An open set of structurally unstable families of vector fields in the two-sphere

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This is the first part of a two parts paper dedicated to global bifurcations in the plane. In this part we construct an open set of three parameter families whose topological classification has a numerical invariant that may take an arbitrary positive value. In the second part we construct an open set of six parameter families whose topological classification has a functional invariant. Any germ of a monotonically increasing function $(\mathbb{R}, a) \to (\mathbb{R}, b)$ for any $a > 0, b > 0$ may be realized as this invariant. Here and below "families" are "families of vector fields in the two-sphere".

1 Introduction

There are zillions of families of planar vector fields whose bifurcations are investigated up to now. All of them are weakly structurally stable in their domains (in a neighborhood of a singular point or a polycycle). Thirty years ago Arnold conjectured that this is the case for generic families of vector fields considered on the whole sphere [1]. The goal of the first part of this paper is to disprove this conjecture[1].

Our first main result is

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[1]In [1] Arnold includes the conjecture mentioned above in the list of six. Four of them were disproved before. The last two are disproved in this paper. Right after the statement of the conjectures Arnold writes: "*Certainly proofs or counterexamples to the above conjectures are necessary for investigating nonlocal bifurcations in generic l-parameter families.*"

Theorem 1. *There exists an open set in the space of infinitely smooth three-parameter families of vector fields in the two sphere such that every family from this set is structurally unstable.*

The infinite smoothness allows us to use a very simple reference to the theory of normal forms below. The result is also true for low smoothness, but it requires more technical details that we prefer to skip. Recall the necessary definitions.

Definition 1. Two vector fields v and w on a manifold M are called *orbitally topologically equivalent*, if there exists a homeomorphism $M \to M$ that links the phase portraits of v and w, that is, sends orbits of v to orbits of w and preserves their time orientation.

Definition 2. Let M be a manifold, not necessarily closed, and B, B' be topological balls in $\mathbb{R}^k$. Two families of vector fields $\{ v_\alpha, \alpha \in B \}$, $\{ w_\beta, \beta \in B' \}$ on M are called *weakly topologically equivalent* if there exists a map

$$H\colon B \times M \to B' \times M, \quad H(\alpha, x) = (h(\alpha), H_\alpha(x)) \tag{1}$$

such that h is a homeomorphism, and for each $\alpha \in B$ the map $H_\alpha\colon M \to M$ is a homeomorphism that links the phase portraits of v_α and $w_{h(\alpha)}$.

Definition 3. We say that a family of vector fields is *weakly structurally stable* if it is weakly topologically equivalent to its small perturbations.

Two families are *topologically equivalent* if the map H in (1) is a homeomorphism. Topological classification of families with a very simple dynamics may have functional invariants that occur due to the requirement of the continuity of H_α in α [4]. That's why we consider weak equivalence instead of topological one.

2 Description of the families

We will first describe the degenerate vector fields, then their unfoldings.

2.1 Class of degeneracies

Consider a vector field v which has a polycycle γ with two vertexes and three edges, see Figure 1. The vertexes are hyperbolic saddles L and M with the characteristic numbers λ and μ. Recall that a *characteristic number* of a saddle is the modulus of the ratio of its eigenvalues, the negative one in the numerator. Suppose that

$$\lambda < 1, \ \lambda^2 \mu > 1. \tag{2}$$

The edges are: a time oriented separatrix loop l of L, and two time oriented saddle connections: LM and ML.

Moreover, suppose that the vector field v has a saddle E outside the polycycle γ and a saddle I inside the separatrix loop l of L. Suppose that one of the unstable separatrixes

