $H^2(\mathbb{CP}^2, \mathbb{Z}) = \mathbb{Z}$, any lagrangian submanifold must present trivial homology class; there are no lagrangian 2 - spheres (M. Gromov), Riemannian surfaces of genus $g > 1$ (M. Audin), Klein bottles (S. Nemirovsky and V. Shevchishin) — all these types are not realized by smooth lagrangian submanifolds of the projective plane; it was believed that well known Clifford tori are unique examples of lagrangian tori in $\mathbb{CP}^2$ since in 1996 Yu. Chekanov proposed a construction of lagrangian torus which is not

Hamiltonian isotopic to a Clifford torus — and nobody knows are there other types of lagrangian tori; nevertheless certain constructions of appropriate categories exist (Fukaya - Seidel).

Thus even for this basic case in dimension 4 the problem is not solved yet.

Why we are interested in lagrangian geometry? Lagrangian geometry is very important in Mathematical physics; f.e. several approaches to Geometric Quantization are based on Lagrangian geometry. In these approaches lagrangian submanifolds represent quantum states so an old idea of P.M. Dirac, stated that the phase space of classical mechanical system should contain the ingredients of a natural quantization procedure, is realized. Thus it is natural to study all possible states so it is reasonable to find all types of lagrangian submanifolds (see, f.e. [1]).

F.e. in ALAG - programme (abelian algebraic lagrangian geometry, see [2]) the Chekanov result ensures that the moduli space of half weighted Bohr - Sommerfeld lagrangian cycles of level 3, $\mathcal{B}^{hw,r}_{S,3}$, has at least two disjoint components, and may be in a future one will find certain connecting space with a tunneling effect between these components.

As well in a popular modern subject of Mathematical physics — Homological Mirror symmetry — one should try to describe all objects in the Fukaya category, so all types of nonisotopic lagrangian tori.

Well known Clifford tori in $\mathbb{CP}^2$ comes from the toric geometry: the projective plane carries two real Morse functions in involution with respect to the Poisson brackets induced by the Kahler form of the standard Fubini - Study metric. These functions can be explicitly expressed as:

$$f_1 = \frac{|z_1|^2 - |z_2|^2}{\sum_{i=0}^2 |z_i|^2}, f_2 = \frac{|z_0|^2 - |z_1|^2}{\sum_{i=0}^2 |z_i|^2}, \{f_1, f_2\}_\omega = 0$$

in homogeneous coordinates $[z_0 : z_1 : z_2]$; the degeneration set

$$\Delta(f_1, f_2) = \{df_1 \wedge df_2 = 0\} \subset \mathbb{CP}^2$$

is formed by three lines $l_i, l_i = \{z_i = 0\}$; the action map $F = (f_1, f_2) : \mathbb{CP}^2 \to P_{\mathbb{CP}^2} \subset \mathbb{R}^2$ sends $\Delta(f_1, f_2)$ to the boundary component $\partial P_{\mathbb{CP}^2}$ of the convex polytop $P_{\mathbb{CP}^2}$, and the preimage of any inner point $p \in P_{\mathbb{CP}^2}$ is a smooth lagrangian torus, labeled by values of f_1, f_2. Thus the Clifford tori are just Liouville tori for this completely integrable system. And it is the standard picture for any toric manifold.

In 1996 Yu. Chekanov in [3] proposed the construction of exotic lagrangian tori by the first version to $\mathbb{R}^4$. The construction looks rather simple: fix a complex structure, so we have $\mathbb{C}^2$ with a coordinate system (z_1, z_2); choose a smooth contractible loop $\gamma \subset \mathbb{C}^*$ which lies in a half plane so $\mathrm{Re}\gamma > 0$; consider two - dimensional subset given in the coordinates by the explicit formula $(z_1, z_2) = (e^{i\phi}\gamma, e^{-i\phi}\gamma)$ — and it is a lagrangian torus! Note however that if the loop γ is not contractible, we get a torus which is equivalent to the standard one. Furthemore, since the projective plane without projective line $\mathbb{CP}^2 \backslash l$ is symplectomorphic to an open ball in $\mathbb{R}^4$ one implements the construction to the projective plane. Using certain special Hofer's capacity technique, Chekanov proved this torus is not equivalent to the standard one.

