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# A dichotomy for the dominating set problem for classes defined by small forbidden induced subgraphs

D.S. Malyshev

*National Research University Higher School of Economics, 25/12 Bolshaya Pecherskaya Ulitsa, 603155, Nizhny Novgorod, Russia*

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ABSTRACT

We completely determine the complexity status of the dominating set problem for hereditary graph classes defined by forbidden induced subgraphs with at most five vertices.



## 1. Introduction

A *coloring* is an arbitrary mapping of colors to vertices of some graph. A graph coloring is said to be *proper* if no two adjacent vertices have the same color. The *chromatic number* $\chi(G)$ of a graph $G$ is the minimal number of colors in proper colorings of $G$. The *coloring problem*, for a given graph and a number $k$, is to determine whether its chromatic number is at most $k$ or not. The *vertex k-colorability problem* is to verify whether vertices of a given graph can be properly colored with at most $k$ colors. The edge $k$-colorability problem is defined by analogy.

An *independent set* and a *clique* of a graph are sets of pairwise non-adjacent and adjacent vertices, respectively. The *independent set problem* is to determine whether a given graph contains an independent set with a given number of elements. The clique problem is defined by analogy.

For a graph $G$, a subset $V' \subseteq V(G)$ *dominates* $V'' \subseteq V(G)$ if each of the vertices of $V''\backslash V'$ has a neighbor in $V'$. A *dominating set* of a graph $G$ is a subset dominating all its vertices. The size of a minimum dominating set of $G$ is said to be the *domination number* of $G$ denoted by $\gamma(G)$. For a graph $G$ and a number $k$, the *dominating set problem* is to decide whether $\gamma(G) \leq k$ or not.

A *class* is a set of simple unlabeled graphs closed under isomorphism. A class of graphs is *hereditary* if it is closed under deletion of vertices. It is well-known that any hereditary (and only hereditary) graph class $\mathcal{X}$ can be defined by a set of its forbidden induced subgraphs $\mathcal{Y}$. We write $\mathcal{X} = Free(\mathcal{Y})$ in this case, and the graphs in $\mathcal{X}$ are said to be $\mathcal{Y}$*-free*. If $\mathcal{Y} = \{G\}$, then we will write "$G$-free" instead of "$\{G\}$-free". If a hereditary class can be defined by a finite set of forbidden induced subgraphs, then it is said to be *finitely defined*.

The coloring problem for $G$-free graphs is polynomial-time solvable if $G$ is an induced subgraph of a $P_4$ or a $P_3 + K_1$, and it is NP-complete in all other cases [13]. A similar result is known for the dominating set problem. Namely, the problem is polynomial-time solvable for $Free(\{G\})$ if $G = P_i + O_k$, where $i \leq 4$ and $k$ is arbitrary, and it is NP-complete for all other choices of $G$ [11]. The situation for the vertex $k$-colorability problem is not clear, even when only one induced subgraph is forbidden.

*E-mail address:* dsmalyshev@rambler.ru.

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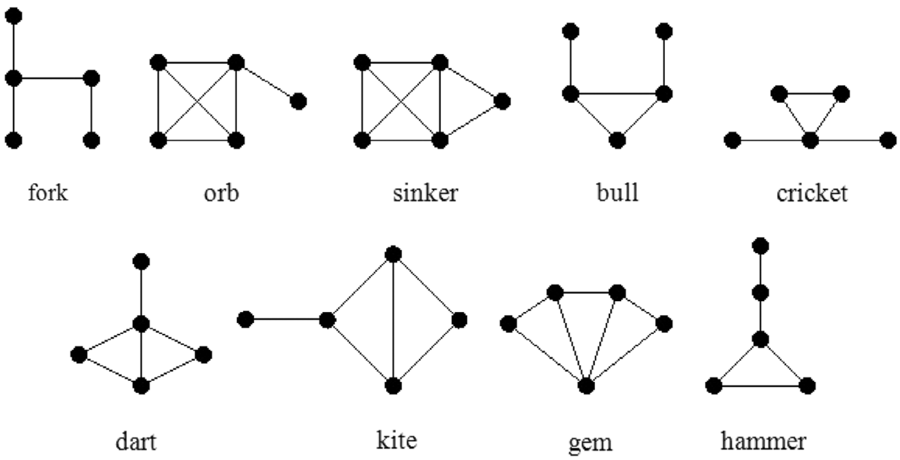


**Fig. 1.** The graphs *fork*, *orb* and, etc.

The complexity of the vertex 3-colorability problem is known for all the classes of the form $Free(\{G\})$ with $|V(G)| \leq 6$ [5]. A similar result for $G$-free graphs with $|V(G)| \leq 5$ was recently obtained for the vertex 4-colorability problem [10]. On the other hand, for fixed $k$, the complexity status of the vertex $k$-colorability problem is open for $P_7$-free graphs ($k = 3$), for $P_6$-free graphs ($k = 4$), and for $P_3 + P_2$-free graphs ($k = 5$).

The independent set problem is polynomial-time solvable for a hereditary class defined by forbidden induced subgraphs with at most five vertices if and only if a forest is one of the subgraphs, unless $P = NP$ [14,16]. A similar complete complexity dichotomy was obtained in [19] for the edge 3-colorability problem. For the coloring problem, a complete classification for pairs of forbidden induced subgraphs is open, even if they have at most four vertices. Although, the complexity is known for some such pairs [9,15,20,21,28].

In the paper, we present a complete dichotomy for the dominating set problem in the family of hereditary classes defined by forbidden induced subgraphs with at most five vertices.

## 2. Notation

We use the standard notation $P_n, O_n, K_n$ for a simple path, an empty graph, and a complete graph with $n$ vertices, respectively. A graph $K_{p,q}$ is a complete bipartite graph with $p$ vertices in the first part and $q$ in the second. The graphs *fork*, *orb*, *sinker*, *bull*, *cricket*, *dart*, *kite*, *gem*, *hammer* are drawn in Fig. 1.

A formula $N(x)$ denotes the neighborhood of a vertex $x$. A *sum* $G_1 + G_2$ is the disjoint union of $G_1$ and $G_2$ with non-intersected sets of vertices. A *graph join* $G_1 \times G_2$ of graphs with non-intersected sets of vertices is a graph $(V(G_1) \cup V(G_2), E(G_1) \cup E(G_2) \cup \{(v, u)|\ v \in V(G_1), u \in V(G_2)\})$. For a graph $G$ and $V' \subseteq V(G)$, a graph $G[V']$ is the subgraph of $G$ induced by $V'$. The symmetric difference of sets $A$ and $B$ is denoted by $A \otimes B$.

We refer to textbooks in graph theory for any terminology undefined here [4,7].

## 3. Boundary graph classes for the dominating set problem and their applications

To answer the question when an NP-complete graph problem becomes easier, a natural idea coming to mind is to consider a phase transition between easy and hard hereditary classes under some natural statements of the easiness and hardness. We use the following formal definitions. For a given NP-complete graph problem $\Pi$, a hereditary class is said to be *$\Pi$-easy* if $\Pi$ can be polynomially solved for its graphs. A hereditary class is *$\Pi$-hard* if $\Pi$ is NP-complete for it. Unfortunately, the phase transition approach seems to be unsuccessful.

Maximal $\Pi$-easy and minimal $\Pi$-hard classes are natural boundary elements in the lattice of hereditary classes. It turns out that the boundary may be absent at all. First, there are no maximal $\Pi$-easy classes, as any $\Pi$-easy class $\mathcal{X}$ can be extended by adding a graph $G \notin \mathcal{X}$ and all the proper induced subgraphs of $G$. Clearly, the resultant class is also $\Pi$-easy. Second, minimal hard classes exist for some problems and do not exist for some others. For a given graph and a positive length function on its edges, the *traveling salesman problem* is to check whether the minimum length of its cycles once visiting each vertex is at most a given number or not. It is NP-complete in the class of all complete graphs. Each proper hereditary subclass of the class is finite. Hence, the class of all complete graphs is a minimal hard case for the problem. On the other hand, for the vertex and edge variants of the $k$-colorability problem, any hard class contains a proper hard subclass. Indeed, if $\mathcal{Y}$ is a minimal hard case for the problem, then it must contain a graph $H$ that cannot be properly colored in $k$ colors. Therefore, $\mathcal{Y} \setminus Free(\{H\})$ contains only graphs that also cannot be properly colored in $k$ colors. There is a trivial polynomial-time algorithm to test whether a given graph in $\mathcal{Y}$ belongs to $\mathcal{Y} \cap Free(\{H\})$. Hence, $\mathcal{Y} \cap Free(\{H\})$ must be hard for the