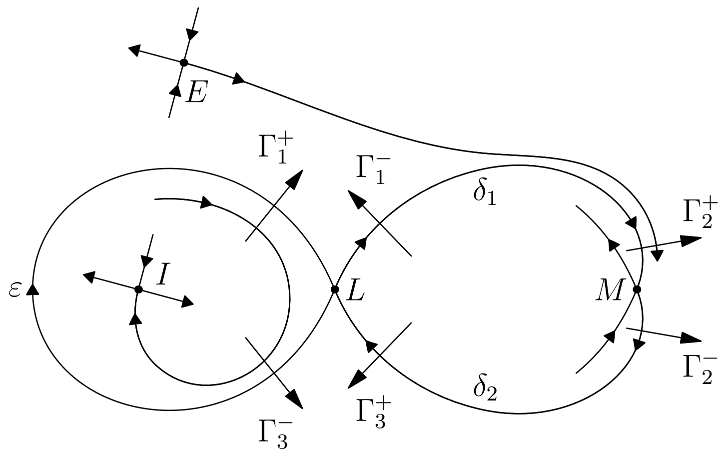

Figure 1: The phase portrait of the degenerate vector field v

of E winds onto γ, and one of the stable separatrixes of I winds onto l in the negative time. Denote by U_E and S_I these separatrixes. In Section 4 we prove that inequalities (2) imply the possibility of this winding. Polycycles described above may occur in generic three-parameter families. Existence of separatrixes U_E and S_I winding to γ and from l does not increase the codimension of the degeneracy.

Theorem 2. *Let v be a vector field described above. Consider a generic three-parameter unfolding of v. Then this family is structurally unstable. Moreover, the ratio $\frac{\log \lambda}{\log \mu}$ is a topological invariant of this family.*

Theorem 1 follows from Theorem 2, and the first statement Theorem 2 follows from the second one. So, it is enough to prove the second statement of Theorem 2.

2.2 Genericity assumptions for the unfolding

Genericity assumptions for the unperturbed vector field v are: $\lambda \neq 1$, $\lambda\mu \neq 1$: they follow from (2).

Let us describe the genericity assumptions for the unfolding of v.

Let U_L and U_M be neighborhoods of L and M to be chosen later. Fix points $O_j^\pm$ at γ so that $O_1^\pm \in U_L$, $O_2^\pm \in U_M$ and $O_3^\pm \in U_L$, and the order of the points on γ is the following: L, O_1^-, O_2^+, M, O_2^-, O_3^+, L, O_3^-, O_1^+. Fix cross sections $\Gamma_j^\pm$ to γ through $O_j^\pm$ oriented from inside to outside of γ, $j = 1, 2, 3$.

Let $\alpha \in (\mathbb{R}^3, 0)$ be the parameter and v_α be the corresponding vector field of the family, $v = v_0$. For α close to zero, the vector field v_α has two saddles $L(\alpha)$ and $M(\alpha)$ smoothly depending on α, together with their separatrixes. Let $O_j^-(\alpha)$ be the intersection points of the outgoing separatrixes of these saddles with the cross sections Γ_j^- smoothly depending on α and such that $O_j^-(0) = O_j^-$. In a similar way the points $O_j^+(\alpha) \in \Gamma_j^+$ are defined. Choose smooth orientation preserving charts $x_{j,\alpha}^\pm \colon (\Gamma_j^\pm, O_j^\pm(\alpha)) \to (\mathbb{R}, 0)$. Other requirements for coordinates $x_{j,\alpha}^\pm$ will be introduced later.

Define the regular maps $\Delta^{reg}_{j,\alpha}$ from the cross section Γ^-_{j-1} to Γ^+_j along the orbits of v_α. The numeration is cyclic modulo 3, so $\Gamma^+_4 \equiv \Gamma^+_1$, $\Gamma^-_0 \equiv \Gamma^-_3$. These are smooth maps both in coordinate and the parameter because the points $O^-_{j-1}(0)$ and $O_j(0)$ are connected by an arc of a phase curve of the field v_0. In coordinates introduced above, the maps $\Delta^{reg}_{j,\alpha}$ become functions, that we denote by the same symbol. We can now state the genericity assumption on the unfolding v_α. Consider the map $\sigma\colon (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ given by

$$\sigma\colon \alpha \mapsto (\Delta^{reg}_{1,\alpha}(0), \Delta^{reg}_{2,\alpha}(0), \Delta^{reg}_{3,\alpha}(0)).$$

It may be called "*separatrix splitting*" map. We require:

$$\det \frac{\partial \sigma}{\partial \alpha}(0) \neq 0. \tag{3}$$

This is the only genericity assumption needed. If this assumption holds for one choice of smooth coordinates $x^\pm_{j,\alpha}$, then it holds for any.