This exotic torus was called **the Chekanov torus**; the forthcoming paper by Yu. Chekanov and F. Schlenk contains the details how to construct these nonstandard tori in the projective space $\mathbb{CP}^n$ for certain n, the products $S^1 \times ... \times S^1$, and some other cases, see [4].

An alternative description of the Chekanov tori based on the notion of pseudotoric structure. We can produce the torus taking the pencil $\{Q_w\}$ such that $Q_w = \{z_1 z_2 = w\} \subset \mathbb{C}^2$ — one dimensional complex family of quadratic surfaces given in the coordinate system (z_1, z_2) by the quadratic equation which depends on complex parameter $w \in \mathbb{C}$. Then one takes real Morse function $F = |z_1|^2 - |z_2|^2$ and observes that the Hamiltonian vector field X_F of this function F preserves each quadric Q_w from the family. Then one fixes a smooth contractible loop $\gamma' \subset \mathbb{C}^*_w$

where $\mathbb{C}_w$ parameterizes our family $\{Q_w\}$. The choice of the value for our function F marks the level set on each quadratic surface which is a loop, so taking smooth loops $S_w = \{F = 0\} \cap Q_w$ on every quadratic surface $Q_w, w \in \gamma'$ and collecting all these loops along γ' one gets a torus:

$$T(\gamma') = \bigcup_{w \in \gamma'} S_w;$$

it is not hard to see, that we again get the Chekanov torus from the previous construction, if we put $\gamma = \sqrt{\gamma'}$.

Let's repeat the construction for the projective plane. To do this consider pencil of quadrics $\{Q_p\}$, $p \mapsto [\alpha : \beta] \subset \mathbb{CP}^1_{\alpha,\beta}$ where $Q_p = \{\alpha z_1 z_2 = \beta z_0^2\} \subset \mathbb{CP}^2$. Take real Morse function F explicitly given by the formula

$$F = \frac{|z_1|^2 - |z_2|^2}{\sum_{i=0}^2 |z_i|^2}$$

in homogenoues coordinates $[z_0 : z_1 : z_2]$. It can be checked directly that its Hamiltonian vector field X_F preserves each element of the pencil, so we can proceed as in the previous noncompact case. Let's choose a smooth contractible loop $\gamma \subset \mathbb{CP}^1_{\alpha,\beta} \backslash \{[1 : 0], [0 : 1]\}$ since the last points are covered by singular quadrics; then on each quadric $Q_p, p \in \gamma$ we can take the level set $S_p = \{F = 0\} \cap Q_p$, and this level set is a smooth loop. Then we collect the level sets S_p along the loop γ getting again a lagrangian torus $T(\gamma) = \bigcup_{p \in \gamma} S_p$. The point is that the resulting torus is exactly the Chekanov torus, given by the identification of symplectic ball in $\mathbb{R}^4$ and $\mathbb{CP}^2 \backslash$line. On the other hand if $\gamma \subset \mathbb{CP}^1_{\alpha,\beta}$ was taken non contractible then the resulting torus would be equivalent to a Clifford torus.

Therefore we get certain correspondence between the equivalence classes of lagrangian tori and the fundamental group of the punctured projective line $\pi_1(\mathbb{CP}^1_{\alpha,\beta} \backslash \{[0 : 1], [1 : 0]\})$ without the north and the south poles.

What is the difference between toric and pseudo toric considerations?

We illustrate it on the ideal level by the following diagramme:

$\mathbb{R}$	$\mathbb{C}$
real Morse function f	Lefschetz pencil $\{Q_p\}$
$\parallel$	$\parallel$
$f : X \to \mathbb{R}$	$\psi : X \backslash B \to \mathbb{CP}^1, Q_p = \overline{\psi^{-1}(p)}$
$\downarrow$	$\downarrow$
toric case	pseudtoric case
(f_1, f_2) on $\mathbb{CP}^2$	$(f, \{Q_p\})$ on $\mathbb{CP}^2$
such that $\{f_1, f_2\}_\omega = 0$	such that $X_f \parallel Q_p$
$\uparrow$	$\uparrow$
standard commutation rel.	new commutation rel.