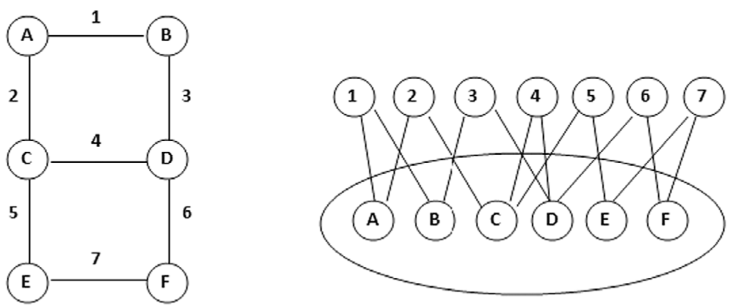


**Fig. 2.** The graph and the image of $Q(\cdot)$.

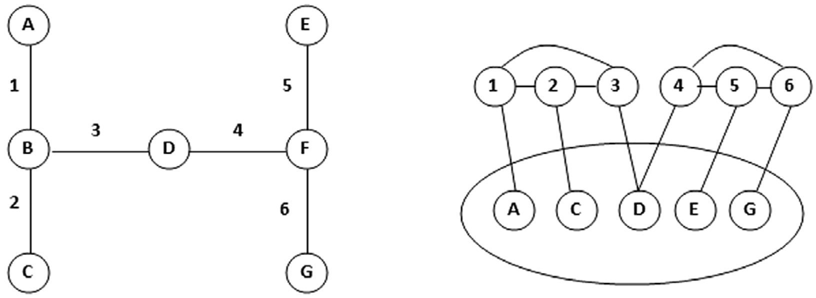


**Fig. 3.** The graph and the image of $Q^*(\cdot)$.

problem, and we have a contradiction. The phenomena of the absence of the boundary we just considered were noticed in [22].

So, to classify hereditary classes, we have to take into account that the sets of easy and hard classes can be open with respect to the inclusion relation. In other words, there may be infinite monotonically decreasing sequences of hard classes. Intuitively, the limits of such chains should play a special role in the analysis of the complexity. This observation leads to the notion of a boundary graph class. A class $\mathcal{X}$ is *Π-limit* if there is an infinite sequence $\mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \ldots$ of $\Pi$-hard classes such that $\mathcal{X} = \bigcap_{k=1}^{\infty} \mathcal{X}_k$. A $\Pi$-limit class that is minimal under inclusion is said to be *Π-boundary*. The following theorem shows the significance of the boundary class notion.

**Theorem 1** ([1,2]). *A finitely defined class is Π-hard if and only if it includes some Π-boundary class.*

The notion of a boundary graph class was originally introduced by V.E. Alekseev for the independent set problem [1]. It was later applied for the dominating set problem [3]. Nowadays, boundary classes are known for several algorithmic graph problems [1–3,12,18,22–25,27].

Assuming $P \neq NP$, four concrete graph classes are known to be boundary for the dominating set problem [3,25]. The first of them is $\mathcal{S}$. It constitutes all the forests with at most three leaves in each connected component. The second one is $\mathcal{T}$, which is the set of all the line graphs of the graphs in $\mathcal{S}$. To define the two remaining classes, we need to define two operators acting on graphs.

For a graph $G = (V, E)$, a graph $Q(G)$ has vertex set $V \cup E$ and edge set $\{(v_i, v_j)|\ v_i, v_j \in V\} \cup \{(v, e)|\ v \in V, e \in E, v \text{ is incident to } e\}$. The class $\mathcal{Q}$ is the set $\{G|\ \exists H \in \mathcal{S}, G = Q(H)\}$ plus the set of all the induced subgraphs of all its graphs. Let $G = (V, E)$ be a graph having degrees of vertices at most three. Let $V'$ be the set of all the degree three vertices of $G$ and $V'' \triangleq V(G)\backslash V'$. We define a graph $Q^*(G)$ as follows. The set $V(Q^*(G))$ coincides with $V'' \cup E$. A vertex $x \in V'$ is incident to edges $e_1(x), e_2(x), e_3(x)$ in the graph $G$. The set $E(Q^*(G))$ coincides with $\{(v_i, v_j)|\ v_i, v_j \in V''\} \cup \{(v, e)|\ v \in V'', e \in E, v \text{ is incident to } e\} \cup \bigcup_{x \in V'} \{(e_1(x), e_2(x)), (e_1(x), e_3(x)), (e_2(x), e_3(x))\}$. The class $\mathcal{Q}^*$ is the set $\{G|\ \exists H \in \mathcal{S}, G = Q^*(H)\}$ plus the set of all the induced subgraphs of all its graphs. Figs. 2 and 3 clarify the operators $Q(\cdot)$ and $Q^*(\cdot)$.

Unfortunately, a complete description of all the boundary classes for the dominating set problem is not known. At the same time, there is no a complete complexity dichotomy for the problem in the family of all finitely defined classes. A natural idea arises is to consider some "simple" its subfamily, where one can hope for obtaining a dichotomy. For example, to focus on a family of hereditary classes defined by a small number of forbidden induced fragments or small forbidden induced structures. D.V. Korobitsyn has considered in [11] hereditary classes defined by a single forbidden induced subgraph. He proved that the problem is polynomial-time solvable for $G$-free graphs if $G = P_i + O_k$, where $i \leq 4$ and $k$ is arbitrary, otherwise the problem is NP-complete. The complexity of the problem for hereditary subclasses of $P_5$-free graphs is considered in [25]. It has been proved in [25] that if $\mathcal{Y}$ consists of some graphs with five vertices and $P_5 \in \mathcal{Y}$, then the dominating set problem for $Free(\mathcal{Y})$ is polynomial whenever $\mathcal{Y} \cap \mathcal{Q} \neq \emptyset$, otherwise it is NP-complete for $Free(\mathcal{Y})$. The two mentioned results can be reformulated by means of boundary classes in a form similar to Theorem 1.

**Theorem 2.** *A hereditary class defined by a single forbidden induced subgraph is hard for the dominating set problem if it includes at least one of the classes $\mathcal{S}$, $\mathcal{T}$, $\mathcal{Q}$. Otherwise, it is easy.*

**Theorem 3.** *Let $\mathcal{Y}$ be a set of graphs with at most five vertices and $P_5 \in \mathcal{Y}$. Then the dominating set problem is polynomial-time solvable for Free($\mathcal{Y}$) if Free($\mathcal{Y}$) $\not\supseteq \mathcal{Q}$, otherwise it is NP-complete for Free($\mathcal{Y}$).*

Of course, any result on a complexity dichotomy in a family of hereditary classes defined by small forbidden induced subgraphs can be formulated in terms of an explicit description of "easy" prohibitions not in terms of boundary classes. It was done in [5,10,11,13,19,25,26] and many other papers. At the same time, the size of an answer can quickly grow with the size of the prohibitions. The notion of a boundary class helps to represent the answer more compactly.

In this paper, based on Theorems 1 and 3, we extend Theorem 3 by presenting a dichotomy for all possible subsets of forbidden induced subgraphs with at most five vertices. Namely, such a class is hard for the problem if it includes $\mathcal{S}$ or $\mathcal{T}$ or $\mathcal{Q}$, otherwise it is easy. Our way to prove this fact is to take an arbitrary five-vertex graph in $\mathcal{S} \cup \mathcal{T}$ that is not an induced subgraph of a $P_5$, an arbitrary five-vertex graph in $\mathcal{Q}$, forbid them as induced subgraphs, and show polynomial-time solvability of the problem for the resultant class.

## 4. The basic idea and the first steps of its implementation

An *independent dominating set* of a graph $G$ is a subset $V' \subseteq V(G)$, which is an independent set of $G$ and a dominating set of $G$, simultaneously. The size of a minimum independent dominating set of $G$ is said to be the *independent domination number* of $G$ denoted by $i(G)$.

Let $G$ be a connected $P_3 + P_2$-free graph, $x$ and $y$ be its adjacent vertices. Let $G_{xy}$ be the induced subgraph of $G$ obtained by deleting $x$ and $y$, simultaneously. Its vertex set can be partitioned into two parts $A_{xy}$ and $B_{xy}$, where $A_{xy} \triangleq \{z \in V(G_{xy}) | \ z \in N(x) \cup N(y)\}$. Let $\gamma'(G_{xy})$ be the minimum cardinality of the subsets of $V(G_{xy})$ that dominate $B_{xy}$. Clearly, $\gamma(G) = \min(i(G), 2 + \min_{xy \in E(G)} \gamma'(G_{xy}))$.