2.3 Reparametrization

We may now make a new choice of parameters and set:

$$\Delta^{reg}_{1,\alpha}(0) = -\varepsilon, \quad \Delta^{reg}_{2,\alpha}(0) = \delta_1, \quad \Delta^{reg}_{3,\alpha}(0) = \delta_2. \tag{4}$$

Each of the parameters ε, δ_1, δ_2 equals to the size of the separatrix splitting for the loop of L and connections LM, ML respectively. In particular, an equality $\varepsilon = 0$, or $\delta_j = 0$ implies that the corresponding connection stays unbroken.

The class of families described in this section with the parametrization (4) will be called class T.

3 Proof of Theorem 2 modulo auxiliary lemmas

We study first bifurcations in a family V of class T, then consider two equivalent families.

3.1 Some bifurcations in a family of class T

Consider a line $\mathcal{E} = \{\, \delta_1 = \delta_2 = 0 \,\}$. It is parametrized by ε, and corresponds to vector fields v_α with the two connections between $L(\alpha)$ and $M(\alpha)$ unbroken.

Consider the bifurcations in a family V that occur when the parameter α changes along $\mathcal{E}$. There are two sequences of sparkling saddle connection:

- the outcoming separatrix U_E of the saddle E makes n circuits around the whole polycycle γ, and coincides with the incoming separatrix of the saddle L; denote by e_n the corresponding value of $\varepsilon \in \mathcal{E}$;
- the outcoming separatrix of the saddle L makes m circuits inside the loop of L around the saddle I, and then enters the saddle I, thus coinciding with the incoming separatrix S_I; denote by i_m the corresponding value of $\varepsilon \in \mathcal{E}$.

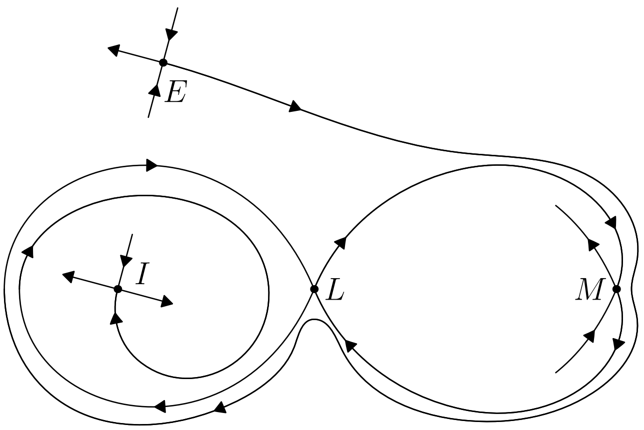

Figure 2: Sparkling saddle connections of an unfolding of v

These connections (that do not in general occur simultaneously), are shown in Figure 2. For simplicity, they are shown in one and the same figure.

The following lemma describes the asymptotics of the sequences e_n and i_m as $n \to \infty$ and $m \to \infty$.

Lemma 3. *In a neighborhood of zero the following holds:*

$$\log(-\log i_m) = -m \log \lambda + o(m), \tag{5}$$

$$\log(-\log e_n) = n \log(\lambda^2 \mu) + o(n). \tag{6}$$

Remark. In fact, $o(m)$ in (5) and $o(n)$ in (6) can be replaced by $a + r_m$ and $b + \rho_n$, respectively, where a and b are constants and r_m and ρ_n tend to 0 exponentially. We do not need this fact for our analysis in the first part, and its proof requires more technical details, so we will skip it.

This Lemma 3 is the most technical part of the proof. We postpone its proof to the end of the first part. Let us deduce the theorem from it.