A complex analog of a real Morse function is a Lefschetz pencil, roughly speaking it is just a complex (or symplectic) map to the compactified complex space. The question is how to relate the real data and new complex data, so what does it mean that a real function and a Lefschetz pencil commute? We propose the following new commutation relation: pencil $\{Q_p\}$ commutes with real function f if the Hamiltonian vector field X_f is parallel to each element Q_p of the pencil at each point. Geometrically (or dynamically) this means that the Hamiltonian flow generated by f preserves the "level sets" of the Lefschetz pencil — but it is exactly the same as for the real functions!

Leaving aside other speculative arguments, we summurize with the following

Definition ([5]): Pseudotoric structure on a compact symplectic manifold (X, ω_X) consists of

· *(real data)* $\{f_1, ..., f_k\}$ — algebraically independent almost everywhere real Morse functions in involution, $\{f_i, f_j\}_\omega = 0$;
· *(complex data)* family of compact symplectic $2k$ -dimensional submanifolds $\{Q_p\}, Q_p \subset X$, parameterized by a compact toric symplectic manifold $(Y, \omega_Y) \ni p$ (or, equivalently, a map with symplectic fibers

$$\psi : X \backslash B \to Y, \quad Q_p = \overline{\psi^{-1}(p)}, \quad B = \text{base set}$$

such that the following *commutation relations* hold:
· the Hamiltonian vector field X_{f_i} of each Morse function f_i from (*real data*) is parallel to each Q_p at each point (or, equivalently, each f_i preserves the fibers of ψ by the Hamiltonian action);
· for each smooth function $h \in C^\infty(Y, \mathbb{R})$ bi- vector field
$X_{\psi^* h} \wedge \nabla_\psi X_h \equiv 0$ — identically vanishes on $X \backslash B$.

In the last expression $X_{\psi^* h} \wedge \nabla_\psi X_h$ we take two vector fields for a function $h \in C^\infty(Y, \mathbb{R})$ taken on the base manifold Y, namely for the lifted function $\psi^* h \in C^\infty(X \backslash B, \mathbb{R})$ on $X \backslash B$ one takes the Hamiltonian vector field with respect to the symplectic form ω_X; on the other hand one takes the lift $\nabla_\psi X_h$ of the Hamiltonian vector field X_h defined by the symplectic form ω_Y on Y, and ∇_ψ is the symplectic connection defined by ψ since this map has symplectic fibers:

$$\nabla_\psi : \Gamma(TY) \to \Gamma(T(X \backslash B)).$$

The last condition looks too horrible but in practice one avoids all the difficulties, taking in mind the following remark: if X and Y are complex, $k = n - 1$, and ψ is complex then the last commutation relation is automatically satisfied.

It's easy to see that the base set B of the family $\{Q_p\}$ must be contained by the degeneration locus $\Delta(f_1, ..., f_k) = \{df_1 \wedge ... \wedge df_k = 0\}$; the singular points of any fiber Q_p must be contained by the degeneration locus $\Delta(f_1, ..., f_k)$ as well; any fiber $Q_p = \psi^{-1}(p) \cup B$ endowed with restrictions $(f_1|_{Q_p}, ...f_k|_{Q_p})$ — is a completely integrable system (= toric (perhaps non smooth) symplectic manifold). Therefore pseudotoric structures supply us with the solutions of the following problem: for non completely integrable Hamiltonian system with the integrals $(f_1, ..., f_k)$ find toric leaves of the Hamiltonian action.

The simplest (and the trivial) example of pseudotoric structure arises if one takes the direct product of two toric manifolds $Y_1 \times Y_2$. This structure is topologically trivial, as the product vector bundle. In analogy with the theory of vector bundles we introduce the following

Definition ([5]): *the number* $n - k = \frac{1}{2} dim Y$ *is called the rank of pseudotoric structure* $(f_1, ..., f_k, \psi, Y)$.

Clearly it is parallel to the notion of the rank of vector bundle.

If singular points of fiber Q_p lies in the base set B we say that the fiber Q_p is regular; if generic fiber Q_p is regular then we say that pseudotoric structure is regular; it's not hard to see that in the regular case the image $\psi(\Delta(f_1, ..., f_k) \backslash B) = D_{\text{sing}} \subset Y$ is a proper compact symplectic submanifold. This submanifold measures topological non triviality of the pseudotoric structure; this subset is empty if and only if the peudotoric structure is topologically trivial so it is the product of toric manifolds.