The independent domination number can be computed in polynomial time for $P_3 + P_2$-free graphs [17]. Therefore, to show polynomial-time solvability of the dominating set problem in a subclass $\mathcal{X} \subseteq Free(\{P_3+P_2\})$, it is sufficient to compute $\gamma'(G_e)$ in time bounded by a concrete polynomial on $|V(G)|$ for each $G \in \mathcal{X}$ and $e \in E(G)$. This reduction is our basic idea.

Clearly, $G[B_{xy}]$ is $P_3$-free, i.e. it is the disjoint union of complete graphs. If a vertex $v \in B_{xy}$ has no neighbors in $A_{xy}$, then any dominating set of $G$ must contain an element of the clique of $G[B_{xy}]$ containing $v$. Removing this clique produces an induced subgraph $H$ of the graph $G$ such that $\gamma'(H_{xy}) = \gamma'(G_{xy}) - 1$, where $H_{xy}$ is obtained from $H$ by deleting $x$ and $y$. This is why we shall always assume that each of the elements of $B_{xy}$ has a neighbor in $A_{xy}$, since computing $\gamma'(G_{xy})$ can be polynomially reduced to this case. Let $\gamma''(G_{xy})$ be the minimum cardinality of the subsets of $A_{xy}$ that dominate $B_{xy}$, and let $k_{xy}$ be the number of the connected components of $G[B_{xy}]$.

**Lemma 1.** *If $\gamma(G) \geq 4$, then $\gamma'(G_{xy}) = \min(\gamma''(G_{xy}), k_{xy})$.*

**Proof.** As $\gamma(G) \geq 4$, $G[B_{xy}]$ has at least two connected components. Clearly, $\gamma'(G_{xy}) \leq \min(\gamma''(G_{xy}), k_{xy})$. Let $D_{xy}$ be a minimum subset of $V(G_{xy})$ dominating $B_{xy}$. It must have at least two elements. If this set contains no elements of $A_{xy}$, then $|D_{xy}| = k_{xy} = \gamma'(G_{xy})$. Therefore, we may assume that $D_{xy} \cap A_{xy} \neq \emptyset$. If some vertex of a connected component of $G[B_{xy}]$ is not dominated by $D_{xy} \cap A_{xy}$, then we consider the subclique of the connected component induced by all its vertices non-dominated by $D_{xy} \cap A_{xy}$. Let $G_1, G_2, \dots, G_s$ be all the subcliques of this type. We will show that each of them has exactly one vertex. Assume that $G_i$ has at least two vertices for some $i \in \overline{1, s}$. As $D_{xy}$ dominates $B_{xy}$, it contains an element belonging to the clique of $G[B_{xy}]$ including $V(G_i)$. Recall that $D_{xy} \cap A_{xy} \neq \emptyset$ and $D_{xy}$ is minimum. By the facts from the last two sentences, there is a vertex $z = z(i) \in D_{xy} \cap A_{xy}$ and a connected component $K = K(i)$ of $G[B_{xy}]$ such that $V(G_i) \cap V(K) = \emptyset$ and $z$ has a neighbor $z' \in V(K)$. The vertices $z', z, x$ or $y$, and any two elements of $V(G_i)$ induce a $P_3 + P_2$. Hence, $G_i$ has only one vertex denoted by $b_i$. As $D_{xy}$ dominates $B_{xy}$, $b_i$ belongs to $D_{xy}$. The vertex $b_i$ has a neighbor $a_i \in A_{xy}$. Therefore, $(D_{xy} \setminus \bigcup_{i=1}^{s} \{b_i\}) \cup \bigcup_{i=1}^{s} \{a_i\}$ is a subset of $A_{xy}$ with $\gamma'(G_{xy})$ vertices dominating $B_{xy}$. Hence, $\gamma''(G_{xy}) \leq \gamma'(G_{xy})$, i.e. $\gamma''(G_{xy}) = \gamma'(G_{xy})$. ■

Let $A'_{xy}$ be the set of those elements $z \in A_{xy}$ having a neighbor in $B_{xy}$ that $B_{xy} \setminus N(z)$ is independent, $H^z \triangleq G \setminus (\{z\} \cup N(z) \cap B_{xy})$ for $z \in A'_{xy}$, $H^z_{xy}$ is obtained from $H^z$ by deleting $x$ and $y$.

**Lemma 2.** *If $\gamma(G) \geq 4$, then $\gamma''(G_{xy}) = \min_{z \in A'_{xy}} \gamma''(H^z_{xy}) + 1$.*

**Proof.** Since $\gamma(G) \geq 4$, the graph $G[B_{xy}]$ has at least two connected components. Let $D_{xy}$ be a minimum subset of $A_{xy}$ dominating $B_{xy}$. If it contains an element of $A'_{xy}$, then $\gamma''(G_{xy}) = \min_{z \in A'_{xy}} \gamma''(H^z_{xy}) + 1$. Hence, we may assume that none of the elements of $D_{xy}$ is an element of $A'_{xy}$. Since $\gamma(G) \geq 4$, $|D_{xy}| \geq 2$. To avoid an induced $P_3 + P_2$, $z$ must have neighbors only in one of the connected components of $G[B_{xy}]$ for any $z \in D_{xy}$. Indeed, any two adjacent elements of $B_{xy} \setminus N(z)$ must belong to the same clique of $G[B_{xy}]$. Hence, they, the vertex $z$, some its neighbor in $V(G[B_{xy}])$, $x$ or $y$ induce a $P_3 + P_2$. Let $z_1 \in D_{xy}$ and $z_2 \in D_{xy}$ be vertices having neighbors $z'_1$ and $z'_2$ in distinct connected components of $G[B_{xy}]$. If $a$ and $b$ are adjacent elements of $B_{xy} \setminus N(z_1)$, then $z'_1, a, b$ belong to the same clique of $G[B_{xy}]$. Otherwise, $G$ is not $P_3 + P_2$-free. Clearly, $(a, z_2) \notin E(G)$ and $(b, z_2) \notin E(G)$. Hence, $a$ and $b$, $z'_2, z_2, x$ or $y$ induce a $P_3 + P_2$. We have a contradiction with the assumption. ■

Now, let $A_{xy}$ and $B_{xy}$ mean the corresponding sets of the graph $H^z$ for the edge $xy$. Clearly, $B_{xy}$ is independent. We will assume that $H^z$ is connected, $A_{xy}$ and $B_{xy}$ are non-empty, and any element of $A_{xy}$ has a neighbor in $B_{xy}$. Additionally, we will also assume that $B_{xy}$ has no degree one vertices and $A_{xy}$ has no two vertices $u$ and $v$ such that $N(u)\backslash(A_{xy} \cup \{x, y\}) \subseteq N(v)\backslash(A_{xy} \cup \{x, y\})$. Computing $\gamma''(H^z_{xy})$ can be easily reduced to the case in polynomial time.

**Lemma 3.** *For the graph $H^z$, any element of $N(x)\backslash(\{y\} \cup N(y))$ is adjacent to any element of $N(y)\backslash\{x\}$.*

**Proof.** Assume that there are non-adjacent vertices $a \in N(x)\backslash N(y)$, $a \neq y$ and $b \in N(y)$, $b \neq x$. By the properties of $H^z$ above, there are vertices $a', b' \in B_{xy}$ such that $a' \in N(a)\backslash N(b)$ and $b' \in N(b)\backslash N(a)$. Then $a', a, b', b, y$ induce a $P_3 + P_2$. We have a contradiction. ■

## 5. Auxiliary results

### 5.1. Properties of irreducible graphs

By some results of the previous section, the dominating set problem for a hereditary class $\mathcal{X} \subseteq Free(\{P_2 + P_3\})$ can be polynomially reduced to a similar-type problem for graphs in $\mathcal{X}$, whose vertex sets were partitioned into two non-empty subsets. If $G$ is such a graph and $(A, B)$ is its partition, then we write $G \triangleq G(A, B)$. Moreover, $B$ is independent, $B$ has no degree one vertices, $A$ has no two vertices $u$ and $v$ such that $N(u)\backslash A \subseteq N(v)\backslash A$, and none of the elements of $B$ is adjacent to all the elements of $A$. The last requirement is explained by the following two reasons. If each of the elements of $B$ is adjacent to all the elements of $A$, then any element of $A$ dominates $B$. Otherwise, any subset of $A$ dominating $\tilde{B}$ also dominates $B$, where $\tilde{B} \triangleq \{x \in B |\ N(x) \neq A\}$. Moreover, $A$ is split into three subsets $A_1, A_2, A_3$ such that adding vertices $x$ and $y$, and all the edges in $\{(x, x') |\ x' \in A_1 \cup A_3\} \cup \{(y, y') |\ y' \in A_2 \cup A_3\}$ to $G$ produce a graph $G' \in \mathcal{X}$. A graph with the properties mentioned above is said to be *irreducible*.