3.2 Two equivalent families of class T

Consider another family V' of class T, and suppose that it is weakly topologically equivalent to V. For the family V' the analogues of the objects α, δ_1, δ_2, $\mathcal{E}$, I_m, e_n, λ, μ, ν are defined; they are denoted by α', ε', δ_1', δ_2', $\mathcal{E}'$, i_m', e_m', λ', μ', ν'.

Let H be the conjugacy (1) that links the families V and V', h be the corresponding map of the parameter spaces. The line $\mathcal{E}$ is topologically distinguished: it corresponds to the connections between $L(\alpha)$ and $M(\alpha)$ unbroken. Hence, $h(\mathcal{E}) = \mathcal{E}'$.

Denote by $\phi\colon (\mathbb{R}_+, 0) \to (\mathbb{R}_+, 0)$ the restriction of h to $\mathcal{E}$. Denote by $\Phi\colon (\mathbb{R}_+, \infty) \to (\mathbb{R}_+, \infty)$ the map ϕ written in the chart $\log(-\log \varepsilon)$. Due to Lemma 3,

$$\begin{aligned} \alpha_m &:= \log(-\log \varepsilon(i_m)) = -m \log \lambda + o(m), \\ \beta_n &:= \log(-\log \varepsilon(e_n)) = n \log(\lambda^2 \mu) + o(n). \end{aligned} \tag{7}$$

Same definitions and relations hold for α'_m and β'_n.

Note that Φ maps the germ of the set $\{ \alpha_m \}$ to the germ of $\{ \alpha'_m \}$ at infinity and preserves the order relation. Hence for some constant k and for m large enough we have $\Phi(\alpha_m) = \alpha'_{m+k}$. Similarly, for some constant l and n large enough, $\Phi(\beta_n) = \beta'_{n+l}$. Therefore,

$$\lim_{m\to\infty} \frac{\#\{ n \mid \beta'_n < a'_m \}}{m} = \lim_{m\to\infty} \frac{\#\{ n \mid \beta_n < a_m \}}{m}.$$

Finally, (7) implies that

$$\begin{aligned} \#\{ n \mid \beta_n < \alpha_m \} &= \#\left\{ n \;\middle|\; n\log(\lambda^2\mu) + o(n) < -m\log\lambda + o(m) \right\} \\ &= \frac{\log\lambda}{\log(\lambda^2\mu)} m + o(m), \end{aligned}$$

and similarly for α'_m, β'_n. Thus $\frac{-\log\lambda}{\log(\lambda^2\mu)} = \frac{-\log\lambda'}{\log((\lambda')^2\mu')}$, hence $\frac{\log\lambda}{\log\mu} = \frac{\log\lambda'}{\log\mu'}$, $\nu = \nu'$.

Theorem 2, modulo Lemma 3, is proved. We switch to the proof of the lemma. It is given in the next two sections.

4 The Poincaré maps for the loop and the polycycle

Let us consider first the Poincaré map $\Delta_{l,\varepsilon}$ for the loop l defined on the interior part $x_1^+ \le 0$ of the cross section Γ_1^+; denote this part by Γ^-. (See Section 2.2 and Figure 1 for the definition of Γ_1^+.)

Lemma 4. *The Poincaré map for the loop l in special coordinates is defined for small positive x and has the form:*

$$\Delta_{l,\varepsilon}(x) = C(\varepsilon)x^{\lambda(\varepsilon)}(1 + o(x^\kappa)) + \varepsilon + r(x,\varepsilon), \tag{8}$$

where functions $C(\varepsilon)$, $\lambda(\varepsilon)$ are positive and smooth, $\lambda(0) = \lambda$, the charasteristic number of the saddle L, number κ is some positive constant, $r = o(x^{\frac{3}{2}\lambda(\varepsilon)}, \varepsilon^{3/2})$.

Remark. This lemma implies that a stable separatrix of the saddle I may wind onto the separatrix loop l of the unperturbed vector field v in the negative time. Indeed, the fixed point zero is repelling for the map (8) because $\lambda < 1$ by assumption. Similarly, the next lemma implies that an unstable separatrix of the saddle E may wind onto the polycycle γ of the unperturbed vector field v in the positive time.