The main reason for the introduction of this new structure is the possibility to construct lagrangian fibrations on whole X starting with lagrangian fibrations on the base toric manifolds and using the toric nature of the fibers. If we choose a system $(h_1, ..., h_{n-k})$ of commuting moment maps on Y (since Y by the definition is toric) we get a lagrangian fibration on the base Y but at the same time we have the following

Theorem ([6]): *Choice of moment maps* $(h_1, ..., h_{n-k})$ *on the base* Y *of a regular pseudotoric structure* $(f_1, ..., f_k, \psi, Y)$ *on a given* X *defines a lagrangian foliation on* X *whose generic fiber is a smooth lagrangian torus.*

The dimensional reduction which happens on $\Delta(h_1, ..., h_{n-k}) \subset Y$ is reflected by the fact that the collection of fibers over $\Delta(h_1, ..., h_{n-k})$ must be cutted from X, and then the resting part

$X \backslash (\bigcup_{p \in \Delta(h_1,...,h_{n-k})} Q_p)$ carries lagrangian fibration. This lagrangian fibration is only generically smooth (so generic fiber is a smooth lagrangian torus), but the singular fibers have singularities which are not of generic type. The type of the singularities is controlled and can be described as follows. A Liouville torus in a completely integrable system carries periodic orbits and unbounded real lines (if we consider irrational motion along the torus). Our singular tori admit additionally trajectories of the separatrix type: take a periodic loop on a torus and contract it to a point — then the torus turns to be singular and instead of periodic loop one gets a stable point. This is the type of singularities which appear in our lagrangian fibration.

General scheme can be summarized by the following diagramme

$$\begin{array}{lccccl}
(f_1,...,f_k): & B & \longrightarrow & \partial P_{Q_p} & & -\text{boundary component} \\
 & \cap & & \cap & & \\
(f_1,...,f_k): & X & \longrightarrow & P_{Q_p} & & -\text{moment pol.for gen.fib.} \\
 & \psi\downarrow & & \times & & \\
(h_1,...,h_{n-k}): & Y & \longrightarrow & P_Y & & -\text{moment pol.for base} \\
 & \cup & & \cup & & \\
(h_1,...,h_{n-k}): & D_{\text{sing}} & \longrightarrow & N & & -\text{hypersurface in} P_Y
\end{array}$$

— here singular lagrangian tori in the fibration are parameterized by certain "incidence cycle" appears from irregular singular fibers of the pseudotoric structure.

Now we slightly generalize the discussion concerning not lagrangian fibrations but lagrangian tori. For this case we can say about possible lifting of lagrangian tori from the base manifold the of pseudotoric structure, namely

Theorem ([7]) *Let $(f_1,...,f_k,\psi,Y)$ be a regular pseudotoric structure on a compact symplectic manifold X. Let $S \subset Y$ be a smooth lagrangian torus which doesn't intersect $D_{sing} \subset Y$. Then the choice of non critical values $(c_1,...c_k)$ of $f_1,...,f_k$ defines a smooth lagrangian torus $T(S,c_1,...,c_k) \subset X$.*

Shortly, the proof of the theorem based on the same procedure we've applied above to construct the Chekanov tori. Taking a lagrangian torus on the base, we collect lagrangian tori from the toric fiber — it can be done simulteneously thanks to the global functions $f_1,...,f_k$ — and the commutation relations from the very definition of pseudotoric structure ensure that the resulting figure is a lagrangian torus in X.

The last theorem shows that lagrangian tori from the base manifold after lifting could give different types of lagrangian tori in whole X. We can take the homology group $H_{n-k}(Y \backslash D_{\text{sing}}, \mathbb{Z})$ of the "punctured" base manifold and then attach to smooth lagrangian tori in the punctured base manifold different classes from the group. Conjecturelly the different classes from $H_{n-k}(Y \backslash D_{\text{sing}}, \mathbb{Z})$ can give different types of lagrangian tori in X. For example, it is true for the projective plane — as we've seen for Clifford and Chekanov tori in $\mathbb{CP}^2$ the following alternative appears:

$$H_1(\mathbb{CP}^1 \backslash ([1:0],[0:1]),\mathbb{Z}) \begin{array}{l} \nearrow \ \text{Clifford type} = \text{primitive elem.} \\ \searrow \ \text{Chekanov type} = \text{trivial elem.} \end{array}$$