Let $N_B(a) \triangleq \{b \in B |\ (a, b) \in E(G)\}$ for a vertex $a \in A$, and let $N_B(A') \triangleq \bigcup_{a \in A'} N_B(a)$ for a subset $A'$ of $A$. Let $G^*$ be the graph obtained from $G$ by adding the minimum possible number of edges to make $A$ to be a clique. Let $\gamma''(G)$ be the minimum cardinality of the subsets of $A$ that dominate $B$. Clearly, $\gamma(G^*) = \gamma''(G)$, as there is a minimum dominating set of $G^*$ contained in $A$.

**Lemma 4.** *Let $A' \triangleq \{a_1, a_2, \ldots, a_k\}$ be an independent subset of $A$ and $b_i \in N_B(a_i)\backslash \bigcup_{j=1, j\neq i}^{k} N_B(a_j)$. Then each of the elements of $N' \triangleq N_B(A')\backslash\{b_1, b_2, \ldots, b_k\}$ is adjacent to all the elements of $A'$.*

**Proof.** If there is an element $a_p \in A'$ having a neighbor $b \in B$, $b \neq b_p$ and an element $a_q \in A'$, $(a_q, b) \notin E(G)$, then $b, b_p, a_p, a_q, b_q$ induce a $P_3 + P_2$. Hence, every element of $N'$ must be adjacent to all the elements of $A'$. ■

**Lemma 5.** *For any three vertices $a_1, a_2, a_3 \in A$ such that $(a_1, a_2) \in E(G)$, $(a_1, a_3) \notin E(G)$, $(a_2, a_3) \notin E(G)$, we have $N_B(a_3) \subseteq N_B(a_1) \cup N_B(a_2)$. If $D$ is a minimal subset of $A$ dominating $B$, then the graph $G[D]$ is complete multipartite.*

**Proof.** Assume that there is a vertex $b \in N_B(a_3)\backslash(N_B(a_1) \cup N_B(a_2))$. To avoid an induced $P_3 + P_2$ in $G$, any element of $N_B(a_1) \otimes N_B(a_2)$ is adjacent to $a_3$. Any element of $N_B(a_1) \cap N_B(a_2)$ is adjacent to $a_3$, otherwise an element of the set, $a_1$, any element of $N_B(a_2)\backslash N_B(a_1)$, $a_3$, and $b$ induce a $P_3 + P_2$ in $G$. We obtain that $N_B(a_1) \cup N_B(a_2) \subseteq N_B(a_3)$, which is impossible by the definition of an irreducible graph.

If $G[D]$ is not complete multipartite, then there are elements $a_1, a_2, a_3$ of $D$ such that $(a_1, a_2) \in E(G)$, $(a_1, a_3) \notin E(G)$, $(a_2, a_3) \notin E(G)$. As $D$ is minimal, then there is a vertex in $N_B(a_3)\backslash(N_B(a_1) \cup N_B(a_2))$, which is impossible by the previous paragraph. ■

**Lemma 6.** *Let $A^* = A_1$ if $A_1 \neq \emptyset$, otherwise $A^* = A_3$. Let $\mathcal{K}$ be the set of all the connected components of $G[A^*]$. Then $A^*$ is independent or $\gamma''(G) = \min_{\{K \in \mathcal{K} |\ N_B(V(K) \cup (A\backslash A^*)) = B\}} \{\gamma''(G_K) |\ G_K \triangleq G[V(K) \cup (A\backslash A^*) \cup B]\}$.*

**Proof.** Let $D$ be a minimum subset of $A$ dominating $B$. We may assume that $D \cap A^*$ has at least two elements (otherwise, the equality is obvious) and $A^*$ is not independent. By Lemma 5, $N_B(A^*) = N_B(V(K))$ for any connected component $K \in \mathcal{K}$ with at least two vertices. Let $K'$ be a connected component of $G[A^*]$ with at least two vertices. If $(D \cap A^*)\backslash V(K')$ has at least two elements, then $D \cap (A\backslash A^*) \cup D \cap V(K') \cup \{a', a''\}$ also dominates $B$ by Lemma 5, where $a'$ and $a''$ are arbitrary adjacent vertices of $K'$. If $(D \cap A^*)\backslash V(K')$ has only one element $a^*$, then $D \cap V(K')$ has an element $a^{**}$, as $|D \cap A^*| \geq 2$. The set $D\backslash\{a^*\} \cup \{a\}$ dominates $B$, where $a \in V(K')$ is an arbitrary vertex adjacent to $a^{**}$. Hence, $\gamma''(G)$ must be equal to the minimum in both cases. The equality is obvious whenever $D \cap A^* \subseteq V(K')$. ■

According to Lemma 6, we may assume that each of the graphs $G[A_1]$ and $G[A_2]$ is either connected or empty. By $\gamma''_k(G)$ we denote the size of a minimum subset of $A$ dominating $B$ and inducing a complete multipartite subgraph with at most $k$ parts if one exists. If there is no such a subset, then $\gamma''_k(G) = +\infty$.

**Lemma 7.** *For each fixed $k$, $\gamma''_k(G)$ can be computed in $O(|A|^k |V(G)|^{O(1)})$ time.*

**Proof.** Let $D$ be a minimal subset of $A$ dominating $B$. By Lemma 5, $G[D]$ is complete multipartite. By Lemma 4, any element of $B$ having a neighbor in a part of $G[D]$ must be adjacent to all the elements of this part or to exactly one of its element. If $G[D]$ has at most $k$ parts, then a subset $A'$ of $A$ containing exactly one element of every part and $N_B(A')$ can be removed from $G$ such that any element of $B^*$ in the resultant graph $G_{A'}(A^*, B^*)$ has only one neighbor in $A^*$. A subset $A'$ of this type is said to be *admissible*. If there is no an admissible set, then $\gamma''_k(G) = +\infty$. Otherwise, $\gamma''_k(G)$ is equal to the minimal of the sums $|A'| + |B^*|$ over all the admissible subsets. This optimal sum can be computed in $O(|A|^k|V(G)|^{O(1)})$ time. ∎

*5.2. The classes Free({$P_3 + P_2$, orb}), Free({$P_3 + P_2$, $K_5$}), Free({$P_3 + P_2$, gem}), and Free({$P_3 + P_2$, sinker})*

**Lemma 8.** *If $G(A, B)$ is an irreducible $\{P_3 + P_2, orb\}$-free or $\{P_3 + P_2, K_5\}$-free graph, then $\gamma''(G) = \gamma''_4(G)$.*

**Proof.** Let $D$ be a minimum subset of $A$ dominating $B$. As $D$ is minimum, for each $a \in D$, there is a vertex $b_a \in N_B(a)\setminus \bigcup_{v\in D\setminus\{a\}} N_B(v)$. Hence, $G[D]$ is complete multipartite with at most four parts by Lemma 5. Therefore, $\gamma''(G) = \gamma''_4(G)$. ∎

**Lemma 9.** *The dominating set problem for $\{P_3 + P_2, gem\}$-free graphs can be polynomially reduced to the same problem for $\{P_5, gem\}$-free graphs*

**Proof.** Let $G(A, B)$ be an irreducible $\{P_3 + P_2, gem\}$-free graph. By Lemma 6, we may assume that $G[A_1]$ is connected and $G[A_2]$ is connected or an empty graph. There is no a vertex in $B$ adjacent to a vertex in $A_3$ and to a vertex in $A_1 \cup A_2$. It is a corollary of Lemma 3 and the fact that $G'$ is *gem*-free. Let $b$ be an arbitrary vertex of $B$ adjacent to a vertex in $A_1$ and to a vertex in $A_2$. By Lemma 3 and the fact that $G'$ is gem-free, $A_3$ must be empty. The vertex $b$ must be adjacent to all the elements of $A_1$. Otherwise, by the connectivity of $G[A_1]$, there are adjacent vertices $a_1^1, a_1^2 \in A_1$ and $a_2 \in A_2$ such that $(b, a_1^1) \in E(G), (b, a_2) \in E(G), (b, a_1^2) \notin E(G)$. By Lemma 3, $(a_1^1, a_2)$ and $(a_2^1, a_2)$ are some edges of $G$. Then $b, a_1^1, a_1^2, a_2, x$ induce a *gem* in $G'$. The set $A_2$ is independent, otherwise $G[A_2]$ is connected and $b$ must be adjacent to all the vertices of $A$, which is impossible by the definition of an irreducible graph. So, any element of $B$ having a neighbor in $A_1$ and a neighbor in $A_2$ must be adjacent to all the elements of $A_1$.