Proof. The theory of normal forms for local families provides a finitely smooth normal form for the family v_ε near a saddle L, see [2]. If λ is irrational, then the normal form is linear. If λ is rational, then the normal form is polynomial and integrable. In both cases the correspondence map $\Delta_{l,\varepsilon}^{sing} \colon \Gamma^- \to \Gamma_3^-$ may be explicitly calculated [2]. For our purposuses it is sufficient to know that there exist smooth ε-dependent coordinates x, y on Γ_1^+, Γ_3^- such that, for $\lambda \ne 1$,

$$y(\Delta_{l,\varepsilon}^{sing}(x)) = x^{\lambda(\varepsilon)}(1 + o(x^\kappa)), \tag{9}$$

where κ is some positive number, and $\lambda(\varepsilon)$ is the charasteristic number of the saddle $L(\varepsilon)$. Hence $\lambda(\varepsilon)$ is a smooth function, $\lambda(0) = \lambda < 1$.

The coordinates x, y induce an orientation of Γ_1^+, Γ_3^- opposite to that induced by x_1^+, x_3^-. Another requirement on x_1^+, x_3^- is: $x_1^+ = -x$, $x_3^- = -y$.

Denote by $f_{3,\varepsilon}^-$ the correspondence map $\Delta_{3,\varepsilon}^{reg}$ for $\alpha = (\varepsilon, 0, 0)$ written in the charts x, y. By definition of ε (see (4)), $f_{3,\varepsilon}^-(0) = \varepsilon$. Hence,

$$f_{3,\varepsilon}^-(x) = \varepsilon + C(\varepsilon)x + O(x^2, \varepsilon^2), \text{ where } C(\varepsilon) \neq 0. \tag{10}$$

The Poincaré map $\Delta_{l,\varepsilon}$ is the composition:

$$\Delta_{l,\varepsilon} = f_{3,\varepsilon}^- \circ \Delta_{l,\varepsilon}^{sing}.$$

The last three displays imply (8). □

In what follows, we will be interested in the inverse Poincaré map of the polycycle γ, rather than in the Poincar'e map itself. It is considered on the cross section Γ_3. In more detail, this is the map

$$\Delta_{\gamma,\varepsilon}^{-1} \colon \Gamma^+ \to \Gamma_3^-,$$

where Γ^+ is a subset of Γ_3^-, a half interval oriented as Γ_3^- (from inside to outside γ), with the the vertex 0.

Lemma 5. *The inverse Poincaré map for the polycycle γ in special coordinates is defined for small positive y and has the form:*

$$\Delta_{\gamma,\varepsilon}^{-1}(y) = C_1(\varepsilon) y^{\lambda_1(\varepsilon)}(1 + o(y^\kappa)) + C_2\varepsilon + \rho(y, \varepsilon), \tag{11}$$

where $C_1(\varepsilon) \neq 0$, $C_2 > 0$ and

$$\lambda_1(\varepsilon) = \frac{1}{\lambda^2(\varepsilon)\mu(\varepsilon)} < 1, \quad \rho(y, \varepsilon) = o(x^{\frac{3}{2}\lambda_1(\varepsilon)}, \varepsilon^{\frac{3}{2}}) \tag{12}$$

and κ is some positive constant.

Proof. There are singular correspondence maps $\Delta_j^{sing} \colon \Gamma_j^+ \to \Gamma_j^-$ defined on the parts of Γ_j^+ located outside γ. They may be described by means of the normal forms theory, but we will focus on maps inverse to them.

The singular maps written in the normalizing coordinates have the form:

$$(\Delta_{j,\varepsilon}^{sing})^{-1} = x^{\mu_j(\varepsilon)}(1 + o(x^\kappa)), \tag{13}$$

where κ is some positive constant,

$$\mu_1(\varepsilon) = \mu_3(\varepsilon) = \frac{1}{\lambda(\varepsilon)}, \quad \mu_2(\varepsilon) = \frac{1}{\mu(\varepsilon)}.$$

This is proved as formula (9).