But is it possible to construct such exotic lagrangian tori for any compact toric variety? The answer is an affirmative in view of the following

Theorem ([8]): *Any smooth compact toric symplectic manifold admits regular pseudotoric structure $(f_1,...,f_{n-1},\psi,\mathbb{CP}^1)$ of rank one. For this structure the singular divisor $D_{sing} \subset \mathbb{CP}^1$*

consists of exactly two distinct points, $p_N, p_S \subset \mathbb{CP}^1$. The primitive and the trivial elements of $H_1(\mathbb{CP}^1 \backslash (p_N \cup p_S), \mathbb{Z})$ generates lagrangian tori of the standard type and of the Chekanov type respectively.

Suppose additionaly that our given toric (X, ω_X) is monotone, this means that the cohomology class of the symplectic form is proportional to the canonical class of the associated almost complex structure on X: $K_X = k[\omega_X] \subset H^2(X, \mathbb{Z})$; so f.e. Fano varieties in algebraic geometry are monotone. Then the main conjecture we would like to propose in this circumstances is based on the following remark: if there is a standard monotone lagrangian torus then there exists a monotone lagrangian torus of the Chekanov type. And for the future work we have as the major aim the following

Main conjecture. *These monotone tori are not Hamiltonian isotopic.*

The Theorem above which states the existence of pseudotoric structures on toric symplectic manifolds can be proved as follows ([8]). Let's take for a given toric X the set of commuting Morse moment maps $(f_1, ... f_n)$, which give the action map by "action coordinates" $F = (f_1, ..., f_n) : X \to P_X$ to convex moment polytop $P_X \subset \mathbb{R}^n$; then for the components D_i of the boundary divisor $D = F^{-1}(\partial P_X)$ one can find an integer combination $\sum \lambda_i D_i$ equals to zero. This sum can be rearranged to the form

$$\sum_{\lambda_i > 0} \lambda_i D_i = \sum_{\lambda_j < 0} |\lambda_j| D_j, \quad D_i \neq D_j;$$

therefore we have two divisors from the same linear system

$$D_+ = \sum_{\lambda_i > 0} \lambda_i D_i, \quad D_- = \sum_{\lambda_j < 0} |\lambda_j| D_j \in |\sum_{\lambda_i > 0} \lambda_i D_i|.$$

Then one takes the pencil $< D_+, D_- >$ spanned by two divisors $D_\pm$ with the base set $B = D_+ \cap D_-$, and it would be our pencil ψ from the definition of pseudotoric structure, and for generic point $p \in \mathbb{CP}^1$, $p \neq [1:0](\mapsto D_+)$, $[0:1](\mapsto D_-)$, the divisor $\overline{\psi^{-1}(p)} \subset X$ is smooth outside the base set B. The same linear combination $\sum \lambda_i D_i$ after substitution of linear forms l_i which correspond to D_i in $\mathbb{R}^n$ gives a linear relation on x_i — and this relation derive our real data $f'_1, ... f'_{n-1}$ from $f_1, ..., f_n$ just implying the corresponding linear condition.

Example: $\mathbb{CP}^2_3$ — del Pezzo surface of degree 6 is usually given by the blow up procedure applied to the projective plane $\mathbb{CP}^2$ at three points. Explicitly it can be realized in the direct product

$$\mathbb{CP}^2_x \times \mathbb{CP}^2_y \supset \mathcal{U} = \{x_0 y_0 = x_1 y_1 = x_2 y_2\}$$

by the last equation, then the projection to the projective plane is given by $p_x : \mathcal{U} \to \mathbb{CP}^2_x$, $p_x(x_i, y_j) = [x_0 : x_1 : x_2]$. Thus $p^0_x : \mathcal{U} \backslash \text{three lines} \simeq \mathbb{CP}^2_x \backslash \text{three points}$, but $(p^0_x)^{-1}(T_{\mathrm{Ch}}) \subset \mathcal{U}$ is **not lagrangian** — so we can not lift the Chekanov torus using this standard algebro geometric construction. But fortunately we can lift the corresponding pseudotoric structure!