We may assume that $|A_1| \geq 2$, otherwise $G[A]$ is bipartite and $\gamma''(G)$ can be computed in polynomial time by Lemma 7. As $G$ is irreducible, there are two elements $a', a'' \in A_1$ such that $N_B(a')\setminus N_B(a'')$ is not empty. Hence, any element of $N_B(a')\setminus N_B(a'')$ can be adjacent to none of the elements of $A_2$. Therefore, any subset of $A$ dominating $B$ must contain an element of $A_1$. Hence, $\gamma''(G) = \gamma''(G[A_1 \cup N_B(A_1)]) + \gamma''(G[A_2 \cup (N_B(A_2)\setminus N_B(A_1))]) + \gamma''(G[A_3 \cup N_B(A_3)])$. This equality also holds if there is no an element of $B$ adjacent to an element of $A_1$ and to an element of $A_2$. Further, we will explain how computing $\gamma''(G[A_1 \cup N_B(A_1)]), \gamma''(G[A_2 \cup (N_B(A_2)\setminus N_B(A_1))]), \gamma''(G[A_3 \cup N_B(A_3)])$ can be reduced to solving the dominating set problem for $\{P_5, gem\}$-free graphs. Without loss of generality, we will consider the graph $H \triangleq G[A_1 \cup N_B(A_1)]$.

It is well-known that for any connected $P_4$-free graph $H'$ with at least two vertices there are its induced subgraphs $H''$ and $H'''$ such that $H' = H'' \times H'''$ [6]. Hence, there is an unique decomposition $H' = H'_1 \times H'_2 \times \cdots H'_p$, where $H'_1, \ldots, H'_k$ are disconnected and each of the graphs $H'_{k+1}, \ldots, H'_p$ is the one-vertex graph. Moreover, this decomposition can be computed in polynomial time [6].

We may assume that $H$ is irreducible. By Lemma 6, we may also assume that $G[A_1]$ is connected. This graph is $P_4$-free, otherwise some four its vertices and $x$ induce a *gem* in $G'$. Any graph whose vertex set can be partitioned into a clique and an independent set is $P_5$-free [8]. Hence, $H$ is $\{P_5, gem\}$-free whenever $A_1$ is a clique. Assume that $A_1$ is not a clique. There is a decomposition $H = H_1 \times H_2 \times \cdots \times H_{k+1}$ for $k > 1$, where $H_1, \ldots, H_k$ are disconnected, $H_{k+1}$ is either disconnected or a clique.

Let us show that if there is a vertex $b \in B$ adjacent to a vertex $a' \in V(H_i)$ with $i \leq k$ and to a vertex $a'' \in A_1\setminus V(H_i)$, then $b$ is adjacent to all the vertices of $H_i$. Let $K$ be an arbitrary connected component of $H_i$ such that $a' \notin K$. The vertex $b$ must be adjacent to all the vertices of $K$, otherwise $b, a', a'', x$, and some element of $K$ induce a *gem* in $G'$. Hence, $b$ is adjacent to all the vertices of $K$. As $H$ is irreducible, for each $i \leq k$, there is a vertex in $N_B(V(H_i))$ that is not adjacent to each of the elements of $A_1\setminus V(H_i)$. Hence, $\gamma''(H) = \sum_{i=1}^{k+1} \gamma''(G[V(H_i) \cup V_i])$, where, for each $i$, $V_i = N_B(V(H_i))\setminus \bigcup_{j=1, j\neq i}^{k+1} N_B(V(H_j))$. This formula and Lemma 6 give a polynomial-time reduction to $\{P_5, gem\}$-free graphs. ∎

**Lemma 10.** *The dominating set problem for $\{P_3 + P_2, sinker\}$-free graphs can be solved in polynomial time.*

**Proof.** Let $G(A, B)$ be an irreducible $\{P_3 + P_2, sinker\}$-free graph. By Lemma 3 and the fact that $G$ is *sinker*-free, at least one of the sets $A_1$ and $A_2$ is independent. Let $A_2$ be independent. If $A_3 \neq \emptyset$, then $H \triangleq G[A_1 \cup A_3]$ is connected by Lemma 3. If $A_3$ is empty and $A_1$ is independent, then $G[A]$ is bipartite and $\gamma''(G) = \gamma''_2(G)$. If $A_3$ is empty and $A_1$ is not independent, then $H$ is also connected by Lemma 6. If the graph $G[A]$ is $K_5$-free, then $\gamma''(G) = \gamma''_4(G)$ by Lemma 5. Hence, by Lemma 7, $\gamma''(G)$ can be computed in polynomial time. We will assume that $H$ is a connected graph containing a $K_4$.

Let $Q$ be a maximum clique of $H$ and $|Q| \geq 4$. Any element of $V(H)\setminus Q$ adjacent to an element of $Q$ must have exactly $|Q| - 1$ neighbors in $Q$, as $G'$ is *sinker*-free. Since $G'$ is *sinker*-free and $H$ is connected, there is no an element of $V(H)\setminus Q$ that has no neighbors in $Q$. Hence, if $a_1$ and $a_2$ belong to $V(H)\setminus Q$, then they are adjacent if and only if $N(a_1) \cap Q \neq N(a_2) \cap Q$. Thus, $H$ is a complete multipartite graph with at least four parts.

There is no a vertex in $B$ adjacent to an element of $A_1 \cup A_3$ and to an element of $A_2$, simultaneously. Indeed, if such a vertex $b$ and its neighbors $a_1 \in A_1 \cup A_3$, $a_2 \in A_2$ exist, then there is a clique $Q'$ of $H$ with at least three vertices such that $a_1 \in Q'$. By Lemma 3, $a_2$ is adjacent to all the vertices of $Q'$. To avoid an induced *sinker* in $G$, the vertex $b$ must be adjacent to at least two vertices of $Q'$. Hence, $b$, $a_2$, two vertices in $Q' \cap N(b)$, and $x$ induce a *sinker* in $G'$. We have a contradiction with the existence of $b$. Hence, $\gamma''(G) = \gamma''(G[A_1 \cup A_3 \cup N_B(A_1 \cup A_3)]) + \gamma''(G[A_2 \cup N_B(A_2)])$.

We may assume that the subgraph induced by $A_1 \cup A_3 \cup N_B(A_1 \cup A_3)$ is also irreducible. Let us show that there is no an element of $B$ adjacent to vertices in distinct parts of $H$. Let $b'$ be a vertex of this type, $a'_1$ and $a'_2$ be its neighbors in distinct parts of $H$. There is a clique $Q''$ of $H$ containing exactly one representative of each of the parts of $H$ that also contains $a'_1$ and $a'_2$. Clearly, $|Q''| \geq 4$. To avoid an induced *sinker* in $G'$, $b'$ must be adjacent to all the elements of $Q''$. If two vertices $a' \notin Q''$ and $a'' \in Q''$ belong to the same part of $H$, then $b'$ must be adjacent to $a'$. Otherwise, $b'$, any two elements of $Q''\backslash\{a''\}$, $a'$, and $x$ induce a *sinker* in $G'$. Therefore, $b'$ must be adjacent to all the vertices of $A_1 \cup A_3$, which is impossible by the definition of an irreducible graph. So, $\gamma''(G[A_1 \cup A_3 \cup N_B(A_1 \cup A_3)]) = \sum_{i=1}^{k} \gamma''(G[V_i \cup N_B(V_i)])$, where $V_1, \ldots, V_k$ are all the parts of $H$. By Lemma 7, the sum and $\gamma''(G[A_2 \cup N_B(A_2)])$ can be computed in polynomial time, as $\gamma''(G[A_2 \cup N_B(A_2)]) = \gamma''_1(G[A_2 \cup N_B(A_2)])$ and $\gamma''(G[V_i \cup N_B(V_i)]) = \gamma''_1(G[V_i \cup N_B(V_i)])$ for each $i$. ■

### 5.3. The classes $Free(\{P_3 + P_2, K_{1,4}\})$, $Free(\{P_3 + P_2, fork\})$, $Free(\{P_3 + P_2, cricket\})$, $Free(\{P_3 + P_2, bull\})$, $Free(\{P_3 + P_2, kite\})$, $Free(\{P_3 + P_2, dart\})$

**Lemma 11.** *The dominating set problem for $Free(\{P_3 + P_2, K_{1,4}\})$ can be polynomially reduced to the same problem for $Free(\{P_5, K_{1,4}\})$.*