Another requirement for the charts $x^{\pm}_{j,\alpha}$ is: they are restrictions of the normalizing charts to the corresponding cross sections $\Gamma^{\pm}_j$.

Denote by $f_{j,\varepsilon}$ the maps $f_{j,\varepsilon} = \Delta^{reg}_{j,\alpha}$ written in the charts $x^{\pm}_{j,\alpha}$ for $\alpha = (\varepsilon, 0, 0)$, $j = 1, 2, 3$. These maps are smooth in x and ε and bring Γ^-_j to Γ^+_{j+1}. For $j = 1, 2$, $f_{j,\varepsilon}(0) = 0$; moreover, by (4), $f_{3,\varepsilon} = -\varepsilon$.

The Poincaré map $\Delta_{\gamma,\varepsilon}$ has the form:

$$\Delta_{\gamma,\varepsilon} = \Delta^{sing}_{3,\varepsilon} \circ f_{2,\varepsilon} \circ \Delta^{sing}_{2,\varepsilon} \circ f_{1,\varepsilon} \circ \Delta^{sing}_{1,\varepsilon} \circ f_{3,\varepsilon}.$$

The inverse map has the form:

$$\Delta^{-1}_{\gamma,\varepsilon} = f^{-1}_{3,\varepsilon} \circ (\Delta^{sing}_{1,\varepsilon})^{-1} \circ f^{-1}_{1,\varepsilon} \circ (\Delta^{sing}_{2,\varepsilon})^{-1} \circ f^{-1}_{2,\varepsilon} \circ (\Delta^{sing}_{3,\varepsilon})^{-1} =: f^{-1}_{3,\varepsilon} \circ \widetilde{\Delta}. \tag{14}$$

The map $\widetilde{\Delta}$ is the same as in (14). It is a composition of the maps (13), and smooth maps preserving 0. Hence, by (14), it has the form:

$$\widetilde{\Delta}(y) = C(\varepsilon) y^{\lambda_1(\varepsilon)} (1 + o(y^{\kappa})) \tag{15}$$

for some smooth $C(\varepsilon) > 0$, $\kappa > 0$.

Moreover, for $x = x^-_3$,

$$f_{3,\varepsilon}(x) = -\varepsilon + C(\varepsilon)x + O(x^2, \varepsilon^2).$$

Hence, for $y = x^+_1$,

$$f^{-1}_{3,\varepsilon}(y) = \frac{y + \varepsilon}{C(\varepsilon)} + O(y^2, \varepsilon^2). \tag{16}$$

Combining (14), (15) and (16) we obtain (11), (12), and prove Lemma 5. □

We can now turn to the proof of Lemma 3.

5 Asymptotics of sparkling connections

Let us first consider the sparkling saddle connection with the saddle I involved. Recall that the corresponding values of the parameter ε are denoted by i_m. Assume that the stable separatrix of I intersects the cross section Γ^+_1 at a point with the coordinate $x = S_1(\varepsilon)$, where $S_1(\varepsilon)$ is a smooth function, x is the same chart as in (8). This chart orients Γ^+_1 inside the loop, so $S_1(\varepsilon) > 0$. Note that in the normalizing charts the unstable separatrix of L intersects the same cross section at a point with the coordinate $x = 0$. Therefore, the saddle connection occurs when

$$\Delta^m_{l,\varepsilon}(0) = S_1(\varepsilon). \tag{17}$$

The solutions of this equation are $\varepsilon = i_m$.

Consider now in a similar way the sparkling connection with saddle E involved. The corresponding values of the parameter ε are denoted by e_n. Denote by $S_2(\varepsilon)$ the intersection point of the unstable separatrix of E with Γ_3^-. Now we orient Γ_3^- outside of the loop, so $S_2(\varepsilon) > 0$. The corresponding equation takes the form:

$$\Delta_{\gamma,\varepsilon}^{-n}(0) = S_2(\varepsilon). \tag{18}$$

Note that we consider here the inverse Poincaré map. The solutions of (18) are $\varepsilon = e_n$.