Indeed, let's take the pencil

$$\{Q_{\alpha,\beta}\} = \{\alpha x_0 x_1 y_2^2 = \beta x_2^2 y_0 y_1\} \subset \mathbb{CP}^2_x \times \mathbb{CP}^2_y,$$

on the whole direct product, and then the intersections $Q_{\alpha,\beta} \cap \mathcal{U}$ would give us the Lefschetz pencil ψ on $\mathcal{U}$. Further, the real Morse function

$$F = \frac{|x_0|^2 - |x_1|^2}{\sum_{i=0}^2 |x_i|^2} + \frac{|y_1|^2 - |y_0|^2}{\sum_{i=0}^2 |y_i|^2}$$

preserves by the Hamiltonian action our surface $\mathcal{U}$ and at the same time does do it for each element $Q_{\alpha,\beta}$ of the pencil, therefore $f = F|_{\mathcal{U}}$ must preserve the intersections and thus would give the real data of the desired psedudotoric structure. The choice of a smooth loop $\gamma \subset \mathbb{CP}^1 \backslash ([1:0],[0:1])$ gives a lagrangian torus $T(0,\gamma) = \bigcup_{p\in\gamma}\{f|_{\overline{\psi^{-1}(p)}} = 0\}$, and if γ is contractible, we get a Chekanov torus in $\mathbb{CP}^2_3$

At the end I would like to mention several applications of this generalized notion, pseudotoric structure.

Lots of methods in Mathematical Physics are invented and realized with great success in the case of toric varities. In Geometric Quantization (see, f.e., [9]) we know what one should do in the case when the phase space of classical mechanical system carries a real polarization, namely one takes the Bohr - Sommerfeld fibers and spann the Hilbert space. In Homological Mirror Symmetry (see, f.e., [10]) one takes the canonical fibration on lagrangian tori, counts the fibers with non trivial Floer cohomologies — and then builds on these fibers the corresponding Fukaya category. But all these methods can not be applied in non toric case since if one takes a non toric variety — nobody knows in general how to slice it on lagrangian fibers.

Pseudotoric structure on a symplectic manifold X gives way to apply these methods in more general setup. Indeed, the theorems above ensure that we can construct aslmost canonically certain lagrangian fibrations in the presence of pseudotoric structures. What is the difference with the "regular" toric case? It is in the appearence of singular lagrangian fibers. But as we've seen the types of singularities which appear in the fibers are very special: for example the notion of Bohr - Sommerfeld lagrangian cycle is still valid for these singular lagrangian tori without any modification. One hopes that the definition of the Floer cohomology can be modified as well. Then we can use the powerfull methods not only in toric geometry but in much more wider context — since there are compact symplectic manifolds which are not toric but nevertheless which admit pseudotoric structures. It is natural to call such a manifold **pseudotoric**: the examples are complex quadrics and certain complete intersections in $\mathbb{CP}^n$, the flag variety F^3 and conjecturelly certain complex Grasmmanians. And coming back to the main subject of the present talk we shoud say that in all these cases it is possible to construct lagrangian tori of different type using the pseudtoric structure.

Thus the natural problem arises: **which symplectic manifolds are pseudotoric?** Toric geometry itself concerns this question since in the framework of toric geometry one meets the problem with induced objects. F.e. if one takes the projectivization of the (co)tangent bundle of a toric variety – it is not longer toric (F^3 — the flag variety — is the projectivization of the tangent bundle of the "most toric" one — the projective plane) but it admits the Hamitonian action of an incomplete set of integrals lifted from the toric base. As well it could happen for certain moduli spaces over toric varieties. The construction of a pseudotoric structure is a solution of the toric leaves problem in general; and if this solution exists we can adopt the strong methods from toric geometry to this case.

As a byproduct we've touched a classical problem from mechanics — the study of non completely integrable systems. Again as we've seen the problem could be solved in the case when the phase space admits a psuodotoric structure. Then the solutions can be described in terms of the "action - angle" variables of the base manifold and the toric fibers. The difference with the completely integrable case is in the appearance of singular lagrangian tori of the Liouville type — and it is not problematic since it just means that some additional types of trajectories — separatrices — are presented in the story.

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