**Proof.** Let $G(A, B)$ be an irreducible $\{P_3 + P_2, K_{1,4}\}$-free graph. Let us show that $G^*$ is $\{P_5, K_{1,4}\}$-free. Clearly, $G^*$ is $P_5$-free. Suppose that $G^*$ has a $K_{1,4}$ induced by vertices $a, b_1, b_2, b_3, b_4$, where $(a, b_1), (a, b_2), (a, b_3), (a, b_4)$ are the edges of this $K_{1,4}$. The vertex $a$ belongs to $A$, otherwise non-adjacent vertices $b_1$ and $b_2$ belong to $A$, which is impossible, as $A$ is a clique of $G^*$. There are at least three vertices among $b_1, b_2, b_3, b_4$ belonging to $B$. These three vertices, $a$, $x$ or $y$ induce a $K_{1,4}$ in $G'$. We have a contradiction. ■

**Lemma 12.** *The dominating set problem for $Free(\{P_3 + P_2, fork\})$ can be polynomially reduced to the same problem for $Free(\{P_5, fork\})$.*

**Proof.** Let $G(A, B)$ be an irreducible $\{P_3+P_2, fork\}$-free graph. To prove the lemma, we only need to show that $G^*$ is *fork*-free. Suppose that $G^*$ has a *fork* induced by vertices $x_1, x_2, x_3, y_1, y_2$, where $(x_1, y_1), (x_2, y_1), (y_1, y_2), (y_2, x_3)$ are the edges of the *fork*. The vertices $x_1, x_2, x_3$ must belong to $B$ and the vertices $y_1, y_2$ must belong to $A$, as $B$ is an independent set and $A$ is a clique of $G^*$. The graph $G$ must have the edge $(y_1, y_2)$, otherwise $x_1, x_2, x_3, y_1, y_2$ induce a $P_3+P_2$ in $G'$. Then $G$ is not *fork*-free. We have a contradiction. ■

**Lemma 13.** *For each of the classes $Free(\{P_3 + P_2, cricket\})$ and $Free(\{P_3 + P_2, bull\})$, the dominating set problem can be polynomially reduced to the same problem for $Free(\{P_5, fork\})$.*

**Proof.** Let $\mathcal{X}$ be one of the two classes, $G(A, B)$ be an irreducible graph in $\mathcal{X}$. Let $a_1$ and $a_2$ be arbitrary elements of $A$ having a common neighbor $b \in B$. We will show that $a_1$ and $a_2$ belong to exactly one of the sets $A_1, A_2, A_3$. Assume the opposite. We may also assume that $(a_1, x)$ and $(a_2, y)$ are the edges of $G$. By Lemma 3, $a_1$ and $a_2$ are adjacent. There are elements $b' \in N_B(a_1)\backslash N_B(a_2)$ and $b'' \in N_B(a_2)\backslash N_B(a_1)$. Hence, $a_1, a_2, b', b, b''$ induce a *bull* in $G$. If $(a_1, y) \notin E(G)$ or $(a_2, x) \notin E(G)$, then either $b, b'', a_1, a_2, y$ or $b, b', a_1, a_2, x$ induce a *cricket* in $G'$. The case $(a_1, y) \in E(G)$ and $(a_2, x) \in E(G)$ is impossible by our assumption. Hence, the neighborhood of each of the elements of $B$ is included in one of the sets $A_1, A_2, A_3$. Hence, $\gamma''(G) = \gamma''(G_1) + \gamma''(G_2) + \gamma''(G_3)$, where $G_i$ is a subgraph of $G$ induced by $A_i \cup N_B(A_i)$. Similar to the reasonings of the previous lemma, it is easy to check that all the graphs $G_1^*, G_2^*, G_3^*$ are $\{P_5, fork\}$-free. So, the lemma holds. ■

**Lemma 14.** *For each of the classes $Free(\{P_3+P_2, kite\})$ and $Free(\{P_3+P_2, dart\})$, the dominating set problem can be polynomially reduced to the same problem for $Free(\{P_5, kite\})$ and $Free(\{P_5, dart\})$, respectively.*

**Proof.** Let $G(A, B)$ be an irreducible $\{P_3 + P_2, kite\}$-free or $\{P_3 + P_2, dart\}$-free graph. We will show that $G[A_1 \cup A_3]$ and $G[A_2 \cup A_3]$ are $P_3$-free. Assume that $G[A_1 \cup A_3]$ contains vertices $a_1, a_2, a_3 \in A$ such that $(a_1, a_2) \in E(G)$, $(a_2, a_3) \in E(G)$, and $(a_1, a_3) \notin E(G)$. We will show that $N_B(a_1) \cap N_B(a_2) = \emptyset$. Assume the opposite. Then an element of $N_B(a_1)\backslash N_B(a_2)$, an element of $N_B(a_1) \cap N_B(a_2)$, $a_1, a_2, x$ induce a *dart* in $G'$. The equality $N_B(a_1)\backslash N_B(a_2) = N_B(a_3)\backslash N_B(a_2)$ holds, otherwise an element of $N_B(a_1)\backslash(N_B(a_2) \cup N_B(a_3))$ or an element of $N_B(a_3)\backslash(N_B(a_2) \cup N_B(a_1))$, $a_1, a_2, a_3, x$ induce a *kite* in $G'$. If $(a_1, y)$ and $(a_2, y)$ are not edges of $G'$, then an element of $N_B(a_1) \cap N_B(a_2)$, $a_1, a_2, x, y$ induce a *kite* in $G'$. If $(a_1, y) \notin E(G')$ and $(a_2, y) \in E(G')$, then an element of $N_B(a_1)\backslash N_B(a_2)$, $a_1, a_2, x, y$ induce a *kite* in $G'$. If $(a_1, y) \in E(G')$, then $(a_3, y) \in E(G')$ by Lemma 3. There is an element $b^* \in N_B(a_1) \cap N_B(a_2)$ that is not adjacent to $a_3$, otherwise $N_B(a_1) \subseteq N_B(a_3)$, which is impossible by the definition of an irreducible graph. Then $b^*, a_1, a_3, x, y$ induce a *kite* in $G'$. So, $N_B(a_1) \cap N_B(a_2)$ must be empty. Similarly, $N_B(a_2) \cap N_B(a_3)$ must be empty. Hence, the set $N_B(a_2)\backslash(N_B(a_1) \cup N_B(a_3))$ is not empty and any its element, $a_1, a_2, a_3, x$ induce a *dart*. The set $N_B(a_1)\backslash(N_B(a_2) \cup N_B(a_3))$ is also not empty, and any its element, $a_1, a_2, a_3, x$ induce a *kite*. We have a contradiction with the initial assumption.

Let $A_3$ be non-empty. Then $A_1$ and $A_2$ must be cliques by Lemma 3 and the fact that $G[A_1 \cup A_3]$ and $G[A_2 \cup A_3]$ are $P_3$-free. Similarly, $A_3$ is a clique whenever $A_1 \cup A_2 \neq \emptyset$. Hence, if $A_1 \cup A_2 \neq \emptyset$, then $A$ is a clique and $G$ is $\{P_5, dart\}$-free or $\{P_5, kite\}$-free. Let $A_1 \cup A_2$ be empty. By Lemmas 6 and 7, we may assume that $G[A_3]$ is connected. As $G[A_3]$ is $P_3$-free, then this graph is complete. Hence, $G$ is also $\{P_5, dart\}$-free or $\{P_5, kite\}$-free.

Let $A_3$ be empty. If $G[A]$ is bipartite, then $\gamma''(G) = \gamma_2''(G)$ and $\gamma''(G)$ can be computed in polynomial time by Lemma 7. Otherwise, by Lemma 6, we may assume that $G[A_1]$ is complete and $G[A_2]$ is complete or an empty graph. If $G[A_2]$ is complete, then $G$ is $\{P_5, dart\}$-free or $\{P_5, kite\}$-free by Lemma 3. If $A_2$ is independent and $\min(|A_1|, |A_2|) = 1$, then $G[A]$ is also bipartite or complete. Assume that $A_1$ and $A_2$ have at least two vertices and $A_2$ is independent.

Let us show that there is no a vertex in $B$ adjacent to a vertex in $A_1$ and a vertex in $A_2$. Let $b$ be such a vertex. Those neighbors of $b$ must be adjacent by Lemma 3. If $G'$ is *dart*-free, then $b$ is adjacent to all the vertices of $A_1$. Then, for each $a', a'' \in A_1$, an element of $N_B(a') \backslash N_B(a'')$, $b$, $a'$, $a''$, $x$ induce a *dart* in $G'$. If $G'$ is *kite*-free, then $b$ must also be adjacent to all the elements of $A_1$, otherwise there are vertices $a_1, a_2 \in A_1$ such that $(a_1, b) \notin E(G)$, $(a_2, b) \in E(G)$. The set $A_2$ contains an element $a_3$ adjacent to $b$. Clearly, $N_B(a_1) \cap N_B(a_2) = \emptyset$, otherwise an element of $N_B(a_1) \cap N_B(a_2)$, $a_1, a_2, x, y$ induce a *kite* in $G'$. Hence, an element of $N_B(a_1) \backslash N_B(a_3)$, $a_1, a_2, a_3, b$ induce a *kite* in $G$.