Due to (8) and (11), both $\Delta_{l,\varepsilon}$ and $\Delta_{\gamma,\varepsilon}^{-1}$ have the same asymptotic expansion (with different $\lambda(\varepsilon)$). One can then apply the following Lemma 6 to equations (17) and (18) to prove Lemma 3.

Lemma 6. *Let $f_\varepsilon\colon (\mathbb{R}^+, 0) \to \mathbb{R}$ be a map of the form:*

$$f_\varepsilon(x) = C_1(\varepsilon)x^{\Lambda(\varepsilon)}(1 + o(x^\kappa)) + C_2\varepsilon + \rho(x, \varepsilon), \tag{19}$$

where $C_1(\varepsilon) > 0$, $C_2 > 0$, $\kappa > 0$, $\Lambda(0) \in (0, 1)$ and

$$\rho(x, \varepsilon) = o(x^{\frac{3}{2}\Lambda(\varepsilon)}, \varepsilon^{\frac{3}{2}}).$$

Let $\varepsilon_n > 0$ satisfy the following equation:

$$f_{\varepsilon_n}^n(0) = S(\varepsilon_n), \tag{20}$$

where $S(\varepsilon)$ is a smooth function and $S(0) > 0$. Let $\lambda = \Lambda(0)$.

Then

$$\log(-\log \varepsilon_n) = -n \log \lambda + O(1). \tag{21}$$

Proof. The idea is to estimate the function f_ε with simpler functions on some segment, bounded away from 0 for fixed ε and to use the monotonicity of the map. Let I_ε be a segment

$$I_\varepsilon = [C_2\varepsilon/2, 1], \tag{22}$$

where C_2 is the same as in (19).

Lemma 7. *There exist constants $C > 1$ and $1 > c > 0$ such that*

$$cx^\lambda \leq f_\varepsilon(x) \leq Cx^\lambda \tag{23}$$

for all $x \in I_\varepsilon$ and for all ε is small enough.

Proof of Lemma 7. Let us prove the upper estimate in (23). Note that for ε small enough, $\Lambda(\varepsilon) = \lambda + O(\varepsilon) < 1$ and for $x \in I_\varepsilon$ the following estimate holds:

$$C_2\varepsilon < 2x < 2x^{\Lambda(\varepsilon)}.$$

Therefore, for $x \in I_\varepsilon$,

$$f_\varepsilon(x) < (C_1(0) + 3)x^{\Lambda(\varepsilon)}(1 + o(x^\kappa)) + o(x^{\frac{3}{2}\Lambda(\varepsilon)}).$$

The last term is $o(x^{\Lambda(\varepsilon)+\kappa})$ for $\kappa > 0$ small enough, so it can be neglected. Thus it is sufficient to prove that

$$C' x^{\lambda+O(\varepsilon)}(1+o(x^{\kappa})) < Cx^{\lambda}$$

for some constant C. The last inequality holds because $x^{O(\varepsilon)} = e^{O(\varepsilon)\ln x}$ and $O(\varepsilon)\ln x$ is bounded for $x \in I_\varepsilon$ uniformely in ε for ε small enough. It is obvious that increasing C does not affect the inequality, so one can assume $C > 1$.

The proof of the lower estimate is similar. □

The folowing corollary is based on the fact that iterates of monotonic maps respect monotonicity.

Corollary 8. *The following estimates hold for all ε, x and n such that $x \in I_\varepsilon$ and $C^{\frac{1}{1-\lambda}} x^{\lambda^n} < 1$:*

$$c^{\frac{1}{1-\lambda}} x^{\lambda^n} \le f_\varepsilon^n(x) \le C^{\frac{1}{1-\lambda}} x^{\lambda^n}, \tag{24}$$

where constants c, C and C_2 are the same as in Lemma 7.

Proof of the Corollary. We shall prove the corollary by induction. The base $n = 1$ follows from (23) and the inequalities $c < 1 < C$. Suppose that (24) holds for n. Let us prove it for $n+1$.