So, $\gamma''(G) = \gamma''(G[A_1 \cup N_B(A_1)]) + \gamma''(G[A_2 \cup N_B(A_2)])$ and $\gamma''(G[A_2 \cup N_B(A_2)]) = \gamma_1''(G[A_2 \cup N_B(A_2)])$. Hence, by Lemma 7, computing $\gamma''(G)$ can be polynomially reduced to computing $\gamma''(G[A_1 \cup N_B(A_1)])$, where $G[A_1 \cup N_B(A_1)]$ is $\{P_5, dart\}$- or $\{P_5, kite\}$-free. ∎

### 5.4. The class Free($\{fork, K_3 + K_2\}$)

Two non-adjacent vertices $x$ and $y$ of a graph are said to be *quasi-twins* if $N(x) \subseteq N(y)$. If $x$ and $y$ are quasi-twins of a graph $G$, then $\gamma(G) = \gamma(G \backslash \{y\})$. Hence, the dominating set problem for a hereditary class can be polynomially reduced to the same problem for its graphs without quasi-twins.

**Lemma 15.** *Let $G$ be a connected $\{fork, K_3+K_2\}$-free graph without quasi-twins, and let $G \notin$ Free($\{P_5\}$). Let $P \triangleq (x_1, x_2, \ldots, x_k)$ be a maximum induced path of $G$ if $G \in$ Free($\{P_7\}$), otherwise let $P$ be a maximal induced path of $G$ with at least seven vertices. If a vertex $x \in (\bigcup_{v \in V(P)} N(v)) \backslash V(P)$ has a neighbor $y \notin \bigcup_{v \in V(P)} N(v)$, then $x$ must be adjacent to all the vertices of $P$.*

**Proof.** Assume the opposite. The path $P$ must have at least five vertices. The vertex $x$ cannot have exactly one neighbor in $V(P)$, otherwise this neighbor must be an end of $P$ contradicting the maximality of $P$. If $x$ has more than two neighbors in $V(P)$, then they must be consecutive in $P$ to avoid a *fork* induced by $y, x$, some neighbors $a'$ and $a''$ of $x$ on $P$, and a neighbor $a''' \in V(P) \backslash N(x)$ of $a''$. Nevertheless, $G$ contains an induced *fork*, as $x$ cannot be adjacent to all the vertices of $P$. Hence, $x$ must have exactly two neighbors on $P$. They must be adjacent, as $G$ is *fork*-free. Moreover, $k \leq 6$, as $G$ is $K_3+K_2$-free. Hence, $P$ is maximum. We may assume that $x_2$ and $x_3$ are the neighbors in the case $k = 5$, $x_3$ and $x_4$ are the neighbors for $k = 6$, since $G$ is $K_3 + K_2$-free. Suppose that $k = 5$. As $G$ has no quasi-twins, there is a vertex $x' \in N(x_5) \backslash N(x_3)$. As $P$ is maximum, $x'$ must have a neighbor in $V(P) \backslash \{x_5\}$. As $G$ is $\{fork, K_3 + K_2\}$-free, $N(x') \cap V(P) = \{x_1, x_5\}$ or $N(x') \cap V(P) = \{x_1, x_2, x_5\}$ or $N(x') \cap V(P) = \{x_1, x_2, x_4, x_5\}$ or $N(x') \cap V(P) = \{x_1, x_4, x_5\}$. Hence, $x'$ and $y$ cannot be adjacent. Due to the maximality of $P$, $x'$ must be adjacent to $x$ in the case, when $N(x') \cap V(P) = \{x_1, x_2, x_5\}$. It is also true in all the three remaining cases, as $G$ is $K_3 + K_2$-free. Hence, $G$ contains an induced *fork*. We have a contradiction. The case $k = 6$ can be considered similarly. ∎

**Lemma 16.** *The dominating set problem for $\{fork, K_3 + K_2\}$-free graphs can be polynomially reduced to the same problem for $\{P_5, fork, K_3 + K_2\}$-free graphs*

**Proof.** Let $G$ be a connected $\{fork, K_3+K_2\}$-free graph without quasi-twins containing an induced $P_5$. Let $P \triangleq (x_1, \ldots, x_k)$ be a maximum induced path of $G$ if $G \in$ *Free*($\{P_7\}$), otherwise let $P$ be a maximal induced path of $G$ with at least seven vertices. It can be computed in polynomial time. Assume that $V(P)$ is a dominating set of $G$. If $|V(P)| \leq 8$, then $\gamma(G) \leq 8$. Suppose that $|V(P)| \geq 9$ and $G$ is distinct from a simple path and a cycle. Hence, there is a vertex $x \in (\bigcup_{v \in V(P)} N(v)) \backslash V(P)$. Since $G$ is *fork*-free and $P$ is maximal, $x$ has at least two neighbors on $P$. If it has exactly two neighbors, then $x$ must be adjacent to the ends of $P$. As $G$ is $\{fork, K_3 + K_2\}$-free and it is not a cycle, an element of $V(G) \backslash V(P)$ has at least three neighbors on $P$. We may assume that $x$ has at least three neighbors on $P$. Let $x_s$ be the first neighbor of $x$ on $P$ counting from $x_1$. Clearly, $s \leq 2$, otherwise $x, x_s, x_{s+1}, x_{s-1}, x_{s-2}$ or $x, x_s, x_{s+1}, x_1, x_2$ induce a *fork* or a $K_3 + K_2$, respectively. If $s = 2$, then $N(x) \cap \{x_4, \ldots, x_k\}$ has at most two vertices and they must be adjacent, as $G$ is *fork*-free. It is impossible, as $G$ is $\{fork, K_3 + K_2\}$-free. If $s = 1$ and $(x, x_2) \notin E(G)$, then $N(x) \cap \{x_4, \ldots, x_k\}$ is a clique with at most two vertices, as $G$ is *fork*-free. It is also impossible, as $G$ is $\{fork, K_3 + K_2\}$-free. If $(x, x_1) \in E(G)$ and $(x, x_2) \in E(G)$, then no two vertices of $V(P) \backslash (N(x) \cup \{x_3\})$ are adjacent, as $G$ is $K_3 + K_2$-free. Hence, this set has at most two elements, as $G$ is $\{fork, K_3 + K_2\}$-free. In other words, $x$ is adjacent to at least $|V(P)| - 3$ vertices of $P$. Therefore, each of the elements of $V(G) \backslash V(P)$ is adjacent to only the ends of $P$ or to at least $|V(P)| - 3$ its vertices. Hence, $\{x_1, x_2, x_3, x_4, x\}$ is a dominating set of $G$.

Now, assume that $V(P)$ is not a dominating set of $G$ and $\gamma(G) \geq 11$. Let $V_1$ be the set of those elements in $\bigcup_{v \in V(P)} N(v) \backslash V(P)$ that are adjacent to all the vertices of $P$ and have a neighbor outside $\bigcup_{v \in V(P)} N(v)$. By the previous lemma, $V_1$ is not empty. Let $V_2$ be the set of those elements in $\bigcup_{v \in V(P)} N(v) \backslash V(P)$ that are not adjacent to all the vertices of $P$, $V_3 \triangleq V(G) \backslash \bigcup_{v \in V(P)} N(v)$. The set $V_3$ is not empty. By the previous lemma, none of the elements of $V_2$ has a neighbor

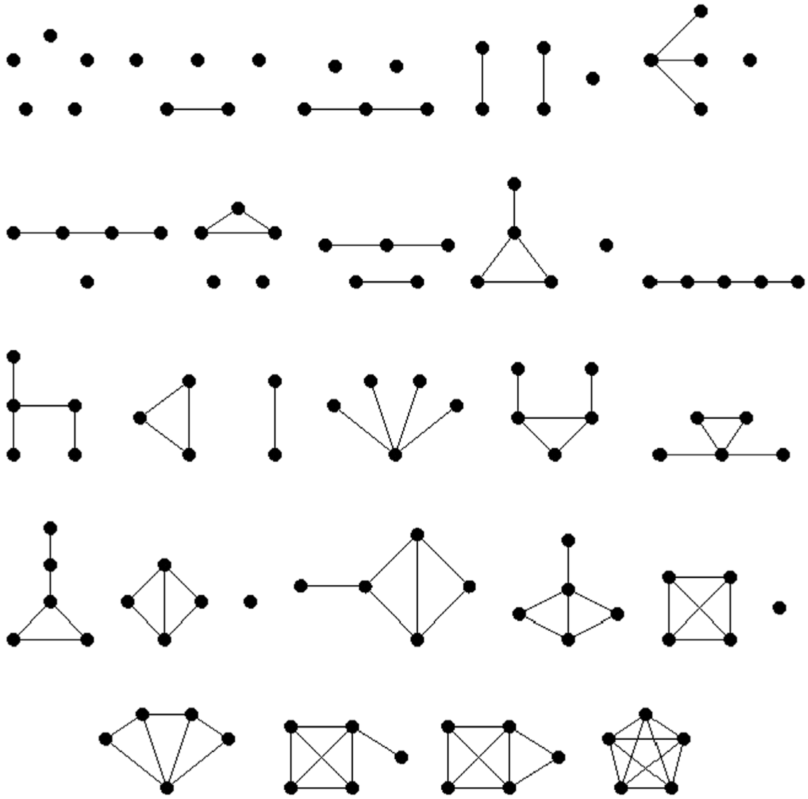

**Fig. 4.** All the five-vertex graphs in $\mathcal{S} \cup \mathcal{T} \cup \mathcal{Q}$.

outside $\bigcup_{v\in V(P)} N(v)$. It is easy to check that each of the elements of $V_1$ is adjacent to each of the elements of $V_2$, as $G$ is $\{fork, K_3 + K_2\}$-free. As $G$ is connected and *fork*-free, any element of $V_3$ has a neighbor in $V_1$.

Let $H$ be a graph obtained from $G$ by removing any vertex of $P$. We will show that there is a minimum dominating set of $H$ containing an element of $V_1$. Hence, this set must be a dominating set of $G$. Therefore, $\gamma(H) = \gamma(G)$. Let $D$ be a minimum dominating set of $H$ containing no elements of $V_1$. The set $D \cap \bigcup_{v\in V(P)} N(v)$ has at most one element, otherwise any element of $V(P) \cap V(H)$, any element of $V_1$, and $V_3 \cap D$ form a dominating set of $G$ containing an element of $V_1$. To avoid an induced *fork* in $G$, $N(z') \cap V_1 = N(z'') \cap V_1$ for any two adjacent vertices $z' \in V_3$ and $z'' \in V_3$. Hence, each of the elements of $N(z) \cap V_1$ is adjacent to each of the elements of $N(z) \cap V_3$ for any $z \in V_3$. Hence, we may assume that $V_3 \cap D$ is independent. Otherwise some its element can be replaced by any its neighbor in $V_1$ such that the resultant set is a dominating set of $G$ containing an element of $V_1$. Let $V_3 \cap D \triangleq \{z_1, \ldots, z_p\}$. For any $i$, there is a vertex $y_i \in V_1$ such that $y_i \in N(z_i) \setminus \bigcup_{j=1, j\neq i}^{p} N(z_j)$, otherwise the idea of a replacement also works. As $\gamma(G) \geq 11$, then $|V_3 \cap D| \geq 9$. By Ramsey's theorem, some three vertices among $y_1, \ldots, y_p$ are pairwise non-adjacent or some four vertices form a clique. The first alternative is impossible, as $G$ is *fork*-free. Suppose that $y_1, y_2, y_3, y_4$ constitute a clique of $H$. Let us show that $(D \setminus \{z_1, z_2, z_3, z_4\}) \cup \{y_1, y_2, y_3, y_4\}$ is a dominating set of $G$. Clearly, $V_3 \cap \bigcup_{i=1}^{4} N(z_i) \subseteq \bigcup_{i=1}^{4} N(y_i)$. If $y \in V_1 \setminus \{y_1, \ldots, y_p\}$ has a neighbor in $\{z_1, z_2, z_3, z_4\}$, then $y$ must have a neighbor in $\{y_1, y_2, y_3, y_4\}$, otherwise $G$ is not $K_3 + K_2$.

So, deleting vertices in long induced paths in $\{fork, K_3 + K_2\}$-free graphs gives a polynomial-time reduction to $\{P_5, fork, K_3 + K_2\}$-free graphs. ■

## 6. Main result

The following result was proved in [25].

**Lemma 17.** *The dominating set problem for a hereditary class $\mathcal{X} \subseteq Free(\{G + O_1\})$ can be polynomially reduced to the same problem for $\mathcal{X} \cap Free(\{G\})$.*

Recall that the classes $\mathcal{S}, \mathcal{T}, \mathcal{Q}, \mathcal{Q}^*$ defined in the third section are boundary for the dominating set problem. To simplify understanding the proof of the last theorem, a complete list of all the five-vertex graphs in $\mathcal{S} \cup \mathcal{T} \cup \mathcal{Q}$ is presented in the figure below. Its completeness can be verified by the list of all five-vertex graphs in [29] and the fact that any graph in $\mathcal{Q}$ can contain none of the graphs $C_4, C_5, K_2 + K_2$, none of the complements of $K_2 + O_3, K_3 + O_2$ as an induced subgraph (see Fig. 4).

**Theorem 4.** *Let $\mathcal{X}$ be defined by a set of forbidden induced subgraphs with at most five vertices. The dominating set problem is NP-complete for $\mathcal{X}$ if it includes at least one of the classes $\mathcal{S}$, $\mathcal{T}$, $\mathcal{Q}$. Otherwise, the problem can be solved in polynomial time for $\mathcal{X}$.*

**Proof.** Let $\mathcal{Y}$ be a minimal set such that $\mathcal{X} = Free(\mathcal{Y})$. By Theorem 1, the dominating set problem is NP-complete for $\mathcal{X}$ if it includes $\mathcal{S}$ or $\mathcal{T}$ or $\mathcal{Q}$. Assume that $\mathcal{S} \not\subseteq \mathcal{X}, \mathcal{T} \not\subseteq \mathcal{X}, \mathcal{Q} \not\subseteq \mathcal{X}$. Hence, $\mathcal{Y}$ contains a forest. It must be an induced subgraph of a $P_5 + O_2$ or a $P_3 + P_2$ or a $fork + O_1$. If $\mathcal{Y}$ contains an induced subgraph of a $P_5 + O_2$, then $\mathcal{X}$ is easy for the problem by Theorem 3 and Lemma 17. Suppose that a $P_3 + P_2$ belongs to $\mathcal{Y}$. Let $G$ be a graph in $\mathcal{Q}$ containing at most five vertices. Taking into account the presented list of all the five-vertex graphs in $\mathcal{S} \cup \mathcal{T} \cup \mathcal{Q}$, it is easy to check that $G$ is an induced subgraph of at least one of the graphs $O_5, P_4 + O_2, K_5 + O_2, orb, sinker, kite, dart, cricket, fork + O_1, K_{1,4}, gem + O_1, bull$. By Lemmas 7–14 and 17, the problem for $\mathcal{X}$ can be polynomially reduced to the same problem for the classes $Free(\{P_4\}), Free(\{P_5, O_5\}), Free(\{P_5, K_5\}), Free(\{P_5, orb\}), Free(\{P_5, sinker\}), Free(\{P_5, sinker\}), Free(\{P_5, kite\}), Free(\{P_5, dart\}), Free(\{P_5, cricket\}), Free(\{P_5, fork\}), Free(\{P_5, K_{1,4}\}), Free(\{P_5, gem\}), Free(\{P_5, bull\})$. Hence, by Theorem 3, $\mathcal{X}$ is easy for the problem. Assume that $\mathcal{Y}$ contains an induced subgraph of a $fork + O_1$ and it contains none of the induced subgraphs of a $P_5 + O_2$. As $\mathcal{T} \not\subseteq \mathcal{X}$, $\mathcal{Y}$ contains a graph in $\mathcal{T}$, which is an induced subgraph of a $K_3 + K_2$ or a $hammer + O_1$ or a $bull$. The classes $Free(\{fork, bull\})$ and $Free(\{fork, hammer\})$ are easy for the problem [25]. By these facts, Lemmas 16 and 17, and Theorem 3, the problem is polynomial-time solvable for $\mathcal{X}$. ■

There is an interesting detail concerning the previous theorem. Namely, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{Q}$ do not provide a dichotomy in the family of all the hereditary classes defined by at most six vertices. That is, assuming $P \neq NP$, it is not true that such a class is hard for the dominating set problem if and only if it includes $\mathcal{S}$ or $\mathcal{T}$ or $\mathcal{Q}$. Indeed, none of the classes $\mathcal{S}$, $\mathcal{T}$, $\mathcal{Q}$ is contained in $Free(\{K_{1,4}, P_3 + P_3\})$. Hence, it should be an easy case for the dominating set problem assuming the correctness of those fact. But, by Theorem 1, the class is a hard case for the problem, since $Free(\{K_{1,4}, P_3 + P_3\}) \supseteq \mathcal{Q}^*$.

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