Since $C^{\frac{1}{1-\lambda}} x^{\lambda^n} < 1$, estimates (23) and monotonicity of f_ε imply that

$$f_\varepsilon^{n+1}(x) \le f_\varepsilon\left(C^{\frac{1}{1-\lambda}} x^{\lambda^n}\right) \le C \times \left(C^{\frac{1}{1-\lambda}} x^{\lambda^n}\right)^{\lambda} = C^{\frac{1}{1-\lambda}} x^{\lambda^{n+1}}.$$

The lower estimate can be proved in the same way. □

Let us now deduce Lemma 6 from Corollary 8. In (24) let us replace $c^{\frac{1}{1-\lambda}}$ by c and $C^{\frac{1}{1-\lambda}}$ by C. As shown below, Corollary 8 allows us to replace equation (20) by a simpler equation of the form

$$cx^{\lambda^n} = S. \tag{25}$$

Its solution satisfies the relation

$$\log(-\log x) = -n\log\lambda + \log|\log\frac{S}{c}|. \tag{26}$$

Let us now prove (21). The saddle connection equation (20) can be rewritten in the following form:

$$f_{\varepsilon_n}^{n-1}(f_{\varepsilon_n}(0)) = S(\varepsilon_n). \tag{27}$$

Let $x_n := f_{\varepsilon_n}(0)$. Expansion (19) implies that $x_n = C_2\varepsilon_n + o(\varepsilon_n)$ and therefore $x_n \in I_{\varepsilon_n}$ for ε_n small enough. This allows us to apply Corollary 8. Let us fix some $S > S(0)$ and $s < S(0)$ such that $S(\varepsilon) \in (s, S)$ for ε small enough. Let x_+ and x_- be solutions of the equations:

$$cx_+^{\lambda^{n-1}} = S, \quad Cx_-^{\lambda^{n-1}} = s.$$

Corollary 8 and monotonicity of the maps in (24) implies: $x_- < x_n < x_+$. From this and (26) we have:

$$-n\log\lambda + \log\lambda + \log|\log\frac{s}{C}| < \log(-\log x_n) < -n\log\lambda + \log\lambda + \log|\log\frac{S}{c}|.$$

This completes the proof of the Lemma. □

Together with the Lemma, Theorems 1 and 2 are proved.

6 Conclusion

After these theorems are proved, a vast realm of problems opens.

Problem 1. *Prove that all the generic one-parameter families are weakly structurally stable.*

Problem 2. *Is the same correct for generic two-parameter families? If yes, give a topological classification of these families.*

Problem 3. *Distinguish structurally unstable generic three parameter families from the structurally stable ones.*

Remark. In [3], a complete list of polycycles that may occur in generic two and three parameter families was presented. It looks natural to study "sparkling saddle suspensions" over these polycycles.

Problem 4. *Find ALL the topological invariants of the family described in Theorem 2.*

7 Announcement

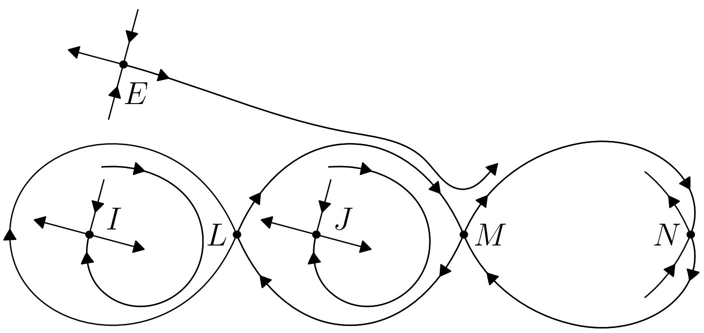

Figure 3: A codimension 5 vector field such that its generic 6-parametric unfolding has functional moduli of weak equivalence classification

The following theorem is the subject of the second part.

Theorem 9. *Consider a class of vector fields with a polycycle shown in Figure 3. There exists an open set of vector fields from this class such that a generic* 6*-parametric unfolding of any vector field from this set has a* functional modulus *of weak equivalence.*

Writing of the proof of this statement is now in process.